Computational Analysis of generalized zeta functions

by

using difference equations

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Abstract: In this article, author performs computational analysis for the generalized zeta functions by using computational software Mathematica. To achieve the purpose recently obtained difference equations are used. These difference equations have a computational power to compute these functions accurately while they can not be computed by using their known integral represenations. Several authors investigated such functions and their analytic properties, but no work has been reported to study the graphical representations and zeors of these functions. Author performs numerical computations to evaluate these functions for different values of the involved parameters. Taylor series expansions are also presented in this research.

Keywords: computational analysis; difference equations; analytic number theory; generalized zeta function; plots; zeros; Taylor Series

1. **Introduction**

Riemann zeta function and its generalizations are very important to investigate the primirality of an algorithm and pattern of prime numbers in Cryptography. More recently Tassaddiq [1] has established some difference equations involving the generalized Hurwitz zeta functions. Analysis of such functions has always been remained important due their fundamental applications in different applied sciences like Physics, Engineering and Computer Science. Main motivation for this article is to plot the graphs and compute the zeros of these functions. In our present investigation we use computational software Mathematica for the analysis of generalized zeta functions and we obtain different forms of series representations for these functions. Before going on our research results, we review the literature to present some preliminaries and basic definitions.

In this paper, we practice the customary symbolizations:

$$N := \{1, 2, \cdots\}; \ N_0 := N \cup \{0\}; \ Z^- := \{-1, -2, \cdots\}; \ Z_0^- := Z^- \cup \{0\}, \tag{1}$$

where Z^- is the set of integers. The involved symbols R, R^+, and C represent the set of real, positive real, and complex numbers, consistently.

The Hurwitz–Lerch zeta function has always been a topic of motivation for several researchers due to its impact in analytic number theory and other applied sciences. Recently, Srivastava presented a considerably new universal family of Hurwitz–Lerch zeta functions defined by [2] (p. 1487, Equation (1.14)):

$$\Phi_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a;b,\lambda)$$

$$= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\exp\left(-at-\frac{b}{t^\lambda}\right){}_p\Psi^*{}_q\left[\begin{matrix}(\lambda_1,\rho_1),\ldots, & (\lambda_p,\rho_p)\\ (\mu_1,\sigma_1),\ldots, & (\mu_p,\sigma_p)\end{matrix};ze^{-t}\right]dt; \tag{2}$$

$$(min[R(a),R(s)]>0;R(b)\geqq 0;\lambda\geqq 0)$$

so that, evidently, one can get the subsequent connection with the extended Hurwitz–Lerch zeta functions $\Phi_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a)$ defined by the authors of [3] (p. 503, Equation (6.2)) (see also References [4] and [5]):

$$\Phi_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a;0,\lambda) = \Phi_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a)$$

$$= e^b\Phi_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a;b,0) \tag{3}$$

In the above Equation (2), ${}_p\Psi^*{}_q$ where (p, $q\in\mathbb{N}_0$) is the standard Fox–Wright function defined by the authors of [5] (p. 2219, Equation (1)) (see also References [4] (p. 516, Equation (1)) and [3] (p. 493, Equation (2.1)):

$$_p\Psi^*{}_q\left[\begin{matrix}(\lambda_1,\rho_1),\ldots, & (\lambda_p,\rho_p)\\ (\mu_1,\sigma_1),\ldots, & (\mu_q,\sigma_q)\end{matrix};z\right] = \sum_{\chi=0}^\infty \frac{([\lambda_p])_{\rho_p\chi}}{([\mu_q])_{\sigma_q\chi}}\frac{z^\chi}{\chi!}. \tag{4}$$

Pochhammer symbols $([\lambda_p])_{\rho_p n} := [\lambda_1]_{\rho_p n}\cdots[\lambda_p]_{\rho_p n}$ symbolize the shifted factorial defined in terms of the basic Gamma function as follows:

$$(\lambda)_\rho = \frac{\Gamma(\lambda+\rho)}{\Gamma(\lambda)} = \begin{cases}1 & (\rho=0,\lambda\in\mathbb{C}\setminus\{0\})\\ \lambda(\lambda+1)\ldots(\lambda+\chi-1) & (\rho=\chi\in\mathbb{N};\lambda\in\mathbb{C}),\#\end{cases} \tag{5}$$

$$\Delta := \sum_{j=1}^q \sigma_j - \sum_{j=1}^p \rho_j \text{ and } \nabla := \left(\prod_{j=1}^p \rho_j^{-\rho_j}\right)\cdot\left(\prod_{j=1}^q \sigma_j^{\sigma_j}\right)$$

The series given by Equation (4) converges in the entire complex z-plane for $\Delta>-1$; and if $\Delta=0$, the series (Equation (4)) converges only for $|z|<\nabla$. For more detailed discussion of such functions, we refer the interested reader to also see References [6–10].

The analysis of Srivastava's λ-generalized Hurwitz–Lerch zeta functions and its different forms have attracted noteworthy concern, and many papers have subsequently appeared on this subject. Jankov et al. [11] and Srivastava et al. [4] discussed some inequalities for different cases of λ-generalized Hurwitz–Lerch zeta functions. Srivastava et al. [12] introduced a nonlinear operator related with the λ-generalized Hurwitz–Lerch zeta functions to analyze the inclusion properties of definite subclass of special type of meromorphic functions. Srivastava and Gaboury [13] deliberated on new expansion formulas for such functions (see, for details, References [14] and [15]; see also the further thoroughly associated studies cited in each of these publications). Luo and Raina [5] discussed some new inequalities involving Srivastava's λ-generalized Hurwitz–Lerch zeta functions and obtained the following series representation [5] (p. 2221, Equation (6)):

$$\Phi_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a;b,\lambda) = \frac{1}{\lambda\Gamma(s)}\sum_{\chi=0}^\infty \frac{([\lambda_p])_{\rho_p\chi}}{([\mu_q])_{\sigma_q\chi}}Z_{\frac{1}{\lambda}}^{\frac{s}{\lambda}}(a+\chi)^\lambda b\frac{z^\chi}{\chi!(\chi+a)^s}, \tag{6}$$

$$(\lambda j \in R(j = 1, .., p) \text{ and } \mu_j \in R\backslash Z - 0 \ (j = 1, ..., q); \rho j > 0(j, ..., p); \sigma j > 0(j$$

Srivastava beautifully described important results about the zeta and related functions in an expository article [16]. Choi et al. [17] further discussed these functions by introducing one more variable. Srivastava et al. [18] presented an innovative integral transform connected with the λ-extended Hurwitz–Lerch zeta function. More recently, Tassaddiq [19] obtained a new representation for this family of the $\boldsymbol{\lambda}$-generalized Hurwitz–Lerch zeta functions in terms of complex delta functions such that the definition of these functions is formalized over the space of entire test functions denoted by Z. The author also listed and discussed all the possible special cases of Srivastava's λ-generalized Hurwitz–Lerch zeta functions [19] (p.4) in the form of a table. For the purposes of our present investigation, this table is given on the next page. For any use of the special cases of the generalized Hurwitz–Lerch zeta functions, the reader is referred to this table. For more detailed study of zeta and related functions, we refer the interested reader to References [20–41] and further bibliography cited therein.

Table 1. Different special cases of λ-generalized Hurwitz–Lerch zeta functions [19].

$min[\Re(a), \Re(s)] > 0; \Re(b) \geqq 0; \lambda \geqq 0;$ $\boldsymbol{\rho} =$ $\boldsymbol{\rho_{1,...},\rho_{p;}}, \boldsymbol{\sigma} = \boldsymbol{\sigma_1, ..., \sigma_q}; \boldsymbol{\lambda^*} = \boldsymbol{\lambda_1, ..., \lambda_p}; \boldsymbol{\mu} = \boldsymbol{\mu_1, ..., , \mu_q}$			(p−1 = q = 0; λ_1 = μ; ρ_1 =1)				(p−1 = q = 0; λ_1 = μ; ρ_1 = 1)	
			$\lambda = 1$	$\mu = 1$	$\lambda = \mu = 1$	$b = 0$	$b = 0$	$\mu = 1;\ b = 0$
λ-Generalized Hurwitz–Lerch Zeta Functions	$\Phi^{(\rho;\sigma)}_{\lambda^*,\mu}(\pm z, s, a; b,$ [2], (p. 1487), Equation (1.14)	$\Theta^\lambda_\mu(\mp z, s, a; b$ [42], (p. 90), Equation (6) and [43]	$\Phi^*_\mu(\pm z, s, a, b)$	$\Phi_b(\pm z, s, a,)$	$\Phi_b(\pm z, s, a)$	$\Phi^{(\rho;\sigma)}_{\lambda^*,\mu}(\pm z, s, a)$ ([2], p. 1486, Equation (1.11)) & [3]	$\Phi^*_\mu(\pm z, s, a)$ ([44], p. 100, Equation (1.5))	$\Phi(\pm z, s, a)$ ([45], p. 27, Equation (1.11))
$\Phi^{(\rho;\sigma)}_{\lambda^*,\mu}(\pm e^{-x}, s, a; b, \lambda)$ λ-Generalized Extended Fermi–Dirac and Extended Bose–Einstein Functions	$\Theta^{(\rho;\sigma)}_{\lambda^*,\mu}(x, s, a; b, \lambda)$	$\Theta^\lambda_\mu(x, s, a; b)$	$\Theta^*_\mu(x, s, a, b)$	$\Theta_b(x, s, a; \lambda)$	$\Theta_b(x, s, a)$	$\Theta^{(\rho;\sigma)}_{\lambda^*,\mu}(x, s, a)$	$\Theta^*_\mu(x, s, a)$ ([44], p. 12, Equation (45))	$\Theta_a(x; s)$ ([46], p. 9, Equation (3.14))
	$\Psi^{(\rho;\sigma)}_{\lambda^*,\mu}(x, s, a; b, \lambda)$	$\Psi^\lambda_\mu(x, s, a; b)$	$\Psi^*_\mu(x, s, a, b)$	$\Psi_b(x, s, a, \lambda)$	$\Psi_b(x, s, a)$	$\Psi^{(\rho;\sigma)}_{\lambda^*,\mu}(x, s, a)$	$\Psi^*_\mu(x, s, a)$ ([44], p. 12, Equation (45))	$\Psi_a(x; s)$ [46], p. 115, Equation (4.4)
$\Phi^{(\rho;\sigma)}_{\lambda^*,\mu}(\pm z, s, 1; b, \lambda)$ λ-Generalized Polylogarithm Functions	$\mathrm{Li}^{(\rho;\sigma)}_{\lambda^*,\mu}(\pm z, s; b, \lambda)$	$\mathrm{Li}^\lambda_\mu(\mp z, s, a; b$	$\mathrm{Li}^*_\mu(z, s, b)$	$\mathrm{Li}_b(z, s, \lambda)$	$\mathrm{Li}_b(z, s)$	$\mathrm{Li}^{(\rho;\sigma)}_{\lambda^*,\mu}(z, s)$	$\mathrm{Li}^*_\mu(z, s)$ ([44], p. 12, Equation (47))	$\mathrm{Li}_s(z)$ [45], (Chapter 1)
$\Phi^{(\rho;\sigma)}_{\lambda^*,\mu}(\pm e^{-x}, s + 1,1; b, \lambda)$ λ-Generalized Fermi–Dirac and Bose–Einstein Functions	$F^{(\rho;\sigma)}_{\lambda^*,\mu}(x, s; b, \lambda)$	$F^\lambda_\mu(x, s, a; b)$	$F^*_\mu(x, s, b)$	$F_b(x, s, \lambda)$	$F_b(x, s)$	$F^{(\rho;\sigma)}_{\lambda^*,\mu}(x, s)$	$F^*_\mu(x, s)$ ([44], p. 12, Equation (47))	$F_s(x)$ ([46], p. 109, Equation (1.12)
	$B^{(\rho;\sigma)}_{\lambda^*,\mu}(x, s; b, \lambda)$	$B^\lambda_\mu(x, s, a; b)$	$B^*_\mu(x, s, b)$	$B_b(x, s, \lambda)$	$B_b(x, s)$	$B^{(\rho;\sigma)}_{\lambda^*,\mu}(x, s)$	$B^*_\mu(x, s)$ ([44], p. 12, Equation (45))	$B_s(x)$ ([46], p. 109, Equation (1.12))
$\Phi^{(\rho;\sigma)}_{\lambda^*,\mu}(\pm 1, s, a; b, \lambda)$ λ-Generalized Hurwitz zeta Functions	$\zeta^{(\rho;\sigma)}_{\lambda^*,\mu}(s, a; b, \lambda)$	$\zeta^\lambda_\mu(s, a; b)$	$\zeta^*_\mu(s, a, b)$	$\zeta_b(s, a, \lambda)$	$\zeta_b(s, a)$ [47], p. 308	$\zeta^{(\rho;\sigma)}_{\lambda^*,\mu}(s, a)$	$\zeta^*_\mu(s, a)$ [44]	$\zeta(s, a)$ [45], (Chapter 1)
$\Phi^{(\rho;\sigma)}_{\lambda^*,\mu}(\pm 1, s, 1; b, \lambda)$ λ-Generalized Riemann Zeta Functions	$\zeta^{(\rho;\sigma)}_{\lambda^*,\mu}(s)$	$\zeta^\lambda_\mu s; b)$	$\zeta^*_\mu(s, b)$	$\zeta_b(s, \lambda)$	$\zeta_b(s)$ [47], p. 308	$\zeta^{(\rho;\sigma)}_{\lambda^*,\mu}(s)$	$\zeta^*_\mu(s)$ [44]	$\zeta(s)$ [45], (Chapter 1)

More recently Tassaddiq [1] has established some new difference equations for the family of λ-generalized Hurwitz–Lerch zeta functions and its special cases by following the approach of Tassaddiq and Qadir [34]. From the above discussion and Table 1, we can notice that several authors presented and studied worthwhile generalizations of the Hurwitz–Lerch zeta functions. They obtained various analytic formulas, integral, and series representations. However, as we deeply study Riemann zeta functions, we know their values, their graphs, and several other important aspects. We could not develop this approach for these generalizations. For our interest, we focus on the following form of the generalized zeta function [44]

$$\zeta_\mu^*(s,a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}e^{-at}}{(1-e^{-t})^\mu} dt \qquad (\Re(s) > \Re(\mu); \Re(a) > 0). \tag{7}$$

Bayad and Chikhi [44] obtained reduction and duality formulas of the generalized Hurwitz–Lerch zeta functions. Their results contain the earlier obtained results of Choi [48]. These reduction formulas were concerned with the reduction of one parameter that represent the generalized Hurwitz–Lerch zeta $\Phi_\mu^*(z,s,a)$ and Hurwitz zeta functions $\zeta_\mu^*(s,a)$ in terms of Hurwitz–Lerch zeta $\Phi(z,s,a)$ and Hurwitz zeta $\zeta(s,a)$ functions, respectively. The difference equations presented by Tassaddiq [1] have the advantage of reducing the generalized Hurwitz–Lerch zeta $\Phi_\mu^*(z,s,a)$ and the generalized Hurwitz zeta functions $\zeta_\mu^*(s,a)$ in terms of basic polylogarithm $\text{Li}_s(z)$ and zeta functions $\zeta(s)$, respectively. That one more parameter has reduced by [1] and her results are simple enough to evaluate these functions for different values of the involved parameters. By following the approach developed in this paper, we can initiate a deeper analysis of these functions that will enhance their applications. The Riemann hypothesis is a well-known unsolved problem in analytic number theory [23]. It states that "all the non-trivial zeros of the zeta function exist on the real line $s = \frac{1}{2}$." These zeros seem to be complex conjugates and hence symmetric on this line. The integrals of the zeta function and its generalizations are vital in the study of Riemann hypothesis and for the investigation of zeta functions themselves. The study of distributions in statistical inference and reliability theory [49-50] also involves such integrals. The computational analysis presented in this investigation is worthwhile to evaluate these generalized Hurwitz zeta functions that are consistent with the existing results. For such related studies please see [51-54].

The plan of the paper as follows: We present our compuattaional analysis of the generalized Hurwitz functions in Section 2 for special values of $\mu = 2,3,4,5$. We plot graphs of these functions and compute zeros. Taylor series for these functions are also obtained here. We conclude our results in the last Section 3 by highlighting some future directions of this work.

Throughout this investigation, conditions on the parameters will be considered standard as given in Equations (1)–(6) and Table 1 unless otherwise stated.

2. Results

In this section, we consider some interesting special cases of difference equations. On one side, these are useful to know the values of generalized Hurwitz zeta functions $\zeta_\mu^*(s,a)$ in terms of zeta functions, and on the other, they lead to the computation of some elementary integrals that are nontrivial to obtain for small values of $\mu = 3,4,5$ and the large values of s.

2.1 Compuational Analysis of the Generalized zeta functions; $\mu = 3$

Consider the equation [1, Equation (34)]

$$\zeta_3^*(s,3) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}e^{-3t}}{(1-e^{-t})^3} dt = \frac{1}{2!}[\zeta(s-2) - 3\zeta(s-1) + 2\zeta(s)]$$

$$(s \neq 1,2,3) \tag{4}$$

$$\Gamma(s)\zeta_3^*(s,3) = \int_0^\infty \frac{t^{s-1}e^{-3t}}{(1-e^{-t})^3} dt = \frac{\Gamma(s)}{2!}[\zeta(s-2) - 3\zeta(s-1) + 2\zeta(s)]$$

Similarly, by considering different values of s, we can produce the following Table 2 of values. These computations show that Mathematica is unable to compute the involved integral on a commonly available computer for the large values of s, but it can be done using these new difference equations. For small values of s, our results are 100% accurate with the directly computed results
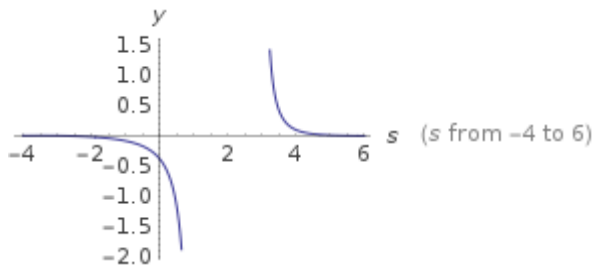
Table 1. Computation of $\int_0^\infty \frac{t^{s-1}e^{-3t}}{(1-e^{-t})^3} dt$.

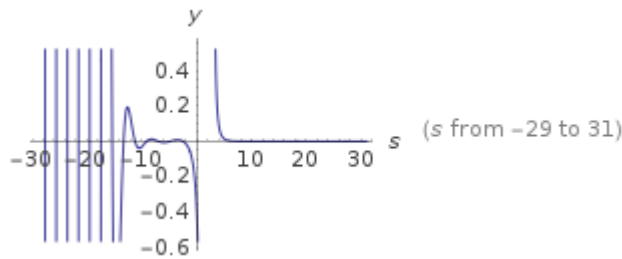| s	Direct Evaluation by Mathematica	Using Recurrence Relation (4)
70	6.83627×10^{64}	6.83627×10^{64}
110	4.74446×10^{123}	4.74446×10^{123}
170	Unable to compute	1.70547×10^{271}
230	Unable to compute	1.347618×10^{409}
500	Unable to compute	1.538829×10^{964}

Plots of the function $\zeta_\mu^*(s,a); \mu = a = 3$

$(2\,Zeta[s] + Zeta[s-2] - 3\,Zeta[s-1])/2$

Plot[(Zeta[-2 + s] - 3 Zeta[-1 + s] + 2 Zeta[s])/2, {s, -4, 6}]

Plot[(Zeta[-2 + s] - 3 Zeta[-1 + s] + 2 Zeta[s])/2, {s, -29, 31}]

Roots of the function $\zeta_\mu^*(s, a)$; $\mu = a = 3$

$(2\,Zeta[s]\ +\ Zeta[s\ -\ 2]\ -\ 3\,Zeta[s\ -\ 1])/2$
FindRoot[(Zeta[-2 + s] - 3 Zeta[-1 + s] + 2 Zeta[s])/2 == 0, {s, -4.82226, -4.50115},
WorkingPrecision -> 15]
s ≈ -4.77097371398911...
FindRoot[(Zeta[-2 + s] - 3 Zeta[-1 + s] + 2 Zeta[s])/2 == 0, {s, -6.97091, -6.95239},
WorkingPrecision -> 15]
s ≈ -6.96715467676022...
FindRoot[(Zeta[-2 + s] - 3 Zeta[-1 + s] + 2 Zeta[s])/2 == 0, {s, -9.11904, -9.10075},
WorkingPrecision -> 15]
s ≈ -9.11014906805379...
FindRoot[(Zeta[-2 + s] - 3 Zeta[-1 + s] + 2 Zeta[s])/2 == 0, {s, -11.2268, -11.2068},
WorkingPrecision -> 15]
s ≈ -11.2190934707418...
FindRoot[(Zeta[-2 + s] - 3 Zeta[-1 + s] + 2 Zeta[s])/2 == 0, {s, -13.3236, -13.284},
WorkingPrecision -> 15]
s ≈ -13.3048228562250...
FindRoot[(Zeta[-2 + s] - 3 Zeta[-1 + s] + 2 Zeta[s])/2 == 0, {s, -17.4409, -17.421},
WorkingPrecision -> 15]
s ≈ -17.4309378745727...
FindRoot[(Zeta[-2 + s] - 3 Zeta[-1 + s] + 2 Zeta[s])/2 == 0, {s, -20, -30}, WorkingPrecision
-> 15]
{s->-25.5839411946583}

(Taylor series) Series of the function $\zeta_\mu^*(s, a)$; $\mu = a = 3$

[(Zeta[-2 + s] - 3 Zeta[-1 + s] + 2 Zeta[s])/2, {s, 0, 4}]

$$-\frac{3}{8} + \frac{1}{8} s \left(12 \log(A) + 4\, \zeta'(-2) - 1 - 4 \log(2\,\pi)\right) + \frac{1}{48} s^2 \left(-12\left(-\zeta''(-2)\right.\right.$$
$$+ 3\, \zeta''(-1) - 2\,\gamma_1 + \log^2(2) + \log^2(\pi) + \log(4)\log(\pi)) + 12\,\gamma^2$$
$$- \pi^2) + \frac{1}{48} s^3 \left(-4\left(-\zeta^{(3)}(-2) + 3\,\zeta^3(-1) - 3\,\gamma_2 - 6\,\gamma_1 \log(2\,\pi)\right.\right.$$
$$+ 2\,\zeta(3) + \log^3(2\,\pi)) + 24\,\gamma\,\gamma_1 + 8\,\gamma^3 + 12\,\gamma^2 \log(2\,\pi)$$
$$- \pi^2 \log(2\,\pi)) + \frac{1}{48}\left(\zeta^{(4)}(-2) - 3\,\zeta^{(4)}(-1) + 2\,\zeta^{(4)}(0)\right) s^4$$
$$+ \frac{1}{240}\left(\zeta^{(5)}(-2) - 3\,\zeta^{(5)}(-1) + 2\,\zeta^{(5)}(0)\right) s^5 + O(s^6)$$

Series[(Zeta[-2 + s] - 3 Zeta[-1 + s] + 2 Zeta[s])/2, {s, Infinity, 4}]

$$3^{-s} + 5\,2^{1-s}3^{-s} + 3\,4^{-s} + 6\,5^{-s} + 9\,2^{2-s}5^{-s} + 15\,7^{-s} + 21\,8^{-s} + 28\,9^{-s}$$
$$+ 45\,11^{-s} + 55\,12^{-s}$$

2.2 Compuational Analysis of the Generalized zeta functions; $\mu = 4$

Consider the equation [1, Equation (40)

$$\zeta_4^*(s, 4) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-4t}}{(1 - e^{-t})^4}\,dt$$
$$= \frac{1}{3!}\left[\zeta(s - 3) - 6\zeta(s - 2) + 11\zeta(s - 1) - 6\zeta(s)\right]; \quad s \neq 1,2,3,4$$

(5)

Table 2. Computation of $\int_0^\infty \frac{t^{s-1} e^{-4t}}{(1-e^{-t})^4}\,dt$

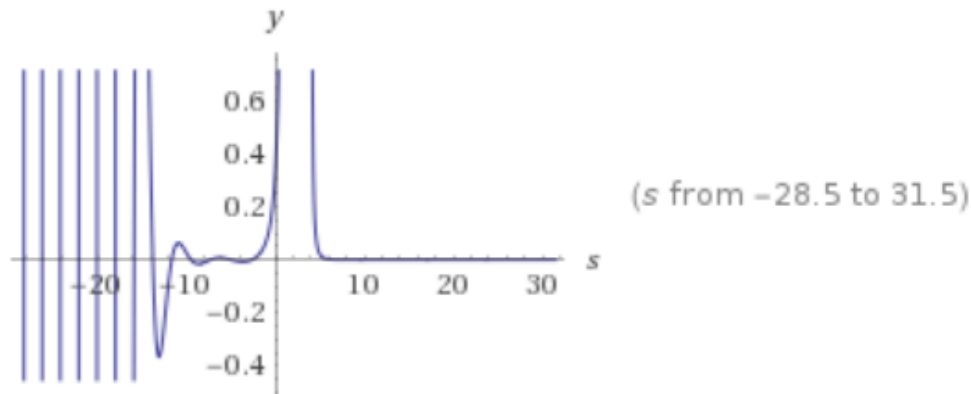| s	Direct Evaluation by Mathematica	Using Recurrence Relation (5)
95	6.9291×10^{88}	6.9291×10^{88}
102	3.6661135×10^{98}	3.6661135×10^{98}
145	Unable to compute	2.79008×10^{162}
165	Unable to compute	1.50289×10^{194}

Plots of the function $\zeta_\mu^*(s, a);\ \mu = a = 4$

(Zeta[s - 3] - 6 Zeta[s - 2] + 11 Zeta[s - 1] - 6 Zeta[s])/6

Plot[(Zeta[-3 + s] - 6 Zeta[-2 + s] + 11 Zeta[-1 + s] - 6 Zeta[s])/6, {s, -3.5, 6.5}]
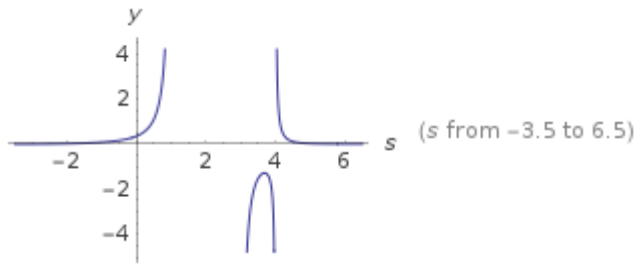
Plot[(Zeta[-3 + s] - 6 Zeta[-2 + s] + 11 Zeta[-1 + s] - 6 Zeta[s])/6, {s, -29, 32}]

Zeros of the function $\zeta_\mu^*(s, a)$; $\mu = a = 4$

FindRoot[(Zeta[-3 + s] - 6 Zeta[-2 + s] + 11 Zeta[-1 + s] - 6 Zeta[s])/6 == 0, {s, -2.6666, -2.59308}, WorkingPrecision -> 15]

s ≈ -2.63545753062799...

FindRoot[(Zeta[-3 + s] - 6 Zeta[-2 + s] + 11 Zeta[-1 + s] - 6 Zeta[s])/6 == 0, {s, -5.12501, -4.96446}, WorkingPrecision -> 15]

s ≈ -5.02439658771749...

FindRoot[(Zeta[-3 + s] - 6 Zeta[-2 + s] + 11 Zeta[-1 + s] - 6 Zeta[s])/6 == 0, {s, -7.34401, -7.26464}, WorkingPrecision -> 15

s ≈ -7.29696425743135...

FindRoot[(Zeta[-3 + s] - 6 Zeta[-2 + s] + 11 Zeta[-1 + s] - 6 Zeta[s])/6 == 0, {s, -9.5199, -9.50028}, WorkingPrecision -> 15]

s ≈ -9.50133546640343...

FindRoot[(Zeta[-3 + s] - 6 Zeta[-2 + s] + 11 Zeta[-1 + s] - 6 Zeta[s])/6 == 0, {s, -11.6674, -11.6489}, WorkingPrecision -> 15

s ≈ -11.6612628793179...

FindRoot[(Zeta[-3 + s] - 6 Zeta[-2 + s] + 11 Zeta[-1 + s] - 6 Zeta[s])/6 == 0, {s, -13.7916, -13.7719}, WorkingPrecision -> 15]

s ≈ -13.7902423170163...

FindRoot[(Zeta[-3 + s] - 6 Zeta[-2 + s] + 11 Zeta[-1 + s] - 6 Zeta[s])/6 == 0, {s, -15.9102, -15.8731}, WorkingPrecision -> 15]

s ≈ -15.8966487406491...

Taylor series of the function $\zeta_\mu^*(s, a)$; $\mu = a = 4$

[(Zeta[-3 + s] - 6 Zeta[-2 + s] + 11 Zeta[-1 + s] - 6 Zeta[s])/6, {s, 0, 4}]

$$\frac{251}{720} + \frac{1}{72} s \left(-132 \log(A) + 12 \, \zeta'(-3) - 72 \, \zeta'(-2) + 11 + 36 \log(2 \, \pi)\right)$$

$$+ \frac{1}{48} s^2 (4 \, (\zeta''(-3) - 6 \, \zeta''(-2) + 11 \, \zeta''(-1) - 6 \, \gamma_1 + 3 \, \log^2(2)$$

$$+ 3 \, log^2(\pi) + log(64) \, log(\pi)) - 12 \, \gamma^2 + \pi^2)$$

$$+ \frac{1}{144} \, s^3 \, (4 \, (\zeta^3(-3) - 6 \, \zeta^3(-2) + 11 \, \zeta^3(-1) - 9 \, \gamma_2$$

$$- 18 \, \gamma_1 \, log(2 \, \pi) + 6 \, \zeta(3) + 3 \, log^3(2 \, \pi)) - 72 \, \gamma \, \gamma_1 - 24 \, \gamma$$

$$- 36 \, \gamma^2 \, log(2 \, \pi) + 3 \, \pi^2 \, log(2 \, \pi)) + \frac{1}{144} \, (\zeta^4(-3) - 6 \, \zeta^4(-2)$$

$$+ 11 \, \zeta^4(-1) - 6 \, \zeta^4(0)) \, s^4 + \frac{1}{720} \, (\zeta^5(-3) - 6 \, \zeta^5(-2)$$

$$+ 11 \, \zeta^5(-1) - 6 \, \zeta^5(0)) \, s^5 + O(s^6)$$

Series[(Zeta[-3 + s] - 6 Zeta[-2 + s] + 11 Zeta[-1 + s] - 6 Zeta[s])/6, {s, Infinity, 4}]

$$35 \, 2^{-3s} + 55 \, 2^{-2s} 3^{1-s} + 56 \, 3^{-2s} + 5 \, 2^{1-s} 3^{-s} + 4^{-s} + 4 \, 5^{-s} + 21 \, 2^{2-s} 5^{-s}$$

$$+ 20 \, 7^{-s} + 120 \, 11^{-s}$$

2.3 Compuational Analysis of the Generalized zeta functions; $\mu = 5$

Consider the equation [1, Equation (46)]

$$\Gamma(s)\zeta_5^*(s, 5) = \int_0^\infty \frac{t^{s-1} e^{-5t}}{(1 - e^{-t})^5} \, dt =$$

$$= \frac{\Gamma(s)}{4!} \begin{bmatrix} \zeta(s-4) - 10\zeta(s-3) + 35\zeta(s-2) \\ -50\zeta(s-1) + 24\zeta(s) \end{bmatrix}; s \neq 1,2,3,4,5.$$
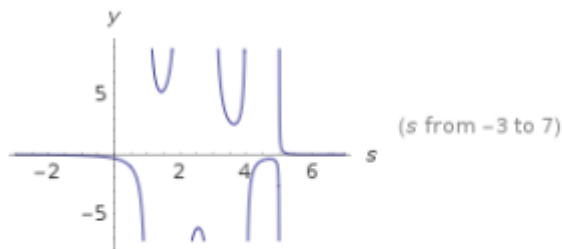
(6)

Table 3. Computation of $\int_0^\infty \frac{t^{s-1}e^{-5t}}{(1-e^{-t})^5}\,dt$

| s	Direct Evaluation by Mathematica	Using Recurrence Relation (6)
96	8.18425×10^80	1.07719×10^{82}
104	5.8077×10^{95}	5.8077×10^{95}
146	Unable to compute	1.78191×10^{156}
166	Unable to compute	1.37955×10^{186}

Plots of the function $\zeta_\mu^*(s,a)$; $\mu = a = 5$

(Zeta[s - 4] - 10 Zeta[s - 3] + 35 Zeta[s - 2] - 50 Zeta[s - 1] + 24 Zeta[s])/4!
Plot[(Zeta[-4 + s] - 10 Zeta[-3 + s] + 35 Zeta[-2 + s] - 50 Zeta[-1 + s] + 24 Zeta[s])/24, {s, -3, 7}]

Plot[(Zeta[-4 + s] - 10 Zeta[-3 + s] + 35 Zeta[-2 + s] - 50 Zeta[-1 + s] + 24 Zeta[s])/24, {s, -28, 32}]
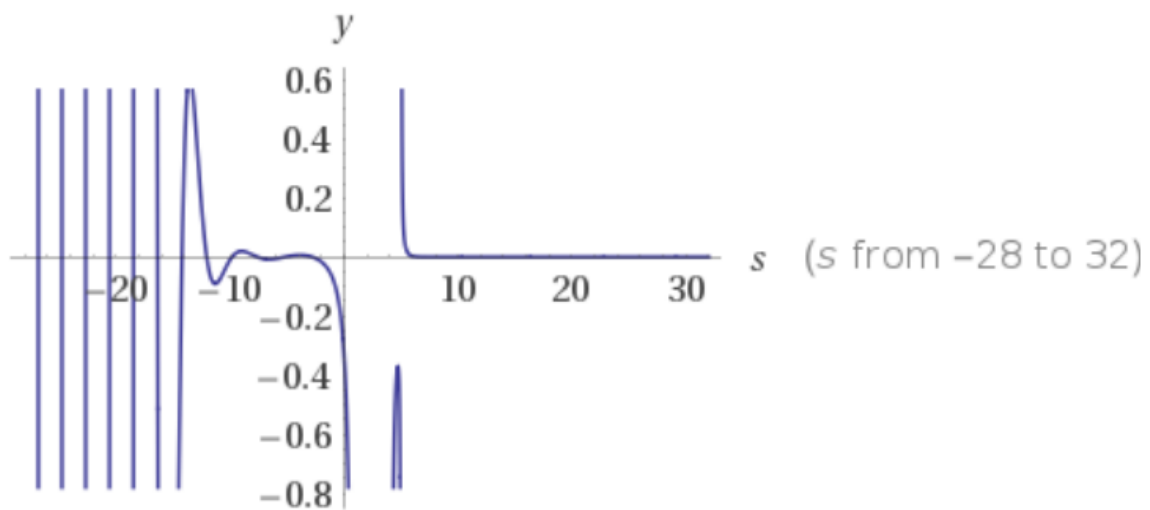
Zeros of the function $\zeta_\mu^*(s,a)$; $\mu = 5 = 5$

FindRoot[(Zeta[-4 + s] - 10 Zeta[-3 + s] + 35 Zeta[-2 + s] - 50 Zeta[-1 + s] + 24 Zeta[s])/24
== 0, {s, -13.7435, 4.3419}, WorkingPrecision -> 15]
 s ≈ -12.0419849804341..

Taylor series of the function $\zeta_\mu^*(s,a)$; $\mu = 5 = 5$

Series[(Zeta[−4 + s] − 10 Zeta[−3 + s] + 35 Zeta[−2 + s] − 50 Zeta[−1 + s]
         + 24 Zeta[s])/24, {s, 0, 4}]

$$
= -\frac{95}{288} + \frac{s}{144} \; (300 \log(A) + 6\zeta'(-4) - 60\,\zeta'(-3) + 210\,\zeta'(-2) - 25
$$
$$
- 72 \log(2\,\pi))
$$
$$
+ \frac{1}{48} s^2 \; (\zeta''(-4) - 10\,\zeta''(-3) + 35\zeta''(-2) - 50\,\zeta''(-1) + 24\,\gamma_1
$$
$$
+ 12\gamma^2 - \pi^2 - 12 \log^2(2) - 12\,\log^2(\pi) - 24 \log(2)\log(\pi))
$$
$$
+ \frac{1}{144}\,s^3\;(\zeta^3(-4) - 10\,\zeta^3(-3) + 35\zeta^3(-2) - 50\,\zeta^3(-1) + 72\,\gamma\,\gamma_1
$$
$$
+ 36\,\gamma_2 + 72\,\gamma_1\log(2\,\pi) - 24\,\zeta(3) + 24\,\gamma^3 - 12\,\log^3(2\,\pi)
$$
$$
+ 36\,\gamma^2\,\log(2\,\pi) - 3\,\pi^2\log(2\,\pi))
$$
$$
+ \frac{1}{576}\left(\zeta^4(-4) - 10\,\zeta^4(-3) + 35\,\zeta^4(-2) - 50\,\zeta^4(-1) + 24\,\zeta^4(0)\right)s^4
$$
$$
+ \frac{1}{2880}\left(\zeta^5(-4) - 10\,\zeta^5(-3) + 35\zeta^5(-2) - 50\,\zeta^5(-1)\right.
$$
$$
+ 24\,\zeta^5(0))s^5 + O(s^6)
$$

Series expansion at $s = \infty$

Series[(Zeta[-4 + s] - 10 Zeta[-3 + s] + 35 Zeta[-2 + s] - 50 Zeta[-1 + s] + 24 Zeta[s])/24, {s, Infinity, 4}]

$$
= 55 \times 2^{1-2s} \times 3^{1-s} + 5^{-s} + 63 \times 2^{1-s} \times 5^{-s} + 5\,6^{-s} + 15\,7^{-s} + 35 \times 8^{-s}
$$
$$
+ 70 \times 9^{-s} + 210 \times 11^{-s}
$$

3. Discussion and Future Directions

   In this study, we have used some recently obtained recurrence relations for the newly defined family of the generalized Hurwitz zeta functions to compute them by using mathematica. This computational analysis proved valuable by plotting the graphs and deriving different series and asymptotic representations, etc. Several authors studied such functions but no work has been reported as the analysis presented here in this research. Actually the main problem to study these functions is that computational softwares are

unable to compute them by using their known integral representations. Some recently obtained representations have become a poweful computational tool to calculate these functions by using computational software Mathematica.

Comparison of graphs and zeros of the generalized zeta functions

By comparing the plots of different cases under discussion for the zeta functions, we can notice that all of them have infinitely many trivial zeros like the original Riemann zeta functions that are transformed according to the involvement of this functions in their expressions. This practice to acquire the outcomes by making use of new difference equations explores the required simplicity that always inspires hope. We have discussed here the direct consequences of our results. It is remarked that the method established in this research is in fact noteworthy for the analysis and study of these higher transcendental functions.

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