ON THE FOURTH COEFFICIENT OF THE INVERSE OF A STARLIKE FUNCTION OF POSITIVE ORDER

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ABSTRACT. We consider the inverse function $z = g(w)$ of a (normalized) starlike function $w = f(z)$ of order α on the unit disk of the complex plane with $0 < \alpha < 1$. Krzyż, Libera and Złotkiewicz obtained sharp estimates of the second and the third coefficients of $g(w)$ in their 1979 paper. Prokhorov and Szynal gave sharp estimates of the fourth coefficient of $g(w)$ as a consequence of the solution to an extremal problem in 1981. We give a straightforward proof of the estimate of the fourth coefficient of $g(w)$ together with explicit forms of the extremal functions.

1. INTRODUCTION

We denote by $\mathcal{A}$ the set of analytic functions $f(z)$ on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ normalized by $f(0) = f'(0) - 1 = 0$. Let $\mathcal{S}$ denote the subset of $\mathcal{A}$ consisting of univalent ones. Each function $f \in \mathcal{S}$ is expanded in the form

$$(1.1) \qquad f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

The Bieberbach conjecture asserts that $|a_n| \leq n$ for $n \geq 2$ and it is finally proved completely by de Branges in 1984 (see, for instance, [4]). We note that equality holds for the Koebe function $K(z) = z/(1-z)^2 = z + 2z^2 + 3z^3 + \cdots$. The Koebe one-quarter theorem implies that $f(\mathbb{D})$ contains the disk $|w| < 1/4$ so that the inverse function $z = g(w)$ of a univalent function $w = f(z)$ in $\mathcal{S}$ is analytic on $|w| < 1/4$. We expand $g = f^{-1}$ in the form

$$(1.2) \qquad g(w) = w + \sum_{n=2}^{\infty} b_n w^n.$$

Löwner [10] showed the sharp inequalities

$$|b_n| \leq \frac{(2n)!}{n!(n+1)!} = \frac{1}{n+1}\binom{2n}{n}, \quad n = 2, 3, \ldots$$

(see also [4, §7.9]). We note that equalities hold for $K^{-1}(w) = [1 - 2w - \sqrt{1-4w}]/(2w)$. A function $f \in \mathcal{A}$ is called *starlike of order* α if

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in \mathbb{D},$$

for a constant $\alpha \in [0, 1)$. We denote by $\mathcal{S}^*(\alpha)$ the set of those functions f. A function in $\mathcal{S}^*(0) =: \mathcal{S}^*$ is simply called starlike and known to map $\mathbb{D}$ univalently onto a starlike domain in $\mathbb{C}$ with respect to the origin. Hence $\mathcal{S}^*(\alpha) \, (\subset \mathcal{S}^*)$ is contained in $\mathcal{S}$ for $0 \leq \alpha < 1$. For a function f in $\mathcal{S}^*(\alpha)$, Robertson [12] proved the sharp inequality

$$|a_n| \leq \frac{\Gamma(2 - 2\alpha + n - 1)}{\Gamma(2 - 2\alpha)\Gamma(n)} = \frac{(2 - 2\alpha)_{n-1}}{(n-1)!},$$

for each $n \geq 2$. Here, $(a)_n$ is the Pochhammer symbol and means $a(a+1)\cdots(a+n-1)$ for $n \geq 1$ and $(a)_0 = 1$. Equalities hold above for the function

$$K_\alpha(z) = \frac{z}{(1-z)^{2(1-\alpha)}} = \sum_{n=1}^{\infty} \frac{(2 - 2\alpha)_{n-1}}{(n-1)!} z^n.$$

It is also known that $f(\mathbb{D})$ contains the disk $|w| < 2^{-2(1-\alpha)}$ for every $f \in \mathcal{S}^*(\alpha)$. However, sharp inequalities are less known for the inverse functions of starlike functions of order $\alpha > 0$. Krzyż, Libera and Złotkiewicz [6] gave the sharp estimates

$$|b_2| \leq 2(1 - \alpha)$$

and

$$|b_3| \leq \begin{cases} (1 - \alpha)(5 - 6\alpha) & \text{for } 0 \leq \alpha \leq \frac{2}{3} \\ 1 - \alpha & \text{for } \frac{2}{3} \leq \alpha < 1. \end{cases}$$

Equality is attained by $g = K_\alpha^{-1}$ for $|b_2|$ and for $|b_3|$ with $0 \leq \alpha \leq \frac{2}{3}$ and by $g = K_{\alpha,2}^{-1}$ for $|b_2|$ with $\frac{2}{3} \leq \alpha < 1$, where

$$K_{\alpha,n}(z) = \{K_\alpha(z^n)\}^{1/n} = \frac{z}{(1 - z^n)^{2(1-\alpha)/n}}, \quad 0 \leq \alpha < 1, \ n = 1, 2, 3, \ldots.$$

Campschroer [2, Theorem 2.IX] proved the following theorem, except for the explicit expression of the bound in the first case and for the second case, both of which were shown by Kapoor and Mishra [5].

Theorem A. *Let* $f^{-1}(w) = w + b_2 w^2 + b_3 w^3 + \cdots$ *for an* $f \in \mathcal{S}^*(\alpha)$. *Then*

$$|b_n| \leq \begin{cases} \dfrac{(n - 2n\alpha + 2)_{n-1}}{n!} & \text{for } 0 \leq \alpha \leq \dfrac{2}{n}, \\[2mm] \dfrac{(n - 2n\alpha + k + 1)_{n-k}}{n(n-1)(n-k-1)!} & \text{for } \dfrac{k}{n} \leq \alpha < \dfrac{k+1}{n}, \ k = 2, \ldots, n-2, \\[2mm] \dfrac{2(1 - \alpha)}{n-1}, & \text{for } \dfrac{n-1}{n} \leq \alpha < 1. \end{cases}$$

Moreover, equality holds when $f = K_\alpha$ *for the first case, and equality holds when* $f = K_{\alpha,n-1}$ *for the last case.*

We remark that the above result was extended to the so-called Janowski starlike functions by Ali and Vasudevarao [1].

Though the sharp bound of $|b_n|$ is not known for $n \geq 5$ and $\frac{2}{n} < \alpha < \frac{n-1}{n}$, Prokhorov and Szynal [11] obtained the sharp bounds of $|b_4|$ for (α, β)-convex functions, which contains the case of starlike functions of order α. Indeed, for given real numbers μ and ν, they solved the extremal problem of finding the maximum $|e_3 + \mu e_1 e_2 + \nu e_1^3|$ over all the

analytic functions $\omega(z) = e_1 z + e_2 z^2 + e_3 z^3 + \cdots$ on $\mathbb{D}$ with $|\omega(z)| \leq |z|$, and then they derived their main results from the solution to this extremal problem (see also [2, Lemma 2.VI]). They also provided the information of extremal functions in terms of the function ω determined by the relation $zf'(z)/f(z) = (1+(1-2\alpha)\omega(z))/(1-\omega(z))$. Therefore, the expression of the extremal functions is not very explicit. We will give a more detailed proof of it with explicit forms of extremal functions as follows. (See also [2, Theorem 2.VIII].)

Theorem 1.1 (*cf.* Prokhorov-Szynal [11, Theorem 2"]). *Let* $f^{-1}(w) = w + b_2 w^2 + b_3 w^3 + b_4 w^4 + \cdots$ *for a function* $f \in \mathcal{S}^*(\alpha)$ *with* $0 \leq \alpha < 1$. *Then the following sharp inequality holds:*

$$
|b_4| \leq \begin{cases}
\dfrac{2}{3}(1-\alpha)(3-4\alpha)(7-8\alpha) & \text{if } 0 \leq \alpha \leq \alpha_1 = \dfrac{3}{5}, \\[2ex]
\dfrac{4(2-3\alpha)^{3/2}(3-4\alpha)(7-8\alpha)}{3\sqrt{5-8\alpha}} & \text{if } \alpha_1 \leq \alpha \leq \alpha_2 = \dfrac{35-\sqrt{33}}{48} \approx 0.609488, \\[2ex]
\dfrac{2}{3}\left(\dfrac{11-12\alpha}{6}\right)^{3/2} & \text{if } \alpha_2 \leq \alpha \leq \alpha_3, \\[2ex]
\dfrac{2}{3}(1-\alpha) & \text{if } \alpha_3 \leq \alpha < 1,
\end{cases}
$$

where

$$
\alpha_3 = \frac{1}{24}\left(21 - \sqrt[3]{13+2\sqrt{11}} - \sqrt[3]{13-2\sqrt{11}}\right) \approx 0.685369.
$$

If $0 \leq \alpha \leq \alpha_1$, *then equality holds for* $f = K_\alpha$.
If $\alpha_1 \leq \alpha \leq \alpha_2$, *then equality holds for*

$$
f(z) = \frac{z}{(1-z)^{2(1-\alpha)\lambda}(1-e^{i\theta_0}z)^{2(1-\alpha)(1-\lambda)}},
$$

where

$$
\theta_0 = \arccos\left[\frac{4\left(72\alpha^2 - 103\alpha + 36\right)}{5-8\alpha}\right],
$$

$$
\lambda = \frac{(5\alpha-3)(7-8\alpha)}{2(1-\alpha)^2(72\alpha-43) - (1-\alpha)\sqrt{(3\alpha-2)(72\alpha-43)(24\alpha^2-33\alpha+11)}}.
$$

If $\alpha_2 \leq \alpha \leq \alpha_3$, *then equality holds for*

$$
f(z) = \frac{z}{(1+z)^{(1-\alpha)(1-s)}(1-z)^{(1-\alpha)(1+s)}}, \quad \text{where } s = \sqrt{\frac{11-12\alpha}{4\sqrt{6}(1-\alpha)}}.
$$

If $\alpha_3 \leq \alpha < 1$, *then equality holds for* $f = K_{\alpha,3}$.

Let $B(\alpha)$ be the sharp bound for $|b_4|$ given in the above theorem. The graph of the function $B(\alpha)$ in the range $1/2 \leq \alpha < 1$ is exhibited in Figure 1.

The organization of the present note is as follows. In the next section, a few preliminary results are given. Some of them may be useful in different situations. Section 3 will be devoted to the proof of the main theorem except for the technical Lemma 3.1, which will be shown in Section 4.
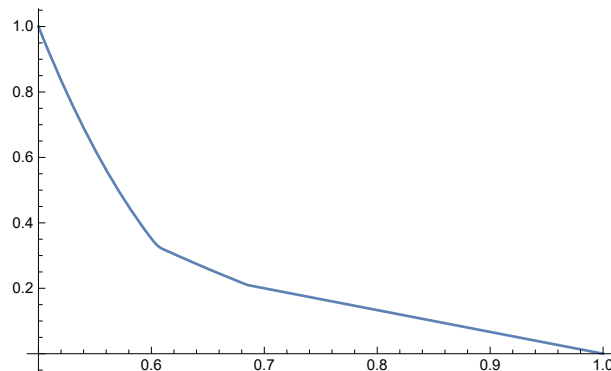
FIGURE 1. The graph of the bound $B(\alpha)$ for $|b_4|$ in the range $1/2 \le \alpha < 1$

As the reader will observe, our proof is quite involved with heavy computations. It is quite surprising that Prokhorov and Szynal were able to show it (even in a much general form) as early as in 1981. Finally, the authors would like to thank Professor Ponnusamy for conveying the paper [11] to their attention. Before that, the authors prepared the manuscript, being unaware of [11]. The authors believe that the present paper has a merit in the sense that it may give new techniques to the reader to attack similar problems.

2. PRELIMINARIES

Recall that an analytic function p on $\mathbb{D}$ with $p(0) = 1$ and $\operatorname{Re} p > 0$ is called a Carathéodory function. We will denote by $\mathcal{P}$ the set of Carathéodory functions. Therefore, we can say that a function $f \in \mathcal{A}$ is starlike of order α if and only if $p(z) = [zf'(z)/f(z) - \alpha]/(1-\alpha)$ belongs to $\mathcal{P}$. In this case, we can write

$$(2.1) \qquad \frac{zf'(z)}{f(z)} = \alpha + (1-\alpha)p(z).$$

It is therefore fundamental to investigate the coefficients of Carathéodory functions in order to study starlike functions of order α. We will introduce several necessary results in this section. The next result is essentially due to Libera and Złotkiewicz [9] (the present version is found in [8]).

Lemma 2.1. *Let* $p(z) = 1 + 2c_1 z + 2c_2 z^2 + 2c_3 z^3 + \cdots$ *be a Carathéodory function. Then*

$$
\begin{aligned}
c_1 &= u, \\
c_2 &= u^2 + v(1 - |u|^2), \\
c_3 &= u^3 + (1 - |u|^2)v(2u - \bar{u}v) + (1 - |u|^2)(1 - |v|^2)w
\end{aligned}
$$

for some $u, v, w \in \overline{\mathbb{D}}$. *Conversely, for complex numbers* c_1, c_2, c_3 *expressed as above, there is a Carathéodory function* p *satisfying* $p(z) = 1 + 2c_1 z + 2c_2 z^2 + 2c_3 z^3 + O(z^4)$.

We will also make use of the next result derived from Carathéodory-Toeplitz theorem (see [7, Lemma 2.6]).

Lemma 2.2. *Let $p(z) = 1 + 2c_1 z + 2c_2 z^2 + \cdots$ be a Carathéodory function with $c_1 = c \in [0,1)$. If $c_2 = c^2 + v(1 - c^2)$ for some $v \in \mathbb{C}$ with $|v| = 1$, then*

$$(2.2) \qquad p(z) = \lambda_1 \frac{1 + \varepsilon_1 z}{1 - \varepsilon_1 z} + \lambda_2 \frac{1 + \varepsilon_2 z}{1 - \varepsilon_2 z},$$

where $\varepsilon_1 = c - q e^{i\varphi}\sqrt{1 - c^2}$ and $\varepsilon_2 = c + q^{-1} e^{i\varphi}\sqrt{1 - c^2}$ with $\varphi = \frac{1}{2}\arg v \in (-\frac{\pi}{2}, \frac{\pi}{2}]$, and $\lambda_1 = 1/(1 + q^2)$, $\lambda_2 = q^2/(1 + q^2)$ and $q = (c\cos\varphi + \sqrt{1 - c^2 \sin^2\varphi})/\sqrt{1 - c^2}$.

We remark that the above condition on $c_1 = c$ and c_2 means that the equality $|c_2 - c_1^2| = 1 - |c_1|^2$ holds. Furthermore, the condition $c = 1$ corresponds to the case when $\lambda_1 = 1$.

Note that $|\varepsilon_1| = |\varepsilon_2| = 1$. Thus, they can be expressed by $\varepsilon_1 = e^{i\theta_1}$ and $\varepsilon_2 = e^{i\theta_2}$ for some $\theta_1, \theta_2 \in \mathbb{R}$. Note also that

$$(2.3) \qquad \lambda_1 = \frac{1 - c^2}{2 - c^2 + c^2\cos 2\varphi + c\sqrt{2(1 + \cos 2\varphi) - c^2 \sin^2 2\varphi}} \quad \text{and} \quad \lambda_2 = 1 - \lambda_1.$$

It is convenient to notice the following fact. The function $f \in \mathcal{A}$ determined by the equation (2.1) for the function p in the form of (2.2) should satisfy the equation

$$\frac{f'(z)}{f(z)} - \frac{1}{z} = \frac{1}{z}\big[\alpha + (1 - \alpha)p(z) - 1\big] = (1 - \alpha)\left[\frac{2\lambda_1\varepsilon_1}{1 - \varepsilon_1 z} + \frac{2\lambda_2\varepsilon_2}{1 - \varepsilon_2 z}\right].$$

Integrating both sides, we obtain

$$\log\frac{f(z)}{z} = -2(1 - \alpha)\lambda_1\log(1 - \varepsilon_1 z) - 2(1 - \alpha)\lambda_2\log(1 - \varepsilon_2 z),$$

and hence, the expression

$$f(z) = \frac{z}{(1 - \varepsilon_1 z)^{2(1-\alpha)\lambda_1}(1 - \varepsilon_2 z)^{2(1-\alpha)\lambda_2}}.$$

By taking the rotation $\varepsilon_1 f(\overline{\varepsilon_1} z) = e^{i\theta_1} f(e^{-i\theta_1} z)$, we can replace $\varepsilon_1, \varepsilon_2$ by $1, \varepsilon = \overline{\varepsilon_1}\varepsilon_2 = e^{i\theta_0}$, respectively, where $\theta_0 = \theta_2 - \theta_1$. Moreover, we may assume that $\operatorname{Im}\varepsilon \geq 0$ by considering $\overline{f(\bar z)}$ if necessary. We now compute

$$\cos\theta_0 = \operatorname{Re}(\overline{\varepsilon_1}\varepsilon_2) = 2c^2 - 1 - c\sqrt{1 - c^2}(q - q^{-1})\cos\varphi$$
$$= 2c^2 \sin^2\varphi - 1 = c^2(1 - \cos 2\varphi) - 1.$$

Thus an extremal function can be given in the form

$$(2.4) \qquad f(z) = \frac{z}{(1 - z)^{2(1-\alpha)\lambda_1}(1 - e^{i\theta_0} z)^{2(1-\alpha)\lambda_2}},$$

where $\theta_0 = \arccos\big[c^2(1 - \cos 2\varphi) - 1\big]$.

For real constants A, B, C we consider the quantity $|A + 2Be^{i\theta} + Ce^{2i\theta}|$ for $\theta \in \mathbb{R}$. Since this is invariant under the transformation $\theta \mapsto -\theta$, it is a function of $x = \cos\theta$. Thus we can define a function $\Phi : [-1, 1] \to [0, +\infty)$ satisfying

$$(2.5) \qquad \Phi(\cos\theta) = |A + 2Be^{i\theta} + Ce^{2i\theta}|.$$

We define $\operatorname{Sgn} x = x/|x|$ for $x \in \mathbb{R}$ with $x \neq 0$. The next elementary lemma (*cf.* [3, Lemma 3.3]) is useful in the sequel.

Lemma 2.3. *Let A, B, C be real constants with $AC < 0$ and set $\xi = -B(A+C)/(2AC)$. Then the function Φ satisfies the inequality*

$$\Phi(x) \leq \begin{cases} \Phi(-1) = |A - 2B + C| = \operatorname{Sgn} B\,(-A + 2B - C) & \text{if } \xi \leq -1, \\[2mm] \Phi(\xi) = \sqrt{1 - \dfrac{B^2}{AC}}\,|A - C| & \text{if } -1 \leq \xi \leq 1, \\[2mm] \Phi(1) = |A + 2B + C| = \operatorname{Sgn} B\,(A + 2B + C) & \text{if } 1 \leq \xi \end{cases}$$

for $-1 \leq x \leq 1$.

3. Proof of the main theorem

We are ready to show Theorem 1.1.

Let $f \in \mathcal{S}^*(\alpha)$ be of the form (1.1). Then we may write

$$\frac{zf'(z)}{f(z)} = \alpha + (1 - \alpha)p(z) = 1 + 2\beta \sum_{n=1}^{\infty} c_n z^n$$

for a Carathéodory function $p(z) = 1 + 2(c_1 z + c_2 z^2 + \cdots)$, where we put $\beta = 1 - \alpha$ for the sake of simplicity.

By a comparison of the coefficients, we have the relations

$$a_2 = 2\beta c_1,$$
$$a_3 = 2\beta^2 c_1^2 + \beta c_2,$$
$$a_4 = \frac{4}{3}\beta^3 c_1^3 + 2\beta^2 c_1 c_2 + \frac{2}{3}\beta c_3.$$

Let $g = f^{-1}$ be of the form (1.2). Then simple computations yield

$$b_2 = -a_2,$$
$$b_3 = -a_3 + 2a_2,$$
(3.1) $$b_4 = -a_4 + 5a_2 a_3 - 5a_2^3.$$

By using these relations, we have

$$b_4 = -\frac{2}{3}\beta\left(c_3 - 12\beta c_1 c_2 + 32\beta^2 c_1^3\right) = -\frac{2}{3}\beta Q,$$

where $Q = c_3 - 12\beta c_1 c_2 + 32\beta^2 c_1^3$. Let $M = M(\beta)$ denote the quantity

$$\sup\left\{ |c_3 - 12\beta c_1 c_2 + 32\beta^2 c_1^3| : p(z) = 1 + 2c_1 z + 2c_2 z^2 + 2c_3 z^3 + O(z^4) \text{ for some } p \in \mathcal{P} \right\}.$$

Thus we need to show

(3.2) $$M(\beta) = \begin{cases} (4\beta - 1)(8\beta - 1) & \text{if } 1 \geq \beta > \beta_1 = \dfrac{2}{5}, \\[3mm] \dfrac{2(3\beta - 1)^{3/2}(4\beta - 1)(8\beta - 1)}{\beta\sqrt{8\beta - 3}} & \text{if } \beta_1 \geq \beta \geq \beta_2 = \dfrac{13 + \sqrt{33}}{48}, \\[3mm] \dfrac{1}{\beta}\left(\dfrac{12\beta - 1}{6}\right)^{3/2} & \text{if } \beta_2 \geq \beta \geq \beta_3 = 1 - \alpha_3, \\[3mm] 1 & \text{if } \beta_3 \geq \beta > 0 \end{cases}$$

with enough information about the extremal functions, in addition. We now use Lemma 2.1 to parametrize the quantity Q as

$$Q = (8\beta - 1)(4\beta - 1)u^3 - 2(6\beta - 1)(1 - |u|^2)uv - (1 - |u|^2)\bar{u}v^2 + (1 - |u|^2)(1 - |v|^2)w$$
$$= Ae^{3i\psi} + 2Be^{i(\psi+\phi)} + Ce^{i(2\phi-\psi)} + (1 - s^2)(1 - t^2)w,$$

where $u = se^{i\psi}$, $v = te^{i\phi}$ and

$$A = (8\beta - 1)(4\beta - 1)s^3,$$
$$B = -(6\beta - 1)(1 - s^2)st,$$
$$C = -(1 - s^2)st^2.$$

Note here that the condition $|c_2 - c_1^2| = 1 - |c_1|^2$ is equivalent to $t = |v| = 1$. Since the argument of w can be taken arbitrarily, we have the sharp inequality

$$|Q| \leq |A + 2Be^{i(\phi-2\psi)} + Ce^{2i(\phi-2\psi)}| + (1 - s^2)(1 - t^2)$$
$$= \Phi(\cos(\phi - 2\psi)) + (1 - s^2)(1 - t^2),$$

where Φ is defined in (2.5). We next divide the proof into two cases.

Case I: When $0 \leq \alpha < \frac{3}{4}$, in this case $\frac{1}{4} < \beta \leq 1$.

We note that $8\beta - 1 > 6\beta - 1 > 4\beta - 1 > 0$. Hence $A > 0, C < 0$ and $B < 0$ as long as $st > 0$. In order to apply Lemma 2.3, we compute

$$(3.3) \qquad \xi = -\frac{B(A + C)}{2AC} = -\frac{(6\beta - 1)\{(8\beta - 1)(4\beta - 1)s^2 - t^2(1 - s^2)\}}{2s^2t(4\beta - 1)(8\beta - 1)}.$$

Therefore, by Lemma 2.3, $M(\beta)$ is the supremum of the following quantity over $s, t \in (0, 1]$:

$$F(s, t) = \begin{cases} \Phi(-1) + (1 - s^2)(1 - t^2) = F_1(s, t) & \text{if } \xi \leq -1, \\ \Phi(\xi) + (1 - s^2)(1 - t^2) = F_2(s, t) & \text{if } -1 \leq \xi \leq 1, \\ \Phi(1) + (1 - s^2)(1 - t^2) = F_3(s, t) & \text{if } 1 \leq \xi. \end{cases}$$

Here,

$$F_1(s, t) = (8\beta - 1)(4\beta - 1)s^3 + 2(6\beta - 1)(1 - s^2)st - (1 - s^2)st^2 + (1 - s^2)(1 - t^2),$$

$$F_2(s, t) = \sqrt{\frac{(6\beta - 1)^2 - 4\beta^2 s^2}{(8\beta - 1)(4\beta - 1)}}\{(8\beta - 1)(4\beta - 1)s^2 + (1 - s^2)t^2\} + (1 - s^2)(1 - t^2),$$
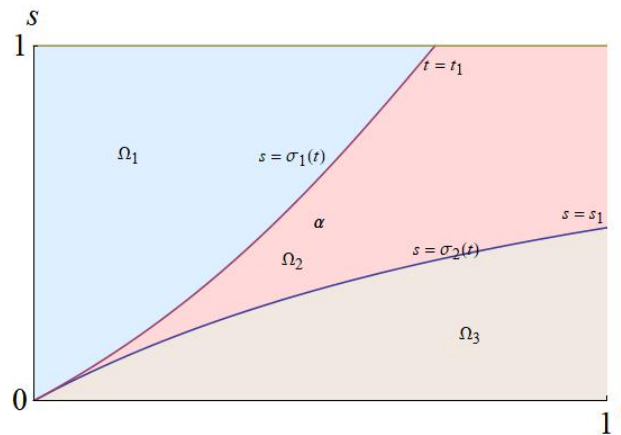
$$F_3(s, t) = -(8\beta - 1)(4\beta - 1)s^3 + 2(6\beta - 1)(1 - s^2)st + (1 - s^2)st^2 + (1 - s^2)(1 - t^2).$$

By its form, the function $F(s, t)$ is continuous on $[0, 1]^2 = [0, 1] \times [0, 1]$. We observe that $\xi \leq -1$ if and only if

$$s^2 \geq \frac{(6\beta - 1)t^2}{(6\beta - 1)t^2 - 2(8\beta - 1)(4\beta - 1)t + (8\beta - 1)(6\beta - 1)(4\beta - 1)} = \sigma_2(t)^2,$$

and that $\xi \geq 1$ if and only if

$$s^2 \leq \frac{(6\beta - 1)t^2}{(6\beta - 1)t^2 + 2(8\beta - 1)(4\beta - 1)t + (8\beta - 1)(6\beta - 1)(4\beta - 1)} = \sigma_1(t)^2.$$

FIGURE 2. The regions $\Omega_1, \Omega_2, \Omega_3$ when $\frac{1}{4} < \beta < \frac{1}{2}$

Here, $0 \le \sigma_1(t) \le \sigma_2(t)$ and $\sigma_1(t) < 1$ for $0 \le t \le 1$. Note that

$$(3.4) \qquad s_1 := \sigma_1(1) = \sqrt{\frac{6\beta - 1}{8\beta^2(24\beta - 5)}} \in (0, 1),$$

which is a decreasing function in $\beta > 0$. In view of the form of $1/\sigma_j(t)^2$, we observe that $\sigma_1(t)$ and $\sigma_2(t)$ are monotone increasing and continuous on $0 \le t \le 1$ and $\sigma_2(t) = 1$ when

$$t = t_1 := \frac{6\beta - 1}{2}.$$

For a fixed β, we divide the set $[0, 1]^2$ into three closed regions $\Omega_1, \Omega_2, \Omega_3$ according as $\xi \le -1, -1 \le \xi \le 1, 1 \le \xi$, respectively. More precisely, we can define them by $\Omega_1 = \{(s, t) \in [0, 1]^2 : \sigma_2(t) \le s\}$, $\Omega_2 = \{(s, t) \in [0, 1]^2 : \sigma_1(t) \le s \le \sigma_2(t)\}$, $\Omega_3 = \{(s, t) \in [0, 1]^2 : s \le \sigma_1(t)\}$.

Then

$$M = M(\beta) = \max_{s,t \in [0,1]} F(s, t) = \max\{M_1, M_2, M_3\},$$

where

$$M_j = \max_{(s,t) \in \Omega_j} F_j(s, t), \quad j = 1, 2, 3.$$

According to the conditions $t_1 \in (0, 1)$ and $t_1 \ge 1$ of t_1, we consider two subcases to find the value $M(\beta)$.

Subcase Ia: When $\frac{1}{4} < \beta < \frac{1}{2}$, we have $t_1 \in (0, 1)$. See Figure 2 for the domains $\Omega_1, \Omega_2, \Omega_3$ in this subcase.

At this point, it is worth noting that $M \ge F_j(0, 0) = 1$ for any $\beta \in (\frac{1}{4}, \frac{1}{2})$. Our strategy to compute M is as follows. In order to find the value of M_j, we first look for critical points of the function F_j in the interior $\text{Int}\,\Omega_j$ of Ω_j. If there is no critical point, the maximum must be taken on the boundary $\partial\Omega_j$. Thus we have only to look at the boundary values of F_j.

On the value M_1 : By the identity

$$\frac{\partial F_1}{\partial t}(s,t) = 2(1-s^2)\big\{(6\beta - 1 + t)s - t\big\},$$

critical points of F_1 in $\operatorname{Int}\Omega_1$ lie on the curve $s = t/(6\beta - 1 + t)$. Since

$$\sigma_2(t)^2 - \left(\frac{t}{6\beta - 1 + t}\right)^2$$

$$= \frac{4\beta^2 t^2(6\beta - 1 - 2t)}{(6\beta - 1 + t)^2\{(6\beta - 1)t^2 - 2(8\beta - 1)(4\beta - 1)t + (8\beta - 1)(6\beta - 1)(4\beta - 1)\}} > 0$$

for $0 < t < t_1$, there is no critical point of $F_1(s,t)$ in $\operatorname{Int}\Omega_1$. Thus the maximum M_1 of F_1 is taken on $\partial\Omega_3$. Simple computations yield

$$F_1(1,t) = (8\beta - 1)(4\beta - 1),$$

$$F_1(s,0) = (8\beta - 1)(4\beta - 1)s^3 + 1 - s^2 =: \varphi(s).$$

Since $\varphi'(s) = \{3(8\beta - 1)(4\beta - 1)s - 2\}s$, the function $\varphi(s)$ is decreasing in $0 < s < 2/[3(8\beta - 1)(4\beta - 1)]$ and increasing in $2/[3(8\beta - 1)(4\beta - 1)] < s$. In particular, the maximum of $F_1(s,0)$ is taken at $s = 0$ or $s = 1$ on $[0,1]$. Noting that the function $F_1(\sigma_2(t),t)$ takes the value $F_1(0,0) = 1$ for $t = 0$ and $F_1(1,t_1) = F_1(1,0)$ for $t = t_1$, the maximum of F_1 on Ω_1 is attained on the common boundary curve $s = \sigma_2(t)$ $(0 \le t \le t_1)$ with Ω_2. Hence, we conclude that $M_1 \le M_2$ so that $M = \max\{M_2, M_3\}$.

On the value M_2 : Since

$$\frac{\partial F_2}{\partial t} = 2t(1-s^2)\left(\sqrt{1 + \frac{4\beta^2(1-s^2)}{(8\beta - 1)(4\beta - 1)}} - 1\right) > 0$$

for $s,t \in (0,1)$, the function $F_2(s,t)$ is increasing in $0 \le t \le 1$ for a fixed $s \in (0,1)$. Therefore, the maximum M_2 is taken on either the segment $[s_1, 1] \times \{1\}$ or the common boundary $s = \sigma_1(t)$ $(0 \le t \le 1)$ with Ω_3. If we put

$$M_4 = \max_{s_1 \le s \le 1} F_2(s,1),$$

then we have $M = \max\{M_3, M_4\}$.

We now look for the maximum of the function

$$\eta(x) = (8\beta - 1)(4\beta - 1)F_2(\sqrt{x}, 1)^2 = \{(6\beta - 1)^2 - 4\beta^2 x\}(32\beta^2 x - 12\beta x + 1)^2$$

on $s_1^2 \le x \le 1$. Here we remark that $\max_{s_1^2 \le x \le 1} \eta(x) = (8\beta - 1)(4\beta - 1)M_4^2$. Since

$$\eta'(x) = 12\beta(32\beta^2 x - 12\beta x + 1)\{192\beta^3 - 136\beta^2 + 29\beta - 2 - 4\beta^2(8\beta - 3)x\},$$

the function $\eta(x)$ has possibly a critical point at

$$x = \theta(\beta) := \frac{192\beta^3 - 136\beta^2 + 29\beta - 2}{4\beta^2(8\beta - 3)}.$$

In view of the formulae

$$\theta(\beta) - 1 = \frac{(8\beta - 1)(4\beta - 1)(5\beta - 2)}{4\beta^2(8\beta - 3)}$$

and
$$\theta(\beta) - s_1^2 = \frac{(8\beta - 1)(4\beta - 1)(288\beta^2 - 156\beta + 17)}{8\beta^2(8\beta - 3)(24\beta - 5)},$$

we see that $s_1^2 < \theta(\beta) < 1$ precisely when $(13 + \sqrt{33})/48 < \beta < 2/5$ for β under consideration. Note here that $(13 + \sqrt{33})/48 \approx 0.390512 > 3/8$. Also, we note that $\eta'(1) = 12\beta(8\beta - 1)^2(4\beta - 1)^2(5\beta - 2)$. According to the above observation, we consider the following three cases.

(i) When $1/4 < \beta \le (13 + \sqrt{33})/48$, $\eta(x)$ is decreasing on $[s_1^2, 1]$ and therefore $F_2(s, 1)$ is decreasing on $[s_1, 1]$. Hence, we have $M_4 = F_2(s_1, 1) \le M_3$ because $(s_1, 1) \in \partial\Omega_1$.

(ii) When $(13 + \sqrt{33})/48 < \beta < 2/5$, $\eta(x)$ is increasing on $[s_1^2, \theta(\beta)]$ and decreasing on $[\theta(\beta), 1]$. Hence $\eta(x)$ takes the maximum at $x = \theta(\beta)$. A straightforward calculation produces
$$\eta(\theta(\beta)) = \frac{4(8\beta - 1)^3(4\beta - 1)^3(3\beta - 1)^3}{\beta^2(8\beta - 3)}$$

and therefore
$$M_4 = \frac{2(8\beta - 1)(4\beta - 1)(3\beta - 1)^{3/2}}{\beta(8\beta - 3)^{1/2}}.$$

(iii) When $2/5 \le \beta < 1/2$, $\eta(x)$ is increasing on $[s_1^2, 1]$ and so is $F_2(s, 1)$ on $[s_1, 1]$. Thus $M_4 = F_2(1, 1) = (8\beta - 1)(4\beta - 1)$.

On the value M_3: The last step is to solve the extremal problem on Ω_3. To find a critical point of F_3, we solve the system of equations
$$\frac{\partial F_3}{\partial t} = 2(1 - s^2)\{s(6\beta - 1 + t) - t\} = 0,$$
$$\frac{\partial F_3}{\partial s} = -3(8\beta - 1 + t)(4\beta - 1 + t)s^2 - 2(1 - t^2)s + t(12\beta - 2 + t) = 0.$$

Solving the first equation, we have the relation $s = t/(6\beta - 1 + t)$. We substitute it into the second equation and simplify to
$$\frac{12\beta t\{(7\beta - 2)t + 2(6\beta - 1)(3\beta - 1)\}}{(6\beta - 1 + t)^2} = 0.$$

Hence a candidate of critical points in $\operatorname{Int}\Omega_1$ is only (s_0, t_0), where
$$s_0 = \frac{2(1 - 3\beta)}{\beta}, \quad t_0 = -\frac{2(6\beta - 1)(3\beta - 1)}{7\beta - 2}.$$

By the requirement $0 < s_0 < 1$, the parameter β should satisfy $2/7 < \beta < 1/3$. After some computations, we have the expression
$$F_3(s_0, t_0) = 145 - \frac{16}{\beta} - 432\beta + 432\beta^2 =: h(\beta).$$

Since $h'(\beta) = 16(6\beta + 1)(3\beta - 1)^2/\beta^2 > 0$ in this range, we have $F_3(s_0, t_0) \le h(1/3) = 1 = F_3(0, 0)$. Therefore, this critical point does not contribute to the maximum of F_3. At any event, we conclude that the maximum M_3 is attained on the boundary of Ω_3.

Next we shall look at the values of F_3 on $\partial\Omega_3$. First, $F_3(0, t) = 1 - t^2 \le 1 = F(0, 0)$ for $t \in [0, 1]$. Second, we will prove the following result in the next section.

Lemma 3.1. *Let* $1/4 < \beta < 1/2$. *Then* $F_3(\sigma_1(t), t) \leq \max\{1, F_3(s_1, 1)\}$ *for* $0 \leq t \leq 1$.

Therefore, we see that the maximum of F_3 on Ω_3 is taken either on the segment $[0, s_1] \times \{1\}$ or at the point $(0, 0)$. We now have

$$F_3(s, 1) = (12\beta - 1)s - 32\beta^2 s^3 =: G(s)$$

for $s \in [0, s_1]$. Let

$$s_2 = \frac{\sqrt{12\beta - 1}}{4\sqrt{6}\beta},$$

which is the positive critical point of $G(s)$. Since

$$s_1^2 - s_2^2 = -\frac{288\beta^2 - 156\beta + 17}{96\beta^2(24\beta - 5)},$$

$s_2 \in [0, s_1]$ if and only if $\beta \leq (13 + \sqrt{33})/48$ for $\beta \in (1/4, 1/2)$. So we divide the problem into two cases.

   (1) When $1/4 < \beta < (13 + \sqrt{33})/48$, we have

$$G(s) \leq G(s_2) = \frac{(12\beta - 1)^{3/2}}{6\sqrt{6}\beta}$$

   for $0 \leq s \leq s_1$. Hence $M_3 = \max\{1, G(s_2)\}$ in this case.

   (2) When $(13 + \sqrt{33})/48 \leq \beta < 1/2$, $G(s)$ is increasing in $[0, s_1]$ so that

$$G(s) \leq G(s_1) = \frac{9(8\beta - 1)(4\beta - 1)\sqrt{6\beta - 1}}{2\sqrt{2}\beta(24\beta - 5)^{3/2}}$$

   for $s \in [0, s_1]$. Hence $M_3 = \max\{1, G(s_1)\}$ in this case.

We now summarize what we have seen. So far, we had $M(\beta) = M = \max\{M_3, M_4\}$ and M_3 is computed in (1) and (2) above. On the other hand M_4 is computed in (i), (ii) and (iii) on the value M_2. Hence, we have the following conclusion.

   (a) When $1/4 < \beta \leq (13 + \sqrt{33})/48 = \beta_2$, we have $M = M_3 = \max\{1, G(s_2)\}$. It is easy to see that $1 \leq G(s_2) = (12\beta - 1)^{3/2}/6\sqrt{6}\beta$ precisely if $\beta \geq \beta_3$, where $\beta_3 \approx 0.31463$ is the unique real solution to the equation $(12\beta - 1)^3 = 6^3\beta^2$. That is,

$$\beta_3 = \frac{1}{24}\left(3 + \sqrt[3]{13 + 2\sqrt{11}} + \sqrt[3]{13 - 2\sqrt{11}}\right),$$

   which is the same as in (3.2).

     Next we shall give extremal functions. First, for $1/4 < \beta \leq \beta_3$, $M = 1$ is attained when $s = 0, t = 0$ and $|w| = 1$, which corresponds to $p(z) = (1 + wz^3)/(1 - wz^3)$. Taking $w = 1$, we get $f = K_{\alpha,3}$. Second, for $\beta_3 \leq \beta \leq \beta_2$, $M = G(s_2)$ is attained when $t = 1, s = s_2$ and $\cos\theta = 1$, where $\theta = \phi - 2\psi$. We may assume $\psi = 0$ so that $\theta = \phi = 0$. Now we will apply Lemma 2.2 with $c = s_2, \varphi = \theta/2 = 0$, where $s_2 = \sqrt{\frac{11 - 12\alpha}{4\sqrt{6}(1 - \alpha)}}$. Then $q = (1 + s_2)/\sqrt{1 - s_2^2}$ so that $\lambda_1 = (1 - s_2)/2$, $\lambda_2 = (1 + s_2)/2$ and $\varepsilon_1 = -1$, $\varepsilon_2 = -1$. Therefore, with the help of (2.4), we obtain the extremal function as in the third case of Theorem 1.1.

(b) When $(13 + \sqrt{33})/48 < \beta < 2/5$, we have $M = \max\{M_3, M_4\}$. Noting $M_4 \geq 1$, we obtain, more precisely,

$$M = \max\left\{\frac{9(8\beta - 1)(4\beta - 1)\sqrt{6\beta - 1}}{2\sqrt{2}\beta(24\beta - 5)^{3/2}}, \frac{2(8\beta - 1)(4\beta - 1)(3\beta - 1)^{3/2}}{\beta(8\beta - 3)^{1/2}}\right\}.$$

Letting m_3 be the first term in this maximum, we have

$$\left(\frac{M_4}{m_3}\right)^2 - 1 = \frac{(144\beta^2 - 78\beta + 13)(288\beta^2 - 156\beta^2 + 17)^2}{81(6\beta - 1)(8\beta - 3)} > 0$$

for β in the current range. Hence, we obtain

$$M = M_4 = \frac{2(8\beta - 1)(4\beta - 1)(3\beta - 1)^{3/2}}{\beta(8\beta - 3)^{1/2}}.$$

Next let us see forms of extremal functions. Recall that M_4 was evaluated according as (i), (ii), (iii) occurs, respectively, in the analysis of the value M_2. First, for $\beta_2 < \beta \leq \beta_1 < 2/5$, $M = M_4$ is attained when $s = \sqrt{\theta(\beta)} =: s_3, t = 1$ and $\cos(\phi - 2\psi) = \xi$, where ξ in (3.3) is now computed as

$$\xi = -\frac{(6\beta - 1)\{(8\beta - 1)(4\beta - 1)\theta(\beta) - (1 - \theta(\beta))\}}{2\theta(\beta)(4\beta - 1)(8\beta - 1)}.$$

We may take $\psi = 0$ so that Lemma 2.2 is applicable with $\varphi = \phi/2 \in (-\pi/2, \pi/2)$ and $c = s_3$. Now we obtain the form of an extremal function in the second case of Theorem 1.1 by (2.3) and (2.4), where we observe

$$\cos\theta_0 = s_3^2(1 - \cos\phi) - 1 = \frac{4(72\beta^2 - 41\beta + 5)}{8\beta - 3} = \frac{4(72\alpha^2 - 103\alpha + 36)}{5 - 8\alpha}$$

and, after some simplifications,

$$\lambda_1 = \frac{(5\beta - 2)(8\beta - 1)}{144\beta^3 - 58\beta^2 - 2\beta\sqrt{(3\beta - 1)(72\beta - 29)(24\beta^2 - 15\beta + 2)}}.$$

(c) When $2/5 \leq \beta < 1/2$, similarly, we have

$$M = \max\left\{\frac{9(8\beta - 1)(4\beta - 1)\sqrt{6\beta - 1}}{2\sqrt{2}\beta(24\beta - 5)^{3/2}}, (4\beta - 1)(8\beta - 1)\right\}.$$

Let

$$H(\beta) = \left(\frac{9\sqrt{6\beta - 1}}{2\sqrt{2}\beta(24\beta - 5)^{3/2}}\right)^2 = \frac{81(6\beta - 1)}{8\beta^2(24\beta - 5)^3}.$$
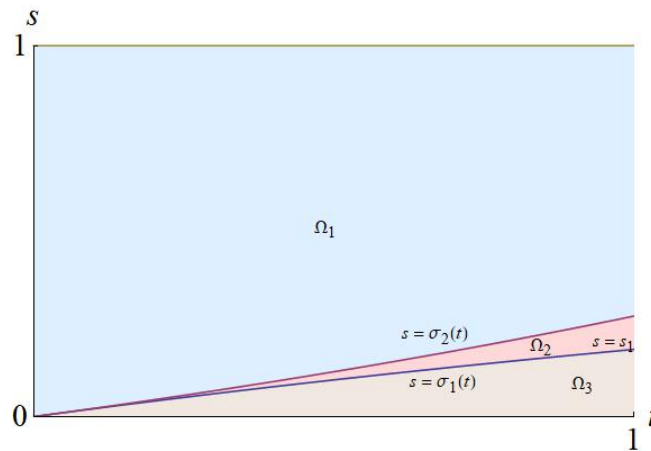
Then

$$H'(\beta) = -\frac{81(288\beta^2 - 75\beta + 5)}{4\beta^3(24\beta - 5)^4} = -\frac{81\{2(12\beta - 2)^2 + 3(7\beta - 1)\}}{4\beta^3(24\beta - 5)^4} < 0$$

for $2/5 \leq \beta < 1/2$. Therefore, the function $H(\beta)$ is decreasing there so that

$$H(\beta) \leq H(2/5) = \frac{354375}{389344} < 1$$

for $2/5 \leq \beta < 1/2$. We thus get $M = (4\beta - 1)(8\beta - 1)$.

FIGURE 3. The regions $\Omega_1, \Omega_2, \Omega_3$ when $\beta = 1$

Since this case corresponds to the condition $s = |u| = 1$, we have the simplest extremal function $f = K_\alpha$.

Subcase Ib: When $\frac{1}{2} \le \beta \le 1$, $t_1 \ge 1$. We focus only on the special case $\beta = 1$ since $\frac{1}{2} \le \beta < 1$ can be proved by the same technique showing in Case II. See Figure 3 for the domains $\Omega_1, \Omega_2, \Omega_3$ when $\beta = 1$. Our aim is to show that $M(1) = 21$.

If $\beta = 1$, F_1, F_2, F_3 have the following forms.

$$F_1(s,t) = 21s^3 + 10(1 - s^2)st - (1 - s^2)st^2 + (1 - s^2)(1 - t^2),$$

$$F_2(s,t) = \sqrt{\frac{25 - 4s^2}{21}}[21s^2 + (1 - s^2)t^2] + (1 - s^2)(1 - t^2),$$

$$F_3(s,t) = -21s^3 + 10(1 - s^2)st + (1 - s^2)st^2 + (1 - s^2)(1 - t^2).$$

A computation shows that

$$\frac{\partial F_1}{\partial s} = 3s^2(21 - 10t + t^2) - 2s(1 - t^2) + (10 - t)t > 0, \quad \text{for } (s,t) \in \Omega_1$$

$$\frac{\partial F_2}{\partial t} = 2\sqrt{\frac{25 - 4s^2}{21}}(1 - s^2)t - 2(1 - s^2)t > 0, \quad \text{for } (s,t) \in \Omega_2$$

$$F_3(s,t) \le -21s^3 + 12 < 21, \quad \text{for } (s,t) \in \Omega_3.$$

Therefore

$$M(1) = \max_{s \in [0,1]}\{F_1(1,t), F_2(s,1), 21\} = \max_{s \in [0,1]}\left\{\sqrt{\frac{25 - 4s^2}{21}}(1 + 20s^2), 21\right\} = 21$$

since $F_1(1,t) \equiv 21$ and $F_2(\sigma_1(t),t) = F_3(\sigma_1(t),t) < 21$ for $t \in [0,1]$. The extremal function is the Koebe function K_0.

Case II: When $\frac{4}{3} \le \alpha < 1$, for any function $f(z) = z + \displaystyle\sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}^*(\alpha)$, we have

$f_\gamma(z) = \frac{1}{\gamma}f(\gamma z) \in \mathcal{S}^*(\alpha) \subset \mathcal{S}^*(\alpha')$ where α' is any constant in $(\alpha_3, 3/4)$ and $\gamma^3 = \frac{1 - \alpha'}{1 - \alpha} > 1$.

In view of (3.1) and the result proved in Case I for f_γ, we have

$$\left|\gamma^3(-a_4 + 5a_2a_3 - 5a_2^3)\right| \le \frac{2}{3}(1 - \alpha')$$

which is

$$\left|-a_4 + 5a_2a_3 - 5a_2^3\right| \le \frac{2}{3}(1 - \alpha).$$

On the other hand the function $K_{\alpha,3}$ makes the equality hold.

   In this way, we have proven (3.2) as required and finish the proof of the main theorem, up to Lemma 3.1.

4. Proof of Lemma 3.1

   The remaining task is to show that $F_3(\sigma_1(t), t) \le \max\{1, F_3(s_1, 1)\}$ for $0 \le t \le 1$, where $s_1 = s_1(\beta)$ is given in (3.4). The standard method using Langrange multipliers seems to involve messy calculations. Thus we will take another approach. It is convenient in the sequel to consider the inverse function $\tau_1 : [0, s_1] \to [0, 1]$ of $\sigma_1 : [0, 1] \to [0, s_1]$. That is,

$$t = \tau_1(s) = \frac{(8\beta - 1)(4\beta - 1)s^2 + s\sqrt{(8\beta - 1)(4\beta - 1)\{(6\beta - 1)^2 - 4\beta^2 s^2\}}}{(6\beta - 1)(1 - s^2)}$$

$$= \frac{\kappa s^2/(6\beta - 1) + s\sqrt{\kappa(1 - \mu s^2)}}{1 - s^2}$$

for $s \in [0, s_1]$, where

$$\kappa = (8\beta - 1)(4\beta - 1) > 0 \quad \text{and} \quad \mu = \frac{4\beta^2}{(6\beta - 1)^2} = \frac{4\beta^2}{\kappa + 4\beta^2} \in (0, 1).$$

Then it is enough to show that the function

$$w(s) = F_3(s, \tau_1(s))$$

satisfies $w(s) \le \max\{1, F_3(s_1, 1)\}$ for $0 \le s \le s_1$. A straightforward computation gives the expression

$$w(s) = 1 + \frac{s^2}{1 + s}v(s),$$

where

$$v(s) = 2\kappa\mu s^2 + (\kappa - 1)s - (1 + \kappa) + 2\kappa(1 + \mu s)\sqrt{\frac{1 - \mu s^2}{1 - \mu}}.$$

We now analyze the behaviour of $v(s)$ over $[0, s_1]$. First we compute

$$v'(s) = 4\kappa\mu s + \kappa - 1 + \frac{2\kappa\mu(1 - s - 2\mu s^2)}{\sqrt{(1 - \mu)(1 - \mu s^2)}}.$$

We note here that $1 - s - 2\mu s^2 \ge 1 - s_1 - 2\mu s_1^2$ for $s \in [0, s_1]$. By using the expression

$$1 - s_1 - 2\mu s_1^2 = 1 - \frac{1}{2}\sqrt{\frac{6\beta - 1}{2\beta(24\beta - 5)}} - \frac{1}{(6\beta - 1)(24\beta - 5)} =: u(\beta),$$

we can check that $u'(\beta) > 0$ on $1/4 < \beta < 1/2$ and thus $u(\beta)$ is increasing there. We compute $u(\frac{17}{52}) = \frac{587}{925} - \frac{6}{17\sqrt{37}} = 0.0060\cdots > 0$. Hence, $1 - s - 2\mu s^2 \ge 0$ over $s \in [0, s_1]$ for

$17/52 \leq \beta < 1/2$. Note that $17/52 \approx 0.326923$. We divide the situation into four cases to show that $w(s) \leq \max\{1, w(s_1)\}$ over $[0, s_1]$.

Case I: When $\beta \in [3/8, 1/2)$, we have $\kappa \geq 1$ and $1 - s - 2\mu s^2 \geq 1 - s_1 - 2\mu s_1^2 \geq 0$ for $s \in [0, s_1]$. Then

$$v'(s) \geq 4\kappa\mu s + \frac{2\kappa\mu(1 - s - 2\mu s^2)}{\sqrt{(1-\mu)(1-\mu s^2)}} > 0$$

for $s \in [0, s_1]$. Hence $w(s)$ is increasing in $[0, s_1]$ and therefore

$$w(s) \leq w(s_1), \quad \text{for } s \in [0, s_1].$$

Case II: When $\beta \in [\frac{17}{52}, \frac{3}{8}]$, as we saw above, $\kappa \leq 1$. Since $\sqrt{1 - \mu s_1^2} \leq \sqrt{1 - \mu s^2} \leq 1$ for $0 \leq s \leq s_1$, we have

$$\sqrt{1 - \mu s^2}\, v'(s)$$

$$= (4\kappa\mu s + \kappa - 1)\sqrt{1 - \mu s^2} + \frac{2\kappa\mu(1 - s - 2\mu s^2)}{\sqrt{1-\mu}}$$

$$\geq 4\kappa\mu s\sqrt{1 - \mu s_1^2} + (\kappa - 1) + \frac{2\kappa\mu(1 - s - 2\mu s^2)}{\sqrt{1-\mu}}$$

$$= \frac{2\kappa\mu}{\sqrt{1-\mu}}\left[-2\mu s^2 + \left(2\sqrt{(1-\mu)(1-\mu s_1^2)} - 1\right)s + 1 + \frac{(\kappa-1)\sqrt{1-\mu}}{2\kappa\mu} \right]$$

$$=: \frac{2\kappa\mu}{\sqrt{1-\mu}} f(s)$$

for $s \in [0, s_1]$. Since $f(s)$ is concave, $f(s) \geq \min\{f(0), f(s_1)\}$ for $s \in [0, s_1]$. Noting the estimate $\sqrt{(4\beta - 1)(8\beta - 1)} \geq \frac{2\sqrt{21}}{13} = 0.7050\cdots > 7/10$, we obtain

$$f(0) = 1 + \frac{(\kappa-1)\sqrt{(1-\mu)}}{2\kappa\mu} = 1 - \frac{(3 - 8\beta)(6\beta - 1)}{2\beta\sqrt{(4\beta - 1)(8\beta - 1)}}$$

$$\geq 1 - \frac{5(3 - 8\beta)(6\beta - 1)}{7\beta} = \frac{3(16\beta - 5)(5\beta - 1)}{7\beta} > 0.$$

We also compute

$$f(s_1) = 1 - 2\mu s_1^2 + \frac{(\kappa-1)\sqrt{(1-\mu)}}{2\kappa\mu} + \left(2\sqrt{(1-\mu)(1-\mu s_1^2)} - 1\right)s_1$$

$$= 1 + \frac{3(4\beta - 1)(8\beta - 1) - 2\beta}{2\beta(6\beta - 1)(24\beta - 5)} + \frac{(8\beta - 3)(6\beta - 1)}{2\beta\sqrt{(4\beta - 1)(8\beta - 1)}} - \frac{1}{2\beta}\sqrt{\frac{6\beta - 1}{48\beta - 10}}$$

$$=: 1 + q_1(\beta) + q_2(\beta) - q_3(\beta).$$

Simple calculations yield

$$q_1'(\beta) = \frac{-3(5 - 108\beta + 956\beta^2 - 3648\beta^3 + 4608\beta^4)}{2\beta^2(24\beta - 5)^2(6\beta - 1)^2}$$

and

$$q_2'(\beta) = \frac{-3 + 54\beta - 300\beta^2 + 544\beta^3}{4\beta[(4\beta - 1)(8\beta - 1)]^{3/2}}.$$

One can easily check that the numerators of $q_1'(\beta)$ and $q_2'(\beta)$ are concave on the interval $[\frac{17}{52}, \frac{3}{8}]$, and that both of them take positive values at the end points $17/52$ and $3/8$. Therefore q_1 and q_2 are increasing on $[\frac{17}{52}, \frac{3}{8}]$. Since q_3 can be written in the form

$$q_3(\beta) = \frac{1}{2\beta}\sqrt{\frac{1}{8} + \frac{1}{4(48\beta - 10)}},$$

it is obvious that it is decreasing on $[\frac{17}{52}, \frac{3}{8}]$. By virtue of the computation

$$1 + (q_1 + q_2 - q_3)\left(\frac{17}{52}\right) = \frac{23083}{15725} - \frac{125}{34\sqrt{21}} - \frac{65}{17\sqrt{37}} = 0.037\cdots > 0,$$

we conclude that $v'(s)$ is positive on $[0, s_1]$. Therefore, no matter whether $v(s_1)$ is positive or negative, we have

$$w(s) = 1 + \frac{s^2}{1+s}v(s) \le \max\{w(0), w(s_1)\} = \max\{1, w(s_1)\}, \quad \text{for } s \in [0, s_1].$$

Case III: When $\beta \in [8/25, 17/52]$, we observe that $\sqrt{1 - \mu} \ge \sqrt{1 - \mu(8/25)} = \sqrt{273}/23 = 0.71838\cdots > 51/71 = 1/A$, because $\mu = \mu(\beta)$ is decreasing in $8/25 \le \beta \le 17/52$, where $A = 71/51$. Note also $s_1(8/25) = \frac{25}{16}\sqrt{\frac{23}{134}}$. Then, for $s \in [0, s_1]$ we have

$$v(s) = 2\kappa\mu s^2 + (\kappa - 1)s - (1 + \kappa) + 2\kappa(1 + \mu s)\sqrt{\frac{1 - \mu s^2}{1 - \mu}}$$

$$\le 2\kappa\mu s^2 + (\kappa - 1)s - (1 + \kappa) + 2A\kappa(1 + \mu s)(1 - \mu s^2/2)$$

$$=: g(s).$$

We will show that $g(s)$ in increasing on $[0, s_1]$ in this case. To do that, we compute

$$\frac{g'(s)}{\kappa\mu} = 4s + \frac{\kappa - 1}{\kappa\mu} + A(2 - 2s - 3\mu s^2)$$

$$= 2(2 - A)s + \frac{\kappa - 1}{\kappa\mu} + A(2 - 3\mu s^2),$$

which is a concave function of s. Thus $g'(s) \ge \min\{g'(0), g'(s_1)\}$ for $s \in [0, s_1]$. Introducing the new variable $x = 16\beta - 3$, we compute

$$\frac{\kappa - 1}{\kappa\mu} = \frac{(6\beta - 1)^2(8\beta - 3)}{\beta(4\beta - 1)(8\beta - 1)} = \frac{(x - 3)(3x + 1)^2}{(x^2 - 1)(x + 3)} =: h(x).$$

Note that the range of x is now $[\frac{53}{25}, \frac{29}{13}]$. Since

$$h'(x) = \frac{16(x^3 + 3)(3x + 1)}{(x^2 - 1)^2(x + 3)^2} > 0,$$

for $x \in [\frac{53}{25}, \frac{29}{13}]$, $h(x)$ is increasing and so that $g'(0)/(\kappa\mu) = h(x) + 2A \geq h(\frac{53}{25}) + 2A = \frac{46}{\sqrt{273}} - \frac{5819}{2184} = 0.1196\cdots > 0$. We also observe that $h(x)$ is convex on $1 < x < +\infty$ by

$$h''(x) = -\frac{16\,(6x^6 + 3x^5 + 9x^4 + 82x^3 + 108x^2 + 27x + 21)}{(x-1)^3(x+1)^3(x+3)^3}.$$

In particular, $(\kappa - 1)/(\kappa\mu)$ is a concave function of β on $(1/4, 1/2)$. Secondly, another computation yields

$$(4.1) \qquad \frac{g'(s_1)}{\kappa\mu} = 2(2-A)s_1(\beta) + \frac{\kappa - 1}{\kappa\mu} + \frac{A}{2} \cdot \frac{576\beta^2 - 216\beta + 17}{(6\beta - 1)(24\beta - 5)}.$$

On the interval $[\frac{8}{25}, \frac{17}{52}]$, we estimate, by convexity,

$$2(2-A)s_1'(\beta) > 2s_1'(1/4) = -20$$

and, by concavity,

$$\frac{d}{d\beta}\left(\frac{\kappa - 1}{\kappa\mu}\right) = 16h'(16\beta - 3) \geq 16h'(7/3) = \frac{1431}{50} > 20.$$

Hence the sum of the first two terms of $g'(s_1)/(\kappa\mu)$ in (4.1) is an increasing function of β on $[8/25, 17/52]$. On the other hand, by using the same change of variables as before, we have

$$\frac{576\beta^2 - 216\beta + 17}{(6\beta - 1)(24\beta - 5)} = \frac{2\,(9x^2 - 13)}{(3x-1)(3x+1)} = 2 - \frac{24}{9x^2 - 1},$$

which is obviously increasing in $x > 1/3$. Therefore, the third term is also increasing in β. In summary, $g'(s_1)/(\kappa\mu)$ is increasing in β so that its minimum over $[\frac{8}{25}, \frac{17}{52}]$ is taken at $\beta = 8/25$ and its value is

$$\frac{g'(s_1(8/25))}{\kappa\mu} = \frac{34312\sqrt{273} + 13650\sqrt{3082} - 575\sqrt{841386} - 779746}{292656} = 0.0599\cdots > 0.$$

Hence we have proved that $g(s)$ is increasing on $[0, s_1]$ for each $\beta \in [\frac{8}{25}, \frac{17}{52}]$. Thus we need only to show that the value $g(s_1)$ is negative. Now we write $g(s_1)$ in the form

$$\begin{aligned}
g(s_1) &= 2\kappa\mu s_1^2 + (\kappa - 1)s_1 - (1 + \kappa) + 2A\kappa(1 + \mu s_1)(1 - \mu s_1^2/2) \\
&= (2 - A)\big[\kappa\mu s_1^2\big] + \big[(\kappa - 1)s_1\big] + \big[2A\kappa - \kappa - 1\big] + 2A\big[\mu\kappa s_1\big] + A\big[-\kappa\mu^2 s_1^3\big] \\
&= (2 - A)T_1 + T_2 + T_3 + 2AT_4 + AT_5.
\end{aligned}$$

By virtue of

$$T_1'(\beta) = \frac{16\beta - 3}{(24\beta - 5)^2(6\beta - 1)^2},$$

$$T_2'(\beta) = \frac{1152\beta^2 - 456\beta + 49}{(24\beta - 5)^2}\sqrt{\frac{48\beta - 10}{6\beta - 1}},$$

$$T_3(\beta) = (2A - 1)(4\beta - 1)(8\beta - 1) - 1,$$

$$T_4(\beta) = \sqrt{2}\beta\sqrt{\frac{(4\beta - 1)(8\beta - 1)}{(6\beta - 1)^2}}\sqrt{2T_1(\beta)},$$

$$T_5(\beta) = -\frac{1}{\sqrt{(48\beta - 10)(6\beta - 1)}}\frac{(4\beta - 1)(8\beta - 1)\beta}{(6\beta - 1)^3(24\beta - 5)}$$

$$=: -\frac{1}{\sqrt{(48\beta - 10)(6\beta - 1)}}t_5(\beta)$$

and

$$t_5'(\beta) = \frac{5 - 60\beta - 24\beta^2 + 1920\beta^2 - 4608\beta^4}{(24\beta - 5)^2(6\beta - 1)^4} < 0, \quad \text{for } \beta \in [\tfrac{8}{25}, \tfrac{17}{52}]$$

by the concavity of the numerator of $t_5'(\beta)$, each $T_j = T_j(\beta)$ is increasing in $8/25 \le \beta \le 17/52$. We thus obtain

$$g(s_1) \le g(s_1(17/52)) = \frac{7670451\sqrt{37} - 54764625}{945465625} = -0.00857\cdots < 0.$$

Hence, we see that $v(s)$ is always negative on $[0, s_1]$ for $\beta \in [\tfrac{8}{25}, \tfrac{17}{52}]$. Therefore in this case

$$w(s) \le 1, \quad \text{for } s \in [0, s_1].$$

Case IV: When $\beta \in [\tfrac{1}{4}, \tfrac{8}{25}]$, we have the estimates

$$v(s) = 2\kappa\mu s^2 + (\kappa - 1)s - (1 + \kappa) + 2\kappa(1 + \mu s)\sqrt{\frac{1 - \mu s^2}{1 - \mu}}$$

$$\le 2\kappa\mu s^2 + (\kappa - 1)s - (1 + \kappa) + 2A\kappa(1 + \mu s)$$

$$=: l(s)$$

$$\le \max\{l(0), l(s_1)\}$$

for $s \in [0, s_1]$, where $A = 71/51$ as before. Next we will show that both $l(0)$ and $l(s_1)$ are negative. First, since $A < 7/5$, we have

$$l(0) = -(1 + \kappa) + 2A\kappa < -1 - \kappa + \frac{14}{5}\kappa = \frac{4}{5}(3\beta - 1)(24\beta - 1) < 0.$$

Second, a computation shows that

$$l(s_1) = 2\kappa\mu s_1^2 + (\kappa - 1)s_1 - (1 + \kappa) + 2A\kappa(1 + \mu s_1)$$

$$= 2[\kappa\mu s_1^2] + [(\kappa - 1)s_1] + [2A\kappa - \kappa - 1] + 2A[\mu\kappa s_1]$$

$$= 2T_1 + T_2 + T_3 + 2AT_4,$$

where T_1, T_2, T_3 and T_4 have the same form as in Case III. The same arguments as in Case III show that all the terms $T_j = T_j(\beta)$ are increasing in β as before. Hence, we obtain

$$l(s_1) \leq l(s_1(8/25)) = \frac{37025\sqrt{3082} - 4054049}{110759375} = -0.0180\cdots < 0.$$

We find that $v(s)$ is always negative on $[0, s_1]$ for $\beta \in [\frac{1}{4}, \frac{8}{25}]$. Therefore in this case too, we have

$$w(s) \leq 1, \quad \text{for } s \in [0, s_1].$$

Considering the above four cases together, we finally proved that $w(s) \leq \max\{1, w(s_1)\}$ over $[0, s_1]$.

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