Multiscale method for solving high-order BVPs

Yingchao Zhang[1], Liangcai Mei[1]*, Yingzhen Lin[1]

[1] **Zhuhai Campus, Beijing Institute of Technology, Zhuhai 519088, China; 16212@bitzh.edu.cn(Yingchao Zhang);12069@bitzh.edu.cn(Yingzhen Lin)**

* **Correspondence: 13201@bitzh.edu.cn**

Abstract: This paper presents a numerical algorithm for solving high-order BVPs. We introduce the construction method of multiscale orthonormal basis in $W_2^m[0,1]$ by multiscale orthonormal basis in $W_2^1[0,1]$. We define $\varepsilon-$approximate solution, and obtain the $\varepsilon-$approximate solution of high-order BVPs by using the approximate theory. Moreover, the convergence and stability of the algorithm are improved. At last, several numerical experiments show the feasibility of the proposed method.

Keywords: Multiscale orthonormal basis; High-order BVPs; Convergence order

1 Introduction

In this article, we construct the multiscale method for the following high-order boundary value problems (BVPs):

$$\begin{cases} u^{(m)} + p_1(x)u^{(m-1)} + \cdots + p_{m-1}(x)u' + p_m(x)u = f(x), & x \in (0,1) \\ B_i u = \alpha_i, \quad i = 1, \cdots, m. \end{cases} \tag{1.1}$$

where $B_i(i = 1, 2, \cdots, m)$ are bounded linear functional on $W_2^m[0,1]$, $p_i(x)(i = 1, 2, \cdots, m)$ have a certain smoothness.

Higher-order BVPs are important mathematical models in the field of electromagnetics, fluid mechanics and material science. For example, the common Cahn-Hilliard equations and Molecular Beam Expital equations are higher-order models [1]. It is difficult to find the analytic solutions of higher-order BVPs because of the complexity of the systems, many numerical algorithms for high-order BVPs have been proposed in recent years. The multistage integration method is an important method to solve the numerical solution of higher-order models by reducing the order gradually [2-5]. Cao [6] solved a class of higher-order fractional ordinary differential equations by the quadratic interpolation function method. The collocation method proposed by Toutounian [7] and orthonormal Bernstein polynomials method proposed by Mirzaee [8] can solve higher-order linear complex differential equations effectively. Raslan et al. [9-11] proposed a variety of numerical algorithms for solving higher-order integro differential equations. Many scholars have also proposed many methods in the field of numerical solution of higher-order partial differential equations, such as Galerkin finite element method [12] and Diethelms method [13].

In this paper, we construct a set of multiscale orthonormal basis in the reproducing kernel space W_2^m. Multiscale is an effective theory for numerical analysis, which is based on the idea of wavelet. As a numerical algorithm for solving boundary value problems, multiscale theory has attracted more and more attention. Zhang [14] constructed an algorithm based on multiscale orthonormal basis for solving second-order BVPs. Multiscale theory combined with wavelet method or reproducing kernel method can also be used to solve BVPs [15,16]. Pezza [17] and Aminikhah [18] proposed a multiscale numerical algorithm for fractional BVPs. In recent years, multiscale finite element method has also been applied to solve the numerical solutions of partial differential equations [19,20]. Reproducing kernel space is an important banach space, which has been used in the field of numerical analysis. Wu and Lin [21] discussed the theory of reproducing kernel in detail, and designed a polynomial reproducing kernel method for solving various operator equations. Niu [22] proposed a new reproducing kernel algorithm for solving nonlinear singular BVPs. Mei and Shen [23-25] have proposed numerical algorithms for solving impulsive differential equations. The reproducing kernel methods are used in the numerical solutions of high-order models, singular BVPs and interface problems [26-30].

2 Multiscale orthonormal basis

In this section, the reproducing kernel space is defined and a set of multiscale orthonormal basis is constructed. These knowledge is very useful in the following article.

Definition 2.1 *The reproducing kernel space* $W_2^m[0,1] = \{u | u^{(m-1)} \in C[0,1],\ u^{(m)} \in L^2[0,1]\ \}$, *and the inner product of* W_2^m *is*

$$\langle u, v\rangle_{W_2^m} = \sum_{i=0}^{m-1} u^{(i)}(0)v^{(i)}(0) + \int_0^1 u^{(m)}v^{(m)}dx, \qquad \|u\|_{W_2^m} = \sqrt{\langle u, u\rangle_{W_2^m}}.$$

Definition 2.2 *The reproducing kernel space*

$$W_{2,0}^m[0,1] = \{u | u \in W_2^m, u(0) = u'(0) = \cdots = u^{(m-1)}(0) = 0, u^{(m-1)}(1) = 0\}.$$

Clearly, $W_{2,0}^m[0,1]$ is the closed subspace of $W_2^m[0,1]$. In Ref. [14], we set up multiscale orthonormal basis in $W_{2,0}^1$:

$$\phi_{i,k}(x) = 2^{\frac{i-1}{2}} \begin{cases} (x - \frac{k}{2^{i-1}}), x \in [\frac{k}{2^{i-1}}, \frac{k+1/2}{2^{i-1}}], \\ (\frac{k+1}{2^{i-1}} - x), x \in [\frac{k+1/2}{2^{i-1}}, \frac{k+1}{2^{i-1}}], \\ 0. \qquad else. \end{cases} \tag{2.1}$$

Where $i = 1, 2, \cdots, k = 0, 1, 2, \cdots, 2^{i-1} - 1$. The graph of $\phi_{i,k}$ are shown in Figure 1.

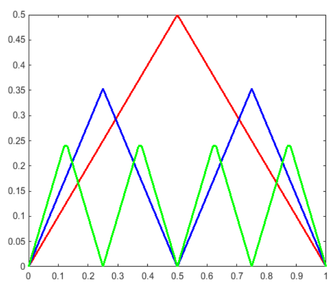

Figure 1: $\phi_{i,k}(x)$.

In order to express the algorithm for solving Eq. (1.1) , this paper constructs a set of orthonormal basis in $W_2^m[0,1]$ by $\{\phi_{i,k}\}_{i=1,k=0}^{\infty,2^{i-1}-1}$. First, let's construct the basis functions in $W_{2,0}^m[0,1]$. Note

$$J_0^{m-1}u = \frac{1}{(m-2)!}\int_0^x (x-s)^{m-2}u(s)ds, (m \in N, m \geq 2).$$

Theorem 2.1 $\{J_0^{m-1}\phi_{1,0}(x), J_0^{m-1}\phi_{2,0}(x), J_0^{m-1}\phi_{2,1}(x), \cdots, J_0^{m-1}\phi_{i,k}(x), \cdots\}$ *is the multiscale orthonormal basis of* $W_{2,0}^m[0,1]$.

Proof We just prove the orthonormality and completeness.

First, orthonormality. obviously,

$$\langle J_0^{m-1}\phi_{i,k}, J_0^{m-1}\phi_{j,m}\rangle_{W_{2,0}^m} = \langle \phi_{i,k}, \phi_{j,m}\rangle_{W_{2,0}^1} = \begin{cases} 1, i=j, k=m, \\ 0, else. \end{cases}$$

Orthonormality is true.

Second, completeness. If $\langle u, J_0^{m-1}\phi_{i,k}\rangle_{W_{2,0}^m} = 0$, then $u \equiv 0$, which means the basis is complete. In fact,

$$\langle J_0^{m-1}\phi_{i,k}, u\rangle_{W_{2,0}^m} = \langle \phi_{i,k}, u^{(m-1)}\rangle_{W_{2,0}^1} = 0 \quad implied \quad to \quad u^{(m-1)} = 0.$$

According to Def. 2.2, $u \equiv 0$. □

Next, we construct the orthonormal basis in $W_2^m[0,1]$. There are $m+1$ more conditions in space $W_{2,0}^m$ than in space W_2^m. If the basis in $W_2^m[0,1]$ is constructed from the basis in $W_{2,0}^m[0,1]$, we need to look for $m+1$ functions $g_k(x) \in W_2^m, k = 0,1,2,\cdots,m$, such that

$$\langle g_i(x), g_j(x)\rangle_{W_2^m} = 0, \tag{2.2}$$
$$\langle g_k(x), g_k(x)\rangle_{W_2^m} = 1, \tag{2.3}$$
$$\langle g_i(x), J_0^{m-1}\phi_{i,k}(x)\rangle_{W_2^m} = 0. \tag{2.4}$$

It is clear that $g_1(x) = 1, g_2(x) = x$ in W_2^m satisfy Eq. (2.2)–Eq. (2.4). Let $g_k(x) = ax^k \in W_2^m, (k = 2,\cdots,m)$. By definition of the inner product, $g_k(x)$ satisfies Eq. (2.2)–Eq. (2.4), and from Eq. (2.3), we can obtain $a = \frac{1}{k!}$.

Based on the above analysis, we get the the orthonormal basis in W_2^m.

Theorem 2.2

$$\{\rho_j(x)\}_{j=1}^{\infty} = \{1, x, \frac{x^2}{2}, \cdots, \frac{x^m}{m!}, J_0^{m-1}\phi_{1,0}(x), J_0^{m-1}\phi_{2,0}(x), J_0^{m-1}\phi_{2,1}(x), \cdots, J_0^{m-1}\phi_{i,k}(x), \cdots\}$$

is the multiscale orthonormal basis of $W_2^m[0,1]$.

Proof According to Th. 2.1 and Eq. (2.2)–Eq. (2.4), it's clear that

$$\langle \rho_i(x), \rho_j(x)\rangle_{W_2^m} = \begin{cases} 1, i=j, \\ 0, i \neq j. \end{cases}$$

So $\{\rho_j(x)\}_{j=1}^{\infty}$ is orthonormal.

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Next, we just need to prove completeness. That is , if $\langle u, \rho_j\rangle_{W_2^m} = 0$, then $u \equiv 0$. In fact,

$$\langle u, \frac{x^k}{k!}\rangle_{W_2^m} = 0 \quad implied \quad to \quad u^{(k)}(0) = 0, \qquad k = 0, 1, 2, \cdots, m-1. \tag{2.5}$$

$$\langle u, \frac{x^m}{m!}\rangle_{W_2^m} = 0 \quad implied \quad to \quad u^{(m-1)}(1) = 0. \tag{2.6}$$

$$\langle u, J_0^{m-1}\phi_{i,k}(x)\rangle_{W_2^m} = \langle u^{(m-1)}, \phi_{i,k}(x)\rangle_{W_{2,0}^1} = 0 \quad implied \quad to \quad u^{(m-1)} \equiv 0. \tag{2.7}$$

From Eq. (2.5)– Eq. (2.7), $u \equiv 0$. □

3 $\varepsilon-$approximate solutions of high-order BVPs

In this section, we give the $\varepsilon-$approximation of (1.1), and get the numerical solution of BVPs by finding the $\varepsilon-$approximate solution of Eq. (1.1).

Put

$$Lu = u^{(m)} + p_1(x)u^{(m-1)} + \cdots + p_{m-1}(x)u' + p_m(x)u,$$

where $L : W_2^m[0, 1] \to L^2[0, 1]$.

Theorem 3.1 $L : W_2^m[0, 1] \to L^2[0, 1]$ is a bounded linear operator.

Proof Because W_2^m is a reproducing kernel space,

$$\begin{aligned} |u^{(k)}(x)| &= |\langle u(\cdot), \frac{\partial^k K(x,\cdot)}{\partial x^k}\rangle_{W_2^m}| \\ &\le \|u(\cdot)\|_{W_2^m} \|\frac{\partial^k K(x,\cdot)}{\partial x^k}\|_{W_2^m}, \quad k = 0, 1, 2, \cdots, m-1. \end{aligned} \tag{3.1}$$

By Eq. (3.1), there exists positive constants M_k such that

$$\begin{aligned} &\|p_{m-k}(x)u^{(k)}(x)\|_{L^2} = \left(\int_0^1 (p_{m-k}(x)u^{(k)}(x))^2 dx\right)^{\frac{1}{2}} \\ &\le \max_{x\in[0,1]} |p_{m-k}(x)| \left(\int_0^1 (u^{(k)}(x))^2 dx\right)^{\frac{1}{2}} \le M_k \|u\|_{W_2^m}, \quad k = 0, 1, 2, \cdots, m-1. \end{aligned} \tag{3.2}$$

therefore

$$\|u^{(m)}\|_{L^2}^2 = \int_0^1 (u^{(m)})^2 dx \le \sum_{i=0}^{2} u^{(i)}(0)u^{(i)}(0) + \int_0^1 (u^{(m)})^2 dx = \|u\|_{W_2^m}^2. \tag{3.3}$$

From Eq. (3.2) and Eq. (3.3), it follows that

$$\|Lu\|_{L^2} \le M\|u\|_{W_2^m},$$

where M is a positive constant. □

Then Eq. (1.1) is equivalent to the following equation

$$\begin{cases} Lu = f(x), & x \in (0, 1) \\ B_i u = \alpha_i, & i = 1, 2, \cdots, m. \end{cases} \tag{3.4}$$

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Zhang [14] proposed the $\varepsilon-$approximate theory of second-order differential equations, now we define the $\varepsilon-$approximate solution of Eq. (3.4) based on the idea.

Definition 3.1 *$\forall\varepsilon > 0$, $\exists N > 0$, when $n > N$, if $\|Lu_n - f\|^2_{L^2} + \sum\limits_{i=1}^{m}(B_i u_n - \alpha_i)^2 < \varepsilon^2$, u_n is called $\varepsilon-$approximate solution of Eq. (3.4).*

lemma 3.1 *$\forall\varepsilon > 0$, $\exists N > 0$, when $n > N$,*

$$u_n = \sum_{k=1}^{n} c_k^* \rho_k \tag{3.5}$$

is the $\varepsilon-$approximate solution of Eq. (3.4), where c_k^* satisfies

$$\begin{aligned} &\| \sum_{k=1}^{n} c_k^* L\rho_k - f\|^2_{L^2} + \sum_{l=1}^{m}(\sum_{k=1}^{n} c_k^* B_l \rho_k - \alpha_l)^2 \\ =& \min_{c_k}[\| \sum_{k=1}^{n} c_k L\rho_k - f\|^2_{L^2} + \sum_{l=1}^{m}(\sum_{k=1}^{n} c_k B_l \rho_k - \alpha_l)^2]. \end{aligned} \tag{3.6}$$

Put J is quadratic form about $c = (c_1, \cdots, c_n)$,

$$J(c_1, \cdots, c_n) = \| \sum_{k=1}^{n} c_k L\rho_k - f\|^2_{L^2} + \sum_{l=1}^{m}(\sum_{k=1}^{n} c_k B_l \rho_k - \alpha_l)^2,$$

c_k^* is the minimum point of J. In fact, in order to find the minimum value of J, that is,

$$\frac{\partial}{\partial c_j} J(c_1, \cdots, c_n) = 0.$$

Because of

$$\frac{\partial}{\partial c_j} J(c_1, \cdots, c_n) = 2\sum_{k=1}^{n} c_k \langle L\rho_k, L\rho_j\rangle_{L^2} - 2\langle L\rho_j, f\rangle_{L^2} + 2\sum_{l=1}^{m}\sum_{k=1}^{n} c_k B_l\rho_k B_l\rho_j - 2\sum_{l=1}^{m} B_l\rho_j\alpha_l,$$

then

$$\sum_{k=1}^{n} c_k \langle L\rho_k, L\rho_j\rangle_{L^2} + \sum_{l=1}^{m}\sum_{k=1}^{n} c_k B_l\rho_k B_l\rho_j = \langle L\rho_j, f\rangle_{L^2} + \sum_{l=1}^{m} B_l\rho_j\alpha_l. \tag{3.7}$$

Let

$$\begin{aligned} \mathbf{A_n} &= \left(\langle L\rho_k, L\rho_j\rangle_{L^2} + \sum_{l=1}^{m} B_l\rho_k B_l\rho_j \right)_{n\times n}, \\ \mathbf{b_n} &= \left(\langle L\rho_k, f\rangle_{L^2} + \sum_{l=1}^{m} B_l\rho_j\alpha_l \right)_{n}. \end{aligned}$$

Then Eq. (3.7) changes to

$$\mathbf{A_n} c = \mathbf{b_n}. \tag{3.8}$$

According to [14], the unique solution of Eq. (3.8) is the minimum point of J.

4 Convergence and stability analysis

In this section, the properties of the algorithm are introduced, such as uniform convergence and stability.

4.1 Convergence analysis

Theorem 4.1 *Assume u is the exact solution of Eq. (1.1), u_n is the $\varepsilon-$approximation of (1.1). If $u^{(m+1)}$ is bounded on [0,1], then $|u-u_n|^2_{W_2^m} \leq 2^{-2n}C$, where C is a constant.*

Proof Assume

$$v_n(x) = \sum_{j=0}^{m-1} c_j \frac{x^j}{j!} + \sum_{i=1}^{n} \sum_{k=0}^{2^{i-1}-1} c_{i,k} J_0^2 \phi_{i,k}(x). \tag{4.1}$$

satisfied Eq. (3.4), where $c_j = \langle u, \frac{x^j}{j!} \rangle_{W_2^m}$, $c_{i,k} = \langle u, J_0^2 \phi_{ik} \rangle_{W_2^m}$. Clearly, $\lim_{n\to+\infty} v_n = u$.

Due to Lemma 3.1, it can get

$$\begin{aligned}
\|u-u_n\|^2_{W_2^m} &\leq \|L^{-1}\|^2 \|Lu - Lu_n\|^2_{L^2} \leq \|L^{-1}\|^2 (\|Lu - Lu_n\|^2_{L^2} + \sum_{i=1}^{m} |B_i u_n - a_i|^2) \\
&\leq \|L^{-1}\|^2 (\|Lu - Lv_n\|^2_{L^2} + \sum_{i=1}^{m} |B_i v_n - a_i|^2) \\
&\leq \|L^{-1}\|^2 (\|Lu - Lv_n\|^2_{L^2} + \sum_{i=1}^{m} |B_i v_n - B_i u|^2) \\
&\leq \|L^{-1}\|^2 (\|L\|^2 \|u - v_n\|^2_{W_2^m} + \sum_{i=1}^{m} \|B_i\|^2 \|u - v_n\|_3 \leq M \|u - v_n\|_{W_2^m}.
\end{aligned} \tag{4.2}$$

That is

$$\|u-u_n\|_{W_2^m} \leq M\|u-v_n\|^2_{W_2^m} = M \left\| u - \sum_{j=0}^{m} c_j \frac{x^j}{j!} - \sum_{i=1}^{n} \sum_{k=0}^{2^{i-1}-1} c_{i,k} J_0^2 \phi_{i,k} \right\|_{W_2^m} = M \sum_{i=n+1}^{\infty} \sum_{k=0}^{2^{i-1}-1} (c_{i,k})^2.$$

We can obtain $(c_{i,k})^2 \leq \left(\frac{1}{2}\right)^{3i} C_1$. In fact,

$$u^{(m)}(x) = u^{(m)}(\frac{k}{2^{i-1}}) + u^{(m+1)}(\xi)(x - \frac{k}{2^{i-1}}),$$

so

$$\begin{aligned}
|c_{i,k}| &= |\langle u, J_0^{m-1} \phi_{i,k} \rangle_{W_{2,0}^m}| = |\langle u^{(m-1)}, \phi_{i,k} \rangle_{W_{2,0}^1}| = |\int_0^1 u^{(m)} (\phi_{i,k})' dx| \\
&\leq |\int_{k/2^{(i-1)}}^{(k+1)/2^{(i-1)}} u^{(m)}(\frac{k}{2^{(i-1)}})(\phi_{i,k})' dx| + |\int_{k/2^{(i-1)}}^{(k+1)/2^{(i-1)}} u^{(m+1)}(\xi)(x - \frac{k}{2^{(i-1)}})(\phi_{i,k})' dx|.
\end{aligned}$$

According to

$$(\phi_{i,k})'(x) = 2^{\frac{i-1}{2}} \begin{cases} 1, & x \in [\frac{k}{2^{i-1}}, \frac{k+1/2}{2^{i-1}}], \\ -1, x \in [\frac{k+1/2}{2^{i-1}}, \frac{k+1}{2^{i-1}}], \\ 0. & else. \end{cases}$$

Then

$$|\int_{k/2^{(i-1)}}^{(k+1)/2^{(i-1)}} u^{(m)}(\frac{k}{2^{(i-1)}})(\phi_{i,k})'dx| = 0.$$

$$|c_{i,k}| \le |\int_{k/2^{(i-1)}}^{(k+1)/2^{(i-1)}} u^{(m+1)}(\xi)(x - \frac{k}{2^{(i-1)}})(\phi_{i,k})'dx| \le \left(\frac{1}{2}\right)^{\frac{3i}{2}} C.$$

So $|c_{i,k}|^2 \le \left(\frac{1}{2}\right)^{3i} C_1$. So then

$$\|u - u_n\|^2_{W_2^m} \le \sum_{i=2n+1}^{\infty} \sum_{k=0}^{2^{i-1}-1} \left(\left(\frac{1}{2}\right)^{3i} C_1\right) = \left(\frac{1}{2}\right)^{2n} C.$$

where C is a constant.

$$|u(x) - u_n(x)|^2 = |\langle u - u_n, K(x,y)\rangle_{W_2^m}|^2 \le (\|u - u_n\|_{W_2^m} \|K(x,y)\|_{W_2^m})^2 \le 2^{-2n}C,$$

where C is a constant, $K(x,y)$ is the reproducing kernel of W_2^m. □

From theorem 4.1, u_n uniformly convergence to u, which is the following theorem

Theorem 4.2 *Assume u is the exact solution of* (1.1), *and* $u_n = \sum_{k=1}^{n} c_k^* \rho_k$ *is the $\varepsilon-$approximate solution of Eq. (1.1). Then u_n uniformly converges to u.*

4.2 Stability analysis

It is well known that if $\mathbf{A}$ is a reversible symmetrical matrix, then the condition number of $\mathbf{A}$ is

$$cond(\mathbf{A})_2 = \left|\frac{\lambda_1}{\lambda_n}\right|,$$

where λ_1 and λ_n are the maximum and minimum eigenvalues of $\mathbf{A}$ respectively.

Obviously, $\mathbf{A}$ of Eq. (3.8) is invertible symmetric matrix. Therefore, in order to prove the stability of the algorithm, we can first prove the boundedness of the eigenvalues.

lemma 4.1 *Suppose* $\lambda\mathbf{x} = \mathbf{A}\mathbf{x}$, $\|\mathbf{x}\| = 1$, *where* $\mathbf{x} = (x_1, \cdots, x_n)^T$ *is related eigenvector of* λ, *then*

$$C^2 \le \lambda \le \|L\|^2 + \sum_{l=1}^{m} \|B_l\|^2.$$

Proof According to $\lambda\mathbf{x} = \mathbf{A}\mathbf{x}$,

$$\lambda x_i = \sum_{j=1}^{n} a_{ij} x_j = \sum_{j=1}^{n} \left(\langle L\rho_i, L\rho_j\rangle_{L^2} + \sum_{l=1}^{m} B_l\rho_i B_l\rho_j\right) x_j,$$

$$\sum_{j=1}^{n}\left(\langle L\rho_i, x_j L\rho_j\rangle_{L^2} + \sum_{l=1}^{m} B_l\rho_i x_j B_l\rho_j\right) = \langle L\rho_i, \sum_{j=1}^{n} x_j L\rho_j\rangle_{L^2} + \sum_{l=1}^{m} B_l\rho_i \sum_{j=1}^{n} x_j B_l\rho_j. \qquad (4.3)$$

Multiply both sides of (4.2) by x_i $(i = 1, 2, \cdots, n)$ and add up, and it get

$$\begin{aligned}\lambda &= \lambda\sum_{j=1}^{n} x_j^2 = \langle\sum_{i=1}^{n} x_i L\rho_i, \sum_{j=1}^{n} x_j L\rho_j\rangle_{L^2} + \sum_{l=1}^{m}\left(\sum_{i=1}^{n} x_i B_l\rho_i \sum_{j=1}^{n} x_j B_l\rho_j\right)\\ &= \|\sum_{i=1}^{n} x_i L\rho_i\|_{L^2}^2 + \sum_{l=1}^{m}\left(B_l\sum_{j=1}^{n} x_j\rho_i\right)^2 \le \|L\|^2\sum_{i=1}^{n} x_i^2 + \sum_{l=1}^{m}\|B_l\|^2\sum_{i=1}^{n} x_i^2\\ &= \|L\|^2 + \sum_{l=1}^{m}\|B_l\|^2.\end{aligned}$$

So

$$\lambda \le \|L\|^2 + \sum_{l=1}^{m}\|B_l\|^2.$$

In addition, $\lambda \ge \|\sum\limits_{i=1}^{n} x_i L\rho_i\|_{L^2}$.

Let $u = \sum\limits_{i=1}^{n} x_i\rho_i$, then $\lambda \ge \|Lu\|_{L^2}^2$. Without loss of generality, put $\|u\|_{W_{2,0}^m} = 1$. According to inverse operator theorem [21], $\|Lu\|_{L^2} \ge C^2\|u\|_{W_2^m}^2$.

So $\lambda \ge \|Lu\|_{L^2} \ge C^2\|u\|_{W_2^m}^2 = C^2$.

To sum up

$$C^2 \le \lambda \le \|L\|^2 + \sum_{l=1}^{m}\|B_l\|^2. \qquad \square$$

From lemma 4.1, we get

$$cond(\mathbf{A})_2 = |\frac{\lambda_1}{\lambda_n}| \le \frac{\|L\|^2 + \sum\limits_{l=1}^{m}\|B_l\|^2}{\frac{1}{C^2}} = (\|L\|^2 + \sum_{l=1}^{m}\|B_l\|^2)C^2.$$

That is, the presented method is stable.

5 Numerical experiments

In this section, we give several numerical experiments to verify the effectiveness of the proposed algorithm. We denote by u_j the approximation to the exact solution $u(x_j)$ obtained by the numerical schemes in the present work, and we measure the errors in the following sense:

$$e(n) = \max_{1<j<n}|u_j - u(x_j)|.$$

where n is the number of bases. C.R. represents the convergence order.

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Example 5.1 Consider the following third-order BVPs [27, 28]

$$\begin{cases} u''' - k^2u' + r = 0, & x \in (0,1) \\ u'(0) = u(\frac{1}{2}) = u'(1) = 0. \end{cases}$$

$u(x) = \dfrac{r(k(2x-1)\sinh(kx) + 2\cosh(kx)\tanh(\frac{k}{2}))}{2k^3}$ is the exact solution of example 5.1, and the numerical results are given in Table 1.

Table 1: Absolute errors of Example 5.2 ($r = 1, k = 5$).

x	Absolute error [27]	Absolute error [28]	Present method $e(131)$	Present method $e(259)$
0	$6.65e-5$	$2.81e-5$	$5.74e-5$	$1.34e-5$
0.1	$6.50e-5$	$2.26e-5$	$3.44e-5$	$7.65e-6$
0.2	$5.25e-5$	$1.41e-5$	$2.02e-5$	$4.08e-6$
0.3	$3.63e-5$	$7.32e-6$	$1.11e-5$	$1.81e-6$
0.4	$1.87e-5$	$2.95e-6$	$4.96e-6$	$2.55e-7$
0.6	$1.73e-5$	$2.95e-6$	$4.94e-6$	$2.23e-6$
0.7	$3.40e-5$	$7.32e-6$	$1.11e-5$	$3.80e-6$
0.8	$4.98e-5$	$1.41e-5$	$2.02e-5$	$6.09e-6$
0.9	$6.20e-5$	$2.26e-5$	$3.44e-5$	$9.68e-6$
1.0	$6.34e-5$	$2.81e-5$	$5.74e-5$	$1.54e-5$

Example 5.2 Our second example is for fourth-order BVPs [30]

$$\begin{cases} u^{(4)} - 2u = -1 - (8\pi^4 - 1)\cos(2\pi x), & x \in (0,1) \\ u(0) = u(1) = u'(0) = u'(1) = 0. \end{cases}$$

Table 2 and figure 2 lists the errors comparison between the multilevel augmentation method [30] and our method for this BVPs

Table 2: Absolute errors of Example 5.2.

n	Absolute error of [30]	C.R. of [30]	Present method $\lvert u - u_n \rvert$	C.R. of present method
35	.	.	$3.693e-2$	.
67	$2.004e-2$	1.99	$9.232e-3$	1.968
131	$5.013e-3$	2.00	$2.307e-3$	2.004
259	$1.253e-3$	1.99	$5.674e-4$	2.038
515	$3.133e-4$	1.99	$1.441e-4$	2.017

Example 5.3 Our third example is a fifth-order BVPs

$$\begin{cases} u^{(5)} + u^{(4)} - 3xu''' - u'' + u' = (2-3x)e^x - 1, & x \in (0,1) \\ u(0) = u'(0) = 0, u''(0) = 1, \\ u'(1) = e - 1, u''(1) = e. \end{cases}$$

The exact solution of this example is $u(x) = e^x - x - 1$. Figure 3 shows the error of example 5.3.

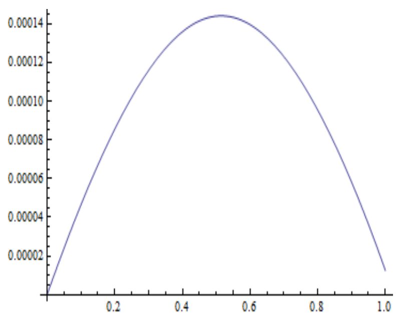

Figure 2: The error for Ex. 5.2(n=515).

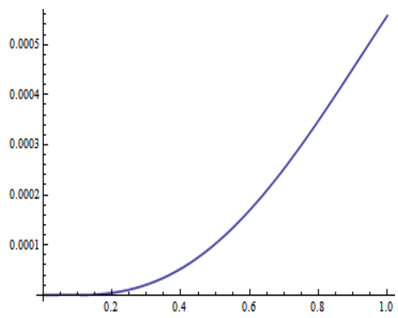

Figure 3: The error for Ex. 5.3(n=67).

6 Conclusion

In summary, this study used a set of multi-scale orthonormal basis to find the $\varepsilon-$approximate solutions of higher-order BVPs. This paper not only demonstrates the convergence and stability in theory, but also demonstrates the feasibility of the method through numerical experiments. Through theoretical analysis and numerical experiments, this method can be extended to solve general linear models, such as linear integral equations, differential equations, fractional differential equations.

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