The optimal tax model

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Abstract:

In this paper we analyze and propose new method and algorithm of selecting the optimal labor time as a function of skills following our main references Mirrlees (1971), Saez (2001) and Stancheva (2014).

The optimal labor time is a situation when the utility function of individual reaches a maximum.

One of the main differences with Saez (2001) is our algorithm for creation the skill distribution.

Saez (2001) proposed and considered the distribution of skills with using empirical distribution of income and the approximation the labor income tax by linear model. The author proved that with using the presented utility function of consumption and labor effort, it will be possible to obtain some level of skills which will be adaptable with observed (public information) taxable revenue which corresponds proposed linear tax schedule

Keywords:

Optimal income taxation; Tax progressivity; non-linear tax, quality criterion, approximation of a convex function.

Introduction. The theory of optimal taxation is an investigation of development of the tax that decreases interferences and deformation (caused by the tax schedule) in the market under some constraints. One of the first applications of this theory is an analysis of the optimal income tax mechanism (Mirrlees). The agents have innate abilities for producing and obtaining income. However, the abilities are known and available only to the agents, but not to the developers of the income tax system. The developers maximize some function of social welfare, where the egalitarian preferences (opportunity to equalize the individual utilities of agents) are taken into account to some extent

Let us introduce the following notation:

1) n is an individual employee productivity = payment for her/him per unit of time. This productivity is considered to be a positive random variable (it is different for different workers) with a probability density distribution $f(x), x > 0;\ f(x) = 0, if\ \ x \leq 0$:

$$P\{a < n < b\} = \int_a^b f(x)dx.$$

2) l is an individual labour time $l \leq L;\ z = nl$ (gross earned income).

3) $T(z)$ is an income tax of z; $c = z - T(z)$ is a net income after tax.

The assumptions on the function $T(z)$ are given below:

4) $u(c, l)$ is a utility function

$$u(c, l) \geq 0, \quad u(0, l) = 0.$$

So $u(c, l)$ is assumed to be the same for all individuals. This function increases and is convex upward (concave) in the first variable c and decreases in the second variable l. The utility function of a taxpayer takes the following form :

$$u(c,l) = u(nl - T(nl), l) \text{ , so it depends on the variables } n, l.$$

The key to the following articles is a proposal for the tax productivity n

Saez (2001) has obtained optimal income tax formulas (linear and nonlinear case) using compensated and uncompensated elasticities, Pareto parameter and social marginal welfare weights. Optimal tax rate can't be negative and marginal tax rate should be zero at the top (if the skill distribution is bounded). Numerical experiment has been performed with simulation of optimal tax rates with calculation of elasticities and main Social Welfare criteria.

Mirrlees(1971) has explained that the distribution of skills is a very general aspect and its relation to the optimal tax rate, which is very difficult to observe and without an incentive; the individuals will not have an interest to declare information about their abilities (it depends the possibility of higher taxes). The Spence Mirrlees condition is to replace skills distribution with income distribution but it is not such a strong condition for the quasi-linear utility function.

Saez and Stantcheva (2016) study a new approach to estimate social marginal welfare weights with using the observed income levels. Every weight at the income level depends on agent's characteristics without taking account of the individual utilities. The authors demonstrated that the wages usually depend on net taxes paid produces and the proposition may considered with horizontal equity concepts.

The article of Diamond (1998) is dedicated to quasi-linear function with no income effect and explaines Mireless framework that could be obtained with a use of labor supply elasticity. The author has got some results for asymptotic tax rates (with unbounded distribution of skills and unbounded growth) and calculated the marginal tax rate for the income increase, which would maximize the welfare.

Saez, Stancheva, Piketty (2014) suggested that every employer chooses his optimal (which depends from his productivity n) labor time $l_n = \varphi(n)$. This choice is obtained by maximizing the utility function with respect to the variable l. Let's introduce:

$$u(nl - T(nl), l) = v(l)$$

and find the global maximum point of the function $v(l)$, where the variable l_n $exists$, so

$v(l_n) > v(l)$ for the other values of l with fixed n.

If this maximum exists , then $l_n \equiv l(n)$ satisfies the equation.

$$v'(l) = n\frac{\partial u}{\partial c}\left(1 - T'(nl)\right) + \frac{\partial u}{\partial l} = 0 \qquad (1)$$

The expression $u(nl_n - T(nl_n), l_n)$ depends only on the random variable n (the productivity of the individual) in these assumptions. The gross income $z_n = nl_n$ and the net income $c_n = z_n - T(z_n)$ are defined implicitly by the equation (1) with the productivity functions n .

The quality criterion is a maximum of the functional

$$W(T) = \mathrm{E}G\left(u(nl_n - T(nl_n), l_n)\right) = \int_0^\infty G\left(u(nl_n - T(nl_n), l_n)\right)f(n)dn$$

Under the budget constraint

$$\mathrm{E}\left(T(nl_n)\right) = \int_0^\infty T(nl_n)f(n)dn \geq K$$

where $G(t)$ is a positive increasing concave function, E is also known as the expectation (mean)), which is corresponding to the random productivity distribution n. Let's assume that $G(t) = t$.

Note that the key assumptions is that choices for the working time are made only by certain group of workers. This assumption leads to find the maximum value in the mathematical model: the equation (1) defines only a local maximum point under the additional condition

$$v''(l) = \frac{d^2}{dl^2}\, u(nl - T(nl), l) = n^2\left(1 - T'\right)^2 \frac{\partial^2 u}{\partial c^2} + 2n\left(1 - T'\right)\frac{\partial^2 u}{\partial c \partial l} +$$

$$+\frac{\partial^2 u}{\partial l^2} - n^2 T'' \frac{\partial u}{\partial c} < 0 \qquad (2)$$

The solution of system (1), (2) requires the application of the implicit function theorem, which is local in nature.

In [1] the existence of a global maximum function $v(l)$ seems problematic even for the simple examples of the utility functions.

This paper is a continuation of research on optimal taxation.

The proposed optimality criterion and statement of problem. The other approaches to calculate the optimal tax are as follows.Let's assume a pair (n, l) as a random vector $n > 0,\ 0 \le l \le L$ with distribution density $f(x, y)$: $\int_0^\infty \int_0^L f(x, y)dydx = 1$. Here L is a maximum labour time for an individual.

The plot of function $T(z)$ presented in Fig. 1 with some simplification.

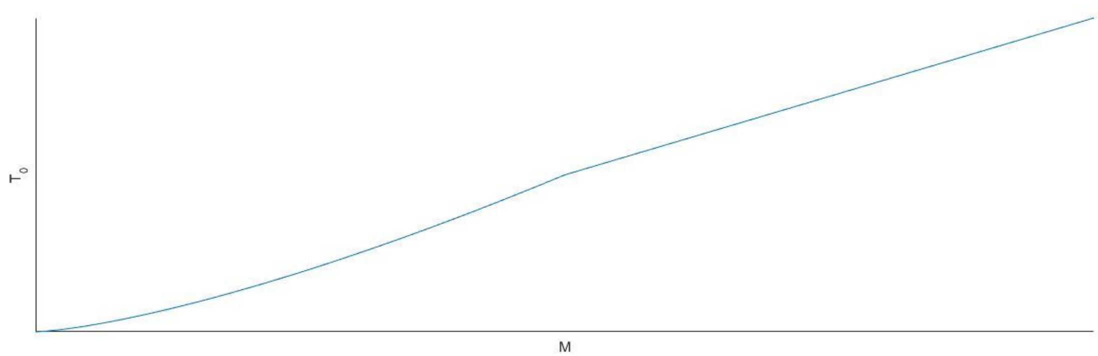


Fig.1. Tax as a function of gross income

If the gross income $z \in [0; M]$, then the tax is progressive , and the curve $T(z)$ is convex. For $z > M$ the function $T(z)$ is linear . Otherwise:

$$0 \leq z \leq M \Leftrightarrow 0 < T'(z) \leq T'(M) \equiv A < 1; \quad T''(z) \geq 0; \; T(z) < T(M) \equiv T_0 \; .$$

$$z > M \Leftrightarrow T(z) = \frac{T_0}{M} z \; .$$

Here $\frac{T_0}{M}$ is a maximum tax rate ($\frac{T_0}{M} = 0{,}45 \; in \; FRance$) and the constraint $A < 1$ is empirical

Next, the situation is considered.$z \in [0; M]$. The quality criterion in determining the formula for $T(z)$ is a maximum of functional

$$W(T) = \iint_D u(xy - T(xy), y) f(x, y) dxdy$$

where the area D defined by inequalities: $0 \leq xy \leq M, x \geq 0, 0 \leq y \leq L$ under some budgetary constraints

$$\iint_D T(xy) f(x, y) dxdy \geq K$$

Thus, the tax optimization is reduced to a variation problem

$$W(T) = \iint_D u(xy - T(xy), y) f(x, y) dxdy \to max \qquad (3)$$

$$D: \{0 \leq xy \leq M, x \geq 0, 0 \leq y \leq L\}$$

under the constraints

$$\iint_D T(xy) f(x, y) dxdy \geq K \qquad 0 < T'(z) \leq A < 1; \quad T''(z) \geq 0 \qquad (4)$$

Let us replace the integration variables in (3) and (4)

$$xy = z, y = y; \text{ Jacobian } J(y,z) = \frac{1}{y}, \ 0 \le z \le M, \ 0 \le l \le L \tag{5}$$

Then the problem is reduced to

$$W(T) = \int_0^M \int_0^L \frac{1}{y} \, u(z - T(z), y) f\left(\frac{z}{y}, y\right) dydz \to max \tag{6}$$

$$\int_0^M \int_0^L \frac{1}{y} \, T(z) f\left(\frac{z}{y}, y\right) dydz \ge K, \quad 0 < T'(z) \le A < 1; \quad T''(z) \ge 0 \tag{7}$$

The aim of this research is to propose a method for solving problem (6), (7) and to perform a numerical experiment for the simulation data.

Let us introduce the notation for the inside integrals:

$$\int_0^L \frac{1}{y} u(z - T(z), y) f\left(\frac{z}{y}, y\right) dy = \Phi\big(z, T(z)\big)$$

$$\int_0^L \frac{1}{y} T(z) f\left(\frac{z}{y}, y\right) dy = \Psi\big(z, T(z)\big)$$

It reduces the original problem to a one-dimensional variational problem:

$$\int_0^M \Phi\big(z, T(z)\big) dz \to max$$

$$\int_0^M \Psi\big(z, T(z)\big) dz \ge K, \quad 0 < T'(z) \le A < 1; \quad T''(z) \ge 0 \ .$$

It is proposed to find a solution to this problem by approximating the tax with convex functions

$$T(z) = \sum_{j=1}^N a_j \varphi_j(z), \quad \varphi_j(0) = 0, \ \varphi_j''(z) \ge 0, \ 0 \le z \le M.$$

where the non-negative coefficients a_j are satisfy by some empirical constraints.

In particular,

$$\varphi_1(z) = z, \varphi_2(z) = (z+1) ln(z+1), \varphi_3(z) = z^{1+\varepsilon}, \ldots,$$

$$\varphi_k(z) = z^{1+\varepsilon(k-2)},\ k \le N. \tag{8}$$

Assuming

$$F(a_1, \dots, a_N) = \int_0^M \Phi\left(z, \sum_{j=1}^N a_j \varphi_j(z)\right) dz$$

$$Q(a_1, \dots, a_N) = \int_0^M \Psi\left(z, \sum_{j=1}^N a_j \varphi_j(z)\right) dz\ ,$$

we get the problem of finding the maximum of the function N of variables $F(a_1, \dots, a_N)$ with thr constraint $Q(a_1, \dots, a_N) \ge K$.

It is assumed that the utility function has the form:

$$u(c, l) = g(c)h(l); \qquad g: R^+ \to R^+;$$

$g\ (0) = 0$, g is increasing and concave$\{g: \uparrow, \frown\}$;

h is decreasing and concave $\{h: \downarrow, \frown\}$.

The condition $g(\infty) = \infty$ is inconsequential, if $c \le M$, so it allows to expand the number of examples.

In this case

$$\Phi\big(z, T(z)\big) = \int_0^L \frac{1}{y} g\big(z - T(z)\big) h(y) f\left(\frac{z}{y}, y\right) dy$$

The **Model example and numerical simulation .** The random values n, l are assumed independent, so

$f(x, y) = f_n(x) f_l(y)$, where the density $f_n(x)$ is Γ-distribution:

$$f_n(x) = \frac{\mu^{p+1}}{\Gamma(p+1)} x^p e^{-\mu x}\ , x > 0;\ \ f(x) = 0, if\ x \le 0.$$

The parameters p, μ are estimated by sample.

Then

$$W(T) = \int_0^M g\big(z - T(z)\big) q(z)\, dz \to max$$

where the function is:

$$\mathrm{q}(z) = \int_0^L h(y) \frac{1}{y} f_n\left(\frac{z}{y}\right) f_l(y) dy$$

Let's choose for this example:

The components of utility function:

$$g(z) = ln(1 + z);$$

$$h(y) = B\left(1 - e^{-\frac{2L}{y}}\right),\ 0 \le y \le L;\ \ h(y) = h(L) = B(1 - e^{-2})\ \ , y > L.$$

The distribution density

$$f_n(x) = \mu^2\, x e^{-\mu x}\ \ , x > 0; \qquad f_l(y) = \frac{1}{L},\ \ 0 \le y \le L;\ \ f_l(y) = 0, y > L\ .$$

This choice allows obtaining explicit analytic expressions for functions in a a direct variation problem . So,

$$\mathrm{q}(z) = \frac{\mu^2 B}{L} \int_0^L \left(1 - e^{-\frac{2L}{y}}\right) \frac{z}{y^2} e^{-\frac{\mu z}{y}} dy$$

and this integral can be computed by replacing $s = \frac{1}{y}$:

$$\mathrm{q}(z) = \frac{\mu B}{L} \frac{2L + \mu\text{я} - \mu z e^{-2}}{L(2L + \mu z)} e^{-\frac{\mu z}{L}}\ .$$

The integral constraint (7) is reduced to its form

$$\frac{\mu}{L}\int_0^M T(z)\, e^{-\frac{\mu z}{L}}\, dz \geq K$$

The final formulation:

to define a convex function

$$T(z) > 0\ ,\ 0 \leq z \leq M\ ,$$

as a solution to an extremal problem for an integral functional:

$$\int_0^M ln\big(1 + z - T(z)\big)\ \frac{2L+\mu я-\mu z e^{-2}}{L(2L+\mu z)}\ e^{-\frac{\mu z}{L}}\, dz \to max \tag{9}$$

which satisfy the below conditions:

$$\frac{\mu}{L}\int_0^M T(z)\, e^{-\frac{\mu z}{L}}\, dz \geq K, \qquad 0 < T'(z) \leq A < 1. \tag{10}$$

For example, let us assume that a standard interval of time is one week.

We may choose the following values of a parameter M, A, L, μ

$$M = 4000\ \left(\frac{euro}{week}\right);\ \ L = 40(hours).$$

Assume $\mu = 1,\ A = 0{,}8.$

The maximum taxable value T_0 is not fixed.

Assume the following (8):

$$T(z) = a_1 z + a_2(z+1)ln(z+1) + a_3 z^{1,1} + a_4 z^{1,2} + a_5 z^{1,3}$$

Let us choose the conditions which are more flexible for the taxpayer in the coefficients:

$$a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 > 0. \qquad (11)$$

The problems (9), (10) can be reduced of finding a conditional extremum of a function of two variables:

$F(a_1, \dots, a_5) = \int_0^M ln(1 + z - (a_1 z + a_2(z + 1)ln(z + 1) + a_3 z^{1,1} + a_4 z^{1,2} + a_5 z^{1,3})) \frac{2L+\mu я-\mu z e^{-2}}{L(2L+\mu z)} e^{-\frac{\mu z}{L}} dz \to max$;

$Q(a_1, \dots, a_5) = \frac{\mu}{L} \int_0^M (a_1 z + a_2(z + 1)ln(z + 1) + a_3 z^{1,1} + a_4 z^{1,2} + a_5 z^{1,3}) \times e^{-\frac{\mu z}{L}} dz = K;$

under the budget constraints (11).

The maximum of function F is finding with a fixed value K,

$1 \leq K \leq K_{max}$, where for $K > K_{max}$. In this case the optimization problem has no solution

The Values Of The Parameters Given In Table 1 ($K_{max} = 9$).

The tables 1and 2 show the coefficient values a_1, a_2, a_3, a_4, a_5;

The figures .2, 3 ,4 show the main tax schedules $T(z)$ and the derivative $T'(z)$ for $K = 1,\ K = 5,\ K = 9$..

Discussion. The given simulation example allows to evaluate the adequacy of the proposed mathematical model. The qualitative considerations are confirmed by results: 1. Tax burden ratios $T(z)$ to rise with the increased budget allocations K . The highest growth is expected with high lift coefficients (for example ,a_5) as the absence of constraint (11). Let us consider a problem of generalization, that is proposed in the [Kleven H J, Kreiner C T, Saez E. 2009], where the object of research is a married couple. Let us assume $\boldsymbol{n}$ and $\boldsymbol{l}$ as the random vectors for each couple: $\boldsymbol{n} = (n_1, n_2)$, $\boldsymbol{l} = (l_1, l_2)$, where n_k is a spouse's productivity, l_k is a labour time. Then the gross family income $z = \langle \boldsymbol{n}, \boldsymbol{l} \rangle = n_1 l_1 + n_2 l_2$, and the problem is reduced of maximizing the integral functional in 4-Dimensional Euclidean Space R^4 .

Conclusion In this paper , a mathematical model of the progressive tax as a solution to the variation problem under the budget constraints has been proposed . The numerical experiment is performed without the errors for a model example. The proposed method of formation tax debate is assuming the practical model with the worked examples of utility function.

APPENDIX

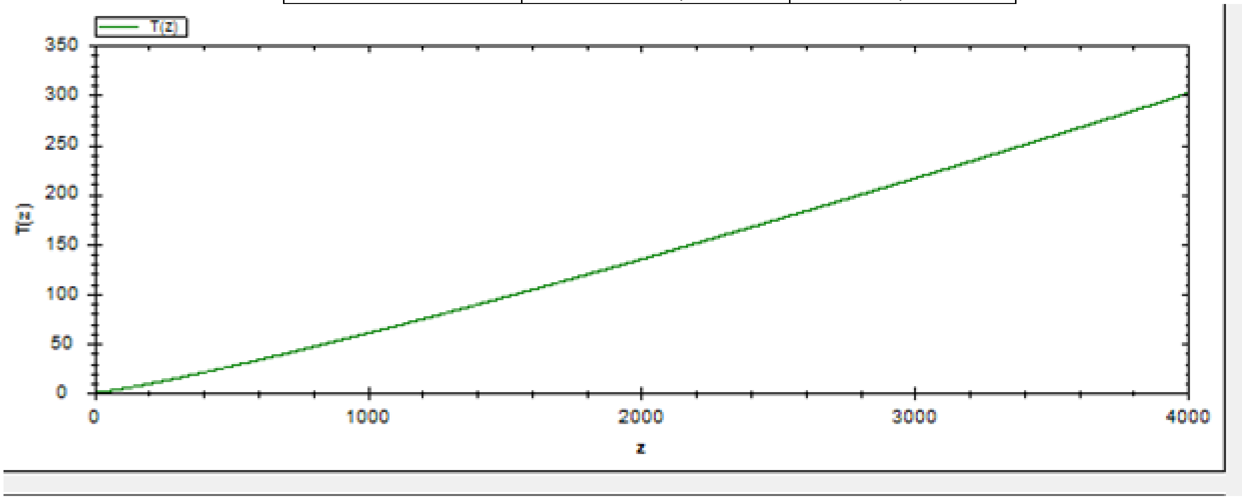
Table 1.the coefficient values $\boldsymbol{a_1, a_2, a_3, a_4, a_5}$

K	a_1	a_2	a_3	a_4	a_5
1	0,00495697	0,00415431	0,00330862	0,00216565	0,00141307
2	0,00991394	0,00830862	0,00661725	0,00433130	0,00282615
3	0,01487092	0,01246293	0,00992587	0,00649695	0,00423923

4	0,01982789	0,01661725	0,01323450	0,00866261	0,00565231
5	0,02478487	0,02077156	0,01654313	0,01082826	0,00706539
6	0,02974184	0,02492587	0,01985175	0,01299391	0,00847846
7	0,03469882	0,02908019	0,02316038	0,01515957	0,00989154
8	0,03965579	0,03323450	0,02646901	0,01732522	0,01130462
9	0,04461277	0,03738881	0,02977763	0,01949087	0,01271770

Table 2. Tax amount

K-value	Tax amount	Tax rate (%)
1	301,6	8,77
2	603,2	17,54
3	904,7	26,30
4	1206,3	35,07
5	1507,9	43,84
6	1809,5	52,61
7	2111,1	61,37
8	2412,7	70,14
9	2714,2	78,91



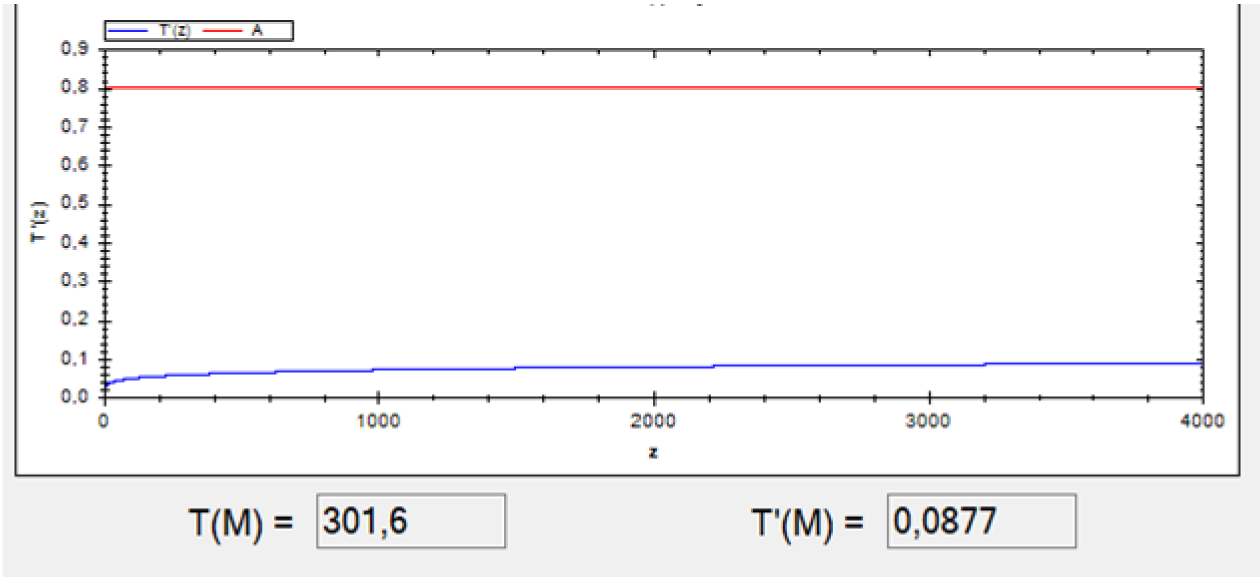


Fig.2 Plot $T(z)$, $T^{'}(z)$, $K = 1$.

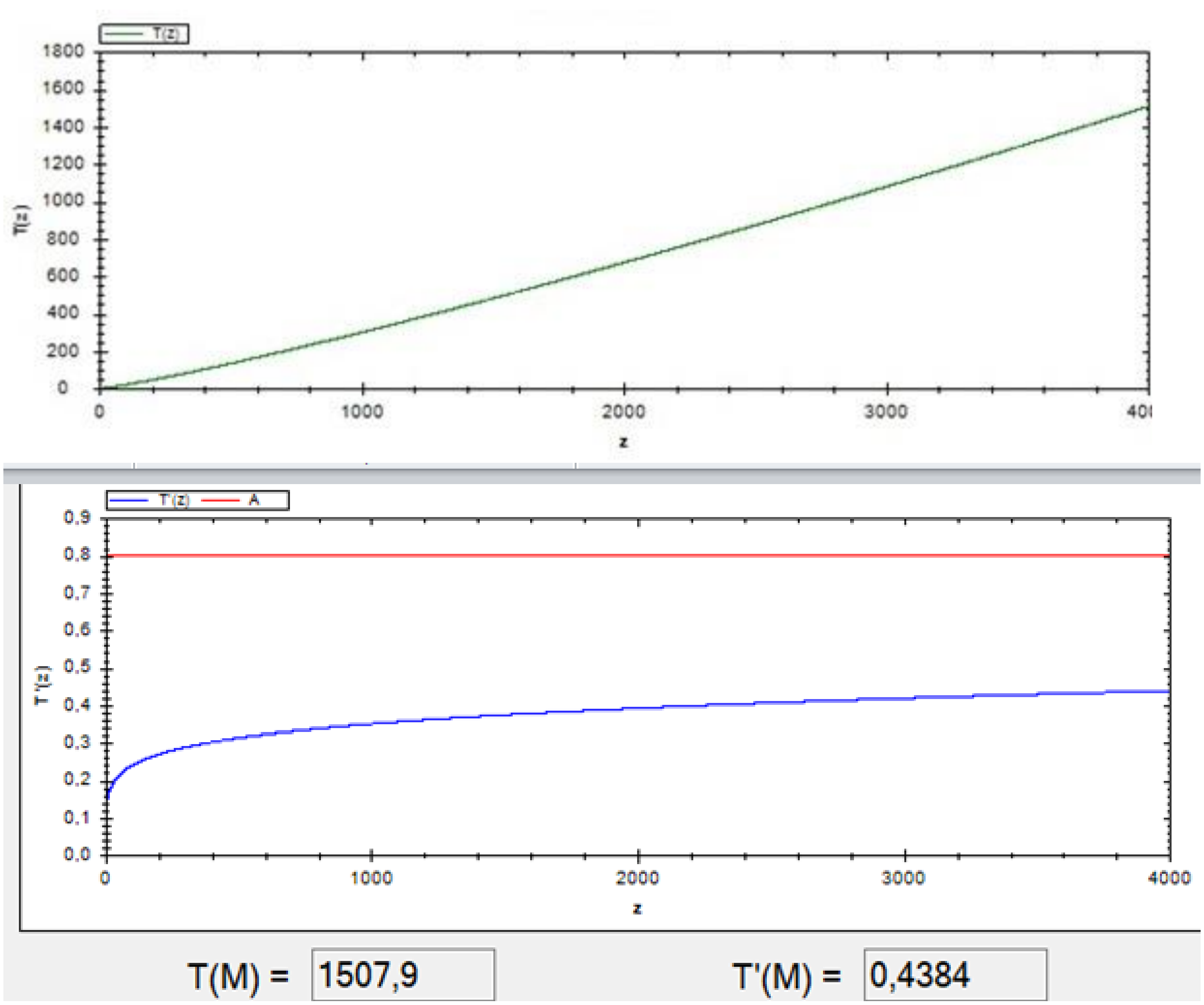


Fig.3 Plot $T(z),\ T^{'}(z),\ K = 5.$

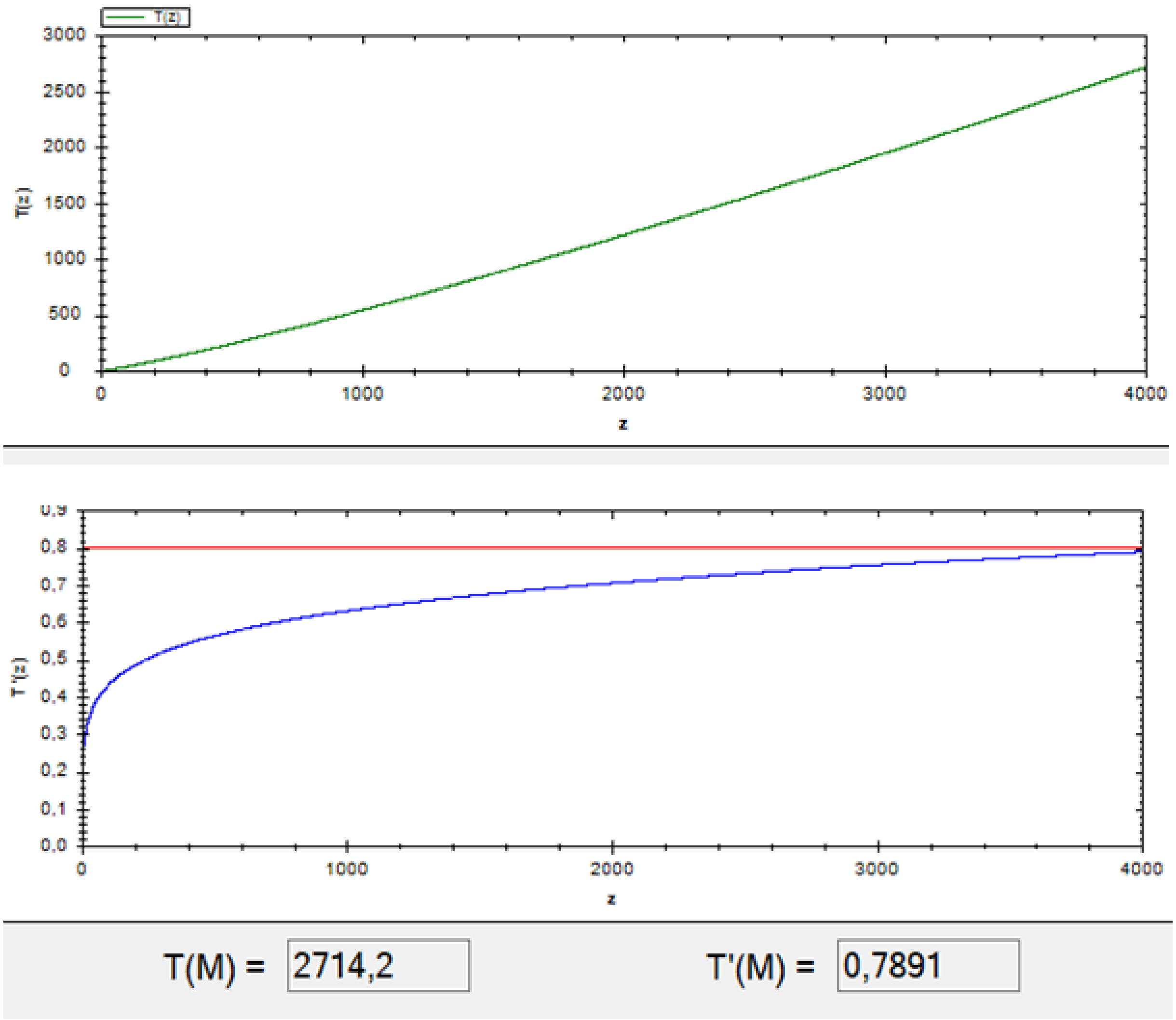


Fig 4. Plot $T(z),\ T^{'}(z),\ K = 9.$

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