

Partial Derivative Approach to the Change of Scale Formula for the Function Space Integral

Young Sik Kim *

Abstract

We investigate the behavior of the partial derivative approach to the change of scale formula and prove relationships among the analytic Wiener integral and the analytic Feynman integral of the partial derivative for the function space integral.

1 Introduction

In [2,3,4], various formulas for linear transformations of Wiener integrals have been given and the behavior of measure and measurability under the change of scale was investigated in [5] on the Wiener space. The scale invariant measurability was proved on the Wiener space in [9]. The relationship between the Wiener integral and the analytic Feynman integral was proved in [6] and [8]. Using these results, they found a change of scale formula for Wiener integrals on $C_0[0, T]$ in [7].

In [10], the author proved the change of scale formula for $F(x) = f((h_1, x)^\sim, \dots, (h_n, x)^\sim)$ where $f \in L_p(\mathbf{R}^n)$ with $1 \leq p \leq \infty$ on abstract Wiener spaces. In [11], the author established those relationships for the Fourier transform of a measure $\hat{\mu}((h_1, x)^\sim, \dots, (h_n, x)^\sim)$ on abstract Wiener spaces. In [13], the author investigated the partial derivative approach to the integral transform for the function space in some Banach algebra on the Wiener space.

In this paper, we prove the change of scale formula for the function space integral about the partial derivative of a Fourier transform on the Wiener space and prove some relationships among the analytic Wiener integral and the analytic Feynman integral and the Wiener integral of the partial derivative for the function space integral.

2 Definitions and Preliminaries

Let $C_0[0, T]$ be the one parameter Wiener space. That is the class of R -valued continuous functions x on $[0, T]$ with $x(0) = 0$. Let M denote the class of all Wiener measurable subsets of $C_0[0, T]$ and let m denote the Wiener measure. $(C_0[0, T], M, m)$ is a complete measure space and we denote the Wiener integral of a functional F by $E_x[F(x)] = \int_{C_0[0, T]} F(x) dm(x)$.

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E-mail : yoskim@hanyang.ac.kr, MSC : 28C20

A subset E of $C_0[0, T]$ is said to be scale-invariant measurable provided $\rho E \in M$ for all $\rho > 0$, and scale invariant measurable set N is said to be scale-invariant null provided $m(\rho N) = 0$ for each $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). If two functionals F and G are equal s-a.e., we write $F \approx G$.

Definition 2.1 Let $\mathbf{C}_+ = \{\lambda | \operatorname{Re}(\lambda) > 0\}$ and $\mathbf{C}_+^\sim = \{\lambda | \operatorname{Re}(\lambda) \geq 0\}$. Let F be a complex-valued measurable function on $C_0[0, T]$ such that the integral

$$J_F(\lambda) = E_x \left(F(\lambda^{-\frac{1}{2}} x) \right) \quad (2.1)$$

exists for all real $\lambda > 0$. If there exists an analytic function $J_F^*(z)$ analytic on \mathbf{C}_+ such that $J_F^*(\lambda) = J_F(\lambda)$ for all real $\lambda > 0$, then we define $J_F^*(z)$ to be the **analytic Wiener integral** of F over $C_0[0, T]$ with parameter z and for each $z \in \mathbf{C}_+$, we write

$$E_x^{anwz} \left(F(x) \right) = E_x \left(F(z^{-\frac{1}{2}} x) \right) = J_F^*(z) \quad (2.2)$$

Let q be a non-zero real number and let F be a function whose analytic Wiener integral exists for each z in \mathbf{C}_+ . If the following limit exists, then we call it the **analytic Feynman integral** of F over $C_0[0, T]$ with parameter q , and we write

$$E_x^{anf_q} \left(F(x) \right) = \lim_{z \rightarrow -iq} E_x^{anwz} \left(F(x) \right), \quad (2.3)$$

where z approaches $-iq$ through \mathbf{C}_+ and $i^2 = -1$.

Definition 2.2 (Ref.[1]) The first variation of a Wiener measurable functional F in the direction $w \in C_0[0, T]$ is defined by the partial derivative as

$$\delta F(x|w) = \frac{\partial}{\partial h} F(x + hw)|_{h=0} \quad (2.4)$$

We will denote it by $[D, F, x, w]$. (See [13]).

The following is a well-known Wiener integration formula for Wiener integrals on the Wiener space $C_0[0, T]$.

Theorem 2.3 Let $C_0[0, T]$ be the Wiener space and let F be a cylinder function on $C_0[0, T]$ of the form $F(x) = f([I, \alpha_1(t), x(t)], \dots, [I, \alpha_n(t), x(t)])$, where $f : \mathbf{R}^n \rightarrow \mathbf{C}$ is a Lebesgue measurable function on \mathbf{R}^n . Then

$$E_x \left(f([I, \alpha_1(t), x(t)], \dots, [I, \alpha_n(t), x(t)]) \right) = \left(\frac{1}{2\pi} \right)^{\frac{n}{2}} \int_{\mathbf{R}^n} f(\vec{u}) \exp \left\{ -\frac{1}{2} \sum_{j=1}^n u_j^2 \right\} d\vec{u} \quad (2.5)$$

where $[I, \alpha_j(t), x(t)] = \int_0^T \alpha_j(t) dx(t)$ for $1 \leq j \leq n$ and " $=$ " mean that if either side exists, then both sides exists and they are equal and $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is an orthonormal class of $L_2[0, T]$.

Remark. For $a \in \mathbf{C}_+$ and $b \in \mathbf{R}$,

$$\int_{\mathbf{R}} \exp \left\{ -au^2 + ibu \right\} du = \sqrt{\frac{\pi}{a}} \exp \left\{ -\frac{b^2}{4a} \right\}. \quad (2.6)$$

3 Results.

Let

$$F(x) = \hat{\mu} \left([I, \alpha_1(t), x(t)], [I, \alpha_2(t), x(t)], \dots, [I, \alpha_n(t), x(t)] \right), \quad (3.1)$$

where $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is an orthonormal class of $L_2[0, T]$ and where

$$\hat{\mu}(\vec{u}) = \int_{R^n} \exp \left\{ i \sum_{j=1}^n u_j v_j \right\} \mu(d\vec{v}), \vec{u} \in R^n \quad (3.2)$$

is the Fourier transform of the measure μ on \mathbf{R}^n and $\vec{u} = (u_1, \dots, u_n)$ and $\vec{v} = (v_1, \dots, v_n)$ are in \mathbf{R}^n and $[I, \alpha_j(t), x(t)] = \int_0^T \alpha_j(t) dx(t)$ for $1 \leq j \leq n$.

Throughout this section, we assume that $w \in C_0[0, T]$ is absolutely continuous in $[0, T]$ with $w' \in L_2[0, T]$.

Theorem 3.1

$$[D, F, x, w] = \int_{\mathbf{R}^n} \left(i \sum_{j=1}^n [I, \alpha_j(t), w(t)] \cdot v_j \right) \cdot \exp \left\{ i \sum_{j=1}^n [I, \alpha_j(t), x(t)] \cdot v_j \right\} \mu(d\vec{v}). \quad (3.3)$$

Proof. By Equation (2.4),

$$\begin{aligned} & [D, F, x, w] \\ &= \frac{\partial}{\partial h} F(x + hw)|_{h=0} \\ &= \frac{\partial}{\partial h} \int_{\mathbf{R}^n} \exp \left\{ i \sum_{j=1}^n [I, \alpha_j(t), x(t)] \cdot v_j + h [I, \alpha_j(t), w(t)] \cdot v_j \right\} \mu(d\vec{v})|_{h=0} \\ &= \int_{\mathbf{R}^n} \left(i \sum_{j=1}^n [I, \alpha_j(t), w(t)] \cdot v_j \right) \cdot \exp \left\{ i \sum_{j=1}^n [I, \alpha_j(t), x(t)] \cdot v_j \right\} \mu(d\vec{v}). \end{aligned} \quad (3.4)$$

The Paley-Wiener-Zygmund integral equals to the Riemann Stieltzes integral

$$\int_0^T \alpha_j(t) dw(t) = \int_0^T \alpha_j(t) w'(t) dt, 1 \leq j \leq n,$$

as w is an absolutely continuous function in $[0, T]$ with $w'(t) \in L_2[0, T]$. Therefore,

$$\begin{aligned} & [D, F, x, w] \\ &= \int_{\mathbf{R}^n} \left(i \sum_{j=1}^n [I, \alpha_j(t), w(t)] \cdot v_j(t) \right) \cdot \exp \left\{ i \sum_{j=1}^n [I, \alpha_j(t), x(t)] \cdot v_j(t) \right\} \mu(d\vec{v}) \\ &= \int_{\mathbf{R}^n} \left(i \sum_{j=1}^n \left(\int_0^T \alpha_j(t) w'(t) dt \right) \cdot v_j(t) \right) \cdot \exp \left\{ i \sum_{j=1}^n [I, \alpha_j(t), x(t)] \cdot v_j(t) \right\} \mu(d\vec{v}) \end{aligned} \quad (3.5)$$

as $w \in C_0[0, T]$ is absolutely continuous on $[0, T]$ with $w' \in L_2[0, T]$. Therefore we have that

$$\left| [D, F, x, w] \right|$$

$$\begin{aligned}
&\leq \int_{\mathbf{R}^n} \left| \sum_{j=1}^n [I, \alpha_j(t), w(t)] \cdot v_j \right| |\mu|(d\vec{v}) \\
&\leq \int_{\mathbf{R}^n} \sum_{j=1}^n \left(\|\alpha_j\|_2 \cdot \|w'\|_2 \cdot |v_j| \right) |\mu|(d\vec{v}) \\
&= \|w'\|_2 \cdot \int_{\mathbf{R}^n} \sum_{j=1}^n |v_j| |\mu|(d\vec{v}) \\
&< \infty,
\end{aligned} \tag{3.6}$$

by a Hölder inequality in $L_2[0, T]$. Therefore $[D, F, x, w]$ exists. \square

Theorem 3.2 For every $z \in C_+$,

$$E_x^{anwz}([D, F, x, w]) = \int_{R^n} \left(i \sum_{j=1}^n [I, \alpha_j(t), w(t)] \cdot v_j(t) \right) \cdot \exp \left\{ -\frac{1}{2z} \sum_{j=1}^n v_j^2 \right\} \mu(d\vec{v}) \tag{3.7}$$

Proof. For $z \in \mathbf{C}^+$,

$$\begin{aligned}
&E_x^{anwz}([D, F, x, w]) \\
&= E_x([D, F, z^{-\frac{1}{2}}x, w]) \\
&= E_x \left(\left(i \sum_{j=1}^n [I, \alpha_j(t), w(t)] \cdot v_j(t) \right) \right. \\
&\quad \cdot \int_{R^n} \exp \left\{ iz^{-\frac{1}{2}} \sum_{j=1}^n [I, \alpha_j(t), x(t)] \cdot v_j(t) \right\} \mu(d\vec{v}) \Bigg) \\
&= \left(\frac{1}{2\pi} \right)^{\frac{n}{2}} \int_{R^n} \left(i \sum_{j=1}^n [I, \alpha_j(t), w(t)] \cdot v_j(t) \right) \\
&\quad \cdot \left(\int_{R^n} \exp \left\{ i \sum_{j=1}^n (z^{-\frac{1}{2}} u_j \cdot v_j) \right\} \mu(d\vec{v}) \right) \exp \left\{ -\frac{1}{2} \sum_{j=1}^n u_j^2 \right\} d\vec{u} \\
&= \left(\frac{1}{2\pi} \right)^{\frac{n}{2}} \int_{R^n} \left(i \sum_{j=1}^n [I, \alpha_j(t), w(t)] \cdot v_j(t) \right) \\
&\quad \cdot \left(\int_{R^n} \exp \left\{ \sum_{j=1}^n \left(-\frac{1}{2} u_j^2 + i z^{-\frac{1}{2}} u_j v_j \right) \right\} d\vec{u} \right) \mu(d\vec{v}) \\
&= \left(\frac{1}{2\pi} \right)^{\frac{n}{2}} \int_{R^n} \left(i \sum_{j=1}^n [I, \alpha_j(t), w(t)] \cdot v_j(t) \right) \cdot \left((2\pi)^{\frac{n}{2}} \exp \left\{ -\frac{1}{2z} \sum_{j=1}^n v_j^2 \right\} \right) \mu(d\vec{v}) \\
&= \int_{R^n} \left(i \sum_{j=1}^n [I, \alpha_j(t), w(t)] \cdot v_j(t) \right) \cdot \exp \left\{ -\frac{1}{2z} \sum_{j=1}^n v_j^2 \right\} \mu(d\vec{v}).
\end{aligned} \tag{3.8}$$

\square

Theorem 3.3 For $z \in C_+$,

$$\exp \left\{ \frac{1-z}{2} \sum_{j=1}^n [I, \alpha_j(t), x(t)]^2 \right\} \cdot [D, F, x, w] \tag{3.9}$$

is a Wiener integrable function of $x \in C_0[0, T]$.

Proof. By the Wiener integration theorem, we have that

$$\begin{aligned}
& E_x \left(\exp \left\{ \frac{1-z}{2} \sum_{j=1}^n [I, \alpha_j(t), x(t)]^2 \right\} \cdot [D, F, x, w] \right) \\
&= E_x \left(\exp \left\{ \frac{1-z}{2} \sum_{j=1}^n [I, \alpha_j(t), x(t)]^2 \right\} \cdot \int_{R^n} \left(i \sum_{j=1}^n [I, \alpha_j(t), w(t)] \cdot v_j(t) \right) \right. \\
&\quad \cdot \exp \left\{ i \sum_{j=1}^n [I, \alpha_j(t), x(t)] \cdot v_j(t) \right\} \mu(d\vec{v}) \Big) \\
&= \int_{R^n} \left(i \sum_{j=1}^n [I, \alpha_j(t), w(t)] \cdot v_j(t) \right) \\
&\quad \cdot E_x \left(\exp \left\{ \sum_{j=1}^n \frac{1-z}{2} [I, \alpha_j(t), x(t)]^2 + i [I, \alpha_j(t), x(t)] \cdot v_j(t) \right\} \right) \mu(d\vec{v}) \\
&= \int_{R^n} \left(i \sum_{j=1}^n [I, \alpha_j(t), w(t)] \cdot v_j(t) \right) \\
&\quad \cdot \left(\left(\frac{1}{2\pi} \right)^{\frac{n}{2}} \int_{R^n} \exp \left\{ \sum_{j=1}^n \frac{1-z}{2} u_j^2 + i u_j v_j \right\} \cdot \exp \left\{ -\frac{1}{2} \sum_{j=1}^n u_j^2 \right\} d\vec{u} \right) \mu(d\vec{v}) \\
&= \left(\frac{1}{2\pi} \right)^{\frac{n}{2}} \int_{R^n} \left(i \sum_{j=1}^n [I, \alpha_j(t), w(t)] \cdot v_j(t) \right) \cdot \left(\int_{R^n} \exp \left\{ \sum_{j=1}^n \left(-\frac{z}{2} u_j^2 + i v_j \cdot u_j \right) \right\} d\vec{u} \right) \mu(d\vec{v}) \\
&= \left(\frac{1}{2\pi} \right)^{\frac{n}{2}} \int_{R^n} \left(i \sum_{j=1}^n [I, \alpha_j(t), w(t)] \cdot v_j(t) \right) \cdot \left(\frac{2\pi}{z} \right)^{\frac{n}{2}} \cdot \exp \left\{ -\frac{1}{2z} \sum_{j=1}^n v_j^2 \right\} \mu(d\vec{v}) \\
&= z^{-\frac{n}{2}} \int_{R^n} \left(i \sum_{j=1}^n [I, \alpha_j(t), w(t)] \cdot v_j(t) \right) \cdot \exp \left\{ -\frac{1}{2z} \sum_{j=1}^n v_j^2 \right\} \mu(d\vec{v}) \tag{3.10}
\end{aligned}$$

As we have that

$$\begin{aligned}
& \left| z^{-\frac{n}{2}} \int_{R^n} \left(i \sum_{j=1}^n [I, \alpha_j(t), w(t)] \cdot v_j(t) \right) \cdot \exp \left\{ -\frac{1}{2z} \sum_{j=1}^n v_j^2 \right\} \mu(d\vec{v}) \right| \\
&\leq z^{-\frac{n}{2}} \int_{R^n} \left| \sum_{j=1}^n [I, \alpha_j(t), w(t)] \cdot v_j(t) \right| |\mu|(d\vec{v}) \\
&\leq z^{-\frac{n}{2}} \|w'\|_2 \int_{R^n} \sum_{j=1}^n |v_j| |\mu|(d\vec{v}) \\
&< \infty, \tag{3.11}
\end{aligned}$$

we have the desired result. \square

Theorem 3.4 For $z \in C_+$,

$$\begin{aligned}
& E_x^{anwz} \left([D, F, x, w] \right) \\
&= z^{\frac{n}{2}} \cdot E_x \left(\exp \left\{ \frac{1-z}{2} \sum_{j=1}^n [I, \alpha_j(t), x(t)]^2 \right\} [D, F, x, w] \right). \tag{3.12}
\end{aligned}$$

Proof. By Theorem 3.2 and Theorem 3.3,

$$\begin{aligned}
 & E_x \left(\exp \left\{ \frac{1-z}{2} \sum_{j=1}^n [I, \alpha_j(t), x(t)]^2 \right\} [D, F, x, w] \right) \\
 &= z^{-\frac{n}{2}} \cdot \int_{R^n} \left(i \sum_{j=1}^n [I, \alpha_j(t), w(t)] \cdot v_j(t) \right) \exp \left\{ -\frac{1}{2z} \sum_{j=1}^n v_j^2 \right\} \mu(d\vec{v}) \\
 &= z^{-\frac{n}{2}} \cdot E_x^{anwz} \left([D, F, x, w] \right). \tag{3.13}
 \end{aligned}$$

□

Theorem 3.5 For real $\rho > 0$,

$$E_x \left([D, F, \rho x, w] \right) = \rho^{-n} \cdot E_x \left(\exp \left\{ \frac{\rho^2 - 1}{2\rho^2} \sum_{j=1}^n [I, \alpha_j(t), x(t)]^2 \right\} [D, F, x, w] \right) \tag{3.14}$$

Proof. For real $z > 0$,

$$\begin{aligned}
 & E_x^{anwz} \left([D, F, x, w] \right) \\
 &= E_x \left([D, F, z^{-\frac{1}{2}} x, w] \right) \\
 &= z^{\frac{n}{2}} \cdot E_x \left(\exp \left\{ \frac{1-z}{2} \sum_{j=1}^n [I, \alpha_j(t), x(t)]^2 \right\} [D, F, x, w] \right) \tag{3.15}
 \end{aligned}$$

Taking $z = \rho^{-2}$, we have the result. □

Theorem 3.6

$$\begin{aligned}
 & E_x^{anf_q} \left([D, F, x, w] \right) \\
 &= \int_{R^n} \left(i \sum_{j=1}^n [I, \alpha_j(t), w(t)] \cdot v_j(t) \right) \exp \left\{ -\frac{i}{2q} \sum_{j=1}^n v_j^2 \right\} \mu(d\vec{v}) \tag{3.16}
 \end{aligned}$$

Proof.

$$\begin{aligned}
 & E_x^{anf_q} \left([D, F, x, w] \right) \\
 &= \lim_{z \rightarrow -iq} E_x^{anwz} \left([D, F, x, w] \right) \\
 &= \lim_{z \rightarrow -iq} \int_{R^n} \left(i \sum_{j=1}^n [I, \alpha_j(t), w(t)] \cdot v_j(t) \right) \exp \left\{ -\frac{1}{2z} \sum_{j=1}^n v_j^2 \right\} \mu(d\vec{v}) \\
 &= \int_{R^n} \left(i \sum_{j=1}^n [I, \alpha_j(t), w(t)] \cdot v_j(t) \right) \cdot \exp \left\{ -\frac{i}{2q} \sum_{j=1}^n v_j^2 \right\} \mu(d\vec{v}) \tag{3.17}
 \end{aligned}$$

whenever $z \rightarrow -iq$ through \mathbf{C}_+ . But we have that

$$\begin{aligned}
 & \left| E_x^{anf_q}([D, F, x, w]) \right| \\
 & \leq \int_{R^n} \left| \sum_{j=1}^n [I, \alpha_j(t), w(t)] \cdot v_j(t) \right| |\mu|(d\vec{v}) \\
 & = \int_{R^n} \left(\sum_{j=1}^n \left| \int_0^T \alpha_j(t) w'(t) dt \right| \cdot |v_j(t)| \right) |\mu|(d\vec{v}) \\
 & \leq \int_{R^n} \sum_{j=1}^n \left(\|\alpha_j\|_2 \cdot \|w'\|_2 \cdot |v_j(t)| \right) |\mu|(d\vec{v}) \\
 & = \|w'\|_2 \cdot \int_{R^n} \sum_{j=1}^n |v_j(t)| |\mu|(d\vec{v}) \\
 & < \infty.
 \end{aligned} \tag{3.18}$$

Therefore the analytic Feynman integral for the partial derivative of the Fourier transform $\hat{\mu}$ of the measure μ on \mathbf{R}^n exists. \square

Theorem 3.7

$$\begin{aligned}
 & E_x^{anf_q}([D, F, x, w]) \\
 & = \lim_{k \rightarrow \infty} z_k^{\frac{n}{2}} \cdot E_x \left(\exp \left\{ \frac{1 - z_k}{2} \sum_{j=1}^n [I, \alpha_j(t), x(t)]^2 \right\} \cdot [D, F, x, w] \right)
 \end{aligned} \tag{3.19}$$

whenever $\{z_k\} \rightarrow -iq$ through C_+ .

Proof. By the theorem 3.4,

$$\begin{aligned}
 & E_x^{anf_q}([D, F, x, w]) \\
 & = \lim_{k \rightarrow \infty} E_x^{anw_{z_k}}([D, F, x, w]) \\
 & = \lim_{k \rightarrow \infty} z_k^{\frac{n}{2}} \cdot E_x \left(\exp \left\{ \frac{1 - z_k}{2} \sum_{j=1}^n [I, \alpha_j(t), x(t)]^2 \right\} \cdot [D, F, x, w] \right)
 \end{aligned} \tag{3.20}$$

whenever $\{z_k\} \rightarrow -iq$ through C_+ . \square

3. Coclusions

We prove the change of scale formula for the function space integral about the partial derivative of a function defined on the function space which is constructed by a Fourier transform of a measure. And we prove some relationships among the analytic Wiener integral and the analytic Feynman integral and the Wiener integral of the partial deriavtive for the function space integral.

Remark 1. In this paper, we prove new results by extending those results [11] to the first variation theory in [1] and to the partial derivative approach developed in [13] and to the change of scale formula in [7] using the Fourier transform of a measure.

Remark 2. In [12], the author extended those results in [11] to the first variation theory in [1] and to the Fourier Feynman transform theory which is an integral transform on the function space.

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Department of Mathematics, College of Natural Sciences, Industry-University Cooperation Foundation, Hanyang University, 222 Wangsimni-ro, Seongdong-gu, Seoul 04763, Republic of Korea