TIMOSHENKO BEAM THEORY

EXACT SOLUTION FOR BENDING,

SECOND-ORDER ANALYSIS, AND STABILITY

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Abstract: This paper presents an exact solution to the Timoshenko beam theory (TBT) for bending, second-order analysis, and stability. The TBT covers cases associated with small deflections based on shear deformation considerations, whereas the Euler–Bernoulli beam theory neglects shear deformations. A material law (a moment–shear force–curvature equation) combining bending and shear is presented, together with closed-form solutions based on this material law. A bending analysis of a Timoshenko beam was conducted, and buckling loads were determined on the basis of the bending shear factor. First-order element stiffness matrices were calculated. Finally second-order element stiffness matrices were deduced on the basis of the same principle.

Keywords: Timoshenko beam; Moment – shear force – curvature equation; Closed-form solutions; Stability; Second-order element stiffness matrix

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1. Introduction

First-order analysis of the Timoshenko beam is routine in practice: the principle of virtual work yields accurate results and is easy to apply. Unfortunately, second-order analysis of the Timoshenko beam cannot be modeled with the principle of virtual work. Pirrotta et al. [1] presented an analytical solution for a Timoshenko beam subjected to a uniform load distribution with various boundary conditions. Sang-Ho et al. [2] presented a nonlinear finite element analysis formulation for shear in reinforced concrete Beams: that formulation utilizes an equilibrium-driven shear stress function. Abbas et al. [3] suggested a two-node finite element for analyzing the stability and free vibration of Timoshenko beam: interpolation functions for displacement field and beam rotation were exactly calculated by employing total beam energy and its stationing to shear strain. Hayrullah et al. [4] performed a buckling analysis of a nano sized beam by using Timoshenko beam theory and Eringen's nonlocal elasticity theory: the vertical displacement function and the rotation function are chosen as Fourier series. Onyia et al. [5] presented a finite element formulation for the determination of the critical buckling load of unified beam element that is free from shear locking using the energy method: the proposed technique provides a unified approach for the stability analysis of beams with any end conditions. Jian-Hua Yin [6] proposed a closed-form solution for reinforced Timoshenko beam on elastic foundation subjected to any pressure loading: the effects of geosynthetic shear stiffness and tension modulus and the location of the pressure loading were investigated. In stability analysis Timoshenko and Gere [7] proposed formulas to account for shear stiffness by means of calculation of buckling loads of the associated Euler–Bernoulli beams. Hu et al.

[8] used matrix structural analysis to derive a closed-form solution of the second-order element stiffness matrix: the buckling loads of single-span beams were also determined.

In this paper a material law combining bending and shear is presented: this material law describes the relationship between the curvature, the bending moment, the bending stiffness, the shear force, and the shear stiffness. Based on this material law closed-form expressions of efforts and deformations are derived, as well as first-order and second-order element stiffness matrices. Stability analysis is conducted, the buckling lengths of single-span systems being determined on the basis of shear factor.

2. Materials and Methods

2.1 Governing equations

2.1.1 Statics

The sign conventions adopted for the loads, bending moments, shear forces, and displacements are illustrated in Figure 1
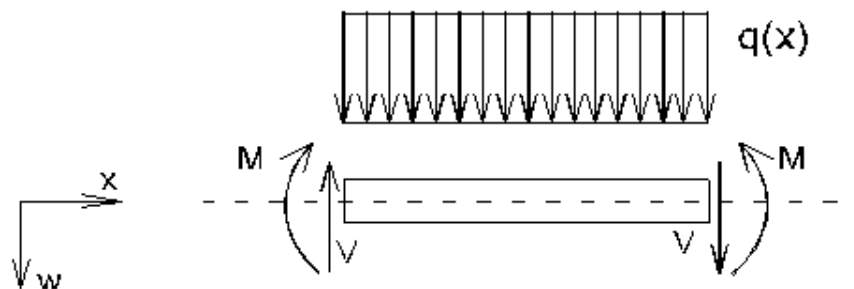
Figure 1.   Sign convention for loads, bending moments, shear forces, and displacements

Specifically, M(x) is the bending moment in the section, V(x) is the shear force, w(x) is the deflection, and q (x) is the distributed load in the positive downward direction.

In first-order analysis the equations of static equilibrium of an infinitesimal element are as follows:

$$V(x) = \frac{dM(x)}{dx} \tag{1}$$

$$\frac{d^2M(x)}{dx^2} = -q(x) \tag{2}$$

According to the Timoshenko beam theory, the bending moment and the shear force are related to the deflection and the rotation (positive in clockwise) of cross section φ(x) as follows:

$$M(x) = -EI\frac{d\varphi(x)}{dx} \tag{3}$$

$$V(x) = \kappa GA \times \left(\frac{dw(x)}{dx} - \varphi(x)\right) \tag{4}$$

In these equations E is the elastic modulus, I is the second moment of area, κ is the shear correction factor, G is the shear modulus, and A is the cross section area.

Equations (1), (3), and (4) also apply in second-order analysis.

2.1.2 Material and geometric equations

Equation (4) can be formulated as follows

$$\frac{dw(x)}{dx} = \varphi(x) + \frac{V(x)}{\kappa GA} \tag{5}$$

Differentiating both sides of Equation (5) with respect to x results in the following equation:

$$\frac{d^2w(x)}{dx^2} = \frac{d\varphi(x)}{dx} + \frac{d}{dx}\left(\frac{V(x)}{\kappa GA}\right) \tag{6}$$

Substitution of Equation (3) into Equation (6) yields the following:

$$\frac{d^2w(x)}{dx^2} = -\frac{M(x)}{EI} + \frac{d}{dx}\left(\frac{V(x)}{\kappa GA}\right) \tag{7}$$

Equation (7) yields the following material law combining bending and shear for the Timoshenko beam:

$$\frac{d^2w(x)}{dx^2} + \frac{M(x)}{EI} - \frac{d}{dx}\left(\frac{V(x)}{\kappa GA}\right) = 0 \tag{8}$$

(8) For a beam with a constant cross section along segments, substituting Equation (1) into Equation (8) yields,

$$\frac{d^2w(x)}{dx^2} + \frac{M(x)}{EI} - \frac{1}{\kappa GA}\frac{d^2M(x)}{dx^2} = 0 \tag{9}$$

For yiel For a continuously varying cross section along segments of the beam, substituting Equation (1) into Equation (8) yields,

$$\frac{d^2w(x)}{dx^2} + \frac{M(x)}{EI(x)} - \frac{1}{\kappa GA(x)}\frac{d^2M(x)}{dx^2} + \frac{1}{(\kappa GA(x))^2}\times\frac{d\kappa GA(x)}{dx}\times\frac{dM(x)}{dx} = 0 \tag{10}$$

In the case of non-uniform heating, the material law (Equation (9)) becomes:

$$\frac{d^2w(x)}{dx^2} + \frac{M(x)}{EI} - \frac{1}{\kappa GA}\times\frac{d^2M(x)}{dx^2} + \frac{\alpha_T\Delta T}{d} = 0 \tag{11}$$

Rotation of the cross section (Equations (1) and (5)) yields the following:

$$\varphi(x) = \frac{dw(x)}{dx} - \frac{1}{\kappa GA}\frac{dM(x)}{dx} \tag{12}$$

The governing equations are the equation of static equilibrium (Equation (2)) and the material law (Equation (9) or Equation (10)).
The shear force – bending moment relationship (Equation (1)), and the rotation of the cross section (Equation (12)) are used to satisfy the boundary conditions and the continuity equations.
Equations (5) to (12) apply as well in first-order analysis and in second-order analysis.

2.1.3 Summary of Timoshenko and Euler-Bernoulli beam equations

Table 1 summarizes the fundamental Timoshenko beam equations and compares them to the Euler–Bernoulli beam equations.

Table 1. Summary of equations for Timoshenko and Euler–Bernoulli beams

	Timoshenko beam	Euler-Bernoulli beam
Static equilibrium	$V(x) = \dfrac{dM(x)}{dx}$	$V(x) = \dfrac{dM(x)}{dx}$

Geometry	$\varphi(x) = \dfrac{dw(x)}{dx} - \dfrac{1}{\kappa GA} \times \dfrac{dM(x)}{dx}$	$\varphi(x) = \dfrac{dw(x)}{dx}$
Material law	$\dfrac{d^2w(x)}{dx^2} + \dfrac{M(x)}{EI} - \dfrac{d}{dx}\left(\dfrac{V(x)}{\kappa GA}\right) = 0$	$\dfrac{d^2w(x)}{dx^2} + \dfrac{M(x)}{EI} = 0$
Non uniform heating	$\dfrac{d^2w(x)}{dx^2} + \dfrac{M(x)}{EI} - \dfrac{1}{\kappa GA}\dfrac{d^2M(x)}{dx^2}$ $+ \dfrac{\alpha_T \Delta T}{d} = 0$	$\dfrac{d^2w(x)}{dx^2} + \dfrac{M(x)}{EI} + \dfrac{\alpha_T \Delta T}{d} = 0$

The bending shear factor is defined as follows:

$$\alpha = \frac{EI}{l^2 \kappa GA} \tag{13}$$

2.2 First-order analysis of the Timoshenko beam
2.2.1 Beam without elastic Winkler foundation

The application of Equation (2) yields the following formulation of the bending moment:

$$M(x) = -\iint q(x)dxdx + C_1 x + C_2 \tag{14}$$

Substitution of Equations (2) and (14) into the material law (Equation (9) for a beam with constant cross section) yields:

$$EI \times w(x) = -\iint \left[\frac{EI}{\kappa GA} q(x) + M(x)\right] dxdx + C_3 x + C_4 \tag{15}$$

The integration constants Ci (i = 1, 2, 3, and 4) are determined using the boundary conditions and continuity equations and combining the deflections, the cross section rotations (Equation (12)), the bending moments, and the shear forces (Equation (1)).

2.2.2 Element stiffness matrix

The sign conventions for bending moments, shear forces, displacements, and rotations adopted for use in determining the element stiffness matrix in local coordinates is illustrated in Figure 2.
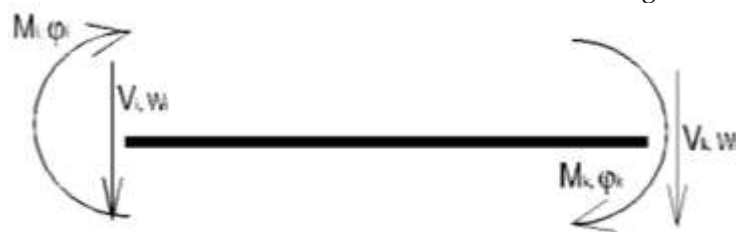
Figure 2. Sign conventions for moments, shear forces, displacements, and rotations for stiffness matrix

Let us define following vectors,

$$\vec{S} = \left[V_i; M_i; V_k; M_k\right]^T \tag{16}$$

$$\vec{V} = \left[w_i; \varphi_i; w_k; \varphi_k\right]^T \tag{17}$$

The element stiffness matrix in local coordinates of the Timoshenko beam is denoted by K_{Tbl}. The relationship between the aforementioned vectors is as follows:

$$\vec{S} = K_{Tbl} \times \vec{V} \tag{18}$$

Application of Equations (12), (14) and (15) yields the following (there is no distributed load):

$$M(x) = C_1 x + C_2 \tag{19}$$

$$EI \times w(x) = -\frac{C_1}{6} x^3 - \frac{C_2}{2} x^2 + C_3 x + C_4 \tag{20}$$

$$EI \times \varphi(x) = -\frac{C_1}{2} x^2 - C_2 x + C_3 - \alpha l^2 C_1 \tag{20a}$$

Considering the sign conventions adopted for bending moments and shear forces in general (see Figure 1) and for bending moments and shear forces in the element stiffness matrix (see Figure 2), we can set following static compatibility boundary conditions in combination with Equations (1) and (19):

$$V_i = -V(x=0) = -C_1 \tag{21}$$

$$M_i = M(x=0) = C_2 \tag{22}$$

$$V_k = V(x=l) = C_1 \tag{23}$$

$$M_k = -M(x=l) = -C_1 l - C_2 \tag{24}$$

Considering the sign conventions adopted for the displacements and rotations in general (see Figure 1) and for displacements and rotations in the member stiffness matrix (see Figure 2), we can set following geometric compatibility boundary conditions in combination with Equations (20) and (20a):

$$w(x=0) = w_i \rightarrow EI \times w_i = C_4 \tag{25}$$

$$\varphi(x=0) = \varphi_i \rightarrow EI \times \varphi_i = -\alpha l^2 C_1 + C_3 \tag{26}$$

$$w(x=l) = w_k \rightarrow EI \times w_k = -l^3/6 \times C_1 - l^2/2 \times C_2 + l C_3 + C_4 \tag{27}$$

$$\varphi(x=l) = \varphi_k \rightarrow EI \times \varphi_k = (-1/2 - \alpha) \times l^2 C_1 - l C_2 + C_3 \tag{28}$$

The combination of Equations (16) to (18) and Equations (21) to (28) yields the first-order element stiffness matrix of the Timoshenko beam:

$$K_{Tbl} = EI \times \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -l & -1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 0 & 1 \\ -\alpha l^2 & 0 & 1 & 0 \\ -l^3/6 & -l^2/2 & l & 1 \\ (-1/2-\alpha)l^2 & -l & 1 & 0 \end{bmatrix}^{-1} \tag{28a}$$

The development of Equation (28a) yields the following widely known formulation of the first-order element stiffness matrix of the Timoshenko beam:

$$K_{Tbl} = \begin{bmatrix} \dfrac{12}{1+\varphi}\dfrac{EI}{l^3} & \dfrac{6}{1+\varphi}\dfrac{EI}{l^2} & \dfrac{-12}{1+\varphi}\dfrac{EI}{l^3} & \dfrac{6}{1+\varphi}\dfrac{EI}{l^2} \\ & \dfrac{4+\varphi}{1+\varphi}\dfrac{EI}{l} & \dfrac{-6}{1+\varphi}\dfrac{EI}{l^2} & \dfrac{2-\varphi}{1+\varphi}\dfrac{EI}{l} \\ & & \dfrac{12}{1+\varphi}\dfrac{EI}{l^3} & \dfrac{-6}{1+\varphi}\dfrac{EI}{l^2} \\ & sym & & \dfrac{4+\varphi}{1+\varphi}\dfrac{EI}{l} \end{bmatrix} \tag{29}$$

Where

$$\varphi = \frac{12EI}{l^2 \kappa GA} \tag{30}$$

Assuming the presence of a hinge at the right end, the sign convention for bending moments, shear forces, displacements, and rotations is illustrated in Figure 3.

Figure 3. Sign conventions for moments, shear forces, displacements, and rotations for stiffness matrix

The vectors of Equations (16) and (17) become

$$\vec{S} = \left[V_i ; M_i ; V_k \right]^T \tag{31}$$

$$\vec{V} = \left[w_i ; \varphi_i ; w_k \right]^T \tag{32}$$

The element stiffness matrix can be expressed as follows:

$$K_{Tbl} = \begin{bmatrix} \dfrac{3}{1+\varphi'}\dfrac{EI}{l^3} & \dfrac{3}{1+\varphi'}\dfrac{EI}{l^2} & \dfrac{-3}{1+\varphi'}\dfrac{EI}{l^3} \\ & \dfrac{3}{1+\varphi'}\dfrac{EI}{l} & \dfrac{-3}{1+\varphi'}\dfrac{EI}{l^2} \\ & sym & \dfrac{3}{1+\varphi'}\dfrac{EI}{l^3} \end{bmatrix} \tag{33}$$

Where

$$\varphi' = \frac{3EI}{l^2 \kappa GA} \tag{34}$$

2.2.3   Beam resting on an elastic Winkler foundation

For a beam resting on a Winkler foundation with stiffness K_w, Equation (2) of static equilibrium becomes,

$$\frac{d^2 M(x)}{dx^2} - K_w w(x) = -q(x) \tag{35}$$

Differentiating Equation (35) twice with respect to x, combined with the material law (Equation (9)), yields the following:

$$\frac{d^4 M(x)}{dx^4} - \frac{K_w}{\kappa G A} \frac{d^2 M(x)}{dx^2} + \frac{K_w}{EI} M(x) = -\frac{d^2 q(x)}{dx^2} \tag{36}$$

The solution of Equation (36) yields the formulation of M(x) with four integration constants. Combining M(x) with Equation (35) yields the following equations:

$$K_w w(x) = \frac{d^2 M(x)}{dx^2} + q(x) \tag{37}$$

$$K_w \frac{dw(x)}{dx} = \frac{d^3 M(x)}{dx^3} + \frac{dq(x)}{dx} \tag{38}$$

The application of Equation (1) to M(x) yields the shear force, and the combination of M(x) with Equations (12) and (38) yields the rotations of the cross section.

2.3   Second-order analysis of the Timoshenko beam
2.3.1   Beam without elastic Winkler foundation

A beam with constant cross section is considered. The axial force N (positive in tension) is assumed to be constant in segments of the beam. The equation of static equilibrium is as follows:

$$\frac{d^2 M(x)}{dx^2} + N \frac{d^2 w(x)}{dx^2} = -q(x) \tag{39}$$

Combining Equation (39) with the material law (Equation (9)) yields the following:

$$(1 + \frac{N}{\kappa G A}) \frac{d^2 M(x)}{dx^2} - \frac{N}{EI} M(x) = -q(x) \tag{40}$$

The solution of Equation (40) yields M(x), which contains the integration constants C_1 and C_2. The combination of M(x) with Equation (39) yields following equations:

$$N \frac{dw(x)}{dx} = -\frac{dM(x)}{dx} - \int q(x)dx + C_3 \tag{41}$$

$$Nw(x) = -M(x) - \iint q(x)dxdx + C_3 x + C_4 \tag{42}$$

The transverse force T(x) is determined as follows:

$$T(x) = V(x) + N \frac{dw(x)}{dx} \tag{43}$$

By applying Equations (1) and (41), the transverse force T(x) can be expressed as follows:

$$T(x) = -\int q(x)dx + C_3 \tag{44}$$

The combination of M(x), Equations (12) and (41) yields the rotation of the cross section as follows:

$$N\varphi(x) = -(1 + \frac{N}{\kappa G A}) \frac{dM(x)}{dx} - \int q(x)dx + C_3 \tag{44a}$$

2.3.2 Element stiffness matrix

The sign conventions for bending moments, transverse forces, displacements, and rotations adopted for use in determining the element stiffness matrix in local coordinates is illustrated in Figure 4.
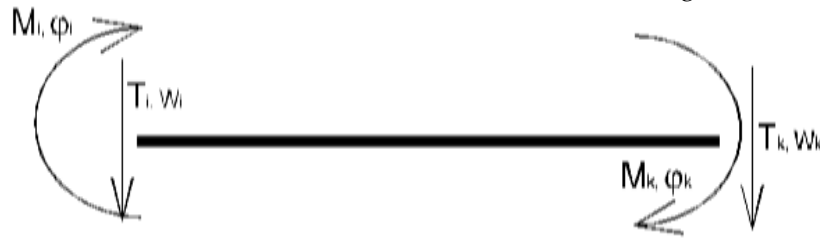
Figure 4. Sign conventions for bending moments, transverse forces, displacements, and rotations for stiffness matrix

Let us define following vectors,

$$\vec{S} = \left[T_i ; M_i ; T_k ; M_k \right]^T \tag{45}$$

$$\vec{V} = \left[w_i ; \varphi_i ; w_k ; \varphi_k \right]^T \tag{46}$$

we set

$$N = k \frac{EI}{l^2} \tag{47}$$

The combination of Equation (40), the bending shear factor (Equation (13)), Equation (47), and the absence of a distributed load yields the following:

$$(1 + k\alpha) \frac{d^2 M(x)}{dx^2} - \frac{k}{l^2} M(x) = 0 \tag{48}$$

Case 1: Compressive force with kα > -1

The solution of Equation (48) is as follows, with the parameter ξ1 defined as shown:

$$M(x) = A_1 \cos \xi_1 \frac{x}{l} + B_1 \sin \xi_1 \frac{x}{l} \tag{49}$$

$$\xi_1 = \sqrt{\frac{-k}{1 + k\alpha}} \tag{50}$$

Combining Equations (39) and (49) yields the following:

$$Nl \frac{dw(x)}{dx} = A_1 \xi_1 \sin \xi_1 \frac{x}{l} - B_1 \xi_1 \cos \xi_1 \frac{x}{l} + NlC_1 \tag{51}$$

$$Nw(x) = -A_1 \cos \xi_1 \frac{x}{l} - B_1 \sin \xi_1 \frac{x}{l} + NC_1 x + ND_1 \tag{52}$$

The combination of Equation (12) for the rotation of the cross section with Equations (49) and (51) yields the following, with the parameter ξ2 defined as shown:

$$Nl\varphi(x) = A_1 \xi_2 \sin \xi_1 \frac{x}{l} - B_1 \xi_2 \cos \xi_1 \frac{x}{l} + NlC_1 \tag{53}$$

$$\xi_2 = \sqrt{-k(1 + k\alpha)} \tag{54}$$

By applying Equations (1), (43), (49), and (51), the transverse force T(x) yields:

$$T(x) = NC_1 \tag{55}$$

Considering the static compatibility boundary conditions (Equations (21) to (34)), whereby the shear forces are replaced by the transverse forces, the geometric compatibility boundary conditions, and Equations (49) to (55), the following equations are obtained:

$$T_i = -T(x=0) \rightarrow T_i = -NC_1 \tag{56a}$$

$$M_i = M(x=0) = A_1 \tag{56b}$$

$$T_k = T(x=l) \rightarrow T_k = NC_1 \tag{56c}$$

$$M_k = -M(x=l) = -A_1 \cos \xi_1 - B_1 \sin \xi_1 \tag{56d}$$

$$w(x=0) = w_i \rightarrow N \times w_i = -A_1 + ND_1 \tag{56e}$$

$$\varphi(x=0) = \varphi_i \rightarrow N \times \varphi_i = -B_1 \xi_2 / l + NC_1 \tag{56f}$$

$$w(x=l) = w_k \rightarrow N \times w_k = -A_1 \cos \xi_1 - B_1 \sin \xi_1 + NlC_1 + ND_1 \tag{56g}$$

$$\varphi(x=l) = \varphi_k \rightarrow N \times \varphi_k = A_1 \xi_2 / l \times \sin \xi_1 - B_1 \xi_2 / l \times \cos \xi_1 + NC_1 \tag{56h}$$

The combination of Equations (18), (45), (46), and (56a) to (56h) yields the element stiffness matrix:

$$K_{Tbl} = EI \frac{k}{l^2} \times \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\cos \xi_1 & -\sin \xi_1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -\frac{\xi_2}{l} & 1 & 0 \\ -\cos \xi_1 & -\sin \xi_1 & l & 1 \\ \frac{\xi_2}{l} \sin \xi_1 & -\frac{\xi_2}{l} \cos \xi_1 & 1 & 0 \end{bmatrix}^{-1} \tag{57}$$

The development of Equation (57) yields the following formulation:

$$K_{Tbl} = \begin{bmatrix} K_{11} \frac{EI}{l^3} & K_{12} \frac{EI}{l^2} & -K_{11} \frac{EI}{l^3} & K_{12} \frac{EI}{l^2} \\ & K_{22} \frac{EI}{l} & -K_{12} \frac{EI}{l^2} & K_{24} \frac{EI}{l} \\ & & K_{11} \frac{EI}{l^3} & -K_{12} \frac{EI}{l^2} \\ sym & & & K_{22} \frac{EI}{l} \end{bmatrix} \tag{58}$$

$K_{11} = -k\xi_2\sin\xi_1/\Delta$
$K_{12} = -k(1-\cos\xi_1)/\Delta$
$K_{22} = k(\cos\xi_1-1/\xi_2\sin\xi_1)/\Delta$
$K_{24} = k(-1+1/\xi_2\sin\xi_1)/\Delta$
$\Delta = 2-2\cos\xi_1-\xi_2\sin\xi_1$

Accordingly, the element stiffness matrix of the beam with a hinge present is as follows:

$$K_{Tbl} = \begin{bmatrix} K_{11} \frac{EI}{l^3} & K_{12} \frac{EI}{l^2} & -K_{11} \frac{EI}{l^3} \\ & K_{12} \frac{EI}{l} & -K_{12} \frac{EI}{l^2} \\ sym & & K_{11} \frac{EI}{l^3} \end{bmatrix} \tag{59}$$

$K_{11} = -k\times\xi_2\cos\xi_1/\Delta$
$K_{12} = -k\times\sin\xi_1/\Delta$
$\Delta = \sin\xi_1-\xi_2\cos\xi_1$

Case 2: Tensile force or compressive force with kα < -1

The solution of Equation (48) is as follows, with the parameter ξ3 defined as shown:

$$M(x) = A_2 \cosh \xi_3 \frac{x}{l} + B_2 \sinh \xi_3 \frac{x}{l} \tag{60}$$

$$\xi_3 = \sqrt{\frac{k}{1+k\alpha}} \tag{61}$$

Combining Equations (39) and (60) and integrating with respect to x yields the following:

$$Nl\frac{dw(x)}{dx} = -A_2\xi_3 \sinh \xi_3 \frac{x}{l} - B_2\xi_3 \cosh \xi_3 \frac{x}{l} + NlC_2 \tag{62}$$

$$Nw(x) = -A_2 \cosh \xi_3 \frac{x}{l} - B_2 \sinh \xi_3 \frac{x}{l} + NC_2 x + ND_2 \tag{63}$$

Combination of Equation (12) for the rotation of the cross section, with Equations (60) and (62) yields the following:

$$Nl\varphi(x) = -A_2\xi_4 \sinh \xi_3 \frac{x}{l} - B_2\xi_4 \cosh \xi_3 \frac{x}{l} + NlC_2 \tag{64}$$

$$\xi_4 = \pm\sqrt{k(1+k\alpha)} \tag{65}$$

The parameter ξ4 (Equation (65)) has a positive value in the case of tension and a negative value in the case of compression with kα < -1.

By applying Equations (1), (43), (60), and (62), the transverse force T(x) yields the following:

$$T(x) = NC_2 \tag{66}$$

Considering the static compatibility boundary conditions (Equations (21) to (24)), whereby the shear forces are replaced by the transverse forces, and the geometric compatibility boundary conditions, and Equations (60) to (66), the element stiffness matrix can be expressed as follows:

$$K_{Tbl} = EI\frac{k}{l^2} \times \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\cosh \xi_1 & -\sinh \xi_1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -\dfrac{\xi_4}{l} & 1 & 0 \\ -\cosh \xi_1 & -\sinh \xi_1 & l & 1 \\ -\dfrac{\xi_4}{l}\sinh \xi_1 & -\dfrac{\xi_4}{l}\cosh \xi_1 & 1 & 0 \end{bmatrix}^{-1} \tag{67}$$

The development of Equation (67) yields the following formulation:

$$K_{Tbl} = \begin{bmatrix} K_{11}\dfrac{EI}{l^3} & K_{12}\dfrac{EI}{l^2} & -K_{11}\dfrac{EI}{l^3} & K_{12}\dfrac{EI}{l^2} \\ & K_{22}\dfrac{EI}{l} & -K_{12}\dfrac{EI}{l^2} & K_{24}\dfrac{EI}{l} \\ & & K_{11}\dfrac{EI}{l^3} & -K_{12}\dfrac{EI}{l^2} \\ sym & & & K_{22}\dfrac{EI}{l} \end{bmatrix}$$

$$K_{11} = k\xi_4\sinh\xi_3/\Delta$$

$$K_{12} = -k(1-\cosh\xi_3)/\Delta$$

$$K_{22} = k(\cosh\xi_3-1/\xi_4\sinh\xi_3)/\Delta \tag{68}$$

$$K_{24} = k(-1+1/\xi_4\sinh\xi_3)/\Delta$$

$$\Delta = 2-2\cosh\xi_3+\xi_4\sinh\xi_3$$

Accordingly, the element stiffness matrix of the beam with a hinge present is as follows:

$$
K_{Tbl} = \begin{bmatrix} K_{11}\dfrac{EI}{l^3} & K_{12}\dfrac{EI}{l^2} & -K_{11}\dfrac{EI}{l^3} \\[2ex] & K_{12}\dfrac{EI}{l} & -K_{12}\dfrac{EI}{l^2} \\[2ex] sym & & K_{11}\dfrac{EI}{l^3} \end{bmatrix}
$$

$$K_{11} = -k\times\xi_4\cosh\xi_3/\Delta \tag{69a}$$
$$K_{12} = -k\times\sinh\xi_3/\Delta \tag{69b}$$
$$\Delta = \sinh\xi_3 - \xi_4\cosh\xi_3 \tag{70}$$

2.3.3 Beam resting on an elastic Winkler foundation with an axial load

In the equation of static equilibrium, the axial force N (the value of which is positive in tension) is assumed to be constant in segments of the beam. The stiffness of the Winkler foundation is denoted by Kw.

$$\frac{d^2M(x)}{dx^2} + N\frac{d^2w(x)}{dx^2} - K_w w(x) = -q(x) \tag{71}$$

The combination of Equation (71) with the material law (Equation (9)) yields the following:

$$K_w w(x) = (1 + \frac{N}{\kappa GA})\frac{d^2M(x)}{dx^2} - \frac{N}{EI}M(x) + q(x) \tag{72}$$

Differentiating twice both sides of Equation (72) with respect to *x* and combining the result with the material law (Equation (9)) yields the following:

$$(1 + \frac{N}{\kappa GA})\frac{d^4M(x)}{dx^4} - (\frac{N}{EI} + \frac{K_w}{\kappa GA})\frac{d^2M(x)}{dx^2} + \frac{K_w}{EI}M(x) = -\frac{d^2q(x)}{dx^2} \tag{73}$$

The solution of Equation (73) yields the formulation of M(x) with four integration constants. From M(x), combined with Equations (1) and (72), the shear force V(x) and the deflection w(x) can be deduced. The application of Equations (12) and (43) yields the transverse forces T(x) and the rotations of the cross section φ(x).

2.3.4 Stability of the Timoshenko beam

The equation of static equilibrium (Equation (39)), the material law (Equation (9) or Equation (10)), the bending moment M(x), the deflection w(x), the rotation of the cross section φ(x), and the transverse force T(x) are used to determine the buckling load, at which the transverse loading q(x) is equal zero.
For kα > -1, Equations (13) and (47) define the parameters α and k, and Equations (49) to (55) are considered to satisfy the boundary conditions and continuity conditions.
For kα < -1, Equations (60) to (66) are considered to satisfy the boundary conditions and continuity conditions.
The resulting eigenvalue problem is solved to determine the buckling loads.

3. Results

3.1 First-order analysis of Timoshenko beams
3.1.1 Beams subjected to uniformly distributed load
Example 1: Let us calculate the responses of a beam with a constant cross section, simply supported at its ends, and subjected to a uniformly distributed load as shown in Figure 5.
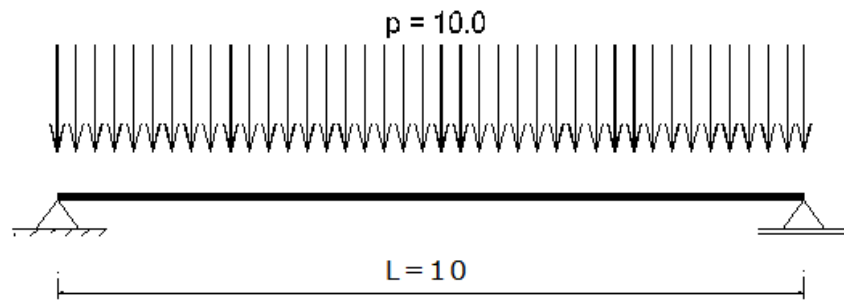
Figure 5. Beam subjected to a uniformly distributed load

The characteristics are as follows: p = 10 kN/m, L = 10 m, b = 0.3 m, H = 0.5m, E = 34.5×106 kN/m², ν = 0.3, and κ = 5/6

The application of Equations (14) and (15), and the appropriate boundary conditions yields the closed-form equation of the deflection curve as follows:

$$EI \times w(x) = \frac{p}{24} x^4 - \frac{pl}{12} x^3 - \alpha \frac{pl^2}{2} x^2 + (\frac{1}{24} + \frac{\alpha}{2}) pl^3 x \qquad (74)$$

The calculation of the deflections with the principle of virtual work is presented in Appendix A.
Details of the results are presented in the supplementary material "Spreadsheet S1."
Closed-form expressions of single-span systems for various support conditions are presented in Appendix B.
Table 2 lists the results obtained with the principle of virtual work and those obtained in the present study.

Table 2. Deflections of the beam: Principle of virtual work, and present study

Node position	Principle of virtual work (exact results)	Present study
0.0	0.00000	0.00000
1.0	0.00382	0.00382
2.0	0.00722	0.00722
3.0	0.00988	0.00988
4.0	0.01157	0.01157
5.0	0.01215	0.01215
6.0	0.01157	0.01157
7.0	0.00988	0.00988
8.0	0.00722	0.00722
9.0	0.00382	0.00382
10.0	0.00000	0.00000

3.1.2 Beams subjected to concentrated load

Example 2:   Let us calculate the responses of a beam with a constant cross section fixed at its left end, simply supported at its right end, and subjected to a concentrated load, as shown in Figure 6.
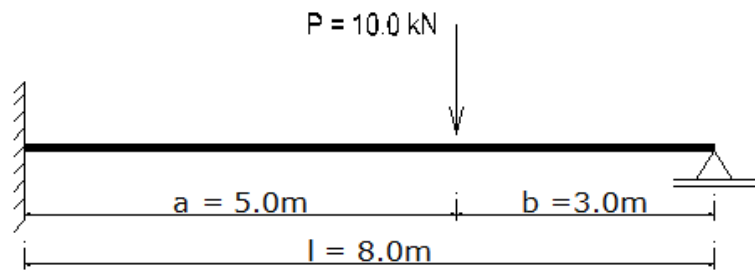
Figure 6. Beam under concentrated load

Details of the analysis and results are listed in Appendix C and in the supplementary material "Spreadsheet S2."

Table 3 lists the moments at the fixed end (M_{FEM}) and under the load (M_{uL}) for different values of the bending shear factor, calculated as described in this paper and according to the principle of virtual work (exact values).

Table 3. Moments in beam subjected to a concentrated load

$\alpha = EI/\kappa GAl^2 =$	0.0000	0.0250	0.0500	0.0750	0.1000	0.1250	0.1500
Calculations as described in this paper							
$M_{FEM} =$	-12.89	-11.99	-11.21	-10.52	-9.92	-9.38	-8.89
$M_{uL} =$	13.92	14.25	14.55	14.80	15.03	15.23	15.42
Calculations according to the principle of virtual work (exact values)							
$M_{FEM} =$	-12.89	-11.99	-11.21	-10.52	-9.92	-9.38	-8.89
$M_{uL} =$	13.92	14.25	14.55	14.80	15.03	15.23	15.42

3.2 Second-order analysis of Timoshenko beams

3.2.1 Stability of beams

We determine the buckling loads of single-span beams with various support conditions for different values of the bending shear factor.

Details of the analysis and results are listed in Appendix D and in the supplementary materials "Spreadsheet S3," "Spreadsheet S4," "Spreadsheet S5," and "Spreadsheet S6."

The buckling load Ncr is defined as follows:

$$N_{cr} = -\pi^2 EI /(\beta l)^2 \tag{75}$$

Values of the buckling factor α are listed in Table 4. Closed-form expressions of the matrices expressing the boundary conditions are presented in Appendix D. To determine the buckling loads the determinants of those matrices are set to zero. Closed-form expressions of the buckling factors for a pinned–pinned beam and a fixed–free beam are also presented in Appendix D.

Table 4. Buckling factors for Timoshenko beam with various support conditions

$\alpha = EI/\kappa GAl^2 =$	0.000	0.025	0.050	0.075	0.100	0.1250	0.150
Both ends simply supported (SS–SS)							
$\beta =$	1.0000	1.1163	1.2220	1.3192	1.4096	1.4946	1.5750
Left end fixed and right end simply supported (F–SS)							
$\beta =$	0.6992	0.8716	1.0146	1.1392	1.2510	1.3530	1.4474
Left end fixed and right end free (F–FR)							
$\beta =$	2.0000	2.0608	2.1198	2.1772	2.2332	2.2877	2.3410
Both ends fixed (F–F)							
$\beta =$	0.5000	0.7048	0.8623	0.9951	1.1122	1.2181	1.3155

We recall that the exact values of the buckling factors β for α= 0.0 (corresponding to the Euler–Bernoulli beam) for the conditions SS–SS, F–SS, F–FR and F–F are 1.00, 0.700, 2.00, and 0.500, respectively.

3.2.2 Beams subjected to uniformly distributed load

Example 3: Let us calculate the responses of a beam with a constant cross section, simply supported at its ends and subjected to a uniformly distributed load and an axial force, as shown in Figure 7.
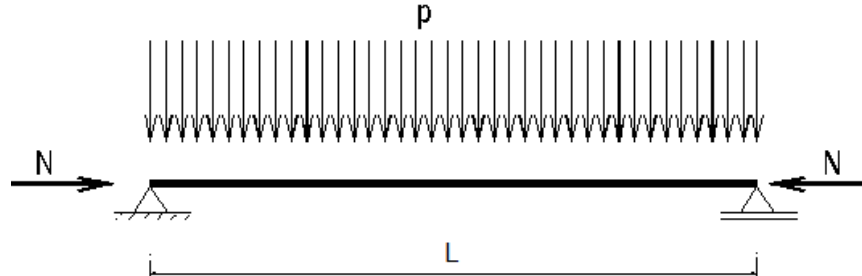
Figure 7.   Beam subjected to a uniformly distributed load and an axial force

The moments at position L/2 are calculated for different values of k (Equation (47)) and α (Equation (13)). Closed-form expressions of the moments are presented in Appendix E.

The formulations for the moments at position L/2 in the case of a compressive force with kα > -1 (Equation (76)) and in the case of a tensile force or compressive force with kα < -1 (Equation (77)) are presented below.

The limits of k corresponding to buckling are listed in the supplementary material "Spreadsheet S3".

Details of the results are presented in the supplementary material "Spreadsheet S7".

Table 5 lists the values of the moments.

$$M(x = l/2) = \left[1 - \cos\frac{\xi_1}{2} + \frac{\cos\xi_1 - 1}{\sin\xi_1}\sin\frac{\xi_1}{2}\right] \times \frac{pl^2}{k} \tag{76}$$

$$M(x = l/2) = \left[1 - \cosh\frac{\xi_3}{2} + \frac{\cosh\xi_3 - 1}{\sinh\xi_3}\sinh\frac{\xi_3}{2}\right] \times \frac{pl^2}{k} \tag{77}$$

Table 5.   Moments at position L/2 for different values of shear factor and axial load

α = 0.025		α = 0.05		α = 0.075		α = 0.10	
k =	MMP /pl² =	k =	MMP /pl² =	k =	MMP /pl² =	k =	MMP /pl² =
-7.50	2.4474	-6.50	7.8536				
-6.00	0.5278	-6.00	1.3947	-5.50	4.2593		
-5.00	0.3453	-5.00	0.5242	-5.00	1.0825	-4.50	1.3635
-4.00	0.2561	-4.00	0.3215	-4.00	0.4316	-4.00	0.6553
-3.00	0.2032	-3.00	0.2313	-3.00	0.2684	-3.00	0.3197
-2.00	0.1683	-2.00	0.1804	-2.00	0.1944	-2.00	0.2108
-1.00	0.1435	-1.00	0.1477	-1.00	0.1522	-1.00	0.1570
0.00	0.1250	0.00	0.1250	0.00	0.1250	0.00	0.1250
1.00	0.1107	1.00	0.1083	1.00	0.1060	1.00	0.1038
2.00	0.0993	2.00	0.0955	2.00	0.0920	2.00	0.0887
3.00	0.0900	3.00	0.0854	3.00	0.0812	3.00	0.0774
4.00	0.0822	4.00	0.0772	4.00	0.0727		

3.2.2a   Beams subjected to concentrated load

Example 4:   Let us calculate the responses of a beam with a constant cross section, fixed at its left end, simply supported at its right end, and subjected to a concentrated load and an axial load, as shown in Figure 8.
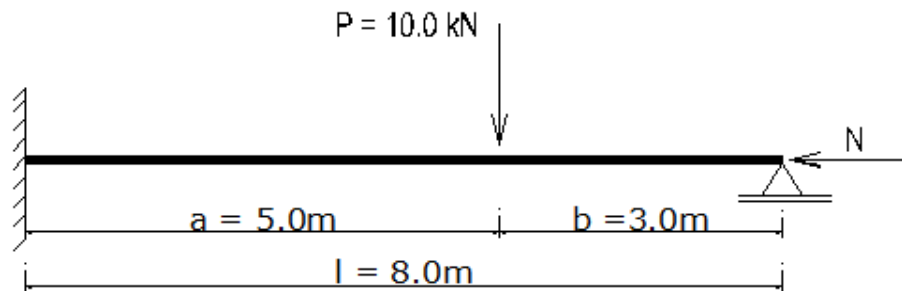
Figure 8. Beam under concentrated load and axial load

Details of the analysis and the results are listed in Appendix F and in the supplementary material "Spreadsheet S8". Table 6 lists the moments at the fixed end (M_{FEM}) and under the load (M_{uL}) for different values of the bending shear factor and axial force.

Table 6. Moments in beam subjected to a concentrated load and an axial force

	Coefficient of axial force k = -4.0			Coefficient of axial force k = -6.0		
$\alpha = EI/\kappa GAl^2 =$	0.0000	0.0250	0.0500	0.0000	0.0250	0.0500
$M_{FEM} =$	-15.65	-16.99	-18.98	-17.60	-21.58	-29.28
$M_{uL} =$	16.73	19.72	23.70	18.72	24.68	35.46
	Coefficient of axial force k = 4.0			Coefficient of axial force k = 6.0		
$\alpha = EI/\kappa GAl^2 =$	0.0000	0.0250	0.0500	0.0000	0.0250	0.0500
$M_{FEM} =$	-11.04	-9.31	-7.98	-10.32	-8.39	-6.98
$M_{uL} =$	12.03	11.27	10.60	11.29	10.22	9.35

3.2.3   Element stiffness matrix

Let us calculate the element stiffness matrix of a beam with the following characteristics:

$k = -1.5$ (Equation (47)), $\alpha = 0.05$ (Equation (13)), and length $L = 4.0$ m.

The matrix is calculated using Equations (57) and (58). Details of the results are presented in the supplementary material "Spreadsheet S9".

The calculation of the element stiffness matrix of this Timoshenko beam yields the following:

$$K_{Tbl} = EI \times \begin{bmatrix} 0.0917 & 0.2303 & -0.0917 & 0.2303 \\ 0.2303 & 0.6759 & -0.2303 & 0.2454 \\ -0.0917 & -0.2303 & 0.0917 & -0.2303 \\ 0.2303 & 0.2454 & -0.2303 & 0.6759 \end{bmatrix} \tag{78}$$

Let us now calculate the element stiffness matrix of the beam with the formula presented by Hu et al. [8]:

$$K_{Tbl} = EI \chi \times \begin{bmatrix} 0 & 0 & (\lambda/L)^2 & 0 \\ (\lambda/L)^2 & 0 & 0 & 0 \\ 0 & 0 & -(\lambda/L)^2 & 0 \\ -(\lambda/L)^2 \cos\lambda & -(\lambda/L)^2 \sin\lambda & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & \chi\lambda/L & 1 & 0 \\ \cos\lambda & \sin\lambda & L & 1 \\ -\chi\lambda/L\sin\lambda & \chi\lambda/L\cos\lambda & 1 & 0 \end{bmatrix}^{-1} \tag{79}$$

The aforementioned characteristics become $P = 1.5 \times EI/L^2$, $\chi = 1 - P/(ksGA) = 1 - 1.5 \times 0.05 = 0.925$,

and $\quad \lambda = \sqrt{PL^2 / \chi EI} = \sqrt{1.5 / 0.925} = 1.273$

Details of the calculations are presented in the supplementary material "Spreadsheet S9".

$$K_{Tbl} = EI \times \begin{bmatrix} 0.0917 & 0.2303 & -0.0917 & 0.2303 \\ 0.2303 & 0.6759 & -0.2303 & 0.2454 \\ -0.0917 & -0.2303 & 0.0917 & -0.2303 \\ 0.2303 & 0.2454 & -0.2303 & 0.6759 \end{bmatrix} \qquad (80)$$

The results are identical.

In fact, both formulas are identical since following equivalences exist between the parameters considered by Hu et al. [8] (χ, λ), and those considered in the present study (k, α, $\xi1$, $\xi2$):

$\chi = 1 + k\alpha$, $\lambda = \xi1$, $\chi\lambda = \xi2$, $\chi(\lambda/L)2 = -k/L2$

The equations for the determination of the buckling loads (see Appendix D) are also identical to those of Hu et al. [8] (Table 1 of [8]).

However, the formula presented by Hu et al. [8] only applies for compressive forces with $k\alpha > -1$.

For the same beam subjected to a tensile force k = 1.5 (Equation (47)), the stiffness matrix is calculated with Equation (67):

$$K_{Tbl} = EI \times \begin{bmatrix} 0.8850 & 1.4784 & -0.8850 & 1.4784 \\ 1.4784 & 4.6884 & -1.4784 & 1.2253 \\ -0.8850 & -1.4784 & 0.8850 & -1.4784 \\ 1.4784 & 1.2253 & -1.4784 & 4.6884 \end{bmatrix} \qquad (81)$$

4. Discussion

The material law developed in this study enables the derivation of closed-form solutions for first-order analysis, second-order analysis, and stability of Timoshenko beams.

The results show that the calculations conducted as described in this paper yield accurate results for first-order and second-order analysis of Timoshenko beams. Closed-form expressions of second-order element stiffness matrices (the axial force being tensile or compressive) in local coordinates were determined.

The determination of element stiffness matrices (ESM) enables the analysis of systems with the direct stiffness method. We showed that ESM can also be determined by the presence of hinges (Equations (33), (59), and (69a) to (70)).

Calculation of bending moments and shear forces:

> Influence of tensile force: With increasing tensile force, bending moments decrease (in absolute values), and with increasing bending shear factor, bending moments decrease (in absolute values).

> Influence of compressive force: With increasing compressive force, bending moments increase (in absolute values), and with increasing bending shear factor, bending moments also increase (in absolute values).

Stability of the beam: With increasing bending shear factor, the buckling load decreases (increase of the buckling length).

The following aspects not addressed in this study could be examined in future research:

- ✓ Analysis of linear structures, such as frames, through the transformation of element stiffness matrices from local coordinates in global coordinates.
- ✓ Second-order analysis of frames free to sidesway with consideration of P-Δ effect.
- ✓ Use of the direct stiffness method, since element stiffness matrices are presented.
- ✓ Closed-form expressions of bending moments and deflections for second-order analysis.
- ✓ Analysis of positions of discontinuity (interior supports, springs, hinges, abrupt change of section), since closed-form expressions of bending moments, shear or transverse forces, rotation of cross sections, and deflection are known.

   ✓ Beams resting on Pasternak foundations, the Pasternak soil parameter can be considered as a tensile force.

5. Conclusions

6. Patents

Supplementary Materials: The following files are uploaded during submission:

Spreadsheet S1: Deflection calculation of a beam with the principle of virtual work.
Spreadsheet S2: Analysis of a fixed-pinned beam under concentrated load
Spreadsheet S3: Buckling analysis of a pinned-pinned beam
Spreadsheet S4: Buckling analysis of a fixed-pinned beam
Spreadsheet S5: Buckling analysis of a fixed-free beam
Spreadsheet S6: Buckling analysis of a fixed-fixed beam
Spreadsheet S7: Bending analysis of a pinned-pinned beam subjected to a uniformly distributed load and an axial force
Spreadsheet S8: Bending analysis of a fixed-pinned beam subjected to a concentrated load and an axial force
Spreadsheet S9: Element stiffness matrix of a beam

Author Contributions:

Funding:

Acknowledgments:

Conflicts of Interest: The author declares no conflict of interest.

Appendix A   **Deflection calculation at position x_0 with the principle of virtual work**

$$\delta(x_0) = \int \frac{M(x) \times m(x)}{EI}\,dx + \int \frac{V(x) \times v(x)}{\kappa GA}\,dx \tag{A1}$$

M(x), V(x) are the bending moment and shear force due to the distributed load, respectively. m(x), v(x) are the bending moment and shear force due to a virtual unit load at the point of interest x_0, respectively.

M(x) = px(l-x)/2 V(x) = p(l/2-x)

m(x) = x(l-x_0)/l for x ≤ x_0	m(x) = x_0(l-x)/l for x ≥ x_0
v(x) = 1-x_0/l for x < x_0	v(x) = -x_0/l for x > x_0
(A2)

The application of Equations (A2) into Equation (A1) yields,

$$\delta(x_0) = \frac{1}{24EI}\,pl^2 x_0(l - x_0)\left[1 + \frac{x_0}{l}\frac{l - x_0}{l}\right] + \frac{1}{2\kappa GA}\,x_0\left(1 - \frac{x_0}{l}\right)pl \tag{A3}$$

Details of the results are presented in the supplementary material "Spreadsheet S1."

Appendix B   **Closed-form expressions of single-span systems for various support conditions and loadings**

Beam with boundary conditions SS–SS (simply supported at both ends)

Application of Equations (14) and (15) and the corresponding boundary conditions yields the following:
- Application of a triangular distributed load (zero at x = 0, and p at x = l)

$$M(x) = -\frac{p}{6l}x^3 + \frac{pl}{6}x \tag{B1}$$

$$EI \times w(x) = \frac{p}{120l}x^5 - \left(\frac{1}{6} + \alpha\right)\frac{pl}{6}x^3 + \left(\frac{7}{360} + \frac{\alpha}{6}\right)pl^3 x \tag{B2}$$

Beam with boundary conditions F–FR (fixed at the left end (x = 0) and free at the right end)
Application of Equations (14) and (15) and the corresponding boundary conditions yields the following:
- Application of a uniformly distributed load p

$$M(x) = -\frac{p}{2}x^2 + plx - \frac{pl^2}{2} \tag{B3}$$

$$EI \times w(x) = \frac{p}{24}x^4 - \frac{pl}{6}x^3 - (\alpha - \frac{1}{2})\frac{pl^2}{2}x^2 + \alpha pl^3 x \tag{B4}$$

Application of a triangular distributed load (zero at x = 0, and p at x = l)

$$M(x) = -\frac{p}{6l}x^3 + \frac{pl}{2}x - \frac{pl^2}{3} \tag{B5}$$

$$EI \times w(x) = \frac{p}{120l}x^5 - \left(\frac{1}{2} + \alpha\right)\frac{pl}{6}x^3 + \frac{pl^2}{6}x^2 + \alpha\frac{pl^3}{2}x \tag{B6}$$

Boundary conditions F–SS (fixed at the left end (x = 0) and simply supported at the right end (x = l))
Application of Equations (14) and (15) and the corresponding boundary conditions yields the following:
- Application of a uniformly distributed load p

$$M(x) = -\frac{p}{2}x^2 + Aplx + Bpl^2 \tag{B7}$$

$$EI \times w(x) = \frac{p}{24}x^4 - A\frac{pl}{6}x^3 - (B + \alpha)\frac{pl^2}{2}x^2 + Cpl^3 x \tag{B8}$$

$$A = \frac{5 + 12\alpha}{8(1 + 3\alpha)} \qquad B = \frac{-1}{8(1 + 3\alpha)} \qquad C = \frac{\alpha(5 + 12\alpha)}{8(1 + 3\alpha)} \tag{B9}$$

- Application of a triangular distributed load (zero at x = 0, and p at x = l)

$$M(x) = -\frac{p}{6l}x^3 + Aplx + Bpl^2 \tag{B10}$$

$$EI \times w(x) = \frac{p}{120l}x^5 - \left(A + \alpha\right)\frac{pl}{6}x^3 - B\frac{pl^2}{2}x^2 + Cpl^3 x \tag{B11}$$

$$A = \frac{9 + 20\alpha}{40(1 + 3\alpha)} \qquad B = \frac{-14}{240(1 + 3\alpha)} \qquad C = \frac{\alpha(9 + 20\alpha)}{40(1 + 3\alpha)} \tag{B12}$$

Boundary conditions F–F (fixed at both ends)
Application of Equations (14) and (15) and the corresponding boundary conditions yields the following:
- Application of a uniformly distributed load p

$$M(x) = -\frac{p}{2}x^2 + \frac{pl}{2}x - \frac{pl^2}{12} \tag{B13}$$

$$EI \times w(x) = \frac{p}{24}x^4 - \frac{pl}{12}x^3 - (\alpha - \frac{1}{12})\frac{pl^2}{2}x^2 + \alpha\frac{pl^3}{2}x \tag{B14}$$

- Application of a triangular distributed load (zero at x = 0, and p at x = l)

$$M(x) = -\frac{p}{6l}x^3 + Aplx + Bpl^2 \tag{B15}$$

$$EI \times w(x) = \frac{p}{120l}x^5 - (A+\alpha)\frac{pl}{6}x^3 - B\frac{pl^2}{2}x^2 + Cpl^3x \tag{B16}$$

$$A = \frac{0.15+2\alpha}{1+12\alpha} \qquad B = -\frac{0.8+12\alpha}{24(1+12\alpha)} \qquad C = \frac{\alpha(0.15+2\alpha)}{1+12\alpha} \tag{B17}$$

Appendix C Fixed-pinned Timoshenko beam subjected to a concentrated load

Equations (19), (20), and (20a) are applied on both sides of the concentrated load, x1 at the left side (x1 ≤ a) and x2 at the right side (x2 ≤ b):

$$M(x_1) = A_1 x_1 + B_1 \tag{C1}$$

$$EI \times w(x_1) = -\frac{A_1}{6}x_1^3 - \frac{B_1}{2}x_1^2 + C_1 x_1 + D_1 \tag{C2}$$

$$EI \times \varphi(x) = -\frac{A_1}{2}x^2 - B_1 x + C_1 - \alpha l^2 A_1 \tag{C3}$$

Similar equations are applied on the right side (with the variable x2).
Following boundary conditions and continuity conditions (see also Equation (1)) are applied:

✓ $w(x_1 = 0) = 0$:	$D_1 = 0$	(C4)
✓ $\varphi(x_1 = 0) = 0$:	$C_1 - \alpha l^2 \times A_1 = 0$	(C5)
✓ $w(x_1 = a) = w(x_2 = 0)$:	$-A_1/6 \times a^3 - B_1/2 \times a^2 + C_1 \times a + D_1 = D_2$	(C6)
✓ $\varphi(x_1 = a) = \varphi(x_2 = 0)$:	$-A_1/2 \times a^2 - B_1 \times a + C_1 - \alpha l^2 \times A_1 = C_2 - \alpha l^2 \times A_2$	(C7)
✓ $M(x_1 = a) = M(x_2 = 0)$:	$A_1 \times a + B_1 = B_2$	(C8)
✓ $Q(x_1 = a) - Q(x_2 = 0) = P$:	$A_1 - A_2 = P$	(C9)
✓ $M(x_2 = b) = 0$:	$A_2 \times b + B_2 = 0$	(C10)
✓ $w(x_2 = b) = 0$:	$-A_2/6 \times b^3 - B_2/2 \times b^2 + C_2 \times b + D_2 = 0$	(C11)

The unknowns are determined by solving of the system of equations, and the moments can be calculated.
For calculation with the principle of virtual work the fixed-end moment and the moment under the load can be expressed as follows:

$$M_{FEM} = -\frac{(1+b/l) \times a/l \times b/l}{6 \times (1/3 + EI/\kappa GAl^2)} \times Pl \tag{C12}$$

$$M_{uL} = b/l \times M_{FEM} + Pab/l \tag{C13}$$

Appendix D Buckling loads of single-span beams with various support conditions

The combination of Equations (50) and (54) yields the following:

$$\xi 1 \times \xi 2 = -k \tag{D1}$$

For the case of a pinned–pinned beam (SS–SS), the results are presented in the supplementary material "Spreadsheet S3". To determine the buckling load, the determinant of the matrix expressing the boundary conditions is set to zero. The matrix, the determinant equation, and the buckling factor β are as follows:

-1,00	0,00	0,00	1,00
1,00	0,00	0,00	0,00
cosξ1	sinξ1	0,00	0,00
-cosξ1	-sinξ1	1,00	1,00

$$\sin\xi_1 = 0 \tag{D2}$$

$$\beta = \sqrt{1 + \alpha\pi^2} \tag{D3}$$

For a fixed–pinned beam (F–SS), the results are presented in the supplementary material "Spreadsheet S4", with the matrix and the equation as follow:

-1,00	0,00	0,00	1,00
0,00	-ξ2	1,00	0,00
cosξ1	sinξ1	0,00	0,00
-cosξ1	-sinξ1	1,00	1,00

$$\xi_2 \times \cos\xi_1 - \sin\xi_1 = 0 \rightarrow \xi_1 \times \sin\xi_1 + k \times \cos\xi_1 = 0 \tag{D4}$$

For the case of a fixed–free beam (F–FR), the results are presented in the supplementary material "Spreadsheet S5", with the matrix, the equation and the buckling factor β as follows

-1,00	0,00	0,00	1,00
0,00	-ξ2	1,00	0,00
cosξ1	sinξ1	0,00	0,00
0,00	0,00	1,00	0,00

$$\cos\xi_1 = 0 \tag{D5}$$

$$\beta = \sqrt{4 + \alpha\pi^2} \tag{D6}$$

For the case of a fixed–fixed beam (F–F), the results are presented in the supplementary material "Spreadsheet S6", with the matrix and equation as follows:

-1,00	0,00	0,00	1,00
0,00	-ξ2	1,00	0,00
-cosξ1	-sinξ1	1,00	1,00
ξ2sinξ1	-ξ2cosξ1	1,00	0,00

$$(1 - \cos\xi_1)^2 - \sin\xi_1 \times (\xi_2 - \sin\xi_1) = 0 \rightarrow 2 - 2\cos\xi_1 + k/\xi_1 \times \sin\xi_1 = 0 \tag{D7}$$

Appendix E   Timoshenko beam subjected to a uniformly distributed load and an axial force

In the case of a compressive force with kα > -1, the bending moment (Equations (40) and (49)) is expressed as follows:

$$M(x) = A_1 \cos\xi_1 \frac{x}{l} + B_1 \sin\xi_1 \frac{x}{l} + \frac{pl^2}{k} \tag{E1}$$

The boundary conditions are described by Equations (E1) and (52). The resulting bending moment is expressed as follows:

$$M(x) = \left[1 - \cos\xi_1 \frac{x}{l} + \frac{\cos\xi_1 - 1}{\sin\xi_1} \sin\xi_1 \frac{x}{l} \right] \times \frac{pl^2}{k} \tag{E2}$$

In the case of a tensile force or a compressive force with $k\alpha < -1$, the bending moment (Equations (40) and (60)) is expressed as follows:

$$M(x) = A_2 \cosh \xi_3 \frac{x}{l} + B_2 \sinh \xi_3 \frac{x}{l} + \frac{pl^2}{k} \tag{E3}$$

The boundary conditions are described by Equations (E3) and (63). The resulting bending moment is expressed as follows:

$$M(x) = \left[1 - \cosh \xi_3 \frac{x}{l} + \frac{\cosh \xi_3 - 1}{\sinh \xi_3} \sinh \xi_3 \frac{x}{l} \right] \times \frac{pl^2}{k} \tag{E4}$$

For $k\alpha = -1$, with Equations (40) and (47), the bending moment is $M(x) = pl^2/k = -\alpha pl^2$.
For $k = 0$ (first-order analysis), the bending moments do not depend on α. The value at the position $L/2$ is $pl^2/8$.

Appendix F   Fixed-pinned Timoshenko beam subjected to a concentrated load and an axial force

Equations (49) to (55) are applied in case of a compressive force with $k\alpha > -1$, and Equations (60) to (66) are applied in case of a tensile force or a compressive force with $k\alpha < -1$.
Case of a compressive force with $k\alpha > -1$: following boundary conditions and continuity conditions are applied:

✓ $w(x_1 = 0) = 0$: $-A_1 + ND_1 = 0$ (F1)

✓ $\varphi(x_1 = 0) = 0$: $-B_1\xi_2 + NC_1 \times l = 0$ (F2)

✓ $w(x_1 = a) = w(x_2 = 0)$: $-A_1\cos\xi_1 a/l - B_1 \sin\xi_1 a/l + NC_1 \times a + ND_1 = -A_2 + ND_2$ (F3)

✓ $\varphi(x_1 = a) = \varphi(x_2 = 0)$: $A_1\xi_2\sin\xi_1 a/l - B_1\xi_2 \cos\xi_1 a/l + NC_1 \times l = -B_2\xi_2 + NC_2 \times l$ (F4)

✓ $M(x_1 = a) = M(x_2 = 0)$: $A_1\cos\xi_1 a/l + B_1 \sin\xi_1 a/l = A_2$ (F5)

✓ $T(x_1 = a) - T(x_2 = 0) = P$: $NC_1 - NC_2 = P$ (F6)

✓ $M(x_2 = b) = 0$: $A_2\cos\xi_1 b/l + B_2 \sin\xi_1 b/l = 0$ (F7)

✓ $w(x_2 = b) = 0$: $-A_2\cos\xi_1 b/l - B_2 \sin\xi_1 b/l + NC_2 \times b + ND_2 = 0$ (F8)

Case of a tensile force or a compressive force with $k\alpha < -1$: following boundary conditions and continuity conditions are applied:

✓ $w(x_1 = 0) = 0$: $-A_1 + ND_1 = 0$ (F9)

✓ $\varphi(x_1 = 0) = 0$: $-B_1\xi_4 + NC_1 \times l = 0$ (F10)

✓ $w(x_1 = a) = w(x_2 = 0)$: $-A_1\cosh\xi_3 a/l - B_1 \sinh\xi_3 a/l + NC_1 \times a + ND_1 = -A_2 + ND_2$ (F11)

✓ $\varphi(x_1 = a) = \varphi(x_2 = 0)$: $-A_1\xi_4\sinh\xi_3 a/l - B_1\xi_4 \cosh\xi_3 a/l + NC_1 \times l = -B_2\xi_4 + NC_2 \times l$ (F12)

✓ $M(x_1 = a) = M(x_2 = 0)$: $A_1\cosh\xi_3 a/l + B_1 \sinh\xi_3 a/l = A_2$ (F13)

✓ $T(x_1 = a) - T(x_2 = 0) = P$: $NC_1 - NC_2 = P$ (F14)

✓ $M(x_2 = b) = 0$: $A_2\cosh\xi_3 b/l + B_2 \sinh\xi_3 b/l = 0$ (F15)

✓ $w(x_2 = b) = 0$: $-A_2\cosh\xi_3 b/l - B_2 \sinh\xi_3 b/l + NC_2 \times b + ND_2 = 0$ (F16)

The unknowns are determined by solving of the system of equations, and the moments can be calculated.

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