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Representations of a Comparison Measure Between Two Fuzzy Sets

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Abstract: This paper analyzes the representation behaviors of a comparison measure between two compared fuzzy sets. Three types of restrictions on two fuzzy sets are considered in this paper: the two disjoint union fuzzy sets, the two disjoint fuzzy sets and the two general fuzzy sets. Differences exist among the numbers of possible representations of a comparison measure for the three types of fuzzy sets restrictions. The value of comparison measure is constant for the two disjoint union fuzzy sets. There are 42 candidate representations of a comparison measure for the two disjoint fuzzy sets. Of which 13 candidate representations with one or two terms can be used to easily calculate and compare a comparison measure.

Keywords: fuzzy set; comparison measure; representation; disjoint.

1. Introduction

Fuzzy sets (FSs) theory, proposed by Zadeh [1], is characterized by a membership function and has successfully been applied in various fields. The paper deals with the well-known notions of comparison measures between two compared FSs. A comparison measure calculates the degree of equality or inequality between two compared FSs. Some related definitions such as similarity, similitude, proximity or resemblance were proposed for the equality measure [2-14], as well as some other dual definitions such as dissimilarity, dissimilitude, divergence or distance for the inequality measure [5, 8, 11, 15-20]. However, the inequality measures have received much less attention in the literature. The degree of comparison measure is an important tool for cluster analysis [7], decision-making [6, 12, 15, 17-19], medical diagnosis [11] and pattern recognition [2, 3, 8]. Recently, many papers [4, 8-10, 13-15, 17-19] have been dedicated to the comparison measures and research on this area is still carried on in the literature.

Couso et al. [5] surveyed a large collection of axiomatic definitions from the literature regarding the notions of comparison measures between two compared FSs. One of the fundamental axioms of a comparison measure is as follows. For the two disjoint FSs A and B, if both comparison measures between A and empty set and that of B and empty set are less, then the degree of a comparison measure between A and B is less. From this axiomatic definition, we analyze the comparison measure behaviors of two FSs A and B in terms of the other simple comparison measures, especially for the intersection and the union of A and B, the empty set and the universal set. The representations of a comparison measure between two FSs can not only present the important components of a comparison measure but also analyze the comparison measure behaviors of two FSs in terms of other simple comparison measures. This paper focuses the representations of a comparison measure between two FSs.

To analyze the representation behaviors of a comparison measure between two FSs, three kinds of two FSs are considered in this paper: the two disjoint union FSs, the two disjoint FSs and the two general FSs. For the two FSs A and B, this paper deals with the representations of a comparison measure for the case that $A \cap B = \emptyset$ and $A \cup B = U$, the case that $A \cap B = \emptyset$ and the general FSs A and B.

The organization of this paper is as follows. Section 2 briefly reviews the FSs and the comparison measures between two compared FSs. We present representations of a comparison measure for the two disjoint union FSs in section 3, the two disjoint FSs in section 4 and the two general FSs in section 5. Finally, some concluding remarks and future research are presented.

2. FSs and Comparison Measures

We firstly review the basic notations of FSs. Let $U = \{x_1, x_2, \ldots, x_n\}$ be a non-empty universal set or referential set of real numbers $\mathcal{R}$.

Definition 1. A FS A over U is defined as

$$A = \{(x_i, \mu_A(x_i)) | 1 \le i \le n\}$$

where the membership function $\mu_A(x_i): U \to [0,1]$ for $1 \le i \le n$. We denote by $\mathcal{F}(U)$ the set of all FSs over U.

Definition 2. For the two FSs $A, B \in \mathcal{F}(U)$, define the membership functions of $A \cap B$, $A \cup B$ and $A \backslash B$ as follows.

1. $\mu_{A \cap B}(x) = min\{\mu_A(x), \mu_B(x)\}$
2. $\mu_{AUB}(x) = max\{\mu_A(x), \mu_B(x)\}$
3. $\mu_{A \backslash B}(x) = min\{\mu_A(x), 1 - \mu_B(x)\}$.

We now recall the definition of comparison measures between two FSs. The following properties are general axioms that may be required in equality measure and inequality measure between two FSs.

Definition 3. A comparison measure $m: \mathcal{F}(\mathrm{U})^2 \to \mathcal{R}$ should satisfy the following properties:

- G1: $0 \le m(A, B) \le 1$, $\forall A, B \in \mathcal{F}(U)$.
- G1*: $0 \le m(A, B) \le 1$, $\forall A, B \in \mathcal{F}(U)$ and there exists two FSs $C, D \in \mathcal{F}(U)$ such that $m(C, D) = 1$.
- G2: $m(A, B) = m(B, A)$, $\forall A, B \in \mathcal{F}(U)$.
- G3: Let $\rho: U \to U$ be a permutation for finite U. Define $A^\rho \in \mathcal{F}(U)$ with membership function $\mu_{A^\rho}(x) = \mu_A(\rho(x))$ for $A \in \mathcal{F}(U)$. Then $m(A, B) = m(A^\rho, B^\rho)$.
- G3*: For finite set U, there exists a function $h: [0,1] \times [0,1] \to \mathcal{R}$ such that $m(A, B) = \sum_{x \in U} h(\mu_A(x), \mu_B(x))$, $\forall A, B \in \mathcal{F}(U)$.
- G4: There exists a function $f: \mathcal{F}(\mathrm{U})^3 \to \mathcal{R}$ such that $m(A, B) = f(A \cap B, A \backslash B, B \backslash A)$, $\forall A, B \in \mathcal{F}(U)$.
- G4*: There exists a function $F: \mathcal{R}^3 \to \mathcal{R}$ and a fuzzy measure $M: \mathcal{F}(U) \to \mathcal{R}$ such that $m(A, B) = F\big(M(A \cap B), M(A \backslash B), M(B \backslash A)\big)$, $\forall A, B \in \mathcal{F}(U)$.
- G5: If $A \cap B = \emptyset$, $A' \cap B' = \emptyset$, $m(A, \emptyset) \le m(A', \emptyset)$ and $m(B, \emptyset) \le m(B', \emptyset)$, then $m(A, B) \le m(A', B')$, $\forall A, B, A', B' \in \mathcal{F}(U)$.

Consider a fuzzy measure $M: \mathcal{F}(U) \to \mathcal{R}$ with $M(A \cap B) = c$, $M(A \backslash B) = a$ and $M(B \backslash A) = b$, $a, b, c \in [0, 1]$. Define a comparison measure as follows.

$$m(A, B) = F\big(M(A \cap B), M(A \backslash B), M(B \backslash A)\big) = F(c, a, b) = \frac{c+1-a+1-b}{3}.$$

For the two disjoint FSs A and B, we have

$$m(A, B) = F(0, a, b) = \frac{2-a-b}{3},$$

$$m(A, \emptyset) = F(0, a, 0) = \frac{2-a}{3}$$

and

$$m(B, \emptyset) = F(0, b, 0) = \frac{2-b}{3},$$

it implies that

$$m(A,B) = -\frac{2}{3} + m(A,\emptyset) + m(B,\emptyset).$$

If $A \cap B = \emptyset$, $A' \cap B' = \emptyset$, $m(A,\emptyset) \leq m(A',\emptyset)$ and $m(B,\emptyset) \leq m(B',\emptyset)$, we obtain

$$m(A,B) = -\frac{2}{3} + m(A,\emptyset) + m(B,\emptyset) \leq -\frac{2}{3} + m(A',\emptyset) + m(B',\emptyset) = m(A',B')$$

which coincides with the result of G5. Therefore, the representation of $m(A,B)$ can not only present its important ingredients but also compare $m(A,B)$ in terms of other measures $m(A,\emptyset)$ and $m(B,\emptyset)$. For two FSs A and B, the adopted components of a comparison measure are A, B, the intersection and the union of A and B, the empty set and the universal set. To represent a comparison measure $m(A,B)$, the adopted comparison measures other than $m(A,B)$ are $m(X,Y)$ of different FSs X and Y, $(X,Y) \neq (A,B)$, $X,Y \in \{\emptyset, A, B, A \cap B, A \cup B, U\}$.

The following sections list the representations of a comparison measure $m(A,B)$ for the two disjoint union FSs A and B, the two disjoint FSs A and B and the two general FSs A and B. More precisely, we consider the case that $A \cap B = \emptyset$ and $A \cup B = U$ for section 3, $A \cap B = \emptyset$ for section 4 and the general FSs A and B for section 5.

3. Representations of a Comparison Measure for the Two Disjoint Union Fuzzy Sets

This section will present the representations of a comparison measure $m(A,B)$ for the two disjoint union FSs A and B. For $A \cap B = \emptyset$ and $A \cup B = U$, we have that $M(A \cap B) = 0$, $M(A \cup B) = 1$, $M(A \backslash B) = a$, $M(B \backslash A) = b$, $a + b = 1$, $a, b \in [0,1]$ and

$$m(A,B) = F\big(M(A \cap B), M(A \backslash B), M(B \backslash A)\big) = F(0,a,b) = \frac{2-a-b}{3} = \frac{1}{3}.$$

Since $A \cap B = \emptyset$ and $A \cup B = U$, the adopted components of a comparison measure are $\{\emptyset, A, B, U\}$. For two different FSs X and Y, $X, Y \in \{\emptyset, A, B, U\}$, the number of the possible forms of $m(X,Y)$ is 6 which are described as follows.

$$m(A,B) = F(0,a,b) = \frac{1}{3},\ m(A,\emptyset) = F(0,a,0) = \frac{2-a}{3},$$
$$m(B,\emptyset) = F(0,b,0) = \frac{2-b}{3},\ m(A,U) = F(a,0,1-a) = \frac{1+2a}{3},$$
$$m(B,U) = F(b,0,1-b) = \frac{1+2b}{3}$$

and

$$m(\emptyset,U) = F(0,0,1) = \frac{1}{3}.$$

From these 6 measures $m(X,Y)$, we obtain 6 equations for the representations of $\frac{a}{3}$ and 6 equations for those of $\frac{b}{3}$ presented as follows.

$$\frac{a}{3} = \frac{1}{2}(m(A,U) - m(A,B)) = m(B,\emptyset) - m(A,B) = \frac{1}{2}(\frac{4}{3} - m(A,B) - m(B,U))$$
$$= \frac{4}{3} - m(B,U) - m(B,\emptyset) = m(A,U) - m(B,\emptyset) = \frac{1}{2}(1 - m(B,U))$$

and

$$\frac{b}{3} = \frac{1}{2}(m(B,U) - m(A,B)) = m(A,\emptyset) - m(A,B) = \frac{1}{2}(\frac{4}{3} - m(A,B) - m(A,U))$$
$$= \frac{4}{3} - m(A,U) - m(A,\emptyset) = m(B,U) - m(A,\emptyset) = \frac{1}{2}(1 - m(A,U)).$$

Since the constant value of

$$m(A,B) = \frac{1}{3}$$

for $A \cap B = \emptyset$ and $A \cup B = U$, we cannot compare the degree of $m(A,B)$ for the two disjoint union FSs A and B.

4. Representations of a Comparison Measure for the Two Disjoint Fuzzy Sets

For the two disjoint FSs A and B, $A \cap B = \emptyset$, we denote $M(A \cap B) = 0$, $M(A \backslash B) = a$, $M(B \backslash A) = b$, $a, b \in [0, 1]$ and

$$m(A, B) = F\big(M(A \cap B), M(A \backslash B), M(B \backslash A)\big) = F(0, a, b) = \frac{2-a-b}{3}.$$

The number of total combinations $m(X, Y)$ of two different FSs X and Y, $X, Y \in \{\emptyset, A, B, A \cup B, U\}$ are 10 which are presented as follows.

$$m(A, B) = F(0, a, b) = \frac{2-a-b}{3},\ m(A, \emptyset) = F(0, a, 0) = \frac{2-a}{3},$$

$$m(B, \emptyset) = F(0, b, 0) = \frac{2-b}{3},\ m(A, A \cup B) = F(a, 0, b) = \frac{2+a-b}{3},$$

$$m(B, A \cup B) = F(b, 0, a) = \frac{2-a+b}{3},\ m(A, U) = F(a, 0, 1-a) = \frac{1+2a}{3},$$

$$m(B, U) = F(b, 0, 1-b) = \frac{1+2b}{3},\ m(A \cup B, \emptyset) = F(0, a+b, 0) = \frac{2-a-b}{3},$$

$$m(A \cup B, U) = F(a+b, 0,1-a-b) = \frac{1+2a+2b}{3}$$

and

$$m(\emptyset, U) = F(0,0,1) = \frac{1}{3}.$$

To represent $m(A, B) = \frac{2-a-b}{3}$, from above 10 measures $m(X, Y)$, we obtain 9 equations for the representations of $\frac{a}{3}$ and 9 equations for those of $\frac{b}{3}$ described as follows.

$$\frac{a}{3}$$

[1] $= \frac{1}{2}(m(A, A \cup B) - m(A, B))$
[2] $= m(B, \emptyset) - m(A, B)$
[3] $= \frac{1}{2}(\frac{4}{3} - m(A, B) - m(B, A \cup B))$
[4] $= \frac{1}{2}(\frac{5}{3} - 2m(A, B) - m(B, U))$
[5] $= \frac{4}{3} - m(B, A \cup B) - m(B, \emptyset)$
[6] $= m(A, A \cup B) - m(B, \emptyset)$
[7] $= \frac{1}{2}(-\frac{5}{3} + m(A \cup B, U) + 2m(B, \emptyset))$
[8] $= \frac{1}{4}(1 + m(A \cup B, U) - 2m(B, A \cup B))$
[9] $= \frac{1}{2}(m(A \cup B, U) - m(B, U))$

and

$$\frac{b}{3}$$

[1] $= \frac{1}{2}(m(B, A \cup B) - m(A, B))$
[2] $= m(A, \emptyset) - m(A, B)$
[3] $= \frac{1}{2}(\frac{4}{3} - m(A, B) - m(A, A \cup B))$
[4] $= \frac{1}{2}(\frac{5}{3} - 2m(A, B) - m(A, U))$
[5] $= \frac{4}{3} - m(A, A \cup B) - m(A, \emptyset)$
[6] $= m(B, A \cup B) - m(A, \emptyset)$
[7] $= \frac{1}{2}(-\frac{5}{3} + m(A \cup B, U) + 2m(A, \emptyset))$
[8] $= \frac{1}{4}(1 + m(A \cup B, U) - 2m(A, A \cup B))$
[9] $= \frac{1}{2}(m(A \cup B, U) - m(A, U))$.

The number of the total combinations of forms of $\frac{a}{3}$ and $\frac{b}{3}$ to represent a comparison measure $m(A, B)$ is $9 \times 9 = 81$. We will denote by [i]-[j], the combination of ith form of $\frac{a}{3}$ and jth form of $\frac{b}{3}$ to represent $m(A, B)$. We classify these 81 combinations into four types (I, II, III, IV). The first type I is the candidate representation of a comparison measure $m(A, B)$. For example, the combination [1]-[2], $\frac{a}{3} = \frac{1}{2}(m(A, A \cup B) - m(A, B))$ and $\frac{b}{3} = m(A, \emptyset) - m(A, B)$, we obtain that

$$m(A, B) = \frac{2}{3} - \frac{1}{2}\big(m(A, A \cup B) - m(A, B)\big) - (m(A, \emptyset) - m(A, B))$$

$$= -\frac{4}{3} + m(A, A \cup B) + 2m(A, \emptyset).$$

Among these 81 combinations, there are 42 candidate representations of $m(A,B)$ for type I. The number of terms $m(X,Y)$, $X,Y \in \{\emptyset, A, B, A \cup B, U\}$ of a candidate representation of $m(A,B)$ is 1, 2, 3 and 4, except for the constant term. There are 1, 12, 23 and 6 candidate representations of $m(A,B)$ for the number of terms being 1, 2, 3 and 4, respectively. The combination [8]-[1] is the one term $m(X,Y)$ of a candidate representation of $m(A,B)$ as follows.

$$\text{[8]-[1]: } m(A,B) = \frac{5}{6} - \frac{1}{2}m(A \cup B, U).$$

If $A \cap B = \emptyset$, $A' \cap B' = \emptyset$ and $m(A \cup B, U) \geq m(A' \cup B', U)$, then $m(A,B) \leq m(A',B')$. The two terms $m(X,Y)$, $X,Y \in \{\emptyset, A, B, A \cup B, U\}$ of candidate representations of $m(A,B)$ are as follows.

[1]-[2]: $m(A,B) = -\frac{4}{3} + m(A, A \cup B) + 2m(A, \emptyset)$

[2]-[1]: $m(A,B) = -\frac{4}{3} + m(B, A \cup B) + 2m(B, \emptyset)$

[1]-[8]: $m(A,B) = -\frac{2}{3} + m(A, \emptyset) + m(B, \emptyset)$

[2]-[3]: $m(A,B) = -m(A, A \cup B) + 2m(B, \emptyset)$

[3]-[2]: $m(A,B) = -m(B, A \cup B) + 2m(A, \emptyset)$

[2]-[4]: $m(A,B) = \frac{1}{6} - \frac{1}{2}m(A, U) + m(B, \emptyset)$

[4]-[2]: $m(A,B) = \frac{1}{6} - \frac{1}{2}m(B, U) + m(A, \emptyset)$

[1]-[4]: $m(A,B) = \frac{1}{3} + m(A, A \cup B) - m(A, U)$

[4]-[1]: $m(A,B) = \frac{1}{3} + m(B, A \cup B) - m(B, U)$

[4]-[4]: $m(A,B) = 1 - \frac{1}{2}m(A, U) - \frac{1}{2}m(B, U)$

[4]-[3]: $m(A,B) = \frac{5}{3} - m(A, A \cup B) - m(B, U)$

and

[3]-[4]: $m(A,B) = \frac{5}{3} - m(B, A \cup B) - m(A, U)$.

The second type II is the relationship between different terms of $m(X,Y)$, $X,Y \in \{\emptyset, A, B, A \cup B, U\}$ other than $m(A,B)$. For example, the combination [1]-[1], $\frac{a}{3} = \frac{1}{2}(m(A, A \cup B) - m(A,B))$ and $\frac{b}{3} = \frac{1}{2}(m(B, A \cup B) - m(A,B))$, we get that

$$m(A,B) = \frac{2}{3} - \frac{1}{2}\big(m(A, A \cup B) - m(A,B)\big) - \frac{1}{2}(m(B, A \cup B) - m(A,B))$$

so

$$m(A, A \cup B) + m(B, A \cup B) = \frac{4}{3}.$$

The combinations [1]-[1], [2]-[5], [2]-[6], [2]-[7], [2]-[8], [2]-[9], [3]-[3], [4]-[5], [4]-[6], [4]-[7], [4]-[8], [4]-[9], [5]-[2], [5]-[4], [6]-[2], [6]-[4], [7]-[2], [7]-[4], [8]-[2], [8]-[4] and [9]-[2] are included in type II. Among these 21 combinations, the number of different relationships between different terms of $m(X,Y)$, $X,Y \in \{\emptyset, A, B, A \cup B, U\}$ other than $m(A,B)$ is 16.

The third type III is the identical equation 0=0. For example, the combination [1]-[3], we obtain that

$$m(A,B) = \frac{2}{3} - \frac{1}{2}\big(m(A, A \cup B) - m(A,B)\big) - \frac{1}{2}(\frac{4}{3} - m(A,B) - m(A, A \cup B))$$

so

$$0 = 0.$$

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The combinations [1]-[3], [3]-[1] and [9]-[4] are listed in the type III.

The fourth type IV is the duplicate representations of a comparison measure $m(A,B)$ which appear in type I. For example, the combination [1]-[5], we obtain that

$$m(A,B) = \frac{2}{3} - \frac{1}{2}\big(m(A, A\cup B) - m(A,B)\big) - \left(\frac{4}{3} - m(A, A\cup B) - m(A,\emptyset)\right)$$

$$= -\frac{4}{3} + m(A, A\cup B) + 2m(A,\emptyset)$$

which is the same as that of combination [1]-[2]. There are 15 combinations in type IV described as follows.

[1]-[8]≡[2]-[2] ≡[5]-[5] ≡[5]-[6] ≡[6]-[5] ≡[6]-[6] ≡[7]-[6],

[2]-[3] ≡[6]-[1] ≡[6]-[3], [7]-[7] ≡[8]-[8] ≡[9]-[9], [1]-[2] ≡[1]-[5],

[2]-[1] ≡[5]-[1], [3]-[2] ≡[3]-[6], [8]-[1] ≡[8]-[3] and [8]-[5] ≡[8]-[6].

The combination [1]-[8]

$$m(A,B) = -\frac{2}{3} + m(A,\emptyset) + m(B,\emptyset)$$

has the largest number (6) of duplicate representations. Both combinations [2]-[3] and [7]-[7] have two duplicate representations.

Therefore, there are 81 combinations of a comparison measure $m(A,B)$ for $A\cap B = \emptyset$. Among these 81 combinations, we obtain 42 candidate representations of $m(A,B)$, 15 duplicate representations of $m(A,B)$, 21 relationships between different terms of $m(X,Y)$, $X,Y \in \{\emptyset, A, B, A\cup B, U\}$ other than $m(A,B)$ and 3 identical equations. There are 1 and 12 candidate representations of $m(A,B)$ for one and two terms $m(X,Y)$, $X,Y \in \{\emptyset, A, B, A\cup B, U\}$, respectively. These 13 candidate representations can be used to easily compare $m(A,B)$ with $A\cap B = \emptyset$.

5. Representations of a Comparison Measure for the Two General Fuzzy Sets

This section lists the representations of a comparison measure $m(A,B)$ for the two general FSs A and B. Let $M(A\cap B) = c$, $M(A\backslash B) = a$, $M(B\backslash A) = b$, $a,b,c \in [0,1]$ and

$$m(A,B) = F\big(M(A\cap B), M(A\backslash B), M(B\backslash A)\big) = F(c,a,b) = \frac{c+1-a+1-b}{3}.$$

The adopted components of a comparison measure are $\{\emptyset, A, B, A\cap B, A\cup B, U\}$. There are 15 combinations $m(X,Y)$ of different FSs X and Y, $X,Y \in \{\emptyset, A, B, A\cap B, A\cup B, U\}$ as follows.

$$m(A,B) = F(c,a,b) = \frac{2-a-b+c}{3},\ m(A,\emptyset) = F(0,a+c,0) = \frac{2-a-c}{3},$$

$$m(B,\emptyset) = F(0,b+c,0) = \frac{2-b-c}{3},\ m(A,A\cap B) = F(c,a,0) = \frac{2-a+c}{3},$$

$$m(B,A\cap B) = F(c,b,0) = \frac{2-b+c}{3},\ m(A,A\cup B) = F(a+c,0,b) = \frac{2+a-b+c}{3},$$

$$m(B,A\cup B) = F(b+c,0,a) = \frac{2-a+b+c}{3},\ m(A,U) = F(a+c,0,1-a-c) = \frac{1+2a+2c}{3},$$

$$m(B,U) = F(b+c,0,1-b-c) = \frac{1+2b+2c}{3},\ m(A\cap B,\emptyset) = F(0,c,0) = \frac{2-c}{3},$$

$$m(A\cup B,\emptyset) = F(0,a+b+c,0) = \frac{2-a-b-c}{3},\ m(A\cap B, A\cup B) = F(c,a+b,0) = \frac{2-a-b+c}{3},$$

$$m(A\cap B,U) = F(c,0,1-c) = \frac{1+2c}{3},\ m(A\cup B,U) = F(a+b+c,0,1-a-b-c) = \frac{1+2a+2b+2c}{3}$$

and $m(\emptyset,U) = F(0,0,1) = \frac{1}{3}$.

One can make several notable observations. Firstly, we have that

$$m(A\cap B, A\cup B) = \frac{2-a-b+c}{3} = m(A,B).$$

So, to calculate the degree of a comparison measure $m(A,B)$ is equivalent to calculate that of $m(A\cap B, A\cup B)$.

Secondly, we have 33 different relationships between different terms of $m(X,Y)$, $X,Y \in \{\emptyset, A, B, A\cap B, A\cup B, U\}$. Since $m(A,B)=\frac{2-a-b+c}{3}$, from these 33 relationships, we obtain 12 equations for the representations of $\frac{a}{3}$, 12 equations for those of $\frac{b}{3}$ and 3 equations for those of $\frac{c}{3}$ presented as follows.

$$\frac{a}{3}=\frac{1}{2}\big(m(A,A\cup B)-m(A,B)\big)=m(B,A\cap B)-m(A,B)=m(A\cap B,\emptyset)-m(A,\emptyset)$$
$$=\frac{1}{2}\left(\frac{4}{3}-m(A,A\cap B)-m(A,\emptyset)\right)=\frac{4}{3}-m(B,A\cup B)-m(B,\emptyset)=m(A,A\cup B)-m(B,A\cap B)$$
$$=m(B,\emptyset)-m(A\cup B,\emptyset)=\frac{1}{2}\left(\frac{4}{3}-m(A\cup B,\emptyset)-m(B,A\cup B)\right)=\frac{1}{2}\left(\frac{5}{3}-2m(A\cup B,\emptyset)-m(B,U)\right)$$
$$=\frac{1}{2}\left(-\frac{5}{3}+m(A\cup B,U)+2m(B,\emptyset)\right)=\frac{1}{4}\left(1+m(A\cup B,U)-2m(B,A\cup B)\right)=\frac{1}{2}\left(m(A\cup B,U)-m(B,U)\right),$$

$$\frac{b}{3}=\frac{1}{2}\left(m(B,A\cup B)-m(A,B)\right)=m(A,A\cap B)-m(A,B)=m(A\cap B,\emptyset)-m(B,\emptyset)$$
$$=\frac{1}{2}\left(\frac{4}{3}-m(B,A\cap B)-m(B,\emptyset)\right)=\frac{4}{3}-m(A,A\cup B)-m(A,\emptyset)=m(B,A\cup B)-m(A,A\cap B)$$
$$=m(A,\emptyset)-m(A\cup B,\emptyset)=\frac{1}{2}\left(\frac{4}{3}-m(A\cup B,\emptyset)-m(A,A\cup B)\right)=\frac{1}{2}\left(\frac{5}{3}-2m(A\cup B,\emptyset)-m(A,U)\right)$$
$$=\frac{1}{2}\left(-\frac{5}{3}+m(A\cup B,U)+2m(A,\emptyset)\right)=\frac{1}{4}\left(1+m(A\cup B,U)-2m(A,A\cup B)\right)=\frac{1}{2}\left(m(A\cup B,U)-m(A,U)\right)$$

and

$$\frac{c}{3}=\frac{1}{2}[m(A,A\cap B)-m(A,\emptyset)]=\frac{1}{2}[m(B,A\cap B)-m(B,\emptyset)]=\frac{1}{2}\left(-\frac{4}{3}+m(A,A\cup B)+m(B,A\cup B)\right).$$

The number of total combinations $m(A,B)$ of forms of $\frac{a}{3}$, $\frac{b}{3}$ and $\frac{c}{3}$ is $12\times 12\times 3=432$. The number of combinations is large. Detailed representations of a comparison measure $m(A,B)$ are available from authors.

From $\frac{a}{3}=m(B,A\cap B)-m(A,B)$, $\frac{b}{3}=m(A,A\cap B)-m(A,B)$ and $\frac{c}{3}=\frac{1}{2}[m(A,A\cap B)-m(A,\emptyset)]$, it implies that

$$m(A,B)=\frac{2}{3}-m(B,A\cap B)+m(A,B)-m(A,A\cap B)+m(A,B)+\frac{1}{2}[m(A,A\cap B)-m(A,\emptyset)]=-\frac{2}{3}+\frac{1}{2}m(A,A\cap B)+m(B,A\cap B)+\frac{1}{2}m(A,\emptyset).$$

Similarly, we have that

$$m(A,B)=-\frac{2}{3}+m(A,A\cap B)+\frac{1}{2}m(B,A\cap B)+\frac{1}{2}m(B,\emptyset),$$
$$m(A,B)=\frac{2}{3}-2m(A\cap B,\emptyset)+\frac{1}{2}m(A,A\cap B)+\frac{1}{2}m(A,\emptyset)+m(B,\emptyset)$$

and

$$m(A,B)=\frac{2}{3}-2m(A\cap B,\emptyset)+\frac{1}{2}m(B,A\cap B)+m(A,\emptyset)+\frac{1}{2}m(B,\emptyset).$$

If $A\cap B=\emptyset$, the above four representations of a comparison measure $m(A,B)$ reduce to

$$m(A,B)=-\frac{2}{3}+m(A,\emptyset)+m(B,\emptyset)$$

which is the representation [1]-[8] of a comparison measure $m(A,B)$ with $A\cap B=\emptyset$ appearing in Section 4. Therefore, for the two disjoint FSs A and B, the representation of a comparison measure $m(A,B)$ with general FSs can be reduced to that of $m(A,B)$ with $A\cap B=\emptyset$.

6. Conclusion and Future Research

For the two FSs A and B, this paper presents the representations of a comparison measure $m(A,B)$ for the two disjoint union FSs, the two disjoint FSs and the two general FSs. The numbers of total combinations $m(A,B)$ are 36, 81 and 432 for the two disjoint union FSs, the two disjoint FSs and the two general FSs, respectively. The fewer the number of restrictions placed on the two FSs, the more the number of possible representations of a comparison measure. For the two disjoint union FSs, the constant value of $m(A,B) = \frac{1}{3}$ implies that we cannot compare the comparison behaviors of the two disjoint union FSs A and B. Among the 81 combinations of the two disjoint FSs A and B, there are 42 candidate representations of $m(A,B)$, 15 duplicate representations of $m(A,B)$, 21 relationships between different $m(X,Y)$, $X,Y \in \{\emptyset, A, B, A \cup B, U\}$ other than $m(A,B)$ and 3 identical equations. There are 1 and 12 candidate representations of $m(A,B)$ for one and two terms $m(X,Y)$, $X,Y \in \{\emptyset, A, B, A \cup B, U\}$, respectively. Applying these 13 candidate representations, we can easily calculate and compare the degree of a comparison measure $m(A,B)$ with $A \cap B = \emptyset$.

In the future, we will analyze the representation behaviors of comparison measures for the generalization of FSs and the general forms of a comparison measure. In particular, the analysis can be extended to the intuitionistic fuzzy sets, hesitant fuzzy sets and neutrosophic sets. Thus, the representation analysis of comparison measures for the intuitionistic fuzzy sets is a subject of considerable ongoing research.

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