# Extended General Relativity for a Curved Universe

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**Abstract**: The recent Planck Legacy release revealed the presence of an enhanced lensing amplitude in the cosmic microwave background, which endorses the early universe positive curvature with a confidence level exceeding 99%. Although general relativity performs accurately in the local/present universe where spacetime is almost flat, the necessity of dark matter/energy and the lost boundary term might be signs of its incompleteness. Utilising Einstein–Hilbert action, I present extended field equations considering the pre-existing universal curvatures. The new extended field equations are inclusive of Einstein field equations in addition to the boundary term and the conformal curvature term contributions.

**Keywords**: General Relativity, Curved Universe.

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## 1. INTRODUCTION

There is strong evidence of early universe positive curvature with a confidence level greater than 99% based on Planck Legacy recent release [1]. In addition, the detected gravitational lensing within several galaxy clusters is an order of magnitude higher than that estimated by the current standard model [2]. Enhancing general relativity by considering the pre-existing universal curvatures and the boundary term contributions might settle down numerous unsolved problems of the universe.

In section 2, the mathematical derivations are presented. Future works are projected in Section 3.

## 2. Extended Field Equations

The pre-existing curvatures signified by $\mathcal{R}_{uv}g^{uv}$ are incorporated into Einstein–Hilbert action:

$$S = \int \left[\frac{E_D}{2}\frac{R_{uv}g^{uv}}{\mathcal{R}_{uv}g^{uv}} + \mathcal{L}_M\right]\sqrt{-g}\, d^4x \tag{1}$$

Yet, to maintain the compatibility of the action, I introduce a new intermediate metric-compatible invariant modulus, $E_D = (T_u^v - T\delta_u^v/2)/R_u^v/\mathcal{R}$, founded on the Theory of Elasticity. $T_u^v$ is the energy-momentum mixed components tensor of trace $T$, $\delta_u^v$ is the Kronecker delta, $R_u^v$ is the Ricci curvature tensor and $\mathcal{R}$ is the scalar curvature. $\mathcal{L}_M$ is the Lagrangian of matter fields and $g$ is determinant of the metric tensor [3]. The action should hold with any variation of the inverse metric $\delta g^{\mu\nu}$

$$S = \int \left[\frac{E_D}{2}\left(\begin{matrix}\dfrac{R_{uv}\delta g^{uv} + g^{uv}\delta R_{uv}}{\mathcal{R}\delta g^{\mu\nu}} - \dfrac{\mathcal{R}_{uv}\delta g^{uv} + g^{uv}\delta\mathcal{R}_{uv}}{\mathcal{R}^2 R^{-1}\delta g^{\mu\nu}} \\ + \dfrac{1}{\sqrt{-g}}\dfrac{\delta\sqrt{-g}}{\mathcal{R}R^{-1}\delta g^{\mu\nu}}\end{matrix}\right) + \frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}}\right]\delta g^{\mu\nu}\sqrt{-g}\, d^4x \tag{2}$$

Regarding the curvature terms in the action, the variation of the Ricci curvature tensor can be written in terms of the covariant derivative of the difference between two Levi-Civita connections [3], the Palatini identity:

$$\delta R_{\sigma\nu} = \nabla_\rho\left(\delta\Gamma^\rho{}_{\nu\sigma}\right) - \nabla_\nu\left(\delta\Gamma^\rho{}_{\rho\sigma}\right) \tag{3}$$

The variation of the Ricci curvature tensor with respect to the inverse metric tensor can be obtained utilising the metric compatibility of the covariant derivative, $\nabla_\sigma g^{\mu\nu} = 0$ with renaming the dummy indices as

$$g^{\sigma\nu}\delta R_{\sigma\nu} = \nabla_\rho(g^{\sigma\nu}\delta\Gamma^\rho{}_{\nu\sigma} - g^{\sigma\rho}\delta\Gamma^\mu{}_{\mu\sigma}) \tag{4}$$

The integration of the total derivative for any tensor density can be transformed as

$$\iiint_V \nabla_\mu \left(\sqrt{-g}\, (g^{\sigma\nu}\delta\Gamma^{\mu}_{\nu\sigma} - g^{\sigma\mu}\delta\Gamma^{\mu}_{\mu\sigma})\right)\, dV = \oiint_S \partial_\mu \left(\sqrt{|h|}\, (g^{\sigma\nu}\delta\Gamma^{\mu}_{\nu\sigma} - g^{\sigma\mu}\delta\Gamma^{\mu}_{\mu\sigma})\right).\hat{n}_u\, dS \tag{5}$$

where $h$ and $\hat{n}_u$ are the determinant of the induced metric on the boundary and the normal respectively. The Lagrangian term in the action is proportional to the energy-momentum tensor [3]:

$$\frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}} = \frac{\delta\mathcal{L}_M}{\delta g^{\mu\nu}} - g_{uv}\,\mathcal{L}_M \tag{6}$$

Utilising Jacobi's formula for $\sqrt{-g}$ as

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta g \tag{7}$$

The extended action is then written

$$\mathrm{S} = \int \left[ \frac{E_D}{2} \left( \begin{array}{c} \dfrac{R_{uv}\delta g^{\mu\nu} + \partial_\mu\left(\sqrt{|h|}\,(g^{\sigma\nu}\delta\Gamma^{\mu}_{\nu\sigma} - g^{\sigma\mu}\delta\Gamma^{\mu}_{\mu\sigma})\right).\hat{n}_u d^{-1}x}{\mathcal{R}\delta g^{\mu\nu}} \\ \dfrac{-\mathcal{R}_{uv}\delta g^{\mu\nu} + \partial_\mu\left(\sqrt{|h|}\,(g^{\sigma\mu}\delta\Gamma^{\mu}_{\mu\sigma} - g^{\sigma\nu}\delta\Gamma^{\mu}_{\nu\sigma})\right).\hat{n}_u d^{-1}x}{\mathcal{R}^2 R^{-1}\delta g^{\mu\nu}} \\ + \dfrac{\delta g}{2\mathcal{R}R^{-1}\, g\delta g^{\mu\nu}} \end{array} \right) \begin{array}{c} - g_{uv}\,\mathcal{L}_M \\ + \dfrac{\delta\mathcal{L}_M}{\delta g^{\mu\nu}} \end{array} \right] \delta g^{\mu\nu}\sqrt{-g}\, d^4x \tag{8}$$

Performing the integration, simplification and utilising the principle of least action, we get

$$R_{uv} - \frac{1}{2}Rg_{uv} - \frac{R}{\mathcal{R}}\mathcal{R}_{uv} + \frac{\mathcal{R}-R}{\mathcal{R}}Kq_{uv} = \frac{8\pi \mathrm{G}}{\mathrm{c}^4}T_{uv} \tag{9}$$

where $Kq_{uv}$ represents the trace of the extrinsic curvature times the induced metric on the spacetime manifold boundary. The boundary term behaviour is compatible with black hole entropy calculations [4] and might remove the singularities from the theory. The conformal curvature term $\mathcal{R}_{uv}$ would be crucial in the understanding of the galaxy rotation curve.

## 3. Future Work

Future works would investigate the evolution of the conformal curvature tensor $\mathcal{R}_{uv}$ and its scalar curvature $\mathcal{R}$ over the conformal time utilising Hamilton's Ricci flow.


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