

COMPLETE MONOTONICITY OF A DIFFERENCE CONSTITUTED BY FOUR DERIVATIVES OF A FUNCTION INVOLVING TRIGAMMA FUNCTION

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Dedicated to people facing and battling COVID-19

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ABSTRACT. In the paper, by virtue of convolution theorem for the Laplace transforms, Bernstein's theorem for completely monotonic functions, and other techniques, the author finds necessary and sufficient conditions for a difference constituted by four derivatives of a function involving trigamma function to be completely monotonic.

CONTENTS

1. Motivations	1
2. Lemmas	4
3. Necessary and sufficient conditions of complete monotonicity	5
4. Remarks	8
References	8

1. MOTIVATIONS

In the literature [1, Section 6.4], the function

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \Re(z) > 0$$

and its logarithmic derivative $\psi(z) = [\ln \Gamma(z)]' = \frac{\Gamma'(z)}{\Gamma(z)}$ are respectively called Euler's gamma function and digamma function. Further, the functions $\psi'(z)$, $\psi''(z)$, $\psi'''(z)$, and $\psi^{(4)}(z)$ are known as trigamma, tetragamma, pentagamma, and hexagamma functions respectively. As a whole, all the derivatives $\psi^{(k)}(z)$ for

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$k \in \{0\} \cup \mathbb{N}$ are known as polygamma functions, where $\mathbb{N}$ denotes the set of all positive integers.

Recall from Chapter XIII in [5], Chapter 1 in [19], and Chapter IV in [21] that, if a function $f(x)$ on an interval I has derivatives of all orders on I and satisfies $(-1)^n f^{(n)}(x) \geq 0$ for $x \in I$ and $n \in \{0\} \cup \mathbb{N}$, then we call $f(x)$ a completely monotonic function on I .

There are a number of papers and mathematicians dedicated to investigation of complete monotonicity of some functions involving the gamma and polygamma functions. For more information and details, please refer to the papers [2, 15, 17, 23] and closely related references therein.

Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \dots, \beta_n) \in \mathbb{R}^n$. A n -tuple α is said to strictly majorize β (in symbols $\alpha \succ \beta$) if $(\alpha_{[1]}, \alpha_{[2]}, \dots, \alpha_{[n]}) \neq (\beta_{[1]}, \beta_{[2]}, \dots, \beta_{[n]})$, $\sum_{i=1}^k \alpha_{[i]} \geq \sum_{i=1}^k \beta_{[i]}$ for $1 \leq k \leq n-1$, and $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i$, where $\alpha_{[1]} \geq \alpha_{[2]} \geq \dots \geq \alpha_{[n]}$ and $\beta_{[1]} \geq \beta_{[2]} \geq \dots \geq \beta_{[n]}$ are rearrangements of α and β in a descending order. A real-valued function ϕ defined on a set $\mathcal{A} \subset \mathbb{R}^n$ is said to be Schur-convex on $\mathcal{A}$ if $\mathbf{x} \prec \mathbf{y}$ for $\mathbf{x}, \mathbf{y} \in \mathcal{A}$ means $\phi(\mathbf{x}) < \phi(\mathbf{y})$. See [4, p. 8, Definition A.1] [4, p. 80, Definition A.1]. There have been a lot of literature such as the papers [3, 18, 20, 22, 24, 25] dedicated to investigation of Schur-convexity.

Let

$$G(x) = x[x\psi'(x) - 1] - \frac{1}{2} = x^2 \left[\psi'(x) - \frac{1}{x} - \frac{1}{2x^2} \right], \quad x \in (0, \infty).$$

For $k \in \{0\} \cup \mathbb{N}$ and $\theta_k, \tau_k \in \mathbb{R}$, let

$$\mathcal{G}_{k, \theta_k}(x) = G^{(2k+1)}(x) + \theta_k [G^{(k)}(x)]^2$$

and

$$\mathfrak{G}_{k, \tau_k}(x) = \frac{G^{(2k+1)}(x)}{[(-1)^k G^{(k)}(x)]^{\tau_k}}$$

on $(0, \infty)$. In [13, Theorem 3.1 and Theorem 4.1], the author discovered that,

- (1) if and only if $\theta_k \geq \frac{3(2k+2)!}{k!(k+1)!}$, the function $\mathcal{G}_{k, \theta_k}(x)$ is completely monotonic on $(0, \infty)$;
- (2) if and only if $\theta_k \leq 0$, the function $-\mathcal{G}_{k, \theta_k}(x)$ is completely monotonic on $(0, \infty)$;
- (3) if and only if $\tau_k \geq 2$, the function $\mathfrak{G}_{k, \tau_k}(x)$ is decreasing on $(0, \infty)$;
- (4) if $\tau_k \leq 1$, the function $\mathfrak{G}_{k, \tau_k}(x)$ is increasing on $(0, \infty)$;
- (5) only if

$$\tau_k \leq \begin{cases} \psi'(1), & k = 0 \\ -\frac{\psi'''(1)}{\psi'(1)\psi''(1)}, & k = 1 \\ \frac{k-1}{k} \frac{\psi^{(k-1)}(1)\psi^{(2k+1)}(1)}{\psi^{(k)}(1)\psi^{(2k)}(1)}, & k \geq 2, \end{cases}$$

the function $\mathfrak{G}_{k, \tau_k}(x)$ is increasing on $(0, \infty)$;

(6) the limits

$$\lim_{x \rightarrow 0^+} \mathfrak{G}_{k, \tau_k}(x) = \begin{cases} -2^{\tau_0}, & k = 0 \\ 6\psi''(1), & k = 1 \\ \frac{2(2k+1)}{(k-1)^{\tau_k} k^{\tau_k-1}} \frac{\psi^{(2k)}(1)}{|\psi^{(k-1)}(1)|}, & k \geq 2 \end{cases}$$

and

$$\lim_{x \rightarrow \infty} \mathfrak{G}_{k, \tau_k}(x) = \begin{cases} -\infty, & \tau_k > 2 \\ -\frac{3(2k+2)!}{k!(k+1)!}, & \tau_k = 2 \\ 0, & \tau_k < 2 \end{cases}$$

are valid;

(7) the double inequality

$$-\frac{3(2k+2)!}{k!(k+1)!} < \mathfrak{G}_{k,2}(x) < \begin{cases} -4, & k = 0 \\ 6\psi''(1), & k = 1 \\ \frac{2(2k+1)}{(k-1)^2 k} \frac{\psi^{(2k)}(1)}{|\psi^{(k-1)}(1)|}, & k \geq 2 \end{cases}$$

is valid on $(0, \infty)$ and sharp in the sense that the lower and upper bounds cannot be replaced by any greater and less numbers respectively.

For $m, n \in \{0\} \cup \mathbb{N}$, let

$$\mathcal{G}_{m,n}(x) = \frac{G^{(m+n+1)}(x)}{G^{(m)}(x)G^{(n)}(x)}$$

and

$$\mathcal{G}_{m,n;\lambda_{m,n}}(x) = G^{(m+n+1)}(x) + \lambda_{m,n} G^{(m)}(x)G^{(n)}(x)$$

on $(0, \infty)$. In [11, Theorems 3.1 and 4.1], the author obtained the following results:

- (1) the function $\mathcal{G}_{m,n}(x)$ is decreasing in $x \in (0, \infty)$ and maps from $(0, \infty)$,
 - (a) if $(m, n) = (0, 0)$, onto the interval $(-6, -4)$;
 - (b) if $(m, n) \in \{(1, 0), (0, 1)\}$, onto the interval $(-12, -4\psi'(1))$;
 - (c) if $(m, n) \in \{(2, 0), (0, 2)\}$, onto the interval $(-18, \frac{6\psi''(1)}{\psi'(1)})$;
 - (d) if $(m, n) = (1, 1)$, onto the interval $(-36, 6\psi''(1))$;
 - (e) if $(m, n) \in \{(2, 1), (1, 2)\}$, onto the interval $(-72, -\frac{6\psi'''(1)}{\psi'(1)})$;
 - (f) if $m, n \geq 2$, onto the interval

$$\left(-\frac{6(m+n+1)!}{m!n!}, \frac{(m+n+1)(m+n)}{mn(m-1)(n-1)} \frac{\psi^{(m+n)}(1)}{\psi^{(m-1)}(1)\psi^{(n-1)}(1)} \right).$$

(2) the double inequality

$$-\frac{6(m+n+1)!}{m!n!} < \mathcal{G}_{m,n}(x)$$

$$\begin{aligned}
& \begin{cases} -4, & (m, n) = (0, 0) \\ -4\psi'(1), & (m, n) \in \{(1, 0), (0, 1)\} \\ \frac{6\psi''(1)}{\psi'(1)}, & (m, n) \in \{(2, 0), (0, 2)\} \\ 6\psi''(1), & (m, n) = (1, 1) \\ -\frac{6\psi'''(1)}{\psi'(1)}, & (m, n) \in \{(2, 1), (1, 2)\} \\ \frac{(m+n+1)(m+n)}{mn(m-1)(n-1)} \frac{\psi^{(m+n)}(1)}{\psi^{(m-1)}(1)\psi^{(n-1)}(1)}, & m, n \geq 2 \end{cases} \\
& <
\end{aligned}$$

is valid on $(0, \infty)$ and sharp in the sense that the lower and upper bounds cannot be replaced by any larger and smaller numbers respectively;

- (3) if and only if $\lambda_{m,n} \leq 0$, the function $(-1)^{m+n+1}\mathcal{G}_{m,n;\lambda_{m,n}}(x)$ is completely monotonic on $(0, \infty)$;
- (4) if and only if $\lambda_{m,n} \geq \frac{6(m+n+1)!}{m!n!}$, the function $(-1)^{m+n}\mathcal{G}_{m,n;\lambda_{m,n}}(x)$ is completely monotonic on $(0, \infty)$.

In this paper, we would like to consider monotonicity of the function

$$\mathbf{G}_{i,j;p,q}(x) = \frac{G^{(i)}(x)G^{(j)}(x)}{G^{(p)}(x)G^{(q)}(x)}$$

and complete monotonicity of the function

$$\mathbf{G}_{i,j;p,q;\Lambda_{i,j;p,q}}(x) = (-1)^{i_j} G^{(i)}(x)G^{(j)}(x) - (-1)^{\ell+m} \Lambda_{i,j;p,q} G^{(p)}(x)G^{(q)}(x) \quad (1.1)$$

on $(0, \infty)$, where $i, j, p, q \in \{0\} \cup \mathbb{N}$ such that $(i, j) \succ (p, q)$. Figure 1 plotted by the software MATHEMATICA hints that the function $\mathbf{G}_{17,11;15,13}(x)$ is not monotonic in $x \in (0, \infty)$. Therefore, in this paper, we will only consider the func-

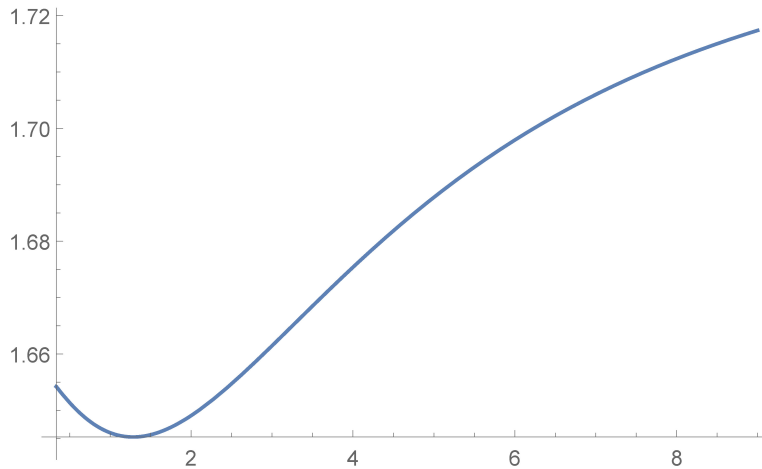


FIGURE 1. The graph of the function $\mathbf{G}_{17,11;15,13}(x)$ on $(\frac{1}{3}, 9)$

tions $\pm \mathbf{G}_{i,j;p,q;\Lambda_{i,j;p,q}}(x)$ and find necessary and sufficient conditions on $\Lambda_{i,j;p,q}$ for $\pm \mathbf{G}_{i,j;p,q;\Lambda_{i,j;p,q}}(x)$ to be completely monotonic on $(0, \infty)$.

2. LEMMAS

The following lemmas are necessary in this paper.

Lemma 2.1 ([11, Lemma 2.3]). *Let*

$$w(t) = \begin{cases} \frac{e^t[(t-2)e^t + t + 2]}{(e^t - 1)^3}, & t \neq 0; \\ \frac{1}{6}, & t = 0. \end{cases}$$

Then the following conclusions are valid:

- (1) *the function $w(t)$ is infinitely differentiable, positive, and even on $(-\infty, \infty)$, is increasing on $(-\infty, 0)$, and is decreasing on $(0, \infty)$;*
- (2) *the function $w(t)$ is logarithmically concave on $(-\infty, \infty)$.*

Lemma 2.2 (Convolution theorem for the Laplace transforms [21, pp. 91–92]). *Let $f_k(t)$ for $k = 1, 2$ be piecewise continuous in arbitrary finite intervals included in $(0, \infty)$. If there exist some constants $M_k > 0$ and $c_k \geq 0$ such that $|f_k(t)| \leq M_k e^{c_k t}$ for $k = 1, 2$, then*

$$\int_0^\infty \left[\int_0^t f_1(u) f_2(t-u) du \right] e^{-st} dt = \int_0^\infty f_1(u) e^{-su} du \int_0^\infty f_2(v) e^{-sv} dv.$$

Lemma 2.3 ([9, Lemma 2.6]). *For $m, n, p, q \in \mathbb{N}$ such that $(p, q) \succ (m, n)$, the function*

$$\frac{s^{m-1}(1-s)^{n-1} + (1-s)^{m-1}s^{n-1}}{s^{p-1}(1-s)^{q-1} + (1-s)^{p-1}s^{q-1}}$$

is increasing in $s \in (0, \frac{1}{2})$.

Lemma 2.4 (Bernstein's theorem [21, p. 161, Theorem 12b]). *A function $f(x)$ is completely monotonic on $(0, \infty)$ if and only if*

$$f(x) = \int_0^\infty e^{-xt} d\sigma(t), \quad x \in (0, \infty), \quad (2.1)$$

where $\sigma(s)$ is non-decreasing and the integral in (2.1) converges for $x \in (0, \infty)$.

Lemma 2.5 ([7, Lemma 2.4]). *For $i, j, \ell, m \in \{0\} \cup \mathbb{N}$ with $(i, j) \succ (\ell, m)$, the inequality $i!j! > \ell!m!$ is valid.*

Lemma 2.6 ([8, Theorem 6.1]). *If $f(x)$ is differentiable and logarithmically concave on $(-\infty, \infty)$, then the product $f(x)f(x_0 - x)$ for any fixed number $x_0 \in \mathbb{R}$ is increasing in $x \in (-\infty, \frac{x_0}{2})$ and decreasing in $x \in (\frac{x_0}{2}, \infty)$.*

3. NECESSARY AND SUFFICIENT CONDITIONS OF COMPLETE MONOTONICITY

In this section, we find necessary and sufficient conditions on $\Lambda_{i,j;p,q}$ for the functions $\pm \mathbf{G}_{i,j;p,q;\Lambda_{i,j;p,q}}(x)$ defined by (1.1) to be completely monotonic on $(0, \infty)$.

Theorem 3.1. *For $i, j, p, q \in \{0\} \cup \mathbb{N}$ such that $(i, j) \succ (p, q)$,*

- (1) *if $\Lambda_{i,j;p,q} \leq 1$, the function $\mathbf{G}_{i,j;p,q;\Lambda_{i,j;p,q}}(x)$ defined by (1.1) is completely monotonic on $(0, \infty)$;*
- (2) *if and only if $\Lambda_{i,j;p,q} \geq \frac{i!j!}{p!q!}$, the function $-\mathbf{G}_{i,j;p,q;\Lambda_{i,j;p,q}}(x)$ is completely monotonic on $(0, \infty)$;*

(3) the double inequality

$$1 < \frac{G^{(i)}(x)G^{(j)}(x)}{G^{(p)}(x)G^{(q)}(x)} < \frac{i!j!}{p!q!} \quad (3.1)$$

is valid on $(0, \infty)$ and the right hand side inequality is sharp in the sense that the number $\frac{i!j!}{p!q!}$ can not be replaced by any smaller one.

Proof. In the proof of [14, Theorem 4], the author derived an integral representation

$$G(x) = \int_0^\infty w(t)e^{-xt}dt, \quad (3.2)$$

where $w(t)$ is defined in Lemma 2.1. Combining (3.2) with Lemma 2.2 gives

$$\begin{aligned} \mathbf{G}_{i,j;p,q;\Lambda_{i,j;p,q}}(x) &= \int_0^\infty w(t)t^i e^{-xt}dt \int_0^\infty w(t)t^j e^{-xt}dt \\ &\quad - \Lambda_{i,j;p,q} \int_0^\infty w(t)t^p e^{-xt}dt \int_0^\infty w(t)t^q e^{-xt}dt \\ &= \int_0^\infty \left[\int_0^t u^i(t-u)^j w(u)w(t-u)du \right] e^{-xt}dt \\ &\quad - \Lambda_{i,j;p,q} \int_0^\infty \left[\int_0^t u^p(t-u)^q w(u)w(t-u)du \right] e^{-xt}dt \\ &= \int_0^\infty \left[\frac{\int_0^t u^i(t-u)^j w(u)w(t-u)du}{\int_0^t u^p(t-u)^q w(u)w(t-u)du} - \Lambda_{i,j;p,q} \right] \\ &\quad \times \left[\int_0^t u^p(t-u)^q w(u)w(t-u)du \right] e^{-xt}dt \\ &= \int_0^\infty \left[\frac{\int_0^1 s^i(1-s)^j w(st)w((1-s)t)ds}{\int_0^1 s^p(1-s)^q w(st)w((1-s)t)ds} - \Lambda_{i,j;p,q} \right] \\ &\quad \times \left[\int_0^t u^p(t-u)^q w(u)w(t-u)du \right] e^{-xt}dt. \end{aligned}$$

By Lemma 2.3, we obtain that the double inequality

$$1 < \frac{s^i(1-s)^j + (1-s)^j s^i}{s^p(1-s)^q + (1-s)^q s^p} < \infty \quad (3.3)$$

is valid and sharp for $s \in (0, \frac{1}{2})$ and $(i, j) \succ (\ell, m)$. Hence, we have

$$\begin{aligned} &\frac{\int_0^1 s^i(1-s)^j w(st)w((1-s)t)ds}{\int_0^1 s^p(1-s)^q w(st)w((1-s)t)ds} \\ &= \frac{\int_0^{1/2} [s^i(1-s)^j + s^j(1-s)^i] w(st)w((1-s)t)ds}{\int_0^{1/2} [s^p(1-s)^q + s^q(1-s)^p] w(st)w((1-s)t)ds} > 1. \quad (3.4) \end{aligned}$$

Consequently, by Lemma 2.4, when $\Lambda_{i,j;p,q} \leq 1$, the function $\mathbf{G}_{i,j;p,q;\Lambda_{i,j;p,q}}(x)$ is completely monotonic on $(0, \infty)$.

By virtue of Lemma 2.1, we acquire

$$\lim_{t \rightarrow 0^+} \frac{\int_0^1 s^i(1-s)^j w(st)w((1-s)t)ds}{\int_0^1 s^p(1-s)^q w(st)w((1-s)t)ds} = \frac{\int_0^1 s^i(1-s)^j ds}{\int_0^1 s^p(1-s)^q ds}$$

$$= \frac{B(i+1, j+1)}{B(p+1, q+1)} = \frac{i!j!}{p!q!}.$$

Let

$$\begin{aligned} S_{i,j;p,q}(t) &= \int_0^{1/2} [s^i(1-s)^j + s^j(1-s)^i] w(st) w((1-s)t) ds \\ &\quad - \frac{i!j!}{p!q!} \int_0^{1/2} [s^p(1-s)^q + s^q(1-s)^p] w(st) w((1-s)t) ds \\ &= \int_0^{1/2} \left([s^i(1-s)^j + s^j(1-s)^i] \right. \\ &\quad \left. - \frac{i!j!}{p!q!} [s^p(1-s)^q + s^q(1-s)^p] \right) w(st) w((1-s)t) ds \\ &= \int_0^{1/2} \left[\frac{s^i(1-s)^j + s^j(1-s)^i}{s^p(1-s)^q + s^q(1-s)^p} - \frac{i!j!}{p!q!} \right] \\ &\quad \times [s^p(1-s)^q + s^q(1-s)^p] w(st) w((1-s)t) ds \end{aligned}$$

for $t \in (0, \infty)$ and $(i, j) \succ (p, q)$. By Lemma 2.3, Lemma 2.5, and the sharp inequality (3.3), we find that the function

$$\frac{s^i(1-s)^j + s^j(1-s)^i}{s^p(1-s)^q + s^q(1-s)^p} - \frac{i!j!}{p!q!}$$

is decreasing in $s \in (0, \frac{1}{2})$ and has a unique zero $s_0 \in (0, \frac{1}{2})$ for $(i, j) \succ (p, q)$. As a result, utilizing Lemmas 2.1 and 2.6, we have

$$\begin{aligned} S_{i,j;p,q}(t) &= \left(\int_0^{s_0} + \int_{s_0}^{1/2} \right) \left[\frac{s^i(1-s)^j + s^j(1-s)^i}{s^p(1-s)^q + s^q(1-s)^p} - \frac{i!j!}{p!q!} \right] \\ &\quad \times [s^p(1-s)^q + s^q(1-s)^p] w(st) w(t-st) ds \\ &< w(s_0 t) w(t-s_0 t) \int_0^{1/2} \left[\frac{s^i(1-s)^j + s^j(1-s)^i}{s^p(1-s)^q + s^q(1-s)^p} - \frac{i!j!}{p!q!} \right] \\ &\quad \times [s^p(1-s)^q + s^q(1-s)^p] ds \\ &= w(s_0 t) w(t-s_0 t) \int_0^{1/2} \left(s^i(1-s)^j + s^j(1-s)^i \right. \\ &\quad \left. - \frac{i!j!}{p!q!} [s^p(1-s)^q + s^q(1-s)^p] \right) ds \\ &= w(s_0 t) w(t-s_0 t) \left(\int_0^{1/2} [s^i(1-s)^j + s^j(1-s)^i] ds \right. \\ &\quad \left. - \frac{i!j!}{p!q!} \int_0^{1/2} [s^p(1-s)^q + s^q(1-s)^p] ds \right) \\ &= w(s_0 t) w(t-s_0 t) \left[B(i+1, j+1) - \frac{i!j!}{p!q!} B(p+1, q+1) \right] \\ &= w(s_0 t) w(t-s_0 t) \left[\frac{i!j!}{(i+j+1)!} - \frac{i!j!}{p!q!} \frac{p!q!}{(p+q+1)!} \right] \\ &= 0. \end{aligned}$$

Accordingly, the inequality

$$\frac{\int_0^{1/2} [s^i(1-s)^j + s^j(1-s)^i] w(st)w((1-s)t)ds}{\int_0^{1/2} [s^p(1-s)^q + s^q(1-s)^p] w(st)w((1-s)t)ds} < \frac{i!j!}{p!q!}$$

is valid and sharp for $t \in (0, \infty)$ and $(i, j) \succ (p, q)$. Therefore, by the equality in (3.4) and Lemma 2.4, for $(i, j) \succ (p, q)$, if and only if $\Lambda_{i,j;p,q} \geq \frac{i!j!}{p!q!}$, the function $-\mathbf{G}_{i,j;p,q;\Lambda_{i,j;p,q}}(x)$ is completely monotonic on $(0, \infty)$.

The double inequality (3.1) follows from complete monotonicity of the functions $\pm \mathbf{G}_{i,j;p,q;\Omega_{i,j;p,q}}(x)$ on $(0, \infty)$. The sharpness of the right hand side inequality in (3.1) follows from the limit

$$\lim_{x \rightarrow \infty} [(-1)^k x^{k+1} G^{(k)}(x)] = \frac{k!}{6}.$$

in [13, Lemma 2.2], where $k \geq 0$. The proof of Theorem 3.1 is complete. $\square$

4. REMARKS

Finally, we list several remarks.

Remark 4.1. What is the necessary and sufficient condition on $\Lambda_{i,j;p,q}$ such that the function $\mathbf{G}_{i,j;p,q;\Lambda_{i,j;p,q}}(x)$ defined in (1.1) is completely monotonic on $(0, \infty)$?

What is the sharp lower bound of the left hand side inequality in (3.1)?

Remark 4.2. For $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, $n > 2$, and two nonnegative integer tuples $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}_0^n$ and $\beta = (\beta_1, \beta_2, \dots, \beta_n) \in \mathbb{N}_0^n$ with $\alpha \succ \beta$, let

$$\mathbf{G}_{\alpha,\beta;C_{\alpha,\beta}}(x) = \prod_{r=1}^n [(-1)^{\alpha_r} G^{(\alpha_r)}(x)] - C_{\alpha,\beta} \prod_{r=1}^n [(-1)^{\beta_r} G^{(\beta_r)}(x)]$$

on $(0, \infty)$. One can discuss necessary and sufficient conditions on $C_{\alpha,\beta} \in \mathbb{R}$ such that the functions $\pm \mathbf{G}_{\alpha,\beta;C_{\alpha,\beta}}(x)$ are respectively completely monotonic on $(0, \infty)$.

Remark 4.3. This paper is the tenth one in a series of articles including [6, 7, 9, 10, 11, 12, 13, 14, 16].

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