

Mathematical Relevance of Polynions

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ABSTRACT

In this note we introduce the notion of polynions and discuss their mathematical relevance. We note that the well-known mathematical structures like quaternions, octonions and sedenions etc. are special cases of polynions.

Keywords: polynion, quaternion, octonion, sedenions, ring, algebra.

MSC2010: 16D25, 16N40.

1. Introduction

The well known quaternions [1-3] were discovered by William Rowan Hamilton. These are very useful in mathematics as well as in physics. Octonions [3-4] were discovered later on the basis of the theory of quaternions. Similarly sedenions [5] are an extention of octonions.

We introduce the notion of polynions, plus and minus polynions. It is noted that the well-known mathematical structures like quaternions, octonions, sedenions are special cases of minus polynions.

The mathematical structures given by minus polynions and plus polynions are quite distinct and have different algebraic properties. It is well-known that Quaternions form a four dimensional non-commutative but associative algebra however the algebra of octonions is neither associative nor commutative. But in the case of the mathematical structures given by plus polynions we retain the both commutativity as well as associativity for multiplication.

In general the notion of plus polynions leads to construct a unital commutative as well as associative algebra of dimension $n+1$ for each n where $n = 2^t - 1$; $t \in N \cup \{0\}$. Here N denotes the set of natural numbers.

In the next section we introduce the notion of polynions. In section 3 we discuss the mathematical relevance of polynions. In the last section we provide conclusions.

2. Polynions

Definition 2.1: Polynion: Let p be an expression of the form $p = a_0 + a_1e_1 + a_2e_2 + \dots + a_ne_n$. If $e_1^2 = e_2^2 = \dots = e_n^2 = -1$ or $e_1^2 = e_2^2 = \dots = e_n^2 = 1$ then p is called a polynion. Here $a_0, a_1, a_2, \dots, a_n$ are real numbers and $n = 2^t - 1; t \in N \cup \{0\}$. Here N denotes the set of natural numbers.

Definition 2.2: Minus Polynion: If $e_1^2 = e_2^2 = \dots = e_n^2 = -1$ and $e_i e_j + e_j e_i = 0$ for each e_i, e_j such that $e_i \neq e_j$, $1 \leq i \leq n, 1 \leq j \leq n$ then $p = a_0 + a_1e_1 + a_2e_2 + \dots + a_ne_n$ is called the minus polynion. Let $U_n^- = \{\pm 1, \pm e_1, \pm e_2, \dots, \pm e_n\}$. Then the set U_n^- of all unit minus polynions forms a group under the multiplication of unit minus polynions for each $n \leq 3$. If $n \geq 7$ then U_n^- satisfies each condition to be a group except associativity.

Definition 2.2: Plus Polynion: If $e_1^2 = e_2^2 = \dots = e_n^2 = 1$ and $e_i e_j + e_j e_i = 2e_i e_j$ for each e_i, e_j such that $e_i \neq e_j$; $1 \leq i \leq n, 1 \leq j \leq n$ then $p = a_0 + a_1e_1 + a_2e_2 + \dots + a_ne_n$ is called the plus polynion. Let $U_n^+ = \{1, e_1, e_2, \dots, e_n\}$. Then the set U_n^+ of all unit plus polynions forms a commutative group of order 2^t under the multiplication of unit plus polynions. In this group each element is self inverse.

It shuld be emphasized that multiplication of unit plus polynions is carried out in the same way as it is done in the case of unit minus polynions but we replace minus sign by plus sign in the case of unit plus polynions. By this we retain commutativity and associtaivity of multiplication for unit plus polynions. We illustrate this by means of the following example.

Let $n = 3$ then in the case of unit minus polynions we have (for $e_i \neq e_j$) $e_1e_2 = -e_2e_1 = e_3$, $e_2e_3 = -e_3e_2 = e_1$ and $e_3e_1 = -e_1e_3 = e_2$. However in the case of unit plus polynions this becomes $e_1e_2 = e_2e_1 = e_3$, $e_2e_3 = e_3e_2 = e_1$ and $e_3e_1 = e_1e_3 = e_2$.

In the next section we discuss about the sets of all minus polynions and plus polynions and provide multiplication rule for e_i and e_j in the case of plus polynions for $n = 1, 3$ and 7 .

3. Mathematical Relevance of Polynoins

We consider the case of minus polynions and plus polynions separately.

Case I: Minus Polynions: Let P_n^- denotes the set of all minus polynion. Let us take $n=0$ then it is easy to see that the set P_n^- of all minus polynions reduces to the set of real numbers. If we take $n=1$ then P_n^- reduces to the set of complex numbers. Similarly one may see that P_n^- reduces to the set of all quaternions and octonions for $n=3$ and 7 respectively. Further if we take $n=15$ then P_n^- gives the set of all sedenions.

It may be noted that real numbers, complex numbers, quaternions, octonions and sedenions are well-known mathematical structures and so the multiplication rule for them can be found in mathematical literatures. Therefore we do not discuss the multiplication rule for e_i and e_j in these cases. For further details on quaternions, octonions and sedenions one may refer [3-5].

We now come to the set of plus polynions, say P_n^+ , and describe the multiplication rule for e_i and e_j in this case.

Case II: Plus Polynions: We see that the set P_n^- of all minus polynions contains the set R of real numbers as well as the set C of complex numbers. However the set P_n^+ of all plus polynions contains the set R of real numbers but not the set C of complex numbers. Below we discuss about the set P_n^+ of plus polynions for $n=1,3$ and 7 (for $n=0$, P_n^+ reduces to the set of real numbers itself).

If we take $n=1$ then we can write $P_1^+ = \{a + e_1 b : a, b \in R\}$. Clearly P_1^+ gives a special class of numbers which is neither real nor complex but it looks like complex numbers.

One may easily verify that P_1^+ is a commutative ring with identity under addition and multiplication of polynions defined below.

Let $p_1 = a + e_1 b \in R_1$ and $p_2 = c + e_1 d \in R_1$ then the sum of p_1 and p_2 is defined by $p_1 + p_2 = (a + e_1 b) + (c + e_1 d) = (a + c) + e_1(b + d)$ and the product of p_1 and p_2 is defined by $p_1 p_2 = (a + e_1 b)(c + e_1 d) = (ac + bd) + e_1(bc + ad)$.

Moreover one may see that P_1^+ is a unital commutative and associative real algebra of dimension two but unlike the algebra of complex numbers this algebra has zero divisors.

If we take $n=3$ then the set of all plus polynions reduces to $P_3^+ = \{a + e_1b + e_2c + e_3d : a, b, c, d \in R\}$. Clearly P_3^+ gives a set which looks like the set of all quaternions. As already explained in this case we have $e_1^2 = e_2^2 = e_3^2 = 1$. Also, $e_1e_2 = e_2e_1 = e_3$, $e_2e_3 = e_3e_2 = e_1$ and $e_3e_1 = e_1e_3 = e_2$. It may be noted that unlike usual Hamilton quaternions P_3^+ forms a unital commutative algebra of dimension four. We have retained commutativity as it is clear from the defining relations of e_i and e_j . It may be verified that this algebra has zero divisors.

One may easily verify that in this case the set $U_3^+ = \{1, e_1, e_2, e_3\}$ of unit plus polynions forms the Klein's group under multiplication.

If we take $n=7$ then the set of all plus polynions reduces to $P_7^+ = \{a_0 + a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6 + a_7e_7 : a_i \in R\}$. Clearly P_7^+ gives a set which looks like the set of all octonions. As already explained in this case we have $e_1^2 = e_2^2 = e_3^2 = \dots = e_7^2 = 1$. The multiplication of e_i and e_j such that $e_i \neq e_j$; $1 \leq i \leq 7$, $1 \leq j \leq 7$ can be given in several ways. For example we may define that

$$\begin{aligned} e_1e_2 = e_2e_1 = e_4, \quad e_1e_3 = e_3e_1 = e_6; \quad e_1e_4 = e_4e_1 = e_2, \quad e_1e_5 = e_5e_1 = e_7, \quad e_1e_6 = e_6e_1 = e_3, \\ e_1e_7 = e_7e_1 = e_5; \quad e_2e_3 = e_3e_2 = e_5, \quad e_2e_4 = e_4e_2 = e_1, \quad e_2e_5 = e_5e_2 = e_3, \quad e_2e_6 = e_6e_2 = e_7, \\ e_2e_7 = e_7e_2 = e_6; \quad e_3e_4 = e_4e_3 = e_7, \quad e_3e_5 = e_5e_3 = e_2, \quad e_3e_6 = e_6e_3 = e_1, \quad e_3e_7 = e_7e_3 = e_4; \\ e_4e_5 = e_5e_4 = e_6, \quad e_4e_6 = e_6e_4 = e_5, \quad e_4e_7 = e_7e_4 = e_3; \quad e_5e_6 = e_6e_5 = e_4, \quad e_5e_7 = e_7e_5 = e_1; \\ e_6e_7 = e_7e_6 = e_2. \end{aligned}$$

It may be noted that unlike usual octonions P_7^+ forms a unital commutative as well as associative algebra of dimension eight. We have retained commutativity and associativity as it is clear from the defining relations of unit plus octonions. It may be verified that this algebra has zero divisors.

One may easily verify that in this case the set $U_7^+ = \{1, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ of unit plus polynions forms a multiplicative group of order eight in which each element is self inverse.

Similarly it may be seen that for $n=15$, the set P_n^+ of all plus polynions gives a mathematical structure corresponding to the sedenions.

Conclusions.

It is seen that quaternions [1-3], octonions [3-4] and sedenions [5] are special cases of minus polynions. The notion of plus polynions leads to define new mathematical structures corresponding to each structure given by the minus polynions. Moreover in the case of plus polynions we retain commutativity and associativity under multiplication and thus the algebra of plus polynions P_n^+ is commutative as well as associative for each $n = 2^t - 1; t \in N \cup \{0\}$ and these algebras have zero divisors whenever $n > 0$. One may consider these notions for further possible research.

References

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