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MODELING COMBINATION OF QUESTION ORDER EFFECT, RESPONSE REPLICABILITY EFFECT, AND QQ-EQUALITY WITH QUANTUM INSTRUMENTS

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ABSTRACT. We continue to analyze basic constraints on human's decision making from the viewpoint of quantum measurement theory (QMT). As has been found, the conventional QMT based on the projection postulate cannot account for combination of the question order effect (QOE) and the response replicability effect (RRE). This was an alarm signal for quantum-like modeling of decision making. Recently, it was shown that this objection to quantum-like modeling can be removed on the basis of the general QMT based on quantum instruments. In the present paper we analyse the problem of combination of QOE, RRE, and the famous QQ-equality (QQE). This equality was derived by Busemeyer and Wang and it was shown (in the joint paper with Solloway and Shiffrin) that statistical data from many social opinion polls satisfies it. Now, we construct quantum instruments satisfying QOE, RRE, and QQE. The general features of our approach are formalized with postulates which generalize *Wang-Busemeyer* postulates for quantum-like modeling of decision making. Moreover, we show that our model closely reproduces the statistics of the famous Clinton-Gore Poll data with a prior belief state independent of the question order. This model successfully removes the order effect from the data to determine the genuine distribution of the opinions in the Poll. The paper also provides a psychologist-friendly introduction to the theory of quantum instruments - the most general mathematical framework for quantum measurements. We hope that this theory will attract attention of psychologists and will stimulate further applications.

keywords: quantum-like models, decision making, social science, quantum instruments, order effect, response replicability effect, QQ-equality

1. INTRODUCTION

Recently the top level experts in cognitive psychology, behavioural economics, political sciences, and molecular biology begin to show interest in quantum formalism

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applications. The majority of applications are based on quantum measurement theory (QMT) that is used to model basic psychological effects.

However, as was shown in [5], the conventional QMT based on the projection postulate (representing the mental state update) confronts with combination of some psychological effects. So, QMT may describe each of the effects individually, but not jointly. In [5], it was shown that combination of *the question order effect* (QOE)[1] and *the response replicability effect* (RRE)[2] cannot be described by the conventional QMT. This was an alarm signal for quantum-like modeling of decision making.

However, since the 1970s QMT has been further developed towards more flexible approach to treating all the physically realizable quantum measurements by abandoning the projection postulate in the conventional QTM, used during the first years of quantum physics. Nowadays, in quantum physics, especially in quantum information theory, one uses QMT based on theory of quantum instruments [15–18].[3] So, say in quantum information theory, nobody would be surprised that in some situations conventional QMT (based on the projection postulate) cannot be applied.

Therefore, the right reply to the challenge presented in [5] should be based on the theory of quantum instruments. The first attempt to proceed in this direction was done in the paper of Basieva and Khrennikov [6]. Still, they used a restricted class of quantum instruments. (At the same time, this class is standard for quantum information theory.) And their paper confirmed the impossibility statement formulated originally in [5]. Only very recently it was shown [8] that by using quantum instruments, it is possible to describe the combination of QOE and RRE. Thus, quantum-like modeling program was secured from the objection of paper [5].

Immediately after publication of article [8], Dzhafarov (the private communication) raised the question whether the model presented in [8] matches the famous QQ-equality (QQE). The latter is one of the most important fruits of the quantum-like approach to decision making. This is a special constraint on probabilities that was derived by Busemeyer and Wang [3] in the quantum probabilistic (QP) framework. So, from the classical probabilistic (CP) viewpoint there is no reasons for QQE to hold, but usage of quantum formalism may require it. In paper [9], it was shown that statistical data from a bunch of social opinion polls satisfies QQE. We stress that this equality was derived on

[1]QOE: dependence of the (sequential) joint probability distribution of answers on the questions' order: $p_{AB} \neq p_{BA}$; see [3, 9] for its modeling with conventional QMT.

[2]RRE [5]: Suppose that after answering the A-question with the "yes", Alice is asked another question B, and gives and answer to it. And then she is asked A again. In the social opinion pools and other natural decision making experiments, Alice would definitely repeat her original answer to A, "yes". This is $A - B - A$ response replicability. (In the absence of B-question, we get $A - A$ replicability). Combination of $A - B - A$ and $B - A - B$ replicability forms RRE.

[3]Generalizing the notion of quantum observables in the form of positive operator valued measures (POVMs) (which are widely used in quantum information theory and started to be used in quantum-like modeling [1, 7]) appears naturally in the framework of theory of quantum instruments (see appendix). But, we shall not use them in the present paper.

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the basis of conventional QMT [3] and that it can be violated in general QMT [15–18]. Thus, combination of QOE and RRE with QQE is a delicate problem: one has to go beyond conventional QMT in a special way.

The purpose of this paper is two-fold:

A). We present the solution of this problem (QOE+RRE+QQE) by constructing the corresponding quantum instruments. This step is important to justify the use of the quantum-like models for decision making. The general features of our approach are formalized with postulates which generalize Wang-Busemeyer postulates for quantum-like modeling of decision making [3]. This is important for continuity of research on quantum-like modeling in psychology.

We emphasize that our model is a complete model for a certain set of data satisfying the QQ-equality including the Clinton-Gore poll in the following sense. Since the Clinton-Gore poll data sets approximately satisfy QQE, there should be a uniform method to make the original data to satisfy QQE with small distortion. Here the main point is that *such a method can be defined independent of the model.* We show that our model completely reproduces the modified data of the Clinton-Gore poll data and this means our model reproduces the original data as accurately as how accurately the data fits QQE. Moreover, we show that this model closely reproduces the statistics of the famous Clinton-Gore poll data with a prior belief state independent of the question order. The model successfully removes the order effect from the data to determine the genuine distribution of the opinions in the Poll.[4]

In this paper we do not discuss alternative theories. We only offer quantum models as viable candidates for the considered experiments, without comparison with possible other state-of the art models conventionally used in psychology. For such discussion we refer the reader to works [21–23].

B). Irrespective of the (QOE+RRE+QQE)-problem, the paper presents the most general formalism of QMT based on theory of quantum instruments. We hope that this theory will attract attention of psychologists and stimulate further applications in psychology.

[4]It should be pointed out that we are not trying to compare the classical [13] and quantum [14] probability models applied for decision making. We remark that in quantum physics comparison of these models started in 1920th (with the Wigner function) and is still continuing, e.g., in the hot discussions on Bell's inequality [21–23]. How far can one proceed with classical probability in quantum physics? This is a very complex problem (see, e.g., [24]- [27]) and it is too early to make a final conclusion. Nevertheless, the quantum formalism has been successfully applied to numerous theoretical and engineering problems. This formalism is powerful and successful, irrespective of the (im)possibility to proceed with classical probability. One of the advantages of quantum formalism is linearity of the state space. This reduces required calculations to simple linear algebra for matrices and vectors. In fact, the linear space structure of mental spaces has been widely used in cognitive science and psychology. Quantum-like modeling continues the tradition of modeling cognitive processes with linear state spaces, with the advantage of enhanced mathematical formalism and methodology elaborated in quantum physics, one of the most successful branches of science.

Finally, we remark that the projective type (von Neumann-Lüders [14, 30]) instruments, i.e., describing the measurement's feedback onto the system's state by orthogonal projections (section 2.2), are widely used in quantum-like modeling in cognition, psychology, and decision making [2, 3], [9]- [12], [28, 29]. Although this model is very attractive due to its simplicity, one has to be careful when applying it. In [5], the role of RRE was elevated. In fact, RRE is a special exhibition of long term memory in combination with keeping consistency in decisions (actions). So, RRE is one of the basic characteristics of *rational cognition.* Quantum instruments constructed in this paper express the *long term memory effect* and decision consistency (based on this effect) that are sufficient to generate RRE. In contrast, the projective type instruments for observables represented by *non-commuting operators* do not. (And one has to consider non-commuting operators to generate the QOE [3, 5].) In the $A-B-A$-measurement scheme, this means that generally the B-measurement washes out the memory about the first A-measurement. For observables given by operators with nondegenerate spectra, memory is washed out completely. In the second A-measurement, the system "remembers" only the output of the B-measurement. For observables, with degenerate spectra memory is washed out essentially, but not completely. Such short term memory decision making can be considered as indication of "irrational" behavior. In social opinion polls, people generally behave rationally (by using long term memory), in spite of the order effect. The latter does not contradict rationality

2. Quantum instruments

We briefly present quantum instruments (see [8] for non-physicist friendly presentation, see also basic papers [15–20]; see appendix for coupling with generalized observables given by POVMs.)

2.1. Quantum states and observables.

In quantum theory, it is postulated that every quantum system S corresponds to a complex Hilbert space $\mathcal{H}$; denote the scalar product of two vectors by the symbol $\langle\psi_1|\psi_2\rangle$. Throughout the present paper, we assume $\mathcal{H}$ is *finite dimensional.* States of the quantum system S are represented by density operators acting in $\mathcal{H}$ (positive semi-definite operators with unit trace). Denote this state space by the symbol $\mathbf{S}(\mathcal{H})$.

In quantum physics (especially quantum information theory), there are widely used notations invented by Dirac: a vector belonging to $\mathcal{H}$ is symbolically denoted as $|\psi\rangle$; orthogonal projector on this vector is denoted as $|\psi\rangle\langle\psi|$, it acts to the vector $|\xi\rangle$ as $\langle\psi|\xi\rangle|\psi\rangle$.

Any density operator ρ of rank one is of the form $\rho = |\psi\rangle\langle\psi|$ with a unit-norm vector $|\psi\rangle$. In this case, $|\psi\rangle$ is called a *state vector*, so $|\psi\rangle \in \mathcal{H}$, $\||\psi\rangle\| = \sqrt{\langle\psi|\psi\rangle} = 1$.

Observables (or physical quantities) of the quantum system S are represented by self-adjoint operators in $\mathcal{H}$. Each observable A can be represented as

$$A = \sum_x x E^A(x), \tag{1}$$

where $E^A(x)$ is the spectral projection of the observable A corresponding to the eigenvalue x if x is an eigenvalue of A; otherwise, let $E^A(x) = 0$. We note that spectral projectors sum up to the unit operator I :

$$\sum_x E^A(x) = I. \tag{2}$$

We also remark that each orthogonal projector E is self-adjoint and idempotent, i.e., $E^* = E$ and $E^2 = E$.

In what follows, we identify the states of the system with density operators and the observables of the system with self-adjoint operators and make no notational distinctions.

According to the standard interpretation of quantum mechanics, we can measure every observable A in any state ρ in principle, and then we obtain one of its eigenvalues as the outcome of the measurement, while quantum mechanics cannot in general predict the precise outcome, but only predict its probability distribution $\Pr\{A = x \| \rho\}$ by the *Born rule* as

$$\Pr\{A = x \| \rho\} = \mathrm{Tr}[E^A(x)\rho]. \tag{3}$$

For a state vector $|\psi\rangle$, this leads to the relation

$$\Pr\{A = x \| |\psi\rangle\} = \|E^A(x)|\psi\rangle\|^2, \tag{4}$$

as $\mathrm{Tr}[E^A(x)|\psi\rangle\langle\psi|] = \|E^A(x)|\psi\rangle\|^2$.

Thus, quantum mechanics determines the probability distribution of a single observable in a given state. Then, how about the joint probabilities of several observables determined? There are two types of joint probabilities in quantum mechanics: one for simultaneous measurement and the other for successive measurement. We shall discuss them for two observables.

In quantum mechanics, two observables A and B are simultaneously measurable in any state ρ if and only if A and B are commuting, i.e., $[A, B] = AB - BA = 0$, and in this case, the joint probability distribution of outcomes of simultaneous measurements of A and B in a state ρ is give by

$$\Pr\{A = x, B = y \| \rho\} = \mathrm{Tr}[E^A(x)E^B(y)\rho]. \tag{5}$$

For a state vector $|\psi\rangle$, this leads to the relation

$$\Pr\{A = x, B = y \| \rho\} = \|E^A(x)E^B(y)|\psi\rangle\|^2. \tag{6}$$

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Since A and B are commuting, in the above expressions we have $E^A(x)E^B(y) = E^B(y)E^A(x)$ and the order of $A = x$ and $B = y$ are interchangeable. Note that Eq. (5) and Eq. (6) can be obtained by the Born rule without a new postulate [14]; see Appendix B.

How does quantum mechanics determine the joint probability distribution of successive measurements? What happens if we measure another observable B just after the A measurement? Does the Born rule determine the joint probability of the outcomes of the A measurement and the subsequent B measurement? Suppose that the observable A is measured in a state ρ and the result $A = x$ is obtained and subsequently the observable B is measured and the result $B = y$ is obtained. To determine the joint probability distribution $\Pr\{A = x, B = y\|\rho\}$ of their outcomes, we consider the conditional probability distribution of B given $A = x$ defined by

$$\Pr\{A = x, B = y\|\rho\} = \Pr\{A = x\|\rho\} \Pr\{B = y|A = x\|\rho\}. \tag{7}$$

The definition of conditional probability is based on the following reasoning. The state of the system just after the A measurement is considered to depend on the initial state ρ and the outcome $A = x$ so that we denote it by $\rho_{\{A=x\}}$. Since the B measurement is carried out in this state, the Born rule applies to this state to obtain

$$\Pr\{B = y|A = x\|\rho\} = \Pr\{B = y\|\rho_{\{A=x\}}\}, \tag{8}$$

and hence the joint probability distribution is given by

$$\begin{aligned}\Pr\{A = x, B = y\|\rho\} &= \Pr\{B = y\|\rho_{\{A=x\}}\} \Pr\{A = x\|\rho\}\\ &= \mathrm{Tr}[E^B(y)\rho_{\{A=x\}}]\mathrm{Tr}[E^A(x)\rho].\end{aligned} \tag{9}$$

The Born rule gives the probability of the output in a given state. But, in order to determine the joint probability distribution $\Pr\{A = x, B = y\|\rho\}$ we need another postulate to determine the state $\rho_{\{A=x\}}$ after the measurement.

2.2. **Von Neumann-Lüders instruments.** In the conventional QMT, the *projection postulate*[5] is posed, stating: in the measurement of an observable A, the input state ρ is changed to the output state

$$\rho_{\{A=x\}} = \frac{E^A(x)\rho E^A(x)}{\mathrm{Tr}[E^A(x)\rho]} \tag{10}$$

[5]For observables given by self-adjoint operators with non-degenerate spectra, this postulate was suggested by von Neumann [14]. Then Lüders extended it even to observables with degenerate spectra [30]. For the latter, von Neumann used a more general rule (in the spirit of theory of quantum instruments).

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provided that the measurement leads to the outcome $A = x$.[6] If the input state is the state vector $|\psi\rangle$, the output state is also the state vector $|\psi_{\{A=x\}}\rangle$ such that

$$|\psi_{\{A=x\}}\rangle = \frac{E^A(x)|\psi\rangle}{\|E^A(x)|\psi\rangle\|}. \tag{11}$$

In this case, i.e., for the density operator $\rho = |\psi\rangle\langle\psi|$, we have the relations

$$\rho_{\{A=x\}} = \frac{E^A(x)\rho E^A(x)}{\mathrm{Tr}[E^A(x)\rho]} = \frac{E^A(x)|\psi\rangle\langle\psi|E^A(x)}{\|E^A(x)|\psi\rangle\|^2} = |\psi_{\{A=x\}}\rangle\langle\psi_{\{A=x\}}|. \tag{12}$$

Note that equality (2) can be rewritten in the form:

$$\sum_x E^A(x)^* E^A(x) = I, \tag{13}$$

and the Born rule is written as

$$\Pr\{A = x\|\rho\} = \mathrm{Tr}[E^A(x)\rho E^A(x)]. \tag{14}$$

According to the projection postulate, if observables A and B are successively measured in this order in the initial input state ρ, the joint probability distribution of their outcomes is given by

$$\Pr\{A = x, B = y\|\rho\} = \mathrm{Tr}[E^B(y)E^A(x)\rho E^A(x)]. \tag{15}$$

For the state vector $|\psi\rangle$ we have

$$\Pr\{A = x, B = y\||\psi\rangle\} = \|E^B(y)E^A(x)|\psi\rangle\|^2. \tag{16}$$

Thus, in the conventional QMT the outcome probability distribution and the state change caused by the measurement are uniquely determined by the observable A.

However, the theory should also reflect the evident fact that the same quantum observable, say energy, can be measured with a variety of quantum measuring instruments. In the quantum formalism, these instruments are characterized by back-actions of measurements to system's states. In modern QMT, a more flexible rule is adopted, in which the state change caused by the measurement is not uniquely determined by the observable to be measured. There are many ways to measure the same observable.

2.3. **The Davis-Lewis-Ozawa quantum instruments.** It has been known that the projection postulate is too restrictive to describe all the physically realizable measurements of the observable A. Davies-Lewis [15] proposed to abandon this postulate and proposed a more flexible approach to QMT.

The space $\mathcal{L}(\mathcal{H})$ of linear operators in $\mathcal{H}$ is a linear space over the complex numbers. Moreover, $\mathcal{L}(\mathcal{H})$ is the complex Hilbert space with the scalar product, $\langle A|B\rangle = \mathrm{TrA^*B}$. We consider linear operators acting on it; they are called *superoperators*. The latter terminology is used to distinguish operators acting in the Hilbert spaces $\mathcal{H}$ and $\mathcal{L}(\mathcal{H})$. Otherwise superoperators are usual linear operators. In particular, for $\mathcal{T} : \mathcal{L}(\mathcal{H}) \to$

[6]We recall that this rule was adopted by Wang and Busemeyer [3].

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$\mathcal{L}(\mathcal{H})$, there is well defined its adjoint operator $\mathcal{T}^* : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$. However, some basic notions are specific for superoperators. A superoperator $\mathcal{T} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ is called *positive* if it maps the set of positive semi-definite operators into itself. A superoperator is called *completely positive* if its natural extension $\mathcal{T} \otimes \mathrm{id}$ to the tensor product $\mathcal{L}(\mathcal{H}) \otimes \mathcal{L}(\mathcal{H}) = \mathcal{L}(\mathcal{H} \otimes \mathcal{H})$ is again a positive superoperator on $\mathcal{L}(\mathcal{H}) \otimes \mathcal{L}(\mathcal{H})$. (This notion is rather complicated technically. Therefore we shall not discuss it in more detail.)

Consider now a general measurement on the system S. The statistical properties of any measurement are characterized by

- (i) the output probability distribution $\Pr\{\mathbf{x} = x \| \rho\}$, the probability distribution of the output $\mathbf{x}$ of the measurement in the input state ρ :
- (ii) the quantum state reduction $\rho \mapsto \rho_{\{\mathbf{x}=x\}}$, the state change from the input state ρ to the output state $\rho_{\{\mathbf{x}=x\}}$ conditional upon the outcome $\mathbf{x} = x$ of the measurement.

According to Davies–Lewis [15] and Ozawa [17], the modern quantum measurement theory postulates that any measurement of the system S is described by a mathematical structure called a *quantum instrument*. This is any map $x \to \mathcal{I}(x)$, where for each real x, the map $\mathcal{I}(x)$ is a completely positive superoperator satisfying the normalization condition $\sum_x \mathrm{Tr}[\mathcal{I}(x)\rho] = 1$ for any state ρ.

Davies–Lewis [15] originally postulated that the superoperator $\mathcal{I}(x)$ should be positive. However, Yuen [31] pointed out that the Davies–Lewis postulate is too general to exclude physically non-realizable instrument. Ozawa [17] introduced complete positivity to ensure that every quantum instrument is physically realizable. Thus, complete positivity is a sufficient condition for an instrument to be physically realizable. On the other hand, necessity is derived as follows [19, p. 369]. Every observable A of a system S is identified with the observable $A \otimes I$ of a system $S + S'$ with any system S' external to S[7]. Then, every physically realizable instrument I_A measuring A should be identified with the instrument $I_{A\otimes I}$ measuring $A \otimes I$ such that $I_{A\otimes I}(x) = I_A(x) \otimes \mathrm{id}$. This implies that $I_A(x) \otimes \mathrm{id}$ is agin a positive superoperator, so that $I_A(x)$ is completely positive. Similarly, any physically realizable instrument $\mathcal{I}(x)$ measuring system S should have its extended instrument $\mathcal{I}(x) \otimes \mathrm{id}$ measuring system $S + S'$ for any external system S'. This is fulfilled only if $\mathcal{I}(x)$ is completely positive. Thus, complete positivity is a necessary condition for I_A to describe a physically realizable instrument.

Given a quantum instrument $\mathcal{I}$, the output probability distribution for the input state ρ is defined by the generalized Born rule in the trace-form,

$$\Pr\{\mathbf{x} = x \| \rho\} := \mathrm{Tr}\,[\mathcal{I}(x)\rho], \tag{17}$$

[7]For example, the color A of my eyes is not only the property of my eyes S but also the property $A \otimes I$ of myself $S + S'$.

and the quantum state reduction is defined by

$$\rho \mapsto \rho_{\{\mathbf{x}=x\}} := \frac{\mathcal{I}(x)\rho}{\mathrm{Tr}[\mathcal{I}(x)\rho]}. \tag{18}$$

According to the Kraus theorem [43], for any instrument $\mathcal{I}$, there exists a family $\{M_{xj}\}_{x,j}$ of operators, called the *measurement operators* for $\mathcal{I}$, in $\mathcal{H}$ such that

$$\mathcal{I}(x)\rho = \sum_j M_{xj}\rho M_{xj}^* \tag{19}$$

for any state ρ. In this case, we have

$$\sum_{xj} M_{xj}^* M_{xj} = I. \tag{20}$$

Conversely, any family $\{M_{xj}\}_{x,j}$ of operators in $\mathcal{H}$ satisfying (20) defines an instrument $\mathcal{I}$.

The above general formulation of quantum instruments reflects variety of real measuring instruments for the same system S that measure an observable A accurately or with some error. A quantum instrument $\mathcal{I}_A$ is called an instrument measuring an observable A, or an A-measuring instrument, if the output probability distribution satisfies Born's rule (3) for the A-measurement, i.e,

$$\mathrm{Tr}[\mathcal{I}_A(x)\rho] = \mathrm{Tr}[E^A(x)\rho]. \tag{21}$$

The projective A-measuring instrument is defined by

$$\mathcal{I}_A(x)\rho := E^A(x)\rho E^A(x) \tag{22}$$

for any state ρ and real number x. Then, this instrument satisfies not only the Born formula (3) or (14), but also the projection postulate (10). Thus, the projection postulate is no longer the requirement for the measurement of the observable A but only one type of the measurement of A.

Measurement operators $\{M_{xj}\}_{x,j}$ for $\mathcal{I}_A$ in $\mathcal{H}$ satisfies the relation

$$E^A(x) = \sum_j M_{xj}^* M_{xj}. \tag{23}$$

Conversely, any family $\{M_{xj}\}_{x,j}$ of operators in $\mathcal{H}$ satisfying (23) defines an A-measuring instrument $\mathcal{I}_A$ by Eq. (19). For the projective instrument $\mathcal{I}_A$ measuring A the measurement operators coincide with the spectral projections, i.e., $M_{xj} = E^A(x)$ with $j \in \{1\}$.

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2.4. **Quantum order effect from the quantum instrument viewpoint.** Finally, we make the probabilistic comment on quantum instruments. Typically the main attention is given to Born's rule (21) as generating probabilities from quantum states, $\rho \mapsto \Pr\{A = x\|\rho\}$. We would like to elevate the role of the state transform (18), measurement back-action. It generates quantum probability conditioning. If after measurement of observable A with outcome $A = x$, one measures observable B, then the probability to get output $B = y$ is given by Born's rule for state $\rho_{\{A=x\}}$:

$$\Pr\{B = y|A = x\|\rho\} = \mathrm{Tr}\,[\mathcal{I}_B(y)\rho_{\{A=x\}}] = \frac{\mathrm{Tr}\,[\mathcal{I}_B(y)\mathcal{I}_A(x)\rho]}{\mathrm{Tr}\,[\mathcal{I}_A(x)]}. \tag{24}$$

Now, by using quantum conditional probability we can define the sequential joint probability distribution of A (first) and B (last),

$$\Pr\{A = x, B = y\|\rho\} = \Pr\{A = x\|\rho\}\Pr\{B = y|A = x\|\rho\} = \mathrm{Tr}\,[\mathcal{I}_B(y)\mathcal{I}_A(x)\rho]. \tag{25}$$

It is clear that if superoperators $\mathcal{I}_A(x)$ and $\mathcal{I}_B(y)$ do not commute, i.e.,

$$[\mathcal{I}_A(x), \mathcal{I}_B(y)] = \mathcal{I}_A(x)\mathcal{I}_B(y) - \mathcal{I}_B(y)\mathcal{I}_A(x) \neq 0, \tag{26}$$

then generally $\Pr\{A = x, B = y\|\rho\} \neq \Pr\{B = y, A = x\|\rho\}$. This is the mathematical representation of QOE.

In the framework of the quantum instrument theory, one has to distinguish noncommutativity of observables and noncommutativity of instruments. The standard noncommutativity condition

$$[A, B] \neq 0, \tag{27}$$

describes incompatibility of observables, in the sense that their JPD, the JPD for their *simultaneous measurements*, is not well defined. So, condition (27) is related to joint measurements, not to sequential measurement and not to QOE. The latter is characterized not via non-existence of JPD, but via noncommutativity of the state updates, formalized via condition (26).

Generally, noncommutativity of instruments (26) does not imply noncommutativity of the observables to be measured (27), whereas the converse statement holds as shown in the following theorem.

Theorem 2.1. *For any A-measuring instrument $\mathcal{I}_A$ and B-measuring instrument $\mathcal{I}_B$, if $[\mathcal{I}_A(x), \mathcal{I}_B(y)] = 0$ for any x, y then $[A, B] = 0$.*

Proof. It is well known that $\mathcal{I}_A(x)^*X = T_A^*(X)E^A(x)$ and $\mathcal{I}_B(y)Y = T_B^*(Y)E^B(y)$ for any $X, Y \in \mathcal{L}(\mathcal{H})$, where $T_A^* = \sum_x \mathcal{I}_A(x)^*$, $T_B^* = \sum_y \mathcal{I}_B(y)^*$ and that $[T_A^*(X), E^A(x)] = 0$ and $[T_B^*(Y), E^B(y)] = 0$ for all $X, Y \in \mathcal{L}(\mathcal{H})$ and x, y [17, Proposition 4.4]. Suppose $\mathcal{I}_A(x)\mathcal{I}_B(y) - \mathcal{I}_B(y)\mathcal{I}_A(x) = 0$ for every x, y. Then, we have $\mathcal{I}_A(x)^*\mathcal{I}_B(y)^*I = \mathcal{I}_B(y)^*\mathcal{I}_A(x)^*I$ and $\mathcal{I}_A(x)^*E^B(y) = \mathcal{I}_B(y)^*E^A(x)$. Summing up all y we have $\mathcal{I}_A(x)^*I = T_B^*(E^A(x))$ and $E^A(x) = T_B^*(E^A(x))$. From $[T_B^*(Y), E^B(y)] = 0$ we have $[E^A(x), E^B(y)] = 0$ for all x, y. Thus, $[A, B] = 0$. $\square$

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We shall in fact construct instruments $\mathcal{I}_A$ and $\mathcal{I}_B$ showing the question order effect with commuting observables A and B, while the commutativity of A and B is necessary for $\mathcal{I}_A$ and $\mathcal{I}_B$ to show the response replicability effect.

2.5. **Quantum instruments as representation of indirect measurements.** The basic model for construction of quantum instruments is based on the scheme of indirect measurements. This scheme formalizes the following situation. As was permanently emphasized by Bohr (one of the founders of quantum mechanics), the results of quantum measurements are generated in the process of interaction of a system S with a measurement apparatus M. This apparatus consists of a complex physical device interacting with S and a pointer that shows the result of measurement, say spin up or spin down. An observer can see only outputs of the pointer and he associates these outputs with the values of the observable A for the system S. So, the observer approaches only the pointer, not the system by it self. Whether the outputs of the pointer can be associated with the "intrinsic properties" of S or not is one of the main problems of quantum foundations, it is still the topic for hot foundational debates [21]- [27]. Thus, the indirect measurement scheme involves:

- the states of the systems S and the apparatus M;
- the operator U representing the interaction-dynamics for the system $S + M$;
- the meter observable M_A giving outputs of the pointer of the apparatus M.

We shall make the following remark on the operator U of the interaction-dynamics. As all operations in quantum mechanics, it is a linear operator. In quantum formalism, dynamics of the state of an isolated system is described by the Schrödinger equation and its evolution operator is unitary. In the indirect measurement scheme, it is assume that (approximately) the compound system $S + M$ is isolated. Hence, its evolution operator U is unitary.

Formally, an *indirect measurement model*, introduced in [17] as a "(general) measuring process", is a quadruple

$$(\mathcal{K}, \sigma, U, M_A)$$

consisting of a Hilbert space $\mathcal{K}$, a density operator $\sigma \in \mathbf{S}(\mathcal{K})$, a unitary operator U on the tensor product of the state spaces of S and M, $U : \mathcal{H} \otimes \mathcal{K} \to \mathcal{H} \otimes \mathcal{K}$, and a self-adjoint operator M_A on $\mathcal{K}$. By this measurement model, the Hilbert space $\mathcal{K}$ describes the states of the apparatus M, the unitary operator U describes the time-evolution of the composite system $S + M$, the density operator σ describes the initial state of the apparatus M, and the self-adjoint operator M_A describes the meter observable of the apparatus M. Then, the output probability distribution $\Pr\{A = x\|\rho\}$ in the system state $\rho \in \mathbf{S}(\mathcal{H})$ is given by

$$\Pr\{A = x\|\rho\} = \mathrm{Tr}[(I \otimes E^{M_A}(x))U(\rho \otimes \sigma)U^*], \tag{28}$$

where $E^{M_A}(x)$ is the spectral projection of M_A for the eigenvalue x.

The change of the state ρ of the system S caused by the measurement for the outcome $A = x$ is represented with the aid of the map $\mathcal{I}_A(x)$ in the space of density operators defined as

$$\mathcal{I}_A(x)\rho = \mathrm{Tr}_{\mathcal{K}}[(I \otimes E^{M_A}(x))U(\rho \otimes \sigma)U^*], \tag{29}$$

where $\mathrm{Tr}_{\mathcal{K}}$ is the partial trace over $\mathcal{K}$. Then, the map $x \mapsto \mathcal{I}_A(x)$ turn out to be a quantum instrument. Thus, the statistical properties of the measurement realized by any indirect measurement model $(\mathcal{K}, \sigma, U, M_A)$ is described by a quantum measurement. We remark that conversely any quantum instrument can be represented via the indirect measurement model [17]. Thus, quantum instruments mathematically characterize the statistical properties of all the physically realizable quantum measurements[8].

Now, we point to a few details which were omitted in the above considerations. The measuring interaction between the system S and the apparatus M turns on at time t_0, the time of measurement, and turns off at time $t = t_0 + \Delta t$. We assume that the system S and the apparatus M do not interact each other before t_0 nor after $t = t_0 + \Delta t$ and that the compound system $S + M$ is isolated in the time interval (t_0, t). The *probe system* P is defined to be the minimal part of apparatus M such that the compound system $S + P$ is isolated in the time interval (t_0, t). Then the above scheme is applied to the probe system P, instead of the whole apparatus M. The rest of the apparatus M performs the pointer measurement on the probe P. In particular, the unitary evolution operator U describing the state-evolution of the system $S + P$ has the form $H = e^{-i\Delta t H}$, where $H = H_S + H_P + H_{SP}$ is Hamiltonian of $S + P$ with the terms H_S and H_P representing the internal dynamics in the subsystems S and P of the compound system and H_{SP} describing the interaction between the subsystems.

Introduction of probe systems may be seen as unnecessary complication of the scheme of indirect measurements. However, it is useful if the apparatus M is a very complex system that interacts (often in parallel) with many systems $S_j, j = 1, 2, ..., m$. Its different probes are involved solely in interaction with the concrete systems, P_j with S_j. And the system $S_j + P_j$ can be considered as an isolated system; in particular, from interactions with other systems S_i and probes P_i.

The indirect measurement scheme is part of theory of open quantum systems [32]. Instead of a measurement apparatus M, we can consider the surrounding environment **E** of the system S (see [33]- [35] for applications to psychology).

[8]It is an interesting problem whether instruments on a more general state space are realizable by an indirect measurement model $(\mathcal{K}, \sigma, U, M_A)$. This problem was studied extensively by Okamura and Ozawa [20]. Any discrete classical instrument is realizable. There exists a non-realizable continuous classical instrument. There also exists a non-realizable instrument for a local system in algebraic quantum field theory, but all instruments are realizable within arbitrary error. A necessary and sufficient condition for any instrument on a von Neumann algebra (which includes classical and quantum mechanics, and algebraic quantum field theory) to be precisely realizable by an indirect quantum measurement model is known as the normal extension property (NEP) [20].

3. Indirect measurements of mental observables: Unconscious as a system and consciousness as a measurement apparatus

The scheme of indirect measurements presented in the previous section was created in quantum physics. Our aim is to adapt it to cognitive psychology. The main question is about cognitive analogues of the system S and the measurement apparatus M. We suggest to use the framework developed by Khrennikov [36] for quantum-like modeling of *the Helmholtz sensation-perception theory*. This scheme can be extended to a general scheme of unconscious-conscious interaction in the process of decision making

The measured system S is a sensation (or generally any state of the unconscious mind). Consciousness as the measurement apparatus to the unconscious interacts with sensations to make the decisions (to generate outcomes of measurements). Consciousness, of course, is not concerned just with a single probe. It is a large environment with many probes interacting with the unconscious. There should be a rule of transformation from a sensation to a conscious decision. This is unitary transformation and measurement of the meter in the probe. The operational description neglects neurophysiological and electrochemical structures of interaction, just like the operational description of quantum computers with unitary transformations and measurements [37]. The unconscious is a black box that is mathematically described by the state of sensation space (= the unconscious state), so that the unconscious state probabilistically determines the decision (by interaction with consciousness), then the unconscious state is changed according to the previous unconscious state and the decision made. Thus, each probe is described by a quantum instrument. Instruments are probe dependent.

For the question-measurements, the question A is transferred into the unconscious, where it plays the role of a sensation, so to say a high mental level sensation. Then, interaction described by the unitary operator U generates a new state of the compound system - the unconscious-conscious. And consciousness performs the final "pointer reading", the measurement of the meter observable. Pointer reading can be treated as generation of a perception, a high mental level perception.

We understand that appealing to the unconscious-conscious description of cognitive processes is not so common in the modern psychology. However, this description well matches the indirect measurement scheme, which is broadly used to describe functions of quantum computers and quantum information devices without details of physical constructions. We can appeal to the authority of James [38] (appealing to Freud [40] might generate a negative reaction), see also Jung [39]. We also can mention the series of works on the use of the unconscious-conscious scheme in quantum-like modeling of cognition [36, 41, 42].

4. Constructions of Instruments

4.1. Observables A and B.

In modeling successive question-response experiments, such as the Clinton-Gore experiment, we consider two questions A and B for a subject

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to answer "yes" (y) or "no" (n) and consider the joint probability $p(AaBb)$ for obtaining the answer a ($a = y$ or n) for the question A and the answer b ($b = y$ or n) for the question B, if the question B is asked after the question A, and the analogous joint probability $p(BbAa)$ if the question A is asked after the question B.

We models the above joint probability distributions by the joint probability distributions of outcomes of successive measurements of 2-valued observables A and B in a quantum system in a given state ρ in the A–B order and in the B–A order. According to the general QMT, the joint probability distributions are well defined by two 2-valued observables (represented by projections) A, B in a fixed Hilbert space $\mathcal{H}$, a quantum state ρ, and an A-measuring instrument $\mathcal{I}_A$ and a B-measuring instrument $\mathcal{I}_B$ by the relations

$$\begin{aligned} p(AaBb) &= \Pr\{A = a, B = b\|\rho\} = \mathrm{Tr}[\mathcal{I}_B(b)\mathcal{I}_A(a)\rho], \\ p(BbAa) &= \Pr\{B = b, A = a\|\rho\} = \mathrm{Tr}[\mathcal{I}_A(a)\mathcal{I}_B(b)\rho]. \end{aligned} \tag{30}$$

Let A and B be projections on a Hilbert space $\mathcal{H}$. They correspond to questions labeled A and B, respectively. The eigenvalue 1 means the answer "yes" to the questions, and the eigenvalue 0 means the answer "no" to the questions.

Let $\mathbf{C}^2 = \{|0\rangle, |1\rangle\}^{\perp\perp}$ and $\mathbf{C}^3 = \{|0\rangle, |1\rangle, |2\rangle\}^{\perp\perp}$, where $^\perp$ stands for the orthogonal complement in $\mathcal{H}$, so that $S^{\perp\perp}$ stands for the subspace spanned by a subset S of $\mathcal{H}$. We suppose that $\mathcal{H}$ consists of three components $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ such that $\mathcal{H}_1 = \mathcal{H}_2 = \mathbf{C}^2$ and $\mathcal{H}_3 = \mathbf{C}^3$. The space $\mathcal{H}$ is called the space of *mind states*, the first component $\mathcal{H}_1$ is called the space of *belief states for question* A, the secant component $\mathcal{H}_2$ is called the space of *belief states for question* B, the third component is called the space of *personality states*.

Denote by I_1, I_2, and I_3 the identity operators on the first, second, and the third component of $\mathcal{H}$, respectively. We let $A = |1\rangle\langle 1| \otimes I_2 \otimes I_3$, and $B = I_1 \otimes |1\rangle\langle 1| \otimes I_3$. Thus, we consider the case where projections A and B commute.

4.2. **Instrument $\mathcal{I}_A$ measuring A.** We construct an instrument $\mathcal{I}_A$ measuring A as follows. The instrument $\mathcal{I}_A$ carries out a measurement of A by a measuring interaction between the object S described by the state space $\mathcal{H}$ and the probe P described by the state space $\mathcal{K} = \mathbf{C}^2 \otimes \mathbf{C}^2$, which is prepared in the state $|00\rangle$ just before the measuring interaction. Denote by I_4 and I_5 the identity operators on the first and second component of $\mathcal{K}$. The time evolution of the composite system $S + P$ during the measuring interaction is described by a unitary operator U_A on $\mathcal{H} \otimes \mathcal{K}$ satisfying

$$U_A : |000\rangle|00\rangle \mapsto |000\rangle|00\rangle, \tag{31}$$
$$U_A : |010\rangle|00\rangle \mapsto |010\rangle|01\rangle, \tag{32}$$
$$U_A : |100\rangle|00\rangle \mapsto |100\rangle|10\rangle, \tag{33}$$
$$U_A : |110\rangle|00\rangle \mapsto |110\rangle|11\rangle, \tag{34}$$
$$U_A : |001\rangle|00\rangle \mapsto |011\rangle|00\rangle, \tag{35}$$
$$U_A : |011\rangle|00\rangle \mapsto |011\rangle|01\rangle, \tag{36}$$
$$U_A : |101\rangle|00\rangle \mapsto |101\rangle|10\rangle, \tag{37}$$
$$U_A : |111\rangle|00\rangle \mapsto |101\rangle|11\rangle, \tag{38}$$
$$U_A : |002\rangle|00\rangle \mapsto |002\rangle|00\rangle, \tag{39}$$
$$U_A : |012\rangle|00\rangle \mapsto |002\rangle|01\rangle, \tag{40}$$
$$U_A : |102\rangle|00\rangle \mapsto |112\rangle|10\rangle, \tag{41}$$
$$U_A : |112\rangle|00\rangle \mapsto |112\rangle|11\rangle, \tag{42}$$

and the outcome of the measurement is obtained by measuring the meter observable

$$M_A = |1\rangle\langle 1| \otimes I_5 \tag{43}$$

of the probe P. Note that both A and M_A have the same spectrum $\{0, 1\}$ and they are projections. We denote by $I_{\mathcal{H}}$ and $I_{\mathcal{K}}$ the identity operators on $\mathcal{H}$ and $\mathcal{K}$, respectively, and denote by $M_A^{\perp}$ the orthogonal complement of the projection M_A, i.e., $M_A^{\perp} = I_{\mathcal{K}} - M_A$.

Equations (31)–(34) describe the change of the mind state + the probe state with the personality state $|0\rangle$. In this case, the belief state is copied to the probe state and the belief state does not change. Thus, in the decision stage, measuring the first state of the probe state results in the decision equal to the belief for question A just before the question-response process.

Equations (35)–(38) describe the change of the mind state + the probe state with the personality state $|1\rangle$. In this case, the belief state is copied to the probe state as well but the belief state for question B changes if the belief states for question A and question B are equal. The decision stage results in the preset belief for question A as well.

Equations (39)–(42) describe the change of the mind state + the probe state with the personality state $|2\rangle$. In this case, the belief state is copied to the probe state as well but the belief state for question B changes if the belief states for question A and question B are not equal. The decision stage results in the preset belief for question A as well.

Suppose that the probe P is prepared in the state $|00\rangle$ just before the measuring interaction, the measuring process described by the indirect measurement model

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$(\mathcal{K}, |00\rangle, U_A, M_A)$ defines the instrument $\mathcal{I}_A$ by

$$\mathcal{I}_A(a)\rho = \mathrm{Tr}_{\mathcal{K}}\left[\left(I_{\mathcal{H}} \otimes P^{M_A}(a)\right) U_A(\rho \otimes |00\rangle\langle 00|)U_A^* \left(I_{\mathcal{H}} \otimes P^{M_A}(a)\right)\right], \tag{44}$$

for any density operator ρ on $\mathcal{H}$, where $\mathrm{Tr}_{\mathcal{K}}$ stands for the partial trace over $\mathcal{K}$ and $P^{M_A}(a)$ denotes the spectral projection of an observable M_A for a, i.e., $P^{M_A}(0) = M_A^{\perp}$ and $P^{M_A}(1) = M_A$. Consequently,

$$\begin{aligned}
\mathcal{I}_A(0)\rho &= \mathrm{Tr}_{\mathcal{K}}[(I_{\mathcal{H}} \otimes M_A^{\perp})U_A(\rho \otimes |00\rangle\langle 00|)U_A^*(I_{\mathcal{H}} \otimes M_A^{\perp})],\\
\mathcal{I}_A(1)\rho &= \mathrm{Tr}_{\mathcal{K}}[(I_{\mathcal{H}} \otimes M_A)U_A(\rho \otimes |00\rangle\langle 00|)U_A^*(I_{\mathcal{H}} \otimes M_A)]
\end{aligned} \tag{45}$$

for any ρ.

The instrument $\mathcal{I}_A$ determines the probability distribution $\Pr\{\mathbf{a} = a\|\rho\}$ of the outcome $\mathbf{a}$ of the measurement, where $a = 0, 1$, and the state change $\rho \mapsto \rho_{\{\mathbf{a}=a\}}$ caused by the measurement is determined by

$$\Pr\{\mathbf{a} = a\|\rho\} = \mathrm{Tr}[\mathcal{I}_A(a)\rho], \tag{46}$$

$$\rho \mapsto \rho_{\{\mathbf{a}=a\}} = \frac{\mathcal{I}_A(a)\rho}{\mathrm{Tr}[\mathcal{I}_A(a)\rho]}. \tag{47}$$

From Appendix C, it follows that the instrument $\mathcal{I}_A$ measures the observable A, i.e.,

$$\begin{aligned}
\Pr\{\mathbf{a} = 0\|\rho\} &= \mathrm{Tr}[\mathcal{I}_A(0)\rho] = \mathrm{Tr}[A^{\perp}\rho],\\
\Pr\{\mathbf{a} = 1\|\rho\} &= \mathrm{Tr}[\mathcal{I}_A(1)\rho] = \mathrm{Tr}[A\rho]
\end{aligned} \tag{48}$$

for any density operator ρ on $\mathcal{H}$, and we obtain

$$\begin{aligned}
\mathcal{I}_A(a)\rho = \sum_{\beta} &\left(|a,\beta,0\rangle \langle a,\beta,0|\rho|a,\beta,0\rangle \langle a,\beta,0|\right.\\
&\left.+|a,a^{\perp},1\rangle \langle a,\beta,1|\rho|a,\beta,1\rangle \langle a,a^{\perp},1| + |a,a,2\rangle \langle a,\beta,2|\rho|a,\beta,2\rangle \langle a,a,2|\right).
\end{aligned} \tag{49}$$

for any density operator ρ on $\mathcal{H}$ and $a = 0, 1$ as a general form of the instrument $\mathcal{I}_A$ obtained by the indirect measurement model $(\mathcal{K}, |00\rangle, U_A, M_A)$.

4.3. **Instrument $\mathcal{I}_B$ measuring B.** The instrument $\mathcal{I}_B$ is constructed with the same probe system P prepared in the state $|00\rangle$ in an analogous manner with the instrument $\mathcal{I}_A$. The unitary operator U_B on $\mathcal{H} \otimes \mathcal{K}$, describing the time evolution of the composite

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system $S+P$ during the measuring interaction, is supposed to satisfy

$$U_B : |000\rangle|00\rangle \mapsto |000\rangle|00\rangle, \tag{50}$$
$$U_B : |010\rangle|00\rangle \mapsto |010\rangle|01\rangle, \tag{51}$$
$$U_B : |100\rangle|00\rangle \mapsto |100\rangle|10\rangle, \tag{52}$$
$$U_B : |110\rangle|00\rangle \mapsto |110\rangle|11\rangle, \tag{53}$$
$$U_B : |001\rangle|00\rangle \mapsto |101\rangle|00\rangle, \tag{54}$$
$$U_B : |011\rangle|00\rangle \mapsto |011\rangle|01\rangle, \tag{55}$$
$$U_B : |101\rangle|00\rangle \mapsto |101\rangle|10\rangle, \tag{56}$$
$$U_B : |111\rangle|00\rangle \mapsto |011\rangle|11\rangle, \tag{57}$$
$$U_B : |002\rangle|00\rangle \mapsto |002\rangle|00\rangle, \tag{58}$$
$$U_B : |012\rangle|00\rangle \mapsto |112\rangle|01\rangle, \tag{59}$$
$$U_B : |102\rangle|00\rangle \mapsto |002\rangle|10\rangle, \tag{60}$$
$$U_B : |112\rangle|00\rangle \mapsto |112\rangle|11\rangle, \tag{61}$$

and the meter observable on $\mathcal{K}$ is given by

$$M_B = I_4 \otimes |1\rangle\langle 1|. \tag{62}$$

Then the instrument $\mathcal{I}_B$ of this measuring process $(\mathcal{K}, |00\rangle, U_B, M_B)$ is defined by

$$\mathcal{I}_B(b)\rho = \mathrm{Tr}_{\mathcal{K}}\left[\Big(I_{\mathcal{H}} \otimes P^{M_B}(b)\Big)U_B(\rho \otimes |00\rangle\langle 00|)U_B^*\Big(I_{\mathcal{H}} \otimes P^{M_B}(b)\Big)\right] \tag{63}$$

for any density operator ρ on $\mathcal{H}$ and $b = 0, 1$. Then we have

$$\begin{aligned}\mathcal{I}_B(b)\rho = &\sum_{\alpha} |\alpha, b, 0\rangle\langle\alpha, b, 0|\rho|\alpha, b, 0\rangle\langle\alpha, b, 0| \\ &+ |b^{\perp}, b, 1\rangle\langle\alpha, b, 1|\rho|\alpha, b, 1\rangle\langle b^{\perp}, b, 1| + |b, b, 2\rangle\langle\alpha, b, 2|\rho|\alpha, b, 2\rangle\langle b, b, 2|.\end{aligned} \tag{64}$$

for any density operator ρ on $\mathcal{H}$ and for any $b = 0, 1$. Thus, the probability distribution of the outcome $\mathbf{b}$ of the instrument $\mathcal{I}_B$ is given by

$$\begin{aligned}\Pr\{\mathbf{b} = 0\|\rho\} &= \mathrm{Tr}[\mathcal{I}_B(0)\rho] = \mathrm{Tr}[B^{\perp}\rho], \\ \Pr\{\mathbf{b} = 1\|\rho\} &= \mathrm{Tr}[\mathcal{I}_B(1)\rho] = \mathrm{Tr}[B\rho]\end{aligned} \tag{65}$$

for any density operator ρ on $\mathcal{H}$, and hence the instrument $\mathcal{I}_B$ measures the observable B.

5. Postulates for quantum models

Our model presented above satisfies the following postulates.

Postulate 1: A subject's mind state is represented by a state vector in or a density operator on a multidimensional feature space (technically, an N-dimensional Hilbert space).

The above postulate extends Postulate 1 in Wang-Busemeyer [3] to include density operators. State vectors $|\psi_j\rangle$ can be mixed in two way. For complex numbers α_j with $\sum_j |\alpha_j|^2 = 1$, we can make a new state vector, called the superposition, $\sum_j \alpha_j|\psi_j\rangle$, and alternatively for positive numbers p_j with $\sum_j p_j = 1$, we can make a new state, called the mixture, represented by the density operator $\sum_j p_j|\psi_j\rangle\langle\psi_j|$. We use density operators if the state is a probability mixture (not a superposition) of state vectors.

Postulate 2: A potential response to a question is represented by a subspace of the multidimensional feature space.

This repeats Postulate 2 in [3]. In our model, we consider two questions A and B corresponding to the two subspaces $\mathcal{A}$ and $\mathcal{B}$, which are represented by the two projections A and B that project the whole space $\mathcal{H}$ onto $\mathcal{A}$ and $\mathcal{B}$, respectively.

Postulate 3: The probability of responding to an opinion question equals the squared length of the projection of the state vector onto the response subspace, or equals the trace of the projection of the density operator onto the response subspace.

This extends Postulate 3 in [3] to include density operators. According to Eq. (48), in our model the probability that a subject in a state vector $|\psi\rangle$ responds "yes" or "no" to the opinion question A is given by

$$\begin{aligned} \Pr\{A = y \| |\psi\rangle\} &= \|A|\psi\rangle\|^2, \\ \Pr\{A = n \| |\psi\rangle\} &= \|A^\perp|\psi\rangle\|^2. \end{aligned} \tag{66}$$

From Eq. (65) analogous relations hold for question B. According to Eq. (48), for the ensemble of subjects represented by a density operator ρ the frequency of responses "yes" or "no" to the opinion question A is given by

$$\begin{aligned} \Pr\{A = y \| \rho\} &= \mathrm{Tr}[A\rho] = \mathrm{Tr}[A\rho A], \\ \Pr\{A = n \| \rho\} &= \mathrm{Tr}[A^\perp\rho] = \mathrm{Tr}[A^\perp\rho A^\perp]. \end{aligned} \tag{67}$$

From Eq. (65) analogous relations hold for question B.

Postulate 4: The updated mind state after deciding an answer to a question is determined by the instrument corresponding to the question.

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This postulate is more general than Postulate 4 in [3]. According to Eq. (19) and Eq. (20) for any A-measuring instrument $\mathcal{I}_A$, there exists a family $\{M_{1j}, M_{0j}\}_j$ of operators such that

$$\mathcal{I}_A(1)\rho = \sum_j M_{1j}\rho M_{1j}^*, \tag{68}$$

$$A = \sum_j M_{1j}^* M_{1j}, \tag{69}$$

$$\mathcal{I}_A(0)\rho = \sum_j M_{0j}\rho M_{0j}^*, \tag{70}$$

$$A^\perp = \sum_j M_{0j}^* M_{0j}. \tag{71}$$

If the state before answering the question A is $\rho = |\psi\rangle\langle\psi|$, the updated mind state $\rho_{\{A=y\}}$ after deciding an answer $A = y$ to the question A is determined by the instrument $\mathcal{I}_A$ as

$$\begin{aligned} \rho_{\{A=y\}} &= \frac{\mathcal{I}_A(1)\rho}{\mathrm{Tr}[\mathcal{I}_A(1)\rho]} = \frac{\sum_j M_{1j}|\psi\rangle\langle\psi|M_{1j}^*}{\|A|\psi\rangle\|^2}, \\ \rho_{\{A=n\}} &= \frac{\mathcal{I}_A(0)\rho}{\mathrm{Tr}[\mathcal{I}_A(0)\rho]} = \frac{\sum_j M_{0j}|\psi\rangle\langle\psi|M_{0j}^*}{\|A^\perp|\psi\rangle\|^2}. \end{aligned} \tag{72}$$

According to Eq. (49), an explicit form of the family $\{M_{1j}, M_{0j}\}_j$ is given with parameter $j = (\beta, \gamma)$ as follows.

$$\begin{aligned} M_{1,(\beta,0)} &= |1,\beta,0\rangle\langle 1,\beta,0|, \ M_{1,(\beta,1)} = |1,0,1\rangle\langle 1,\beta,1|, \ M_{1,(\beta,2)} = |1,1,2\rangle\langle 1,\beta,2|. \\ M_{0,(\beta,0)} &= |0,\beta,0\rangle\langle 0,\beta,0|, \ M_{0,(\beta,1)} = |0,1,1\rangle\langle 0,\beta,1|, \ M_{0,(\beta,2)} = |0,0,2\rangle\langle 0,\beta,2|. \end{aligned} \tag{73}$$

Thus, we reproduce Eqs. (69) and (71):

$$\sum_{\beta.\gamma} M_{1,(\beta,\gamma)}^* M_{1,(\beta,\gamma)} = \sum_{\beta,\gamma} |1,\beta,\gamma\rangle\langle 1,\beta,\gamma| = |1\rangle\langle 1| \otimes I_2 \otimes I_3 = A,$$

$$\sum_{\beta.\gamma} M_{0,(\beta,\gamma)}^* M_{0,(\beta,\gamma)} = \sum_{\beta,\gamma} |0,\beta,\gamma\rangle\langle 0,\beta,\gamma| = |0\rangle\langle 0| \otimes I_2 \otimes I_3 = A^\perp.$$

Wang-Busemeyer [3] posed

Postulate 4 (Wang-Busemeyer [3]*): The updated belief state after deciding an answer to a question equals the normalized projection on the subspace representing the answer.*

This condition corresponds to the projective A-measuring instrument $\mathcal{I}_A^{proj}$ such that $\mathcal{I}_A^{proj}(1)\rho = A\rho A$ and $\mathcal{I}_A^{proj}(0)\rho = A^\perp\rho A^\perp$. This instrument has the measurement

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operators $\{A, A^{\perp}\}$ so that

$$\begin{aligned}\rho_{\{A=y\}} &= \frac{\mathcal{I}_A(1)\rho}{\mathrm{Tr}[\mathcal{I}_A(1)\rho]} = \frac{A|\psi\rangle\langle\psi|A}{\|A|\psi\rangle\|^2},\\ \rho_{\{A=n\}} &= \frac{\mathcal{I}_A(0)\rho}{\mathrm{Tr}[\mathcal{I}_A(0)\rho]} = \frac{A^{\perp}|\psi\rangle\langle\psi|A^{\perp}}{\|A^{\perp}|\psi\rangle\|^2},\end{aligned} \tag{74}$$

or states changes as

$$\begin{aligned}|\psi\rangle \mapsto |\psi_{\{A=1\}}\rangle &= \frac{A|\psi\rangle}{\|A|\psi\rangle\|},\\ |\psi\rangle \mapsto |\psi_{\{A=0\}}\rangle &= \frac{A^{\perp}|\psi\rangle}{\|A^{\perp}|\psi\rangle\|}.\end{aligned} \tag{75}$$

Thus, our model satisfies Postulates 1-3 consistent with those posed by Wang-Busemeyer [3]. Moreover, our model generalizes Postulates 4 posed by Wang-Busemeyer [3] equivalent to the projection postulate, so that we allow any A-measuring instrument for question A,

Last but not least, we should pose an additional postulate on the ensemble of subjects, which is not explicit but satisfied by the Wang-Busemeyer model [3].

Postulate 5: If a sequence of question-response experiments are carried out to the same ensemble of individual subjects, the joint probability of the sequence of responses are probability mixture of those joint probabilities with respect to the frequency of individual mind states in the ensemble.

If a sequence of question-response experiments are carried out to the same ensemble of individual subjects, and a randomly chosen individual subject has the state $|\psi_j\rangle$ with probability p_j, then we suppose that the joint probability of the sequence of responses $X_1 = x_1, \ldots, X_n = x_n$, where $X_k = A$ or $X_k = B$ and $x_k = 0$ or $x_k = 1$ for all $k = 1, \ldots, n$, is given by

$$\Pr\{X_1 = x_1, \ldots, X_n = x_n \| \rho\} = \mathrm{Tr}[\mathcal{I}_{X_n}(x_n) \cdot \mathcal{I}_{X_1}(x_1)\rho], \tag{76}$$

where $\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$. Then, we have

$$\begin{aligned}&\Pr\{X_1 = x_1, \ldots, X_n = x_n \| \rho\}\\ &\quad = \sum_j p_j \Pr\{X_1 = x_1, \ldots, X_n = x_n \| |\psi_j\rangle\}.\end{aligned} \tag{77}$$

Thus, Postulate 5 is satisfied by our model.

The structure of this postulate is discussed in more detail in appendix E.

6. MIND STATES

In modeling successive question-response experiments, such as the Clinton-Gore experiment, we assume that the subject's mind state is represented by the space $\Omega = \{0,1\}^2 \times \{0,1,2\}$. Here we suppose that any individual subject has one of the mind

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states, $\omega \in \Omega$, and any statistical ensemble of the subjects is characterized by a probability distribution on Ω. Thus, the statistical data under consideration should be explained solely by one of the probability distributions on Ω. The mind state α represents the state in which the subject will answer "yes" for the question A if $\alpha = 1$ and answer "no" otherwise. The mind state β represents the analogous state for the question B. Thus, the mind state (α, β) represents the subject's prior belief as long as the questions A and B are concerned. For $(\alpha, \beta, \gamma) \in \Omega$ we write $\delta_{(\alpha,\beta,\gamma)} = |\alpha, \beta, \gamma\rangle\langle\alpha, \beta, \gamma| \in \mathcal{H}$. In this way, we identify the mind state $(\alpha, \beta, \gamma) \in \Omega$ with the quantum state $|\alpha, \beta, \gamma\rangle\langle\alpha, \beta, \gamma| \in \mathcal{H}$, and any probability distribution μ of the mind states $\omega \in \Omega$ with the quantum state $\hat{\mu} = \sum_\omega \mu(\omega)\delta_\omega$. The dynamics of producing the answer to the question and preparing the mind state for the next question is supposed to be described as a process of quantum measurement, or equivalently a mathematical object called a quantum instrument, which we believe is the most flexible framework consistent with the psycho-physical parallelism.

Our previously defined two quantum instruments $\mathcal{I}_A$ and $\mathcal{I}_B$ describe the measurement on the mind state as follows.

Theorem 6.1. *For any* $(\alpha, \beta, \gamma) \in \Omega$ *and* $a, b = 0, 1$, *we have*

$$\mathcal{I}_A(a)\delta_{(\alpha,\beta,0)} = \delta_\alpha(a)\delta_{(\alpha,\beta,0)}, \tag{78}$$

$$\mathcal{I}_A(a)\delta_{(\alpha,\beta,1)} = \delta_\alpha(a)\delta_{(\alpha,\alpha^\perp,1)}, \tag{79}$$

$$\mathcal{I}_A(a)\delta_{(\alpha,\beta,2)} = \delta_\alpha(a)\delta_{(\alpha,\alpha,2)}, \tag{80}$$

$$\mathcal{I}_B(b)\delta_{(\alpha,\beta,0)} = \delta_\beta(b)\delta_{(\alpha,\beta,0)}, \tag{81}$$

$$\mathcal{I}_B(b)\delta_{(\alpha,\beta,1)} = \delta_\beta(b)\delta_{(\beta^\perp,\beta,1)}, \tag{82}$$

$$\mathcal{I}_B(b)\delta_{(\alpha,\beta,2)} = \delta_\beta(b)\delta_{(\beta,\beta,2)}. \tag{83}$$

Proof. The relations follow from Eq. (49) and Eq. (64) by routine computations. □

The mind state γ represents the personality of the subject in such a way that if $\gamma = 1$, the subject changes his/her mind to prepare the answer for the other question to be the opposite to the previous answer, and if $\gamma = 2$, the subject changes his/her mind to prepare the answer for the other question to be the same as the previous answer; on the other hand, if $\gamma = 0$, the subject's mind is so robust that the question will not affect his/her mind.

Thus, the state change is the Bayesian update if γ has only the value $\gamma = 0$, and yet if the value $\gamma = 1$ or $\gamma = 2$ is allowed, the statistics of the answers does not follows the Bayesian update rule.

The following considerations considerably simplify our manipulations of instruments $\mathcal{I}_A$ and $\mathcal{I}_B$ for arbitrary density operators ρ.

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From Appendix D, for any density operator ρ we have

$$\begin{aligned}\mathcal{I}_A(a)\rho &= \sum_{\alpha,\beta,\gamma} \mu(\alpha,\beta,\gamma)\mathcal{I}_A(a)\delta_{(\alpha,\beta,\gamma)} = \mathcal{I}_A(a)\rho',\\ \mathcal{I}_B(b)\rho &= \sum_{\alpha,\beta,\gamma} \mu(\alpha,\beta,\gamma)\mathcal{I}_B(b)\delta_{(\alpha,\beta,\gamma)} = \mathcal{I}_B(b)\rho',\end{aligned} \tag{84}$$

where $\mu(\alpha,\beta,\gamma) = \langle\alpha,\beta,\gamma|\rho|\alpha,\beta,\gamma\rangle$, and

$$\rho' = \sum_{\alpha,\beta,\gamma} \mu(\alpha,\beta,\gamma)\delta_{(\alpha,\beta,\gamma)}. \tag{85}$$

Therefore, the operations of $\mathcal{I}_A$ and $\mathcal{I}_B$ for arbitrary density operators ρ are reduced to only for the density operators ρ' of the form of Eq. (85).

7. Response Replicability Effect (RRE)

Here, we consider two properties of successive measurements of observables A and B.

In A-A and A-B-A paradigms, the response to A is repeated (with probability 1). We call this response replicability effect (RRE).

In A-B vs B-A paradigm, the joint probabilities of the two responses are different on a set of states with a positive Lebesgue measure. This is question order effect (QOE).

The psychological problem raised by the Clinton-Gore experiment is as follows. We are given two observables A and B whose joint probability distributions (JPDs) of successive measurements shows QOE and is naturally considered to satisfy RRE. We cannot represent the JPDs as the JPDs of classical random variables A and B defined by Kolmogorov, which does not show QOE, nor the JPDs of the outcomes of successive measurements of non-commuting quantum observables A and B defined by von Neumann and Lüders, which does not satisfy RRE.

We shall show that instruments $\mathcal{I}_A$ and $\mathcal{I}_B$ have both Response Replicability Effect (RRE) in this section, and Question Order Effect (QOE) in the next section.

Theorem 7.1. *The instruments $\mathcal{I}_A$ and $\mathcal{I}_B$ have the Response Replicability Effect; namely, they satisfy the following relations.*

(i) $\sum_a \mathrm{Tr}[\mathcal{I}_A(a)\mathcal{I}_A(a)\rho] = 1$ *for any density operator* ρ.
(ii) $\sum_b \mathrm{Tr}[\mathcal{I}_B(b)\mathcal{I}_B(b)\rho] = 1$ *for any density operator* ρ.
(iii) $\sum_{a,b} \mathrm{Tr}[\mathcal{I}_A(a)\mathcal{I}_B(b)\mathcal{I}_A(a)\rho] = 1$ *for any density operator* ρ.
(iv) $\sum_{a,b} \mathrm{Tr}[\mathcal{I}_B(b)\mathcal{I}_A(a)\mathcal{I}_B(b)\rho] = 1$ *for any density operator* ρ.

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Proof. From Eq. (84) we assume without any loss of generality that $\rho = \delta_{(\alpha,\beta,\gamma)}$ for some α, β, γ. For any α, β, we have

$$\begin{aligned}\sum_a \mathrm{Tr}[\mathcal{I}_A(a)\mathcal{I}_A(a)\delta_{(\alpha,\beta,0)}] &= \sum_a \delta_\alpha(a)\mathrm{Tr}[\mathcal{I}_A(a)\delta_{(\alpha,\beta,0)}] \\ &= \sum_a \delta_\alpha(a)\mathrm{Tr}[\delta_{(\alpha,\beta,0)}] \\ &= \mathrm{Tr}[\delta_{(\alpha,\beta,0)}] \\ &= 1.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\sum_a \mathrm{Tr}[\mathcal{I}_A(a)\mathcal{I}_A(a)\delta_{(\alpha,\beta,1)}] &= 1, \\ \sum_a \mathrm{Tr}[\mathcal{I}_A(a)\mathcal{I}_A(a)\delta_{(\alpha,\beta,2)}] &= 1.\end{aligned}$$

Thus, relation (i) follows. Relation (ii) follows analogously. For any α, β, we have

$$\begin{aligned}\sum_{a,b} \mathrm{Tr}[\mathcal{I}_A(a)\mathcal{I}_B(b)\mathcal{I}_A(a)\delta_{(\alpha,\beta,0)}] &= \sum_{a,b} \delta_\alpha(a)\mathrm{Tr}[\mathcal{I}_A(a)\mathcal{I}_B(b)\delta_{(\alpha,\beta,0)}] \\ &= \sum_{a,b} \delta_\alpha(a)\delta_\beta(b)\mathrm{Tr}[\mathcal{I}_A(a)\delta_{(\alpha,\beta,0)}] \\ &= \sum_{a,b} \delta_\alpha(a)\delta_\beta(b)\delta_\alpha(a)\mathrm{Tr}[\delta_{(\alpha,\beta,0)}] \\ &= \mathrm{Tr}[\delta_{(\alpha,\beta,0)}] \\ &= 1.\end{aligned}$$

Similarly,

$$\begin{aligned}\sum_{a,b} \mathrm{Tr}[\mathcal{I}_A(a)\mathcal{I}_B(b)\mathcal{I}_A(a)\delta_{(\alpha,\beta,1)}] &= 1, \\ \sum_{a,b} \mathrm{Tr}[\mathcal{I}_A(a)\mathcal{I}_B(b)\mathcal{I}_A(a)\delta_{(\alpha,\beta,2)}] &= 1.\end{aligned}$$

Thus, relation (iii) follows. Relation (iv) follows in a similar way. □

8. Question Order Effect (QOE)

In what follows, we shall show that instruments $\mathcal{I}_A$ and $\mathcal{I}_B$ have the Question Order Effect (QOE).

Let $\hat{\mu} = \sum_{\alpha,\beta,\gamma} \mu(\alpha,\beta,\gamma)\delta_{(\alpha,\beta,\gamma)}$ for any probability distribution μ on Ω, we write

$$p(AaBb) = \mathrm{Tr}[\mathcal{I}_B(b)\mathcal{I}_A(a)\hat{\mu}], \tag{86}$$

$$p(BbAa) = \mathrm{Tr}[\mathcal{I}_A(a)\mathcal{I}_B(b))\hat{\mu}], \tag{87}$$

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where $a, b = 0, 1$. We will write Ay, An, By, Bn instead of $A1, A0, B0, B1$. we have

$$
\begin{aligned}
&p(AaBb)\\
&\qquad = \sum_{\alpha,\beta,\gamma} \mu(\alpha,\beta,\gamma)\mathrm{Tr}[\mathcal{I}_B(b)\mathcal{I}_A(a)\delta_{(\alpha,\beta,\gamma)}]\\
&\qquad = \sum_{\alpha,\beta} \mu(\alpha,\beta,0)\delta_\alpha(a)\delta_\beta(b) + \mu(\alpha,\beta,1)\delta_\alpha(a)\delta_{\alpha^\perp}(b) + \mu(\alpha,\beta,2)\delta_\alpha(a)\delta_\alpha(b)\\
&\qquad = \mu(a,b,0) + \delta_{a^\perp}(b)\sum_\beta \mu(a,\beta,1) + \delta_a(b)\sum_\beta \mu(a,\beta,2).
\end{aligned}
$$

Similarly,

$$p(BbAa) = \mu(a,b,0) + \delta_{b^\perp}(a)\sum_\alpha \mu(\alpha,b,1) + \delta_b(a)\sum_\alpha \mu(\alpha,b,2).$$

Thus,

$$p(AyBy) = \mu(1,1,0) + \mu(1,1,2) + \mu(1,0,2). \tag{88}$$
$$p(AyBn) = \mu(1,0,0) + \mu(1,0,1) + \mu(1,1,1). \tag{89}$$
$$p(AnBy) = \mu(0,1,0) + \mu(0,1,1) + \mu(0,0,1). \tag{90}$$
$$p(AnBn) = \mu(0,0,0) + \mu(0,0,2) + \mu(0,1,2). \tag{91}$$
$$p(ByAy) = \mu(1,1,0) + \mu(1,1,2) + \mu(0,1,2). \tag{92}$$
$$p(BnAy) = \mu(1,0,0) + \mu(1,0,1) + \mu(0,0,1). \tag{93}$$
$$p(ByAn) = \mu(0,1,0) + \mu(0,1,1) + \mu(1,1,1). \tag{94}$$
$$p(BnAn) = \mu(0,0,0) + \mu(0,0,2) + \mu(1,0,2). \tag{95}$$

Theorem 8.1. *The instruments $\mathcal{I}_A$ and $\mathcal{I}_B$ have the Question Order Effect; namely, we have the following statements. Let ρ be a density operator on $\mathcal{H}$.*

(i) *The relation*
$$\mathrm{Tr}[\mathcal{I}_B(1)\mathcal{I}_A(1)\rho] = \mathrm{Tr}[\mathcal{I}_A(1)\mathcal{I}_B(1)\rho] \tag{96}$$
holds if and only if $\langle 1,0,2|\rho|1,0,2\rangle = \langle 0,1,2|\rho|0,1,2\rangle$.

(ii) *The relation*
$$\mathrm{Tr}[\mathcal{I}_B(0)\mathcal{I}_A(1)\rho] = \mathrm{Tr}[\mathcal{I}_A(1)\mathcal{I}_B(0)\rho] \tag{97}$$
holds if and only if $\langle 1,1,1|\rho|1,1,1\rangle = \langle 0,0,1|\rho|0,0,1\rangle$.

(iii) *The relation*
$$\mathrm{Tr}[\mathcal{I}_B(1)\mathcal{I}_A(0)\rho] = \mathrm{Tr}[\mathcal{I}_A(0)\mathcal{I}_B(1)\rho] \tag{98}$$
holds if and only if $\langle 1,1,1|\rho|1,1,1\rangle = \langle 0,0,1|\rho|0,0,1\rangle$.

(iv) *The relation*
$$\mathrm{Tr}[\mathcal{I}_B(0)\mathcal{I}_A(0)\rho] = \mathrm{Tr}[\mathcal{I}_A(0)\mathcal{I}_B(0)\rho] \tag{99}$$
holds if and only if $\langle 1,0,2|\rho|1,0,2\rangle = \langle 0,1,2|\rho|0,1,2\rangle$.

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(v) *One of Eqs. (96), (97), (98), (99) for a general density operator ρ holds only on a set of density operators ρ with Lebesgue measure 0.*

Proof. Eq. (96) holds if and only $p(AyBy) = p(ByAy)$. Thus, assertion (i) follows from Eqs. (84), (88), and (92). Assertions (ii)–(iv) follows similarly. The last assertion follows from the fact that each relation holds on a submanifold of the space of density operators with co-dimension 1. □

9. QQ-EQUALITY

If two questions, adjacent to each other, are asked in different orders, then the quantum model of measurement order makes an a priori and parameter-free prediction, named the QQ equality [2, 3, 9]:

$$\begin{aligned} q &= [p(ByAy) + p(BnAn)] - [p(AyBy) + p(AnBn)] \\ &= [p(AyBn) + p(AnBy)] - [p(ByAn) + p(BnAy)] = 0. \end{aligned} \tag{100}$$

Note that the formula in [9, P. 9435, left column] is not correct in that the first and second lines are not equal but have the opposite signs; the correct definition here is due to [3]. We have

$$p(AyBy) - p(ByAy) = \mu(1,0,2) - \mu(0,1,2), \tag{101}$$
$$p(AnBn) - p(BnAn) = \mu(0,1,2) - \mu(1,0,2), \tag{102}$$
$$p(AyBn) - p(BnAy) = \mu(1,1,1) - \mu(0,0,1), \tag{103}$$
$$p(AnBy) - p(ByAn) = \mu(0,0,1) - \mu(1,1,1). \tag{104}$$

Thus, it is easy to check that the QQ equality holds and we have

Theorem 9.1. *The instruments $\mathcal{I}_A$ and $\mathcal{I}_B$ satisfy the QQ equality.*

10. QQE-RENORMALIZATIONS

We have shown that joint probabilities of responses to questions obtained from our model satisfies QQE, so that if the original data does not satisfy QQE we cannot reproduce those data from our model with arbitrary mind state ρ.

Given the joint probabilities $p(AyBy)$, $p(AyBn)$, $p(AnBy)$, $p(AnBn)$, $p(ByAy)$, $p(ByAn)$, $p(BnAy)$, and $p(BnAn)$, let

$$\begin{aligned} S_1 &= \frac{p(AyBy) + p(AnBn) + p(ByAy) + p(BnAn)}{2}, \\ S_2 &= \frac{p(AyBn) + p(AnBy) + p(ByAn) + p(BnAy)}{2}. \end{aligned} \tag{105}$$

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Their QQE-renormalizations $\bar{p}(AyBy)$, $\bar{p}(AyBn)$, $\bar{p}(AnBy)$, $\bar{p}(AnBn)$, $\bar{p}(ByAy)$, $\bar{p}(ByAn)$, $\bar{p}(BnAy)$, and $\bar{p}(BnAn)$, are defined as follows.

$$
\begin{aligned}
\bar{p}(AyBy) &= S_1 \times \frac{p(AyBy)}{p(AyBy)+p(AnBn)},\\
\bar{p}(AnBn) &= S_1 \times \frac{p(AnBn)}{p(AyBy)+p(AnBn)},\\
\bar{p}(ByAy) &= S_1 \times \frac{p(ByAy)}{p(ByAy)+p(BnAn)},\\
\bar{p}(BnAn) &= S_1 \times \frac{p(BnAn)}{p(ByAy)+p(BnAn)},\\
\bar{p}(AyBn) &= S_2 \times \frac{p(AyBn)}{p(AyBn)+p(AnBy)},\\
\bar{p}(AnBy) &= S_2 \times \frac{p(AnBy)}{p(AyBn)+p(AnBy)},\\
\bar{p}(ByAn) &= S_2 \times \frac{p(ByAn)}{p(ByAn)+p(BnAy)},\\
\bar{p}(BnAy) &= S_2 \times \frac{p(BnAy)}{p(ByAn)+p(BnAy)},
\end{aligned}
$$

Theorem 10.1. *The following statements hold.*

(i) The QQE-renormalizations $\bar{p}(AaBb), \bar{p}(BaAb)$ with $a, b = y, n$ define joint probability distributions; i.e., $\bar{p}(AaBb)$ and $\bar{p}(BaAb)$ satisfy $0 \le \bar{p}(AaBb), \bar{p}(BaAb) \le 1$ for all $a, b = y, n$ and $\sum_{a,b=y,n} \bar{p}(AaBb) = \sum_{a,b=y,n} \bar{p}(BaAb) = 1$.

(ii) The joint probability distributions $\{\bar{p}(AaBb), \bar{p}(BaAb) \mid a, b = y, n\}$ satisfy the QQE, i.e.,

$$
\begin{aligned}
q &= [\bar{p}(ByAy) + \bar{p}(BnAn)] - [\bar{p}(AyBy) + \bar{p}(AnBn)]\\
&= [\bar{p}(AyBn) + \bar{p}(AnBy)] - [\bar{p}(ByAn) + \bar{p}(BnAy)] = 0.
\end{aligned}
$$

(iii) If the joint probability distributions $p(AaBb), p(BaAb)$ with $a, b = y, n$ satisfy the QQE then $\bar{p}(AaBb) = p(AaBb)$ and $\bar{p}(BaAb) = p(BaAb)$ for all $a, b = y, n$.

(iv) The following relations hold.

$$
\begin{aligned}
&\bar{p}(AyBy) + \bar{p}(AnBn) + \bar{p}(ByAy) + \bar{p}(BnAn)\\
&\qquad = p(AyBy) + p(AnBn) + p(ByAy) + p(BnAn),\\
&\bar{p}(AyBn) + \bar{p}(AnBy) + \bar{p}(ByAn) + \bar{p}(BnAy)\\
&\qquad = p(AyBn) + p(AnBy) + p(ByAn) + p(BnAy).
\end{aligned}
$$

(v) The following relations hold.

$$\frac{\bar{p}(AyBy)}{\bar{p}(AyBy)+\bar{p}(AnBn)}=\frac{p(AyBy)}{p(AyBy)+p(AnBn)},$$
$$\frac{\bar{p}(AnBn)}{\bar{p}(AyBy)+\bar{p}(AnBn)}=\frac{p(AnBn)}{p(AyBy)+p(AnBn)},$$
$$\frac{\bar{p}(ByAy)}{\bar{p}(ByAy)+\bar{p}(BnAn)}=\frac{p(ByAy)}{p(ByAy)+p(BnAn)},$$
$$\frac{\bar{p}(BnAn)}{\bar{p}(ByAy)+\bar{p}(BnAn)}=\frac{p(BnAn)}{p(ByAy)+p(BnAn)},$$
$$\frac{\bar{p}(AyBn)}{\bar{p}(AyBn)+\bar{p}(AnBy)}=\frac{p(AyBn)}{p(AyBn)+p(AnBy)},$$
$$\frac{\bar{p}(AnBy)}{\bar{p}(AyBn)+\bar{p}(AnBy)}=\frac{p(AnBy)}{p(AyBn)+p(AnBy)},$$
$$\frac{\bar{p}(ByAn)}{\bar{p}(ByAn)+\bar{p}(BnAy)}=\frac{p(ByAn)}{p(ByAn)+p(BnAy)},$$
$$\frac{\bar{p}(BnAy)}{\bar{p}(ByAn)+\bar{p}(BnAy)}=\frac{p(BnAy)}{p(ByAn)+p(BnAy)}.$$

(v) The following relations hold.

$$\frac{\bar{p}(AyBy)}{p(AyBy)}-1=\frac{q}{2S_1-q},$$
$$\frac{\bar{p}(AnBn)}{p(AnBn)}-1=\frac{q}{2S_1-q},$$
$$\frac{\bar{p}(ByAy)}{p(ByAy)}-1=\frac{-q}{2S_1+q},$$
$$\frac{\bar{p}(BnAn)}{p(BnAn)}-1=\frac{-q}{2S_1+q},$$
$$\frac{\bar{p}(AyBn)}{p(AyBn)}-1=\frac{-q}{2S_2+q},$$
$$\frac{\bar{p}(AnBy)}{p(AnBy)}-1=\frac{-q}{2S_2+q},$$
$$\frac{\bar{p}(ByAn)}{p(ByAn)}-1=\frac{q}{2S_2-q},$$
$$\frac{\bar{p}(BnAy)}{p(BnAy)}-1=\frac{q}{2S_2+q}.$$

Proof. The assertions can be verified straightforward calculations. □

From Theorem 10.1 (iv), if $S_1 = S_2$ and $|q| \ll 1$, the renormalization error $(\bar{p}-p)/p$ is uniformly $|q|$, and if $S_1 < S_2$ and $|q| \ll 1$, it is uniformly $|q|/(2S_1)$.

11. Independence of personality state

In the previous sections, we have not assumed that in the probability distribution μ of the mind state, the personality state is independent from the belief state. According to section 4.2 the personality state determines the type of how the belief state is changed by U_A and U_B. It would be desirable to describe it to be independent from the belief state (α, β).

If the personality state depends on belief states, the same subject responds to questions in various patterns, so that it is difficulty to interpret the role of this parameter, and according to the number of free parameters it is likely that we can always find a suitable input data to explain the QQE renormalized output data by our model. On the other hand, if the personality state is independent of belief states, each individual subject is classified by the personality state and each ensemble of subjects for the same experiment can be classified by the probability distribution of the personality state. Thus, the statistical role of the personality state is quite clear.

However, in this case, the number of free parameter becomes not enough to reproduce all the QQE renormalized output data. Thus, the new assumption that the personality state is independent of the belief state bring about the following interesting problems: (i) What QQE renormalized output data can be reproduced by the current model? (ii) In such output data, what are characteristic features of the probability distribution of the personality state in the input data? (iii) How can we extend the role of the personality state to cover more QQE renormalized output data?

For the case where only personality state is $|0\rangle$, the model reduces to Bayesian update model, which does not have QOE. We studied the model only with personality state $|1\rangle$ and showed QOE and RRE. In this paper, we shall further show that the model with personality state $|0\rangle, |1\rangle, |2\rangle$ explains the Clinton-Gore experiment.

Here, we assume the independence of the personality state. Thus, we suppose

$$\mu(\alpha, \beta, \gamma) = p(\alpha, \beta) q(\gamma), \tag{106}$$

$$p(\alpha, \beta) = \sum_{\gamma} \mu(\alpha, \beta, \gamma), \tag{107}$$

$$q(\gamma) = \sum_{\alpha, \beta} \mu(\alpha, \beta, \gamma). \tag{108}$$

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Then, Eqs. (88)–(95) are written as follows.

$$p(AyBy) = p(1,1)q(0) + p(1,1)q(2) + p(1,0)q(2). \tag{109}$$
$$p(AyBn) = p(1,0)q(0) + p(1,0)q(1) + p(1,1)q(1). \tag{110}$$
$$p(AnBy) = p(0,1)q(0) + p(0,1)q(1) + p(0,0)q(1). \tag{111}$$
$$p(AnBn) = p(0,0)q(0) + p(0,0)q(2) + p(0,1)q(2). \tag{112}$$
$$p(ByAy) = p(1,1)q(0) + p(1,1)q(2) + p(0,1)q(2). \tag{113}$$
$$p(BnAy) = p(1,0)q(0) + p(1,0)q(1) + p(0,0)q(1). \tag{114}$$
$$p(ByAn) = p(0,1)q(0) + p(0,1)q(1) + p(1,1)q(1). \tag{115}$$
$$p(BnAn) = p(0,0)q(0) + p(0,0)q(2) + p(1,0)q(2). \tag{116}$$

Let $p(Ay) = p(AyBy) + p(AyBn)$, etc. Then, $p(Ay) = p(1,1) + p(1,0)$, etc. We have

$$p(AyBy) = p(1,1)q(0) + p(Ay)q(2). \tag{117}$$
$$p(AyBn) = p(1,0)q(0) + p(Ay)q(1). \tag{118}$$
$$p(AnBy) = p(0,1)q(0) + p(An)q(1). \tag{119}$$
$$p(AnBn) = p(0,0)q(0) + p(An)q(2). \tag{120}$$
$$p(ByAy) = p(1,1)q(0) + p(By)q(2). \tag{121}$$
$$p(BnAy) = p(1,0)q(0) + p(Bn)q(1). \tag{122}$$
$$p(ByAn) = p(0,1)q(0) + p(By)q(1). \tag{123}$$
$$p(BnAn) = p(0,0)q(0) + p(Bn)q(2). \tag{124}$$

Thus, if $p(Ay) \neq p(By)$ and $p(Ay) \neq p(Bn)$, we can determine $q(0), q(1), q(2)$ by the experimental data:

$$q(2) = \frac{p(AyBy) - p(ByAy)}{p(Ay) - p(By)}, \tag{125}$$
$$q(1) = \frac{p(AyBn) - p(BnAy)}{p(Ay) - p(Bn)}, \tag{126}$$
$$q(1) = \frac{p(AnBy) - p(ByAn)}{p(An) - p(By)}, \tag{127}$$
$$q(2) = \frac{p(AnBn) - p(BnAn)}{p(An) - p(Bn)}. \tag{128}$$

Note that since the output data $\{p(AaBb), p(BbAa) \mid a, b = y, n\}$ obtained from our model satisfy the QQ equality, the above relations are consistent, so that $q(1)$ and $q(2)$ are uniquely determined from the experimental data and then we determine $q(0)$ by $q(0) + q(1) + q(2) = 1$.

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If we obtain the personality state $q(0), q(1), q(2)$, the belief state is determined by the following relations.

$$p(1,1) = \frac{\bar{p}(AyBy) - \bar{p}(Ay)q(2)}{q(0)}, \tag{129}$$
$$p(1,0) = \frac{\bar{p}(AyBn) - \bar{p}(Ay)q(1)}{q(0)}, \tag{130}$$
$$p(0,1) = \frac{\bar{p}(AnBy) - \bar{p}(An)q(1)}{q(0)}, \tag{131}$$
$$p(0,0) = \frac{\bar{p}(AnBn) - \bar{p}(An)q(2)}{q(0)}, \tag{132}$$
$$p(1,1) = \frac{\bar{p}(ByAy) - \bar{p}(By)q(2)}{q(0)}, \tag{133}$$
$$p(0,1) = \frac{\bar{p}(ByAn) - \bar{p}(By)q(1)}{q(0)}, \tag{134}$$
$$p(1,0) = \frac{\bar{p}(BnAy) - \bar{p}(Bn)q(1)}{q(0)}, \tag{135}$$
$$p(0,0) = \frac{\bar{p}(BnAn) - \bar{p}(Bn)q(2)}{q(0)}. \tag{136}$$

Therefore, if $p(Ay) \neq p(By)$ and $p(Ay) \neq p(Bn)$, and Eqs. (125)–(136) determines the probability distributions $\{q(0), q(1), q(2)\}$ and joint probability distributions $\{p(0,0), \ldots, p(1,1)\}$, the QQE renormalized data $\{\bar{p}(AyBy), \ldots, \bar{p}(AnBn)\}$ can be exactly reproduced by our model. A precise statement for the reproducibility condition is given as follows.

Theorem 11.1. *Suppose that joint probabilities* $p(AyBy)$, $p(AyBn)$, $p(AnBy)$, $p(AnBn)$, $p(ByAy)$, $p(ByAn)$, $p(BnAy)$, *and* $p(BnAn)$ *for* $a, b = y, n$, *i.e.,* $p(Aa, Bb), p(Bb, Aa) \geq 0$, $\sum_{a,b=y,n} p(Aa, Bb) = \sum_{a,b=y,n} p(Bb, Aa) = 1$, *are labeled as*

$$p(Ay) > p(By) \geq p(Bn) \geq p(An),$$

and satisfy the QQ-equality (100). Then, there exist input joint probabilities $q(0), q(1), q(2)$ *and* $p(0,0), p(0,1), p(1,0), p(1,1)$ *uniquely such that the model outputs the joint probabilities* $\{p(AaBb), p(Bb, Ab) \mid a, b = y, n\}$ *if and only if the following*

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conditions hold:

$$
\begin{aligned}
&(i) && p(AyBy) \ge p(ByAy),\\
&(ii) && p(AyBn) \ge p(BnAy),\\
&(iii) && \frac{p(AyBn)-p(ByAn)}{p(Ay)-p(By)} \ge \frac{p(AyBn)-p(BnAy)}{p(Ay)-p(Bn)},\\
&(iv) && \frac{p(ByAy)}{p(By)} \ge \frac{p(AyBy)}{p(Ay)},\\
&(v) && \frac{p(BnAy)}{p(Bn)} \ge \frac{p(AyBn)}{p(Ay)},\\
&(vi) && \frac{p(AnBy)}{p(An)} \ge \frac{p(ByAn)}{p(By)},\\
&(vii) && \frac{p(AnBn)}{p(An)} \ge \frac{p(BnAn)}{p(Bn)}.
\end{aligned}
$$

Proof. Necessity: Relations (i) and (ii) follow from (125) and (126) with the assumption $p(Ay) \ge p(By) \ge p(Bn)$. Relation (iii) follows from relations (125), (126) and $1 - q(1) \ge q(2)$; note that the assumption $p(Ay) > p(By)$ implies the relation $p(Bn) > p(An)$. From (117) and (121) we have

$$
\begin{aligned}
\frac{p(AyBy)}{p(Ay)} &= \frac{q(0)p(1,1)}{p(Ay)} + q(2),\\
\frac{p(ByAy)}{p(By)} &= \frac{q(0)p(1,1)}{p(By)} + q(2),
\end{aligned}
$$

and hence relation (iv) follows from the assumption $p(Ay) > p(By)$ and $p(1,1) \ge 0$. Relations (v), (vi), (vii) follows similarly from conditions $q(0)p(1,0) \ge 0, q(0)p(1,0) \ge 0$, and $q(0)p(0,0) \ge 0$,

Sufficiency: Suppose that $q(0), q(1), q(2), p(0,0), \dots, p(1,1)$ satisfy (117), ..., (124), and $q(0) + q(1) + q(2) = 1$. It suffices to prove that they are non-negative. We have $p(2) \ge 0$ by relation (i) and (125). Similarly, $p(1) \ge 0$ follows from relation (ii) and (126), and $q(0) \ge 0$ follows from relation (iii) and $q(0) + q(1) + q(2) = 1$. Then, $p(0,0) \ge 0$ follows from relation (iv) and (117). The non-negativity of $p(1,0), p(0,1), p(0,0)$ follows similarly. □

12. Clinton-Gore poll

In this section, we shall show that the well-known data from Clinton-Gore experiment can be reproduced within $\pm 0.75\%$ of errors from our model.

Consider the following data from Clinton-Gore experiment [3, 4, 9].

$$
\begin{aligned}
p(AyBy) &= 0.4899, && (137)\\
p(AyBn) &= 0.0447, && (138)\\
p(AnBy) &= 0.1767, && (139)\\
p(AnBn) &= 0.2887, && (140)\\
p(ByAy) &= 0.5625, && (141)\\
p(ByAn) &= 0.1991, && (142)\\
p(BnAy) &= 0.0255, && (143)\\
p(BnAn) &= 0.2129. && (144)
\end{aligned}
$$

The QQ equality is approximately satisfied with good accuracy.

$$
\begin{aligned}
q &= [p(ByAy) + p(BnAn)] - [p(AyBy) + p(AnBn)]\\
&= [p(AyBn) + p(AnBy)] - [p(ByAn) + p(BnAy)] = -0.0032. \qquad (145)
\end{aligned}
$$

Thus, their QQE-renormalization $\bar{p}(Aa, Bb), \bar{p}(Bb, Aa)$ are expected to approximate the original data $p(Aa, Bb), p(Bb, Aa)$ with good accuracy.

For the Clinton-Gore poll, we have

$$
\begin{aligned}
S_1 &= \frac{p(AyBy) + p(AnBn) + p(ByAy) + p(BnAn)}{2} = 0.7770, && (146)\\
S_2 &= \frac{p(AyBn) + p(AnBy) + p(ByAn) + p(BnAy)}{2} = 0.2230, && (147)
\end{aligned}
$$

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and we obtain their QQE-renormalizations as follows.

$$
\begin{aligned}
\bar{p}(AyBy) &= S_1 \times \frac{p(AyBy)}{p(AyBy)+p(AnBn)} = 0.4889,\\
\text{Error} &= \frac{\bar{p}(AyBy)}{p(AyBy)} - 1 - 0.0021.\ (-0.21\%),\\
\bar{p}(AnBn) &= S_1 \times \frac{p(AnBn)}{p(AyBy)+p(AnBn)} = 0.2881,\\
\text{Error} &= \frac{\bar{p}(AyBn)}{p(AyBn)} - 1 = -0.0021.\ (-0.21\%),\\
\bar{p}(ByAy) &= S_1 \times \frac{p(ByAy)}{p(ByAy)+p(BnAn)} = 0.5637,\\
\text{Error} &= \frac{\bar{p}(ByAy)}{p(ByAy)} - 1 = 0.0021.\ (+0.21\%),\\
\bar{p}(BnAn) &= S_1 \times \frac{p(BnAn)}{p(ByAy)+p(BnAn)} = 0.2133,\\
\text{Error} &= \frac{\bar{p}(BnAn)}{p(BnAn)} - 1 = 0.0020.\ (+0.20\%),\\
\bar{p}(AyBn) &= S_2 \times \frac{p(AyBn)}{p(AyBn)+p(AnBy)} = 0.0450,\\
\text{Error} &= \frac{\bar{p}(AyBn)}{p(AyBn)} - 1 = 0.0072.\ (+0.72\%),\\
\bar{p}(AnBy) &= S_2 \times \frac{p(AnBy)}{p(AyBn)+p(AnBy)} = 0.1780,\\
\text{Error} &= \frac{\bar{p}(AnBy)}{p(AnBy)} - 1 = 0.0072.\ (+0.72\%),\\
\bar{p}(ByAn) &= S_2 \times \frac{p(ByAn)}{p(ByAn)+p(BnAy)} = 0.1977,\\
\text{Error} &= \frac{\bar{p}(AnBy)}{p(AnBy)} - 1 = -0.0071.\ (-0.71\%),\\
\bar{p}(BnAy) &= S_2 \times \frac{p(BnAy)}{p(ByAn)+p(BnAy)} = 0.0253,\\
\text{Error} &= \frac{\bar{p}(BnAy)}{p(BnAy)} - 1 = -0.0075.(-0.75\%).
\end{aligned}
$$

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Under the assumption of the independence of the personality state, from Eq. (125) and the relation $q(0)+q(1)+q(2)=1$, we can determine $q(0), q(1), q(2)$ by the QQE-normalized data as follows.

$$q(2) = 0.3288, \tag{148}$$
$$q(1) = 0.0668, \tag{149}$$
$$q(0) = 0.6045. \tag{150}$$

We can determine $p(\alpha,\beta)$ for $\alpha,\beta = 0,1$ by Eqs. (117)–(124) as follows.

$$p(1,1) = 0.5184, \tag{151}$$
$$p(1,0) = 0.0155, \tag{152}$$
$$p(0,1) = 0.2429, \tag{153}$$
$$p(0,0) = 0.2231, \tag{154}$$

Note that the unity of the total probability is satisfied.

$$p(1,1)+p(1,0)+p(0,1)+p(0,0) = 1.0000.$$

Thus, the belief state $p(\alpha,\beta)$ and the personality state $q(\gamma)$ are determined from the experimental data. Then, our quantum model with the belief state $p(\alpha,\beta)$ and the personality state $q(\gamma)$ accurately reconstructs the QQE-renormalized data $\bar{p}(Aa, Bb)$, $\bar{p}(Bb, Aa)$ for $a, b = y, n$ as follows.

$$\begin{aligned}
\bar{p}(AyBy) &= p(1,1)q(0) + [p(1,1)+p(1,0)]q(2) = 0.4889,\\
\bar{p}(AyBn) &= p(1,0)q(0) + [p(1,1)+p(1,0)]q(1) = 0.0450,\\
\bar{p}(AnBy) &= p(0,1)q(0) + [p(0,1)+p(0,0)]q(1) = 0.1780,\\
\bar{p}(AnBn) &= p(0,0)q(0) + [p(0,1)+p(0,0)]q(2) = 0.2881,\\
\bar{p}(ByAy) &= p(1,1)q(0) + [p(0,1)+p(1,1)]q(2) = 0.5637,\\
\bar{p}(ByAn) &= p(0,1)q(0) + [p(0,1)+p(1,1)]q(1) = 0.1977,\\
\bar{p}(BnAy) &= p(1,0)q(0) + [p(0,0)+p(1,0)]q(1) = 0.0253,\\
\bar{p}(BnAn) &= p(0,0)q(0) + [p(0,0)+p(1,0)]q(2) = 0.2133.
\end{aligned}$$

Therefore, all data of the QQR-renormalizations are accurately reproduced, and we conclude that the our quantum model reproduces the statistics of the Clinton-Gore Poll data almost faithfully (within $\pm 0.75\%$ of errors) with a *prior belief state* $\{p(0,0),\dots,p(1,1)\}$ *independent of the question order*. Thus, this model successfully removes the order effect from the data to determine the genuine distribution of the opinion to the Poll.

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APPENDIX A. POSITIVE OPERATOR VALUED MEASURES

We restrict considerations to POVMs with a discrete domain of definition $X = \{x_1, ..., x_N\}$. POVM is a map $x \to \Pi^A(x)$, where A is a symbol for the outcome variable, but does not denote any self-adjoint operator nor observable to be measured. Here, for each $x \in X, \Pi^A(x)$ is a positive contractive self-adjoint operator (i.e., $0 \leq \Pi^A(x) \leq I$) (called an *effect*), and the normalization condition

$$\sum_x \Pi^A(x) = I \tag{155}$$

holds, where I is the unit operator. This map defines an operator valued measure on the algebra of all subsets of X, for $O \subset X$, $\Pi^A(O) = \sum_{x \in O} \Pi^A(x)$. The condition (155) characterizes "probability operator valued measures". From (2), we see that the map $x \to E^A(x)$ is a special sort of POVM, the projection valued measure - PVM.

POVMs Π^A represent statistics of measurements with the outcome variable A of quantum observables with the following generalization of the Born's rule:

$$\Pr\{A = x \| \rho\} = \mathrm{Tr}[\Pi^A(x)\rho].$$

We remark that equality (155) implies that $\sum_x \Pr\{A = x\|\rho\} = 1$.

POVM does not represent state update. The latter is typically determined (non-uniquely) via representation of effects in the form:

$$\Pi^A(x) = V(x)^* V(x), \tag{156}$$

where $V(x)$ is a linear operator in H; a canonical choice is $V(x) = \Pi^A(x)^{1/2}$ and every $V(x)$ is of the form $V(x) = U(x)\Pi^A(x)^{1/2}$ with unitary operators $U(x)$ for any x by polar decomposition. Hence, similarly to (13) the normalization condition has the form $\sum_x V(x)^* V(x) = I$. The Born rule can be written similarly to (14):

$$\Pr\{A = x\|\rho\} = \mathrm{Tr}[V(x)\rho V^*(x)] \tag{157}$$

It is assumed that the post-measurement state transformation is based on the map:

$$\rho \to \mathcal{I}_A(x)\rho = V(x)\rho V^*(x), \tag{158}$$

so

$$\rho \to \rho_{\{A=x\}} = \frac{\mathcal{I}_A(x)\rho}{\mathrm{Tr}[\mathcal{I}_A(x)\rho]}. \tag{159}$$

Now, we remark that the map $x \to \mathcal{I}_A(x)$ given by (158) is a (very special) quantum instrument. We would like to elevate the role of the use of quantum instruments, comparing with the use of just POVMs. An instrument provides both statistics of the

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measurement-outputs and the rule for the state update, but POVM should always be endowed with ad hoc condition (156).

Finally, we remark that any instrument $\mathcal{I}_A$, with the outcome variable A, generates POVM Π^A by the rule:

$$\Pi^A(x) = \mathcal{I}_A(x)^* I. \tag{160}$$

However, its state update need not have the form (158). The general form will be give as follows. Recall that any instrument $\mathcal{I}_A$ has a family $\{M_{xj}\}_{x,j}$ of measurement operators satisfying

$$\mathcal{I}_A(x)\rho = \sum_j M_{xj}\rho M_{xj}^*.$$

Then, from Eq. (160) the POVM Π^A generated by the instrument $\mathcal{I}_A$ is given by

$$\Pi^A(x) = \sum_j M_{xj}^* M_{xj}. \tag{161}$$

Thus, Eq. (156) is a special case of Eq. (161) where $V(x) = M_{xj}$ with $j \in \{1\}$.

Appendix B. Derivation of Eq. (5)

It can be shown that for commuting observables A and B there exist an observable C and polynomials f and g such that $A = f(C)$ and $B = g(C)$ [14]. Then, Eq. (5) follows from the relations

$$\begin{aligned}\Pr\{A = x, B = y\|\rho\} &= \Pr\{f(C) = x, g(C) = y\|\rho\} = \sum_{u:f(u)=x,g(u)=y} \Pr\{C = u\|\rho\} \\ &= \sum_{u:f(u)=x,g(u)=y} \mathrm{Tr}[E^C(u)\rho] = \mathrm{Tr}[\sum_{u:f(u)=x} E^C(u) \sum_{v:g(v)=y} E^C(v)\rho] \\ &= \mathrm{Tr}[E^{f(C)}(x)E^{g(C)}(y)\rho] = \mathrm{Tr}[E^A(x)E^B(y)\rho].\end{aligned}$$

The third last equation follows from $E^C(u)E^C(v) = 0$ if $u \neq v$.

Appendix C. Derivations of Eq. (48) and Eq. (49)

Suppose that the object state $|\psi\rangle$ just before the measurement is arbitrarily given, i.e., $|\psi\rangle = \sum_{\alpha,\beta,\gamma} c_{\alpha,\beta,\gamma}|\alpha,\beta,\gamma\rangle$. Then, by the Born formula the probability distribution of the observable A is defined as

$$\Pr\{A = 0\||\psi\rangle\} = \|A^\perp|\psi\rangle\|^2 = \sum_{\beta,\gamma} |c_{0\beta\gamma}|^2, \tag{162}$$

$$\Pr\{A = 1\||\psi\rangle\} = \|A|\psi\rangle\|^2 = \sum_{\beta,\gamma} |c_{1\beta\gamma}|^2. \tag{163}$$

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By linearity of U_A, it follows from Eqs. (31)–(34) that

$$U_A : |\psi\rangle|00\rangle \mapsto \sum_{\alpha,\beta}\Big(c_{\alpha,\beta,0}|\alpha,\beta,0\rangle + c_{\alpha,\beta,1}|\alpha,\alpha^{\perp},1\rangle + c_{\alpha,\beta,2}|\alpha,\alpha,2\rangle\Big)|\alpha,\beta\rangle. \tag{164}$$

Then, we have

$$\big(I_{\mathcal{H}}\otimes P^{M_A}(a)\big)\,U_A|\psi\rangle|00\rangle = \sum_{\beta}\Big(c_{a,\beta,0}|a,\beta,0\rangle + c_{a,\beta,1}|a,a^{\perp},1\rangle + c_{a,\beta,2}|a,a,2\rangle\Big)|a,\beta\rangle \tag{165}$$

for $a = 0,1$. Consequently,

$$(I_{\mathcal{H}}\otimes M_A^{\perp})U_A|\psi\rangle|00\rangle = \sum_{\beta}\Big(c_{0,\beta,0}|0,\beta,0\rangle + c_{0,\beta,1}|0,1,1\rangle + c_{0,\beta,2}|0,0,2\rangle\Big)|0,\beta\rangle,$$
$$(I_{\mathcal{H}}\otimes M_A)U_A|\psi\rangle|00\rangle = \sum_{\beta}\Big(c_{1,\beta,0}|1,\beta,0\rangle + c_{1,\beta,1}|1,0,1\rangle + c_{1,\beta,2}|1,1,2\rangle\Big)|1,\beta\rangle.$$

The probabilities of obtaining the outcomes $\mathbf{a} = 0$ and $\mathbf{a} = 1$ are given by

$$\Pr\{\mathbf{a}=0|\rho\} = \Pr\{M_A = 0|\,U_A|\psi\rangle|00\rangle\} = \|(I_{\mathcal{H}}\otimes M_A^{\perp})U_A|\psi\rangle|00\rangle\|^2 = \sum_{\beta,\gamma}|c_{0\beta\gamma}|^2,$$
$$\Pr\{\mathbf{a}=1|\rho\} = \Pr\{M_A = 1|\,U_A|\psi\rangle|00\rangle\} = \|(I_{\mathcal{H}}\otimes M_A)U_A|\psi\rangle|00\rangle\|^2 = \sum_{\beta,\gamma}|c_{1\beta\gamma}|^2.$$

This shows

$$\Pr\{\mathbf{a}=0|\rho\} = \Pr\{A=0|\,|\psi\rangle\},$$
$$\Pr\{\mathbf{a}=1|\rho\} = \Pr\{A=1|\,|\psi\rangle\}$$

for any state $|\psi\rangle$ in $\mathcal{H}$. It follows that the instrument $\mathcal{I}_A$ measures the observable A, i.e.,

$$\Pr\{\mathbf{a}=0|\rho\} = \mathrm{Tr}[\mathcal{I}_A(0)\rho] = \mathrm{Tr}[A^{\perp}\rho],$$
$$\Pr\{\mathbf{a}=1|\rho\} = \mathrm{Tr}[\mathcal{I}_A(1)\rho] = \mathrm{Tr}[A\rho]$$

for any density operator ρ on $\mathcal{H}$. Thus, we obtain Eq. (48).

From Eqs. (44) and (165) we have

$$\begin{aligned}
&\mathcal{I}_A(a)|\psi\rangle\langle\psi| \\
&= \sum_{\beta}|c_{a,\beta,0}|^2|a,\beta,0\rangle\langle a,\beta,0| + |c_{a,\beta,1}|^2|a,a^{\perp},1\rangle\langle a,a^{\perp},1| + |c_{a,\beta,2}|^2|a,a,2\rangle\langle a,a,2| \\
&= \sum_{\beta}|a,\beta,0\rangle\,\langle a,\beta,0|\psi\rangle\,\langle\psi|a,\beta,0\rangle\,\langle a,\beta,0| + |a,a^{\perp},1\rangle\,\langle a,\beta,1|\psi\rangle\,\langle\psi|a,\beta,1\rangle\,\langle a,a^{\perp},1| \\
&\quad + |a,a,2\rangle\,\langle a,\beta,2|\psi\rangle\,\langle\psi|a,\beta,2\rangle\,\langle a,a,2|.
\end{aligned}$$

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for $a = 0, 1$. By linearity of $\mathcal{I}_A(a)$ we conclude

$$\mathcal{I}_A(a)\rho = \sum_{\beta} |a,\beta,0\rangle \langle a,\beta,0|\rho|a,\beta,0\rangle \langle a,\beta,0| \tag{166}$$
$$+ |a,a^{\perp},1\rangle \langle a,\beta,1|\rho|a,\beta,1\rangle \langle a,^{\perp},1| + |a,a,2\rangle \langle a,\beta,2|\rho|a,\beta,2\rangle \langle a,a,2|.$$

for any density operator ρ on $\mathcal{H}$ and $a = 0, 1$. Thus, we obtain Eq. (49).

Appendix D. Derivation of Eq. (84)

From Eq. (49), we have

$$\begin{aligned}
\mathcal{I}_A(0)|\alpha,\beta,\gamma\rangle\langle\alpha',\beta',\gamma'| = &\sum_{\beta''} |0,\beta'',0\rangle\langle 0,\beta'',0||\alpha,\beta,\gamma\rangle\langle\alpha',\beta',\gamma'||0,\beta'',0\rangle\langle 0,\beta'',0| \\
&+ \mathrm{Tr}[(A^{\perp}\otimes|1\rangle\langle 1|)|\alpha,\beta,\gamma\rangle\langle\alpha',\beta',\gamma'|]\,|010\rangle\langle 010| \\
&+ \mathrm{Tr}[(A^{\perp}\otimes|2\rangle\langle 2|)|\alpha,\beta,\gamma\rangle\langle\alpha',\beta',\gamma'|]\,|002\rangle\langle 002| \\
=&|0,\beta,0\rangle\langle 0,\beta,0||\alpha,\beta,\gamma\rangle\langle\alpha',\beta',\gamma'||0,\beta,0\rangle\langle 0,\beta,0| \\
&+ \langle\alpha',\beta',\gamma'||0\rangle\langle 0|\otimes I_2\otimes|1\rangle\langle 1|)|\alpha,\beta,\gamma\rangle\,|010\rangle\langle 010| \\
&+ \langle\alpha',\beta',\gamma'||0\rangle\langle 0|\otimes I_2\otimes|2\rangle\langle 2|)|\alpha,\beta,\gamma\rangle\,|002\rangle\langle 002| \\
=&\delta_{\alpha}(0)\delta_{\alpha'}(0)\delta_{\beta}(\beta')\delta_{\gamma}(0)\delta_{\gamma'}(0)|0,\beta,0\rangle\langle 0,\beta,0| \\
&+ \delta_{\alpha}(0)\delta_{\alpha'}(0)\delta_{\beta}(\beta')\delta_{\gamma}(1)\delta_{\gamma'}(1)|010\rangle\langle 010| \\
&+ \delta_{\alpha}(0)\delta_{\alpha'}(0)\delta_{\beta}(\beta')\delta_{\gamma}(2)\delta_{\gamma'}(2)|002\rangle\langle 002|.
\end{aligned}$$

It follows that if $\langle\alpha,\beta,\gamma|\alpha',\beta',\gamma'\rangle = 0$ then

$$\mathcal{I}_A(0)|\alpha,\beta,\gamma\rangle\langle\alpha',\beta',\gamma'| = 0.$$

Similarly, if $\langle\alpha,\beta,\gamma|\alpha',\beta',\gamma'\rangle = 0$ then

$$\mathcal{I}_A(a)|\alpha,\beta,\gamma\rangle\langle\alpha',\beta',\gamma'| = 0,$$
$$\mathcal{I}_B(b)|\alpha,\beta,\gamma\rangle\langle\alpha',\beta',\gamma'| = 0$$

for any a, b. Therefore, for any density operator ρ we have

$$\mathcal{I}_A(a)\rho = \sum_{\alpha,\beta,\gamma} \mu(\alpha,\beta,\gamma)\mathcal{I}_A(a)\delta_{(\alpha,\beta,\gamma)} = \mathcal{I}_A(a)\rho',$$
$$\mathcal{I}_B(b)\rho = \sum_{\alpha,\beta,\gamma} \mu(\alpha,\beta,\gamma)\mathcal{I}_B(b)\delta_{(\alpha,\beta,\gamma)} = \mathcal{I}_B(b)\rho',$$

where $\mu(\alpha,\beta,\gamma) = \langle\alpha,\beta,\gamma|\rho|\alpha,\beta,\gamma\rangle$, and

$$\rho' = \sum_{\alpha,\beta,\gamma} \mu(\alpha,\beta,\gamma)\delta_{(\alpha,\beta,\gamma)}.$$

Thus, we obtain Eq. (84).

Appendix E. On Postulate 5

As we know, a density operator ρ can be decomposed into probabilistic mixtures of pure states in various ways. We show that Postulate 5 is invariant with respect to such decompositions.

Suppose we make experiment for 100 people. Then, we can suppose the subject $j = 1, ...100$ is in a pure state ψ_j and the ensemble is described by the mixed state $\rho_1 = (1/100)\sum_j |\psi_j\rangle\langle\psi_j|$. We make another experiment for 200 people and let and $\rho_2 = (1/200)\sum_k |\phi_k\rangle\langle\phi_k|$. The third experiment is done for 300 people. and $\rho_3 = (1/300)\sum_l |\xi_l\rangle\langle\xi_l|$. Then,

$$P(AaBb||\rho_1) = (1/100)\sum_j P(AaBb||\psi_j), P(BbAa||\rho_1) = (1/100)\sum_j P(BbAa||\psi_j),$$

$$P(AaBb||\rho_2) = (1/200)\sum_j P(AaBb||\phi_j), P(BbAa||\rho_2) = (1/200)\sum_j P(BbAa||\phi_j),$$

$$P(AaBb||\rho_3) = (1/300)\sum_j P(AaBb||\xi_j), P(BbAa||\rho_3) = (1/300)\sum_j P(BbAa||\xi_j).$$

It follows that if $p\rho_1 + p'\rho_2 = \rho_3$, where $p, p' > 0, p' = 1 - p$. Then $pP(AaBb||\rho_1) + p'P(AaBb||\rho_2) = P(AaBb||\rho_3)$. This is consistent with $P(AaBb||\rho) = \mathrm{Tr}[I_B(b)I_A(a)\rho]$. This also holds for any longer sequences and the joint probability distribution depends only on ρ, it is independent of its decomposition into pure states, orthogonal or not.

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