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Article

Warping effects in strongly perturbed metrics

Marco Frasca [1,2,‡], Riccardo Maria Liberati [1,3,‡] and Massimiliano Rossi [1,4,‡*]

[1] MBDA Italia S.p.A., Via Monte Flavio, 45, 00131, Rome (Italy).
[2] marco.frasca@mbda.it
[3] riccardo.liberati@mbda.it
[4] massimiliano.rossi@mbda.it
* Correspondence: massimiliano.rossi@mbda.it.
‡ These authors contributed equally to this work.

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Abstract: A technique devised some years ago permits to study a theory in a regime of strong perturbations. This translate into a gradient expansion that, at the leading order, can recover the BKL solution. We solve exactly the leading order equations in a spherical symmetric case and we show that the 4-velocity in such a case is multiplied by an exponential warp factor when the perturbation is properly applied. This factor is always greater than one. We will give a closed form solution of this factor for a simple case. Some numerical examples are also given.

Keywords: General relativity; Schwarzschild solution; Warp factor

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1. Introduction

The study of Einstein equations in certain regimes is often reduced to solve them numerically [1]. The reason is that they form a set of nonlinear PDEs that are generally difficult to handle with analytical tools for most interesting situations. Often, the reason relies on the fact that no small parameter can be found to apply standard perturbation techniques while analytical solutions are very rare and difficult to find. Some years ago, one of us (M.F.) proposed an approach based on earlier works in strongly perturbed systems [2]. It was shown that, under a strong perturbation in the formal limit running to infinity, the leading order is obtained by neglecting the gradient terms in the Einstein equations. The leading order of this perturbation series was firstly proposed by Belinsky, Kalathnikov, Lifshitz for their famous BKL conjecture [3–5], as is known today.

Some decades ago, Alcubierre proposed a solution of the Einstein equations [6] that describes an observer moving with an unbounded velocity provided the condition of positivity of the energy is violated. A recent paper [7] (see also Refs. therein) yields a short recount about Alcubierre metric and its interaction with dust. Indeed, any kind of pathology has emerged about it and the difficulties arise from the fact that this is an engineered metric that is imposed on the Einstein equations. It would be desirable to have a metric like this one emerging as a solution of the Einstein equations and conserving the positivity of the energy. A recent proposal goes in such a direction [8]. This is possible by introducing a hyperbolic shift vector potential and the author shows how this can emerge from a plasma.

In this paper we will show how a warp factor for the velocity can emerge when a strong perturbation is applied to a spherical symmetric metric. So, any Eulerian observer will get its velocity expanded when such perturbation is acting. This extends and complete our preceding work [2]. We will get the exact solution of the leading perturbation equations and we will show how an exponential factor can emerge that is systematically greater than one.

The paper is so structured. In Sec.2, we will introduce the technique to treat strongly perturbed systems. In Sec.3, we apply this to the Einstein equations for a spherical symmetry metric with a time-dependent perturbation. In Sec.4, we solve the leading order perturbation equations. In Sec.5, we yield the geodesic equations. In Sec.6, we show how the expansion factor enters into the velocity providing some examples and an analytic solution. In Sec.7, conclusions are presented.

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2. Strong perturbations

Let us consider the following non-linear equation as a toy model for the Einstein equations

$$-\Box\phi + \lambda V(\phi) = 0 \tag{1}$$

being $\Box = \nabla^2 - \partial_t^2$ the wave operator (here and in the following $c = 1$), ϕ a scalar field and $V(\phi)$ its self-interaction with a coupling λ. For 2D Einstein equations this would be a Liouville equation. We would like to do perturbation theory in the formal limit of $\lambda \to \infty$. This ends up to obtain a non-trivial series in $1/\lambda$. This can be accomplished by rescaling the time variable [9]. We take $t \to \sqrt{\lambda}t$ and the equation above becomes

$$-\nabla^2\phi + \lambda\partial_t^2\phi + \lambda V(\phi) = 0. \tag{2}$$

Then, we take

$$\phi = \phi_0 + \frac{1}{\lambda}\phi_1 + \frac{1}{\lambda^2}\phi_2 + \dots \tag{3}$$

and substitute this into eq.(2). This gives the set of perturbative equations

$$\begin{aligned}
\partial_t^2\phi_0 &= -V(\phi_0) \\
\partial_t^2\phi_1 &= -V'(\phi_0)\phi_1 + \nabla^2\phi_0 \\
\partial_t^2\phi_2 &= -V'(\phi_0)\phi_2 - \frac{1}{2}V''(\phi_0)\phi_1^2 + \nabla^2\phi_1 \\
&\vdots \quad .
\end{aligned} \tag{4}$$

We see that we have obtained a set of non-trivial equations that define the perturbation series in the formal limit $\lambda \to \infty$. This approach can be applied, exactly in this way, to the Einstein equations. This also shows how consistent was the original BKL approach in [3–5]. Indeed, we have obtained a gradient expansion.

3. Strongly perturbed spherical symmetry metric

We are assuming a spherical symmetry metric in the Arnowitt-Deser-Misner (ADM) formalism given by

$$ds^2 = -\alpha^2 dt^2 + \gamma_{rr}dr^2 + \gamma_{\theta\theta}d\theta^2 + \gamma_{\phi\phi}d\phi^2. \tag{5}$$

This implies a specific choice of the gauge where all the components of the shift vector, normally named β_i, are taken to be zero. Then, the perturbation α_1 is just applied to the lapse function as follows [1]

$$\alpha^2 = \alpha_0^2 + \alpha_1. \tag{6}$$

The other components are written as a perturbation series [2]

$$\gamma_{ij} = \gamma_{ij}^{(0)} + \gamma_{ij}^{(1)} + \dots. \tag{7}$$

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Then, we present them here for completeness, the following set of perturbation equations in a gradient expansion is given

$$
\begin{aligned}
\partial_\tau \gamma_{ij}^{(0)} &= -2\alpha_0 K_{ij}^{(0)} \\
\partial_\tau \gamma_{ij}^{(1)} &= -2\alpha_1 K_{ij}^{(0)} - 2\alpha_0 K_{ij}^{(1)} \\
&\vdots \\
\partial_\tau K_{ij}^{(0)} &= -2\alpha_0 K_{il}^{(0)} K_j^{l(0)} + \alpha_0 K^{(0)} K_{ij}^{(0)} \\
\partial_\tau K_{ij}^{(1)} &= -2\alpha_1 K_{il}^{(0)} K_j^{l(0)} - 2\alpha_0 K_{il}^{(1)} K_j^{l(0)} - 2\alpha_0 K_{il}^{(0)} K_j^{l(1)} \\
&+\alpha_1 K^{(0)} K_{ij}^{(0)} + \alpha_0 K^{(1)} K_{ij}^{(0)} + \alpha_0 K^{(0)} K_{ij}^{(1)} \\
&-\frac{1}{2}\alpha_0 \gamma^{lm(0)} \left\{ \partial_l \partial_m \gamma_{ij}^{(0)} + \partial_i \partial_j \gamma_{lm}^{(0)} - \partial_i \partial_l \gamma_{mj}^{(0)} - \partial_j \partial_l \gamma_{mi}^{(0)} \right. \\
&+\gamma^{np(0)} \left[(\partial_i \gamma_{jn}^{(0)} + \partial_j \gamma_{in}^{(0)} - \partial_n \gamma_{ij}^{(0)}) \partial_l \gamma_{mp}^{(0)} \right. \\
&\left. +\partial_l \gamma_{in}^{(0)} \partial_p \gamma_{jm}^{(0)} - \partial_l \gamma_{in}^{(0)} \partial_m \gamma_{jp}^{(0)} \right] \\
&\left. -\frac{1}{2}\gamma^{np(0)} \left[(\partial_i \gamma_{jn}^{(0)} + \partial_j \gamma_{in}^{(0)} - \partial_n \gamma_{ij}^{(0)}) \partial_p \gamma_{lm}^{(0)} + \partial_i \gamma_{ln}^{(0)} \partial_j \gamma_{mp}^{(0)} \right] \right\} \\
&-\partial_i \partial_j \alpha_0 + \frac{1}{2}\gamma^{lm(0)} (\partial_i \gamma_{jm}^{(0)} + \partial_j \gamma_{im}^{(0)} - \partial_m \gamma_{ij}^{(0)}) \partial_l \alpha_0 \\
&\vdots
\end{aligned}
\tag{8}
$$

with ($r_g = 2GM$ is the Schwarzschild radius). For the exterior solution one has

$$
\begin{aligned}
\alpha_0^2 &= \left(1 - \frac{r_g}{r}\right) \\
\gamma_{rr}^{(0)} &= \frac{1}{1 - \frac{r_g}{r}} \\
\gamma_{\theta\theta}^{(0)} = r^2 &\qquad \gamma_{\phi\phi}^{(0)} = r^2 \sin^2\theta
\end{aligned}
\tag{9}
$$

and for the interior solution

$$
\begin{aligned}
\alpha_0^2 &= \frac{1}{4}\left(3\sqrt{1 - \frac{r_g}{r_s}} - \sqrt{1 - \frac{r^2 r_g}{r_s^3}}\right)^2 \\
\gamma_{rr}^{(0)} &= \left(1 - \frac{r^2 r_g}{r_s^3}\right)^{-1} \\
\gamma_{\theta\theta}^{(0)} = r^2 &\qquad \gamma_{\phi\phi}^{(0)} = r^2 \sin^2\theta.
\end{aligned}
\tag{10}
$$

being r_s is the value of the r-coordinate at the body's surface. We also have, with our gauge's choice $\beta_i = 0$, the general formula

$$K_{ij} = -\frac{1}{2\alpha}\partial_\tau \gamma_{ij}. \tag{11}$$

In our case is

$$\alpha^2 = \alpha_0^2 + \alpha_1 = \alpha_0^2 + Af(r,t), \tag{12}$$

being A the amplitude of the perturbation. This yields

$$\partial_\tau \gamma_{ij}^{(1)} = \alpha_1 \frac{1}{\alpha_0}\partial_\tau \gamma_{ij}^{(0)} - 2\alpha_0 K_{ij}^{(1)}, \tag{13}$$

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that reduces to

$$\partial_\tau \gamma_{ij}^{(1)} = -2\alpha_0 K_{ij}^{(1)}, \tag{14}$$

as $\gamma_{ij}^{(0)}$ does not depend on time variable. Now, one has

$$\begin{aligned}
\partial_\tau K_{ij}^{(1)} &= -2\alpha_1 K_{il}^{(0)} K_j^{l(0)} - 2\alpha_0 K_{il}^{(1)} K_j^{l(0)} - 2\alpha_0 K_{il}^{(0)} K_j^{l(1)} \\
&+\alpha_1 K^{(0)} K_{ij}^{(0)} + \alpha_0 K^{(1)} K_{ij}^{(0)} + \alpha_0 K^{(0)} K_{ij}^{(1)} \\
&+\alpha_1 K^{(0)} K_{ij}^{(0)} + \alpha_0 K^{(1)} K_{ij}^{(0)} + \alpha_0 K^{(0)} K_{ij}^{(1)} \\
&-\frac{1}{2}\alpha_0 \gamma^{lm(0)} \left\{ \partial_l \partial_m \gamma_{ij}^{(0)} + \partial_i \partial_j \gamma_{lm}^{(0)} - \partial_i \partial_l \gamma_{mj}^{(0)} - \partial_j \partial_l \gamma_{mi}^{(0)} \right. \\
&+\gamma^{np(0)} \left[(\partial_i \gamma_{jn}^{(0)} + \partial_j \gamma_{in}^{(0)} - \partial_n \gamma_{ij}^{(0)}) \partial_l \gamma_{mp}^{(0)} \right. \\
&\left. +\partial_l \gamma_{in}^{(0)} \partial_p \gamma_{jm}^{(0)} - \partial_l \gamma_{in}^{(0)} \partial_m \gamma_{jp}^{(0)} \right] \\
&\left. -\frac{1}{2}\gamma^{np(0)} \left[(\partial_i \gamma_{jn}^{(0)} + \partial_j \gamma_{in}^{(0)} - \partial_n \gamma_{ij}^{(0)}) \partial_p \gamma_{lm}^{(0)} + \partial_i \gamma_{ln}^{(0)} \partial_j \gamma_{mp}^{(0)} \right] \right\} \\
&-\partial_i \partial_j \alpha_0 + \frac{1}{2}\gamma^{lm(0)} (\partial_i \gamma_{jm}^{(0)} + \partial_j \gamma_{im}^{(0)} - \partial_m \gamma_{ij}^{(0)}) \partial_l \alpha_0 \\
&\vdots \quad .
\end{aligned} \tag{15}$$

This set of equations, written in this way, are too difficult to manage. As we will see below, we can restate them to find an exact leading order solution.

4. Solving perturbation equations

Let us consider the following rewriting of the ADM equations of motion in exact form. We will get (as already said, our gauge is $\beta_i = 0$)

$$\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} \\
\partial_t K_{ij} &= \alpha \left[R_{ij} - 2K_{il} K_j^l + K K_{ij} \right] - \partial_i \partial_j \alpha.
\end{aligned} \tag{16}$$

The Ricci tensor R_{ij} refers to the γ_{ij} and all Latin indexes run from 1 to 3. We can exploit these equations for the diagonal elements to obtain

$$\begin{aligned}
\partial_t \gamma_{11} &= -2\alpha K_{11}, \\
\partial_t \gamma_{22} &= -2\alpha K_{22}, \\
\partial_t \gamma_{33} &= -2\alpha K_{33}, \\
\partial_t K_{11} &= \alpha \left[R_{11} - 2K_{1l} K_1^l + K K_{11} \right] - \partial_1^2 \alpha, \\
\partial_t K_{22} &= \alpha \left[R_{22} - 2K_{2l} K_2^l + K K_{22} \right] - \partial_2^2 \alpha, \\
\partial_t K_{33} &= \alpha \left[R_{33} - 2K_{3l} K_3^l + K K_{33} \right] - \partial_3^2 \alpha.
\end{aligned} \tag{17}$$

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We notice that $K_1^l = \gamma^{kl}K_{ik}$ and $K = \gamma^{kl}K_{kl}$. **We are assuming, without proof, that off-diagonal elements are zero due to the spherical symmetry**. Then,

$$
\begin{aligned}
\partial_t\gamma_{11} &= -2\alpha K_{11}, \\
\partial_t\gamma_{22} &= -2\alpha K_{22}, \\
\partial_t\gamma_{33} &= -2\alpha K_{33}, \\
\partial_t K_{11} &= \alpha\left[R_{11} - \gamma^{11}K_{11}^2 + (\gamma^{22}K_{22} + \gamma^{33}K_{33})K_{11}\right] - \partial_1^2\alpha, \\
\partial_t K_{22} &= \alpha\left[R_{22} - \gamma^{22}K_{22}^2 + (\gamma^{11}K_{11} + \gamma^{33}K_{33})K_{22}\right] - \partial_2^2\alpha, \\
\partial_t K_{33} &= \alpha\left[R_{33} - \gamma^{33}K_{33}^2 + (\gamma^{11}K_{11} + \gamma^{22}K_{22})K_{33}\right] - \partial_3^2\alpha.
\end{aligned} \tag{18}
$$

These equation can be stated in a single set of equations for the γs as

$$
\begin{aligned}
\partial_t^2\gamma_{11} &= -2\dot{\alpha}K_{11} - 2\alpha\dot{K}_{11} = \\
&\frac{\dot{\alpha}}{\alpha}\dot{\gamma}_{11} - 2\alpha^2\left[R_{11} - \gamma^{11}\frac{1}{4\alpha^2}(\dot{\gamma}_{11})^2 + \frac{1}{4\alpha^2}\gamma^{22}\dot{\gamma}_{22}\dot{\gamma}_{11} + \frac{1}{4\alpha^2}\gamma^{33}\dot{\gamma}_{33}\dot{\gamma}_{11}\right] + 2\alpha\partial_1^2\alpha \\
\partial_t^2\gamma_{22} &= -2\dot{\alpha}K_{22} - 2\alpha\dot{K}_{22} = \\
&\frac{\dot{\alpha}}{\alpha}\dot{\gamma}_{22} - 2\alpha^2\left[R_{22} - \gamma^{22}\frac{1}{4\alpha^2}(\dot{\gamma}_{22})^2 + \frac{1}{4\alpha^2}\gamma^{11}\dot{\gamma}_{11}\dot{\gamma}_{22} + \frac{1}{4\alpha^2}\gamma^{33}\dot{\gamma}_{33}\dot{\gamma}_{22}\right] + 2\alpha\partial_2^2\alpha \\
\partial_t^2\gamma_{33} &= -2\dot{\alpha}K_{33} - 2\alpha\dot{K}_{33} = \\
&\frac{\dot{\alpha}}{\alpha}\dot{\gamma}_{33} - 2\alpha^2\left[R_{33} - \gamma^{33}\frac{1}{4\alpha^2}(\dot{\gamma}_{33})^2 + \frac{1}{4\alpha^2}\gamma^{11}\dot{\gamma}_{11}\dot{\gamma}_{33} + \frac{1}{4\alpha^2}\gamma^{22}\dot{\gamma}_{22}\dot{\gamma}_{33}\right] + 2\alpha\partial_3^2\alpha.
\end{aligned} \tag{19}
$$

and so on for the other components. As said into Sec.3, this set of equations can be solved perturbatively by the change of variable $\tau = \sqrt{\lambda}t$ being λ just an ordering parameter that we will set to 1 to the end of computations. This means that we can neglect spatial gradients at the leading order, yielding

$$
\begin{aligned}
\partial_\tau^2\gamma_{11} &= \frac{\dot{\alpha}}{\alpha}\dot{\gamma}_{11} + \frac{1}{2}\gamma^{11}(\dot{\gamma}_{11})^2 - \frac{1}{2}\gamma^{22}\dot{\gamma}_{22}\dot{\gamma}_{11} - \frac{1}{2}\gamma^{33}\dot{\gamma}_{33}\dot{\gamma}_{11} = \\
&\frac{\dot{\alpha}}{\alpha}\dot{\gamma}_{11} + \frac{1}{2}(\gamma_{11})^{-1}(\dot{\gamma}_{11})^2 - \frac{1}{2}(\gamma_{22})^{-1}\dot{\gamma}_{22}\dot{\gamma}_{11} - \frac{1}{2}(\gamma_{33})^{-1}\dot{\gamma}_{33}\dot{\gamma}_{11}
\end{aligned} \tag{20}
$$

This can be rewritten as

$$
\partial_\tau^2\gamma_{11} = \dot{\gamma}_{11}\frac{d}{d\tau}\left[\ln\alpha + \frac{1}{2}\ln\left(\frac{\gamma_{11}}{\gamma_{22}\gamma_{33}}\right)\right] \tag{21}
$$

Then,

$$
\partial_\tau\ln\dot{\gamma}_{11} = \frac{d}{d\tau}\left[\ln\alpha + \frac{1}{2}\ln\left(\frac{\gamma_{11}}{\gamma_{22}\gamma_{33}}\right)\right] \tag{22}
$$

and finally

$$
\ln\dot{\gamma}_{11} = \left[\ln\left(r_k\frac{\alpha}{\alpha_0}\right) + \frac{1}{2}\ln\left(\frac{\gamma_{11}}{\gamma_{22}\gamma_{33}}\right)\right] \tag{23}
$$

where we have properly fixed the integration constant in such a way that, in absence of perturbation, the contribution from α disappears while dimensions are kept with the constant $r_k = r_g$ for the exterior solution and $r_k = r_s$ for the interior solution. This gives the following set of differential equations

$$
\begin{aligned}
\dot{\gamma}_{11} &= r_k\frac{\alpha}{\alpha_0}\sqrt{\frac{\gamma_{11}}{\gamma_{22}\gamma_{33}}} = r_k\frac{\alpha}{\alpha_0}\gamma_{11}\gamma^{-\frac{1}{2}} \\
\dot{\gamma}_{22} &= r_k\frac{\alpha}{\alpha_0}\sqrt{\frac{\gamma_{22}}{\gamma_{11}\gamma_{33}}} = r_k\frac{\alpha}{\alpha_0}\gamma_{22}\gamma^{-\frac{1}{2}} \\
\dot{\gamma}_{33} &= r_k\frac{\alpha}{\alpha_0}\sqrt{\frac{\gamma_{33}}{\gamma_{11}\gamma_{22}}} = r_k\frac{\alpha}{\alpha_0}\gamma_{33}\gamma^{-\frac{1}{2}}
\end{aligned} \tag{24}
$$

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This set can be solved exactly by multiplying in the following way

$$\begin{aligned}\dot{\gamma}_{11}\gamma_{22}\gamma_{33} &= r_k\frac{\alpha}{\alpha_0}\gamma^{\frac{1}{2}}\\ \dot{\gamma}_{22}\gamma_{11}\gamma_{33} &= r_k\frac{\alpha}{\alpha_0}\gamma^{\frac{1}{2}}\\ \dot{\gamma}_{33}\gamma_{11}\gamma_{22} &= r_k\frac{\alpha}{\alpha_0}\gamma^{\frac{1}{2}}\end{aligned}\tag{25}$$

and summing up the three equations obtained in this way giving

$$\dot{\gamma} = 3r_k\frac{\alpha}{\alpha_0}\gamma^{\frac{1}{2}}\tag{26}$$

that has as a solution

$$\gamma(t) = \left[\frac{3}{2}r_k\alpha_0^{-1}\int_0^t \alpha(t')dt' + \sqrt{\gamma(0)}\right]^2\tag{27}$$

and, e.g. one has

$$\gamma(0) = |\gamma_{11}^{(0)}\gamma_{22}^{(0)}\gamma_{33}^{(0)}| = \frac{r^4\sin^2\theta}{1-\frac{r_g}{r}}.\tag{28}$$

for the exterior solution. This yields the set of equations

$$\begin{aligned}\dot{\gamma}_{11} &= \frac{r_k\alpha_0^{-1}\alpha}{\frac{3}{2}r_k\alpha_0^{-1}\int_0^t \alpha(t')dt' + \sqrt{\gamma(0)}}\gamma_{11}\\ \dot{\gamma}_{22} &= \frac{r_k\alpha_0^{-1}\alpha}{\frac{3}{2}r_k\alpha_0^{-1}\int_0^t \alpha(t')dt' + \sqrt{\gamma(0)}}\gamma_{22}\\ \dot{\gamma}_{33} &= \frac{r_k\alpha_0^{-1}\alpha}{\frac{3}{2}r_k\alpha_0^{-1}\int_0^t \alpha(t')dt' + \sqrt{\gamma(0)}}\gamma_{33}.\end{aligned}\tag{29}$$

These can be solved exactly by

$$\begin{aligned}\gamma_{11}(t) &= \exp\left[r_k\alpha_0^{-1}\int_0^t dt''\frac{\alpha(t'')}{\frac{3}{2}r_k\alpha_0^{-1}\int_0^{t''}\alpha(t')dt' + \sqrt{\gamma(0)}}\right]\gamma_{11}^{(0)}\\ \gamma_{22}(t) &= \exp\left[r_k\alpha_0^{-1}\int_0^t dt''\frac{\alpha(t'')}{\frac{3}{2}r_k\alpha_0^{-1}\int_0^{t''}\alpha(t')dt' + \sqrt{\gamma(0)}}\right]\gamma_{22}^{(0)}\\ \gamma_{33}(t) &= \exp\left[r_k\alpha_0^{-1}\int_0^t dt''\frac{\alpha(t'')}{\frac{3}{2}r_k\alpha_0^{-1}\int_0^{t''}\alpha(t')dt' + \sqrt{\gamma(0)}}\right]\gamma_{33}^{(0)}.\end{aligned}\tag{30}$$

We can derive the volume expansion from the equation [6]

$$\Theta = -\alpha\mathrm{Tr}K = -\alpha\gamma^{ij}K_{ij} = \frac{1}{2}\gamma^{ij}\dot{\gamma}_{ij}\tag{31}$$

and K_{ij} are given by eq.(11). Then,

$$\Theta = \frac{3}{2}\frac{r_k\alpha_0^{-1}\alpha}{\frac{3}{2}r_k\alpha_0^{-1}\int_0^t \alpha(t')dt' + \sqrt{\gamma(0)}}\exp\left[r_k\alpha_0^{-1}\int_0^t dt''\frac{\alpha(t'')}{\frac{3}{2}r_k\alpha_0^{-1}\int_0^{t''}\alpha(t')dt' + \sqrt{\gamma(0)}}\right].\tag{32}$$

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Here we can see the first appearance of the expansion (warp) factor given by

$$U(r,\theta,t) = \exp\left[r_k\alpha_0^{-1}\int_0^t dt''\frac{\alpha(t'')}{\frac{3}{2}r_k\alpha_0^{-1}\int_0^{t''}\alpha(t')dt' + \sqrt{\gamma(0)}}\right]. \tag{33}$$

As we will see, this is always greater than one..

5. Geodesic equations

We give here the geodesic equations in such a perturbed metric. For this aim, We need to consider the metric

$$ds^2 = -\alpha^2(r,t)dt^2 + \gamma_{11}(r,\theta,t)dr^2 + \gamma_{22}(r,\theta,t)d\theta^2 + \gamma_{33}(r,\theta,t)d\phi^2. \tag{34}$$

From this it is easy to derive the Lagrangian

$$L = -\alpha^2(r,t)\dot{t}^2 + \gamma_{11}(r,\theta,t)(\dot{r})^2 + \gamma_{22}(r,\theta,t)\dot{\theta}^2 + \gamma_{33}(r,\theta,t)\dot{\phi}^2 \tag{35}$$

where the dot means derivative with respect to the proper time τ. Then, using the Euler-Lagrange equations one has

$$\begin{aligned}
\frac{d}{d\tau}\frac{\partial t}{\partial \dot{t}} - \frac{\partial L}{\partial t} &= \frac{d}{d\tau}[-2\alpha^2(r,t)\dot{t}] + \frac{\partial\alpha^2}{\partial t}\dot{t}^2 - \frac{\partial\gamma_{11}(r,\theta,t)}{\partial t}\dot{r}^2 - \frac{\partial\gamma_{22}(r,\theta,t)}{\partial t}\dot{\theta}^2 - \frac{\partial\gamma_{33}(r,\theta,t)}{\partial t}\dot{\phi}^2 \\
\frac{d}{d\tau}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} &= \frac{d}{d\tau}[2\gamma_{11}(r,\theta,t)\dot{r}] + \frac{\partial\alpha^2}{\partial r}\dot{t}^2 - \frac{\partial\gamma_{11}(r,\theta,t)}{\partial r}\dot{r}^2 - \frac{\partial\gamma_{22}(r,\theta,t)}{\partial r}\dot{\theta}^2 - \frac{\partial\gamma_{33}(r,\theta,t)}{\partial r}\dot{\phi}^2 \\
\frac{d}{d\tau}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} &= \frac{d}{d\tau}[2\gamma_{22}(r,\theta,t)\dot{\theta}] - \frac{\partial\gamma_{11}(r,\theta,t)}{\partial \theta}\dot{r}^2 - \frac{\partial\gamma_{22}(r,\theta,t)}{\partial \theta}\dot{\theta}^2 - \frac{\partial\gamma_{33}(r,\theta,t)}{\partial \theta}\dot{\phi}^2 \\
\frac{d}{d\tau}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} &= \frac{d}{d\tau}[2\gamma_{33}(r,\theta,t)\dot{\phi}]
\end{aligned} \tag{36}$$

Then, finally

$$\begin{aligned}
\frac{d}{d\tau}[\alpha^2(r,t)\dot{t}] - \frac{1}{2}\frac{\partial\alpha^2}{\partial t}\dot{t}^2 + \frac{1}{2}\frac{\partial\gamma_{11}(r,\theta,t)}{\partial t}\dot{r}^2 + \frac{1}{2}\frac{\partial\gamma_{22}(r,\theta,t)}{\partial t}\dot{\theta}^2 + \frac{1}{2}\frac{\partial\gamma_{33}(r,\theta,t)}{\partial t}\dot{\phi}^2 &= 0 \\
\frac{d}{d\tau}[\gamma_{11}(r,\theta,t)\dot{r}] + \frac{1}{2}\frac{\partial\alpha^2}{\partial r}\dot{t}^2 - \frac{1}{2}\frac{\partial\gamma_{11}(r,\theta,t)}{\partial r}\dot{r}^2 - \frac{1}{2}\frac{\partial\gamma_{22}(r,\theta,t)}{\partial r}\dot{\theta}^2 - \frac{1}{2}\frac{\partial\gamma_{33}(r,\theta,t)}{\partial r}\dot{\phi}^2 &= 0 \\
\frac{d}{d\tau}[\gamma_{22}(r,\theta,t)\dot{\theta}] - \frac{1}{2}\frac{\partial\gamma_{11}(r,\theta,t)}{\partial \theta}\dot{r}^2 - \frac{1}{2}\frac{\partial\gamma_{22}(r,\theta,t)}{\partial \theta}\dot{\theta}^2 - \frac{1}{2}\frac{\partial\gamma_{33}(r,\theta,t)}{\partial \theta}\dot{\phi}^2 &= 0 \\
\frac{d}{d\tau}[\gamma_{33}(r,\theta,t)\dot{\phi}] &= 0
\end{aligned} \tag{37}$$

The case $\theta = \pi/2$ will yield

$$\begin{aligned}
\frac{d}{d\tau}[\alpha^2(r,t)\dot{t}] - \frac{1}{2}\frac{\partial\alpha^2}{\partial t}\dot{t}^2 + \frac{1}{2}\frac{\partial\gamma_{11}(r,t)}{\partial t}\dot{r}^2 + \frac{1}{2}\frac{\partial\gamma_{33}(r,t)}{\partial t}\dot{\phi}^2 &= 0 \\
\frac{d}{d\tau}[\gamma_{11}(r,t)\dot{r}] + \frac{1}{2}\frac{\partial\alpha^2}{\partial r}\dot{t}^2 - \frac{1}{2}\frac{\partial\gamma_{11}(r,t)}{\partial r}\dot{r}^2 - \frac{1}{2}\frac{\partial\gamma_{33}(r,t)}{\partial r}\dot{\phi}^2 &= 0 \\
\frac{d}{d\tau}[\gamma_{33}(r,t)\dot{\phi}] &= 0
\end{aligned} \tag{38}$$

The last equation of the set can be integrated out to give

$$\dot{\phi} = \frac{L}{\gamma_{33}(r,t)} \tag{39}$$

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that can be substituted in the other two to give

$$\begin{aligned} \frac{d}{d\tau}[\alpha^2(r,t)\dot{t}] - \frac{1}{2}\frac{\partial\alpha^2}{\partial t}\dot{t}^2 + \frac{1}{2}\frac{\partial\gamma_{11}(r,t)}{\partial t}\dot{r}^2 + \frac{1}{2}\frac{\partial\gamma_{33}(r,t)}{\partial t}\frac{L^2}{\gamma_{33}^2(r,t)} &= 0 \\ \frac{d}{d\tau}[\gamma_{11}(r,t)\dot{r}] + \frac{1}{2}\frac{\partial\alpha^2}{\partial r}\dot{t}^2 - \frac{1}{2}\frac{\partial\gamma_{11}(r,t)}{\partial r}\dot{r}^2 - \frac{1}{2}\frac{\partial\gamma_{33}(r,t)}{\partial r}\frac{L^2}{\gamma_{33}^2(r,t)} &= 0 \end{aligned} \tag{40}$$

Now, we know from eq.(30) that

$$\begin{aligned} \gamma_{11}(r,t) &= U(r,t)\gamma_{11}^{(0)} \\ \gamma_{33}(r,t) &= U(r,t)\gamma_{33}^{(0)} \end{aligned} \tag{41}$$

and then

$$\begin{aligned} \frac{d}{d\tau}[\alpha^2(r,t)\dot{t}] - \frac{1}{2}\frac{\partial\alpha^2}{\partial t}\dot{t}^2 + \frac{1}{2}\frac{\partial U(r,t)}{\partial t}\left[\dot{r}^2\gamma_{11}^{(0)} + \frac{L^2}{U^2(r,t)\gamma_{33}^{(0)}}\right] &= 0 \\ \frac{d}{d\tau}[\gamma_{11}(r,t)\dot{r}] + \frac{1}{2}\frac{\partial\alpha^2}{\partial r}\dot{t}^2 - \frac{1}{2}\frac{\partial U(r,t)}{\partial r}\left[\dot{r}^2\gamma_{11}^{(0)} + \frac{L^2}{U^2(r,t)\gamma_{33}^{(0)}}\right] &= 0. \end{aligned} \tag{42}$$

This set can be solved only numerically. So, we take another approach to evaluate the radial velocity.

6. Radial Velocity

The definition of momenta is given by

$$p_\alpha = g_{\alpha\beta}p^\beta. \tag{43}$$

This yields the dispersion relation

$$p_\alpha p^\alpha = -m^2. \tag{44}$$

Similarly, we can derive the 4-velocity from this and is given by

$$u_\alpha = (-\alpha^2\dot{t}, \gamma_{11}\dot{r}, \gamma_{22}\dot{\theta}, \gamma_{33}\dot{\phi}). \tag{45}$$

Then, the radial motion will be characterized by

$$v_r = \frac{\gamma_{11}}{\alpha}\frac{dr}{dt} = \frac{U(r,\theta,t)}{\alpha(r,\theta,t)}\gamma_{11}^{(0)}\frac{dr}{dt}. \tag{46}$$

One gets a *warp factor*, arising from the applied perturbation,

$$U(r,t) = \exp\left[r_k\alpha_0^{-1}\int_0^t dt''\frac{\alpha(t'')}{\frac{3}{2}r_k\alpha_0^{-1}\int_0^{t''}\alpha(t')dt' + \sqrt{\gamma(0)}}\right] \tag{47}$$

and we realize that, with this geometry, we can have an exponential growth of the radial velocity depending on the applied perturbation.

We can provide a closed form solution for a very simple case, a toy model. We take for a perturbation

$$\alpha_1(t) = \frac{t}{\theta} \tag{48}$$

that is, a linear time increasing term. Then,

$$\alpha^2(r,t) = \alpha_0^2(r) + \frac{t}{\theta}. \tag{49}$$

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Then,

$$U(r,t)=\exp\left[r_k\alpha_0^{-1}\int_0^t dt''\frac{\sqrt{\alpha_0^2(r)+\frac{t''}{\theta}}}{\frac{3}{2}r_k\alpha_0^{-1}\int_0^{t''}\sqrt{\alpha_0^2(r)+\frac{t'}{\theta}}dt'+\sqrt{\gamma(0)}}\right]. \tag{50}$$

This yields

$$U(r,t)=\exp\left[r_k\alpha_0^{-1}\int_0^t dt''\frac{\sqrt{\alpha_0^2(r)+\frac{t''}{\theta}}}{\frac{3}{2}r_k\alpha_0^{-1}\theta\left[\frac{2}{3}\left(\alpha_0^2(r)+\frac{t''}{\theta}\right)^{\frac{3}{2}}-\frac{2}{3}\alpha_0(r)\right]+\sqrt{\gamma(0)}}\right]. \tag{51}$$

and

$$U(r,t)=\exp\left[r_k\alpha_0^{-1}\frac{3}{2}\theta\int_{\frac{2}{3}\alpha_0(r)}^{\frac{2}{3}\left(\alpha_0^2(r)+\frac{t}{\theta}\right)^{\frac{3}{2}}}dx\frac{1}{\frac{3}{2}r_k\alpha_0^{-1}\theta\left[x-\frac{2}{3}\alpha_0(r)\right]+\sqrt{\gamma(0)}}\right]. \tag{52}$$

Final result is

$$U(r,t)=\frac{a\left(\alpha_0^2(r)+\frac{t}{\theta}\right)^{\frac{3}{2}}-b}{a\alpha_0^3(r)-b}, \tag{53}$$

being $a=r_k\theta\alpha_0^{-1}(r)$ and $b=r_k\theta-\sqrt{\gamma(0)}$. It is to see that this factor is always greater than one (this value is taken for $t=0$) and increasing as time increases.

From the formula for radial velocity we can derive the force. This will be obtained by the first derivative of eq.(46). This yields

$$\frac{dv_r}{d\tau}=\frac{d}{d\tau}\left[\gamma_{11}\frac{dr}{d\tau}\right]. \tag{54}$$

This gives, for a mass M,

$$F=M\frac{dv_r}{d\tau}=M\frac{dt}{d\tau}\frac{d\gamma_{11}}{dt}\frac{dr}{d\tau}+\gamma_{11}\frac{d^2r}{d\tau^2}. \tag{55}$$

This gives,

$$F=M\frac{dv_r}{d\tau}=M\frac{1}{\alpha}\frac{d\gamma_{11}}{dt}\frac{dr}{d\tau}+\gamma_{11}\frac{d^2r}{d\tau^2}. \tag{56}$$

In our toy model, we consider $\alpha_0\approx 1$ and $\gamma_{11}^{(0)}\approx 1$, so that

$$\frac{d\gamma_{11}}{dt}=\frac{\frac{a}{\theta}\left(1+\frac{t}{\theta}\right)^{\frac{1}{2}}-\frac{db}{dt}}{a-b}+\frac{a\left(1+\frac{t}{\theta}\right)^{\frac{3}{2}}-b}{(a-b)^2}\frac{db}{dt}. \tag{57}$$

This yields,

$$\frac{d\gamma_{11}}{dt}\approx\frac{r_k\left(1+\frac{t}{\theta}\right)^{\frac{1}{2}}+2r\frac{dr}{dt}}{r^2}+\frac{r_k\theta\left(1+\frac{t}{\theta}\right)^{\frac{3}{2}}-r_k\theta+r^2}{r^4}2r\frac{dr}{dt}. \tag{58}$$

Then, we get

$$\begin{aligned}F\;&\approx\;M\left(1+\frac{t}{\theta}\right)^{-\frac{1}{2}}\left[\frac{r_k}{r^2}\left(1+\frac{t}{\theta}\right)^{\frac{1}{2}}+\frac{2}{r}\frac{dr}{dt}+\frac{2r_k\theta\left(1+\frac{t}{\theta}\right)^{\frac{3}{2}}-2r_k\theta+2r^2}{r^3}\frac{dr}{dt}\right]\frac{dr}{dt}\\&+M\frac{r_k\theta\left(1+\frac{t}{\theta}\right)^{\frac{3}{2}}-r_k\theta+r^2}{r^2}\frac{d^2r}{d\tau^2}.\end{aligned} \tag{59}$$

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with the simple kinematic law of motion $r(t) = r_0 + v_0 t$ it is easy to get

$$F \approx Mv_0 \frac{r_k}{r^2(t)} + 2M\left(1+\frac{t}{\theta}\right)^{-\frac{1}{2}} \frac{v_0^2}{r(t)} + M\left(1+\frac{t}{\theta}\right)^{-\frac{1}{2}} \frac{2r_k\theta\left(1+\frac{t}{\theta}\right)^{\frac{3}{2}} - 2r_k\theta + 2r^2(t)}{r^3(t)} v_0^2. \quad (60)$$

Force is non-null and dependent on the initial velocity and the sphere radius. It is interesting to note that the force tend to 0 as time increases but this corresponds to the unphysical case of a perturbation never turned off. This equation simplifies a lot if we can neglect the terms dependent on r_k. One has

$$F \approx 3M\left(1+\frac{t}{\theta}\right)^{-\frac{1}{2}} \frac{v_0^2}{r(t)}. \quad (61)$$

This result is independent on the sphere geometry or the Schwarzschild radius. Such a perturbation is not completely physical. So, we considered some others having the characteristic to be practically realizable. Considering the interior solution, for a perturbation like $\alpha_1 = At^2$ we get

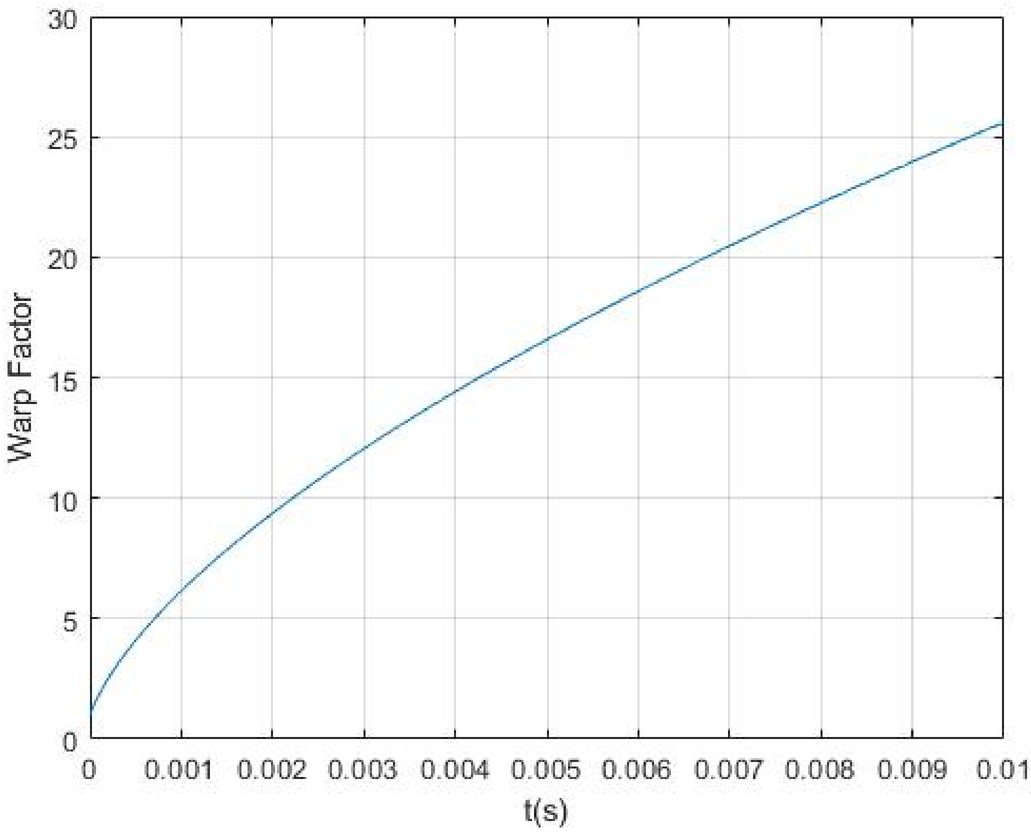

Figure 1. Warp factor for a t^2 perturbation with an equation of motion $r(t) = h_0 + v_0 t$.

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and for a sinusoidal perturbation

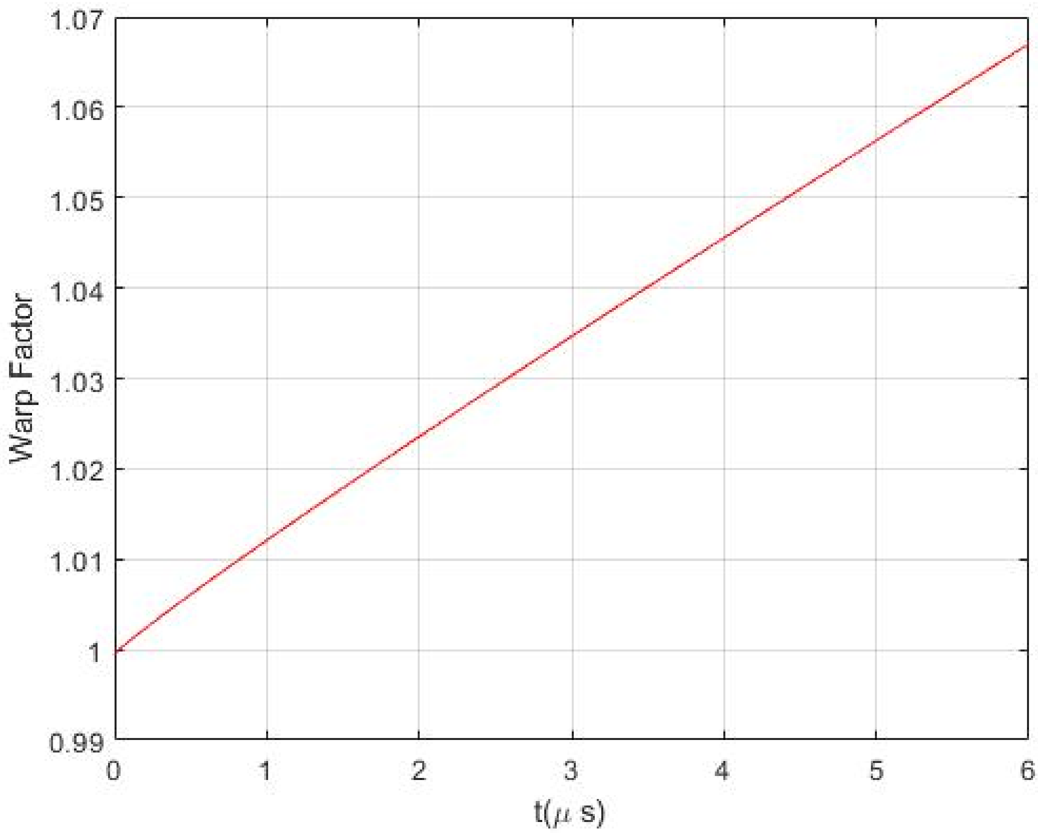

Figure 2. Warp factor for a $\sin(\omega t)$ perturbation with frequency 1 MHz and equation of motion $r(t) = h_0 + v_0 t + kt^2$.

As expected from the toy model, the warp factor is always greater than one and can reach significantly large values depending on the applied perturbation.

7. Conclusions

We have solved the Einstein equations for a strong perturbation in the case of a spherical symmetry solution. In this case, the perturbation series reduces to the case of a gradient expansion and the equations are amenable to an exact analytical treatment. We were able to show that, when a perturbation is properly applied, there appears a multiplicative warp factor on the radial velocity that can, in this way, increase exponentially in time. This warp effect does not require exotic energy and everything is completely in the realm of positive energy solutions of the Einstein equations, even if as a perturbation series.

We hope these results will find some application in the near future.

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