Legendre-Gould Hopper based Sheffer polynomials: properties and applications

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Abstract: In this article, the Legendre-Gould Hopper polynomials are combined with Sheffer sequences to introduce certain mixed type special polynomials. Generating functions, differential equations and certain other properties of Legendre-Gould Hopper based Sheffer polynomials are derived. Further, operational and integral representations providing connections between these polynomials and known special polynomials are established. Certain identities and results for some members of these new mixed polynomials are also obtained. Finally, the determinantal definitions of Legendre-Gould Hopper based Sheffer polynomials are also given.

Keywords: Legendre-Gould Hopper based Sheffer polynomials; Generating relations; Operational method; Monomiality principle; Determinantal definition.

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1. Introduction and preliminaries

Operational methods developed within the context of the monomiality principle have opened new possibilities to treat problems associated with the properties of special functions and polynomials. Even classical problems with well known solutions may acquire a different flavor if viewed from such perspective. Furthermore, the same formalism provides the possibilities of introducing new families of mixed special polynomials. These polynomials have several applications in different branches of mathematics.

Recently, the Legendre-Gould Hopper polynomials (LeGHP) $\frac{{}_RH_n^{(s)}(x,y,z)}{n!}$ and ${}_SH_n^{(s)}(x,y,z)$ are introduced in [30] which are defined by means of the following generating functions

$$exp(zt^s)\ C_0(xt)\ C_0(-yt) = \sum_{n=0}^{\infty} \frac{{}_RH_n^{(s)}(x,y,z)}{n!}\frac{t^n}{n!} \tag{1.1}$$

and

$$exp(yt+zt^s)\ C_0(-xt^2) = \sum_{n=0}^{\infty} {}_SH_n^{(s)}(x,y,z)\frac{t^n}{n!}, \tag{1.2}$$

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respectively, where $C_0(x)$ denotes the Tricomi function of order zero. The n^{th} order Tricomi functions $C_n(x)$ are defined by means of the generating function

$$exp\left(t-\frac{x}{t}\right)=\sum_{n=0}^{\infty}C_n(x)t^n, \tag{1.3}$$

for $t \neq 0$ and for all finite x and are defined by the following series [12, p.150]:

$$C_n(x)=\sum_{s=0}^{\infty}\frac{(-1)^s\ x^s}{s!\ (n+s)!}, \qquad n=0,1,2,...,. \tag{1.4}$$

The 0^{th}-order Tricomi function $C_0(x)$ is also given by the following operational definition:

$$C_0(\alpha x)=exp(-\alpha D_x^{-1})\{1\}, \tag{1.5}$$

where D_x^{-1} denotes the inverse of the derivative operator $D_x := \frac{\partial}{\partial x}$ and

$$D_x^{-n}\{1\}=\frac{x^n}{n!}. \tag{1.6}$$

Also, it is known that [16]

$$-\left(\frac{\partial}{\partial x}x\frac{\partial}{\partial x}\right)C_0(\alpha x)=\alpha\ C_0(\alpha x). \tag{1.7}$$

The LeGHP $\frac{{}_RH_n^{(s)}(x,y,z)}{n!}$ and ${}_SH_n^{(s)}(x,y,z)$ are shown to be quasi-monomial [12, 28] under the action of the following multiplicative and derivative operators [30]:

$$\hat{M}_{RH} := -D_x^{-1}+D_y^{-1}+sz\frac{\partial^{s-1}}{\partial y^{s-1}}, \tag{1.8}$$

$$\hat{P}_{RH} := -\frac{\partial}{\partial x}x\frac{\partial}{\partial x} \tag{1.9}$$

and

$$\hat{M}_{SH} := y+2D_x^{-1}\frac{\partial}{\partial y}+sz\frac{\partial^{s-1}}{\partial y^{s-1}}, \tag{1.10}$$

$$\hat{P}_{SH} := \frac{\partial}{\partial y}, \tag{1.11}$$

respectively.

Consequently, $\hat{M}_{RH}$, $\hat{P}_{RH}$ and $\hat{M}_{SH}$, $\hat{P}_{SH}$ satisfy the following recurrence relations:

$$\hat{M}_{RH}\left\{\frac{{}_RH_n^{(s)}(x,y,z)}{n!}\right\}=\frac{{}_RH_{n+1}^{(s)}(x,y,z)}{(n+1)!}, \tag{1.12}$$

$$\hat{P}_{RH}\left\{\frac{{}_RH_n^{(s)}(x,y,z)}{n!}\right\}=n\frac{{}_RH_{n-1}^{(s)}(x,y,z)}{(n-1)!} \tag{1.13}$$

and

$$\hat{M}_{SH}\{{}_SH_n^{(s)}(x,y,z)\} = {}_SH_{n+1}^{(s)}(x,y,z), \tag{1.14}$$

$$\hat{P}_{SH}\{{}_SH_n^{(s)}(x,y,z)\} = n\ {}_SH_{n-1}^{(s)}(x,y,z), \tag{1.15}$$

respectively, for all $n \in \mathbb{N}$.

In view of the monomiality principle equations

$$\hat{M}_{RH}\hat{P}_{RH}\left\{\frac{{}_RH_n^{(s)}(x,y,z)}{n!}\right\} = n\frac{{}_RH_n^{(s)}(x,y,z)}{n!}, \tag{1.16}$$

$$\hat{M}_{SH}\hat{P}_{SH}\{{}_SH_n^{(s)}(x,y,z)\} = n\ {}_SH_n^{(s)}(x,y,z), \tag{1.17}$$

the differential equations satisfied by $\frac{{}_RH_n^{(s)}(x,y,z)}{n!}$ and ${}_SH_n^{(s)}(x,y,z)$ are [30]:

$$\left(-\frac{\partial}{\partial y} + sz\frac{\partial^{s+1}}{\partial x\partial y^s} + (1-n)\frac{\partial}{\partial x}\right){}_RH_n^{(s)}(x,y,z) = 0 \tag{1.18}$$

and

$$\left(2\frac{\partial^2}{\partial y^2} + sz\frac{\partial^{s+1}}{\partial x\partial y^s} + y\frac{\partial^2}{\partial x\partial y} - n\frac{\partial}{\partial x}\right){}_SH_n^{(s)}(x,y,z) = 0, \tag{1.19}$$

respectively.

Also, $\frac{{}_RH_n^{(s)}(x,y,z)}{n!}$ and ${}_SH_n^{(s)}(x,y,z)$ can be explicitly constructed as:

$$\frac{{}_RH_n^{(s)}(x,y,z)}{n!} = \hat{M}_{RH}^n\{1\}, \qquad {}_RH_0^{(s)}(x,y,z) = 1 \tag{1.20}$$

and

$${}_SH_n^{(s)}(x,y,z) = \hat{M}_{SH}^n\{1\}, \qquad {}_SH_0^{(s)}(x,y,z) = 1, \tag{1.21}$$

respectively.

Identities (1.21) and (1.20) imply that the exponential functions of $\frac{{}_RH_n^{(s)}(x,y,z)}{n!}$ and ${}_SH_n^{(s)}(x,y,z)$ can be given in the forms:

$$exp(t\hat{M}_{RH})\{1\} = \sum_{n=0}^{\infty}\frac{{}_RH_n^{(s)}(x,y,z)}{n!}\frac{t^n}{n!}, \qquad |t| < \infty \tag{1.22}$$

and

$$exp(t\hat{M}_{SH})\{1\} = \sum_{n=0}^{\infty}{}_SH_n^{(s)}(x,y,z)\frac{t^n}{n!}, \qquad |t| < \infty, \tag{1.23}$$

respectively.

For suitable values of the indices and variables, the LeGHP $\frac{{}_RH_n^{(s)}(x,y,z)}{n!}$ and ${}_SH_n^{(s)}(x,y,z)$ give a number of other known special polynomials as special cases. We mention these special cases in Table 1.

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Table 1. List of special cases of LeGHP ${}_SH_n^{(s)}(x,y,z)$ and ${}_RH_n^{(s)}(x,y,z)$

S. No.	Values of the indices and variables	Relation between LeGHP ${}_SH_n^{(s)}(x,y,z)$, ${}_RH_n^{(s)}(x,y,z)$ and its special case	Name of known polynomials	Series definition of known polynomials
I.	$x=0$	${}_SH_n^{(s)}(0,y,z)=H_n^{(s)}(y,z)$	Gould-Hopper [22]	$H_n^{(s)}(x,y)=n!\sum_{k=0}^{[\frac{n}{s}]}\frac{y^k x^{n-sk}}{k!(n-sk)!}$
II.	$z=0$	${}_SH_n^{(s)}(x,y,0)={}_2L_n(x,y)$	2-Variable Legendre type [17]	${}_2L_n(x,y)=n!\sum_{r=0}^{[\frac{n}{2}]}\frac{x^r y^{n-2r}}{(r!)^2(n-2r)!}$
III.	i. $s=m;\ x=0,$ $y\to -D_x^{-1},\ z\to y$ ii.$s=m;\ y=0,\ z\to y$	${}_SH_n^{(m)}(0,-D_x^{-1},y)={}_{[m]}L_n(x,y)$ ${}_RH_n^{(m)}(x,0,y)={}_{[m]}L_n(x,y)$	2-Variable Generalized Laguerre Type [14]	${}_{[m]}L_n(x,y)=n!\sum_{k=0}^{[\frac{n}{m}]}\frac{y^k(-x)^{n-mk}}{k![(n-mk)!]^2}$
IV.	$s=m-1;\ x=0,$ $y\to x,\ z\to y$	${}_SH_n^{(m-1)}(0,x,y)=U_n^{(m)}(x,y)$	Generalized Chebyshev [13]	$U_n^{(m)}(x,y)=\sum_{k=0}^{[\frac{n}{m}]}\frac{(n-k)!\ y^k x^{n-mk}}{k!(n-mk)!}$
V.	i.$s=1,x=0,\ z\to -D_x^{-1}$ ii.$s=1;\ y=0,\ z\to y$	${}_SH_n^{(1)}(0,y,-D_x^{-1})=L_n(x,y)$ ${}_RH_n^{(1)}(x,0,y)=L_n(x,y)$	2-Variable Laguerre [20]	$L_n(x,y)=n!\sum_{r=0}^{n}\frac{(-x)^r y^{n-r}}{(r!)^2(n-r)!}$
VI.	$z=0$	${}_RH_n^{(s)}(x,y,0)=R_n(x,y)$	2-Variable Legendre [19]	$R_n(x,y)=(n!)^2\sum_{r=0}^{n}\frac{y^r(-x)^{n-r}}{(r!)^2[(n-r)!]^2}$
VII.	$x=0,\ y\to x,$ $z\to yD_yy$	${}_SH_n^{(s)}(0,x,yD_yy)=e_n^{(s)}(x,y)$	2-Variable truncated order of s [14]	$e_n^{(s)}(x,y)=n!\sum_{k=0}^{[\frac{n}{s}]}\frac{x^{n-sk}y^k}{(n-sk)!}$
VIII.	$s=2;\ x=0$	${}_SH_n^{(2)}(0,y,z)=H_n(y,z)$	2-Variable Hermite-Kampé de Fériet [2]	$H_n(x,y)=n!\sum_{k=0}^{[\frac{n}{2}]}\frac{y^k x^{n-2k}}{k!(n-2k)}$
IX.	i. $s=2;x=0,$ $y\to D_x^{-1},z\to y$ ii.$s=2;\ x=0,$ $y\to x,z\to y$	${}_SH_n^{(2)}(0,D_x^{-1},y)=G_n(x,y)$ ${}_RH_n^{(2)}(0,x,y)=G_n(x,y)$	Hermite Type [15]	$G_n(x,y)=n!\sum_{k=0}^{[\frac{n}{2}]}\frac{y^k x^{n-2k}}{k![(n-2k)!]^2}$
X.	i.$x\to(\frac{x^2-1}{4}),$ $y\to x,\ z=0$ ii.$s=1;x\to(\frac{1-x}{2}),$ $y\to(\frac{1+x}{2}),\ z=0$	${}_SH_n^{(s)}(\frac{x^2-1}{4},x,0)=P_n(x)$ ${}_RH_n^{(1)}(\frac{1-x}{2},\frac{1+x}{2},0)=P_n(x)$	Legendre [26]	$P_n(x)=n!\sum_{r=0}^{[\frac{n}{2}]}\frac{(x^2-1)^k x^{n-2k}}{2^{2k}(k!)^2(n-2k)!}$
XI.	$s=3;,\ x\to zD_z\ z,$ $y\to x,z\to y$	${}_SH_n^{(3)}(zD_z\ z,x,y)=\ H_n^{(3,2)}(x,y,z)$	Bell-Type [21]	$H_n^{(3,2)}(x,y,z)=n!\sum_{k=0}^{[\frac{n}{3}]}\frac{y^kH_{n-3k}^{(2)}(x,z)}{k!(n-3k)!}$ $=n!\sum_{k=0}^{[\frac{n}{3}]}\sum_{r=0}^{[\frac{n-3k}{2}]}\frac{y^kz^rx^{n-3k-2r}}{k!r!(n-3k-2r)!}$

Next, we recall that the polynomial sequence $\{\mathtt{s}_n(x)\}_{n=0}^{\infty}$ ($\mathtt{s}_n(x)$ being a polynomial of degree n) is called Sheffer A-type zero [26, p.222 (Theorem 72)], (which we shall hereafter call Sheffer-type), if $\mathtt{s}_n(x)$ possesses the exponential generating function of the form

$$A(t)exp(xH(t))=\sum_{n=0}^{\infty}\mathtt{s}_n(x)\frac{t^n}{n!}, \tag{1.24}$$

where $A(t)$ and $H(t)$ have (at least the formal) expansions:

$$A(t)=\sum_{n=0}^{\infty}A_n\frac{t^n}{n!}, \qquad A_0\neq 0 \tag{1.25}$$

and

$$H(t)=\sum_{n=1}^{\infty}H_n\frac{t^n}{n!}, \qquad H_1\neq 0, \tag{1.26}$$

respectively.

Properties of Appell and Sheffer sequences are naturally handled within the framework of modern classical umbral calculus by Roman [27]. In view of the following result [27, p.17], the Sheffer sequences can be alternatively defined as:

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Let $f(t)$ be a delta series and $g(t)$ be an invertible series of the following form:

$$f(t) = \sum_{n=0}^{\infty} f_n \frac{t^n}{n!}, \qquad f_0 = 0, \quad f_1 \neq 0 \tag{1.27}$$

and

$$g(t) = \sum_{n=0}^{\infty} g_n \frac{t^n}{n!}, \qquad g_0 \neq 0. \tag{1.28}$$

Then there exists a unique sequence $\mathtt{s}_n(x)$ of polynomials satisfying the orthogonality conditions

$$\langle g(t)f(t)^k | \mathtt{s}_n(x) \rangle = n!\ \delta_{n,k}, \qquad for\ all\ n, k \geq 0. \tag{1.29}$$

Also, in view of Roman [27, p.18 (Theorem 2.3.4)], the polynomial sequence $\mathtt{s}_n(x)$ is uniquely determined by two (formal) power series given by equations (1.27) and (1.28). The exponential generating function of $\mathtt{s}_n(x)$ is then given by

$$\frac{1}{g(f^{-1}(t))} exp\big(x f^{-1}(t)\big) = \sum_{n=0}^{\infty} \mathtt{s}_n(x) \frac{t^n}{n!}, \tag{1.30}$$

for all $x \in \mathbb{C}$, where $f^{-1}(t)$ is the compositional inverse of $f(t)$. In view of equations (1.24) and (1.30), we have

$$A(t) = \frac{1}{g(f^{-1}(t))} \tag{1.31}$$

and

$$H(t) = f^{-1}(t) \tag{1.32}$$

The sequence $\mathtt{s}_n(x)$ in equation (1.29) is the Sheffer sequence for the pair $(g(t), f(t))$. The Sheffer sequence for $(1, f(t))$ is called the associated Sheffer sequence for $f(t)$ and the Sheffer sequence for $(g(t), t)$ becomes the Appell sequence for $g(t)$ [27, p.17].

The Sheffer class contains very important sequences such as the Hermite, Laguerre, Bernoulli, Poisson-Charlier polynomials etc. These polynomials are important from the view point of applications in physics, number theory and in many other branches of mathematics. We present the lists of some known Sheffer and associated Sheffer families in Tables 2 and 3 respectively.

Table 2. Some known Sheffer polynomials

S. No.	$g(t); A(t)$	$f(t); H(t)$	Generating functions	$\hat{M}(X,D)$ and $\hat{P}(D)$	Polynomials
I.	$e^{(\frac{t}{v})^r}; e^{-t^r}$	$\frac{t}{v}; vt$	$exp(vxt - t^r) = \sum_{n=0}^{\infty} H_{n,r,v}(x) \frac{t^n}{n!}$	$\hat{M} = vX - \frac{r}{v^{r-1}} D^{r-1}$ $\hat{P} = \frac{D}{v}$	Generalized Hermite Polynomials $H_{n,r,v}(x)$ [25]
II.	$(1-t)^{-\nu-1}$ $(1-t)^{-\nu-1}$	$\frac{t}{t-1}; \frac{t}{t-1}$	$\frac{1}{(1-t)^{\nu+1}} exp(\frac{xt}{t-1}) = \sum_{n=0}^{\infty} L_n^{(\nu)}(x) t^n$	$\hat{M} = -XD^2 + (2X - \nu - 1)D - X + \nu + 1,$ $\hat{P} = \frac{D}{D-1}$	Generalized Laguerre Polynomials $n! L_n^{(\nu)}(x)$ [1, 26]
III.	$\frac{2}{e^t-1}; \frac{t}{1-t}$ $(1-t)^{-\nu-1}$	$\frac{e^t-1}{e^t+1}; ln(\frac{1+t}{1-t})$	$\frac{t}{1-t}(\frac{1+t}{1-t})^x = \sum_{n=0}^{\infty} P_n(x) t^n$	$\hat{M} = \frac{X(1+e^D)^2}{2e^D} - \frac{(1+e^D)^2}{2(1-e^D)},$ $\hat{P} = \frac{e^D-1}{e^D+1}$	Pidduck Polynomials $P_n(x)$ [8, 4]

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IV.	$(1-t)^{-\beta}; e^{\beta t}$	$ln(1-t); 1-e^t$	$exp\big(\beta t + x(1-e^t)\big) = \sum_{n=0}^{\infty} a_n^{(\beta)}(x)t^n$	$\hat{M} = XD - X + \beta$, $\hat{P} = ln(1-D)$	Acturial Polynomials $a_n^{(\beta)}(x)$ [8]
V.	$exp\big(a(e^t-1)\big)$; e^{-t}	$a(e^t-1)$; $ln(1+\frac{t}{a})$;	$e^{-t}(1+\frac{t}{a}) = \sum_{n=0}^{\infty} c_n(x;a)\frac{t^n}{n!}$	$\hat{M} = \frac{X}{ae^D} - 1$, $\hat{P} = a(e^D-1)$	Poisson-Charlier Polynomials $c_n(x;a)$ [5, 23]
VI.	$(1+e^{\lambda t})^{\mu}$; $\big(1+(1+t)^{\lambda}\big)^{-\mu}$	e^t-1; $ln(1+t)$	$\big(1+(1+t)^{\lambda}\big)^{-\mu}(1+t)^x = \sum_{n=0}^{\infty} s_n(x;\lambda,\mu)\frac{t^n}{n!}$	$\hat{M} = Xe^{-D} - \frac{\mu\lambda e^{(\lambda-1)D}}{1+e^{\lambda D}}$, $\hat{P} = e^D - 1$	Peters Polynomials $s_n(x;\lambda,\mu)$ [8]
VII.	$\frac{t}{e^t-1}; \frac{t}{ln(1+t)}$	e^t-1; $ln(1+t)$	$\frac{t}{ln(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!}$	$\hat{M} = Xe^{-D} + \frac{D+e^{-D}-1}{D(e^D-1)}$, $\hat{P} = e^D - 1$	Bernoulli Polynomials of the second kind $b_n(x)$ [23]
VIII.	$\frac{1}{2}(1+e^t); \frac{2}{2+t}$	e^t-1; $ln(1+t)$	$\frac{2}{2+t}(1+t)^x = \sum_{n=0}^{\infty} r_n(x)\frac{t^n}{n!}$	$\hat{M} = Xe^{-D} - \frac{1}{(e^D+1)}$, $\hat{P} = e^D - 1$	Related Polynomials $r_n(x)$ [23]
IX.	$sect; \frac{1}{\sqrt{1+t^2}}$	$tant; arctan(t)$ $ln(1+t)$	$\frac{1}{\sqrt{1+t^2}}exp(xarctan(t)) = \sum_{n=0}^{\infty} R_n(x)\frac{t^n}{n!}$	$\hat{M} = (X - tanD)cos^2D$, $\hat{P} = tanD$	Hahn Polynomials $R_n(x)$ [7]
X.	$\frac{1+t}{(1-t)^a}$; $(1-4t)^{\frac{-1}{2}}(\frac{2}{1+\sqrt{1-4t}})^{a-1}$	$\frac{1}{4} - \frac{1}{4}(\frac{1+t}{1-t})^2$; $\frac{-4t}{(1+\sqrt{1-4t})^2}$	$(1-4t)^{\frac{-1}{2}}(\frac{2}{1+\sqrt{1-4t}})^{a-1} \times exp(\frac{-4xt}{(1+\sqrt{1-4t})^2})$ $\sum_{n=0}^{\infty} R_n(a,x)t^n$	$\hat{M} = \frac{-X(1-D)^3}{(1+D)} + \frac{(1-D)^3}{(1+D)^2} + \frac{a(1-D)^3}{(1+D)}$, $\hat{P} = \frac{1}{4} - \frac{1}{4}(\frac{1+D}{1-D})^2$	Shively Pseudo-Laguerre Polynomials $R_n(a,x)$ [26]

Table 3. Some known associated Sheffer polynomials

S. No.	$f(t); H(t)$	**Generating functions**	$\hat{M}(X,D)$ **and** $\hat{P}(D)$	**Polynomials**
I.	$\frac{e^t-1}{e^t+t}; ln(\frac{1+t}{1-t})$	$(\frac{e^t-1}{e^t+t})^x = \sum_{n=0}^{\infty} M_n(x)\frac{t^n}{n!}$	$\hat{M} = \frac{X(1+e^D)^2}{2e^D}$, $\hat{P} = \frac{e^D-1}{e^D+1}$	Mittag-Leffler polynomials $M_n(x)$ [3]
II.	$ln(1+t); e^t-1$	$exp\big(x(e^t-1)\big) = \sum_{n=0}^{\infty} \phi_n(x)\frac{t^n}{n!}$	$\hat{M} = X(1+D)$, $\hat{P} = ln(1+D)$	Exponential polynomials $\phi_n(x)$ [6]
III.	$ln(1+t); e^t-1$	$(1+t)^x = \sum_{n=0}^{\infty}(x)_n\frac{t^n}{n!}$	$\hat{M} = Xe^{-D}$, $\hat{P} = e^D - 1$	Lower factorial polynomials $(x)_n$ [27]
IV.	$t - \frac{1}{2}t^2; 1-\sqrt{1-2t}$	$exp\big(x(1-\sqrt{1-2t})\big) = \sum_{n=0}^{\infty} p_n(x)\frac{t^n}{n!}$	$\hat{M} = \frac{X}{1-D}$, $\hat{P} = -t\frac{1}{2}D^2 + D$	Bessel polynomials $p_n(x)$ [9, 24]

In the present paper, the Legendre-Gould Hopper based Sheffer polynomials are introduced and framed within the context of monomiality principle. Some operational and integral formulas for these polynomials are derived. Further, some results are obtained for some members of the Legendre-Gould Hopper based Sheffer polynomial families. The paper is also concluded with the determinantal definitions of the Legendre-Gould Hopper based Sheffer polynomials(LeGHSP) ${}_{RH^{(s)}}\mathtt{s}_n(x,y,z)$ and ${}_{SH^{(s)}}\mathtt{s}_n(x,y,z)$.

2. Legendre-Gould Hopper based Sheffer polynomials

To generate the Legendre-Gould Hopper based Sheffer polynomials (LeGHSP) denoted by ${}_{RH^{(s)}}\mathtt{s}_n(x,y,z)$ and ${}_{SH^{(s)}}\mathtt{s}_n(x,y,z)$, we prove the following results:

Theorem 2.1. *The Legendre-Gould Hopper based Sheffer polynomials (LeGHSP) denoted by* ${}_{RH^{(s)}}\mathtt{s}_n(x,y,z)$ *and* ${}_{SH^{(s)}}\mathtt{s}_n(x,y,z)$ *are defined by the following generating functions*

$$\frac{1}{g(f^{-1}(t))}exp\big(z(f^{-1}(t))^s\big)\ C_0\big(xf^{-1}(t)\big)\ C_0\big(-yf^{-1}(t)\big) = \sum_{n=0}^{\infty} {}_{RH^{(s)}}\mathtt{s}_n(x,y,z)\frac{t^n}{n!}, \quad (2.1)$$

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or, equivalently

$$A(t)exp\big(z(H(t))^s\big)\ C_0\big(xH(t)\big)\ C_0\big(-yH(t)\big)=\sum_{n=0}^{\infty}{}_{R}H^{(s)}\mathtt{s}_n(x,y,z)\frac{t^n}{n!} \tag{2.2}$$

and

$$\frac{1}{g(f^{-1}(t))}exp\big(yf^{-1}(t)+z(f^{-1}(t))^s\big)\ C_0\big(-x(f^{-1}(t))^2\big)=\sum_{n=0}^{\infty}{}_{S}H^{(s)}\mathtt{s}_n(x,y,z)\frac{t^n}{n!}, \tag{2.3}$$

or, equivalently

$$A(t)exp\big(yH(t)+z(H(t))^s\big)\ C_0\big(-x(H(t))^2\big)=\sum_{n=0}^{\infty}{}_{S}H^{(s)}\mathtt{s}_n(x,y,z)\frac{t^n}{n!}, \tag{2.4}$$

respectively.

Proof. Replacing x in the l.h.s. and r.h.s. of equation (1.30) by the multiplicative operator $\hat{M}_{RH}$ of the LeGHP $\frac{{}_RH_n^{(s)}(x,y,z)}{n!}$, we have

$$\frac{1}{g(f^{-1}(t))}exp\big(\hat{M}_{RH}f^{-1}(t)\big)=\sum_{n=0}^{\infty}\mathtt{s}_n\big(\hat{M}_{RH}\big)\frac{t^n}{n!}. \tag{2.5}$$

Using the expression of $\hat{M}_{RH}$ given in equation (1.8) and then decoupling the exponential operator in the l.h.s. of the resultant equation by using the Crofton-type identity [18, p.12]

$$f\left(y+m\lambda\frac{d^{m-1}}{dy^{m-1}}\right)\{1\}=exp\left(\lambda\frac{d^m}{dx^m}\right)\{f(y)\}, \tag{2.6}$$

we get

$$\frac{1}{g(f^{-1}(t))}exp\left(z\frac{\partial^s}{\partial y^s}\right)exp\left(\left(D_x^{-1}+D_y^{-1}\right)f^{-1}(t)\right)=\sum_{n=0}^{\infty}\mathtt{s}_n\left(-D_x^{-1}+D_y^{-1}+sz\frac{\partial^{s-1}}{\partial y^{s-1}}\right)\frac{t^n}{n!}, \tag{2.7}$$

Now, using the Weyl identity [18]

$$e^{\hat{A}+\hat{B}}=e^{\hat{A}}e^{\hat{B}}e^{-k/2},\quad \big([\hat{A},\hat{B}]=k,\ k\in\mathbb{C}\big), \tag{2.8}$$

we get

$$\frac{1}{g(f^{-1}(t))}exp\left(z\frac{\partial^s}{\partial y^s}\right)exp\big(-D_x^{-1}f^{-1}(t)\big)exp\big(D_y^{-1}f^{-1}(t)\big)=\sum_{n=0}^{\infty}\mathtt{s}_n\left(-D_x^{-1}+D_y^{-1}+sz\frac{\partial^{s-1}}{\partial y^{s-1}}\right)\frac{t^n}{n!}, \tag{2.9}$$

Now, expanding the first exponential in the l.h.s. of equation (2.9) and using definition (1.5), we find

$$\frac{1}{g(f^{-1}(t))}exp\big(z(f^{-1}(t))^s\big)C_0\big(xf^{-1}(t)\big)C_0\big(-yf^{-1}(t)\big)=\sum_{n=0}^{\infty}\mathtt{s}_n\left(-D_x^{-1}+D_y^{-1}+sz\frac{\partial^{s-1}}{\partial y^{s-1}}\right)\frac{t^n}{n!}, \tag{2.10}$$

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Finally, denoting the resultant LeGHSP in the r.h.s. by ${}_{RH^{(s)}}\mathtt{s}_n(x,y,z)$, that is

$$ {}_{RH^{(s)}}\mathtt{s}_n(x,y,z)=\mathtt{s}_n\big(\hat{M}_{RH}\big)=\mathtt{s}_n\bigg(-D_x^{-1}+D_y^{-1}+sz\frac{\partial^{s-1}}{\partial y^{s-1}}\bigg), \tag{2.11}$$

we get assertion (2.1). Also, in view of equations (1.31) and (1.32), generating function (2.1) can be expressed equivalently as equation (2.2). Making use of (1.10) and using a similar argument as in the above proof of (2.1), we can establish the assertions (2.3) and (2.4). □

Next, to show that the LeGHSP ${}_{RH^{(s)}}\mathtt{s}_n(x,y,z)$ and ${}_{SH^{(s)}}\mathtt{s}_n(x,y,z)$ satisfy the monomiality property, we prove the following result:

Theorem 2.2. *The Legendre-Gould Hopper based Sheffer polynomials ${}_{RH^{(s)}}\mathtt{s}_n(x,y,z)$ and ${}_{SH^{(s)}}\mathtt{s}_n(x,y,z)$ are quasi-monomial with respect to the following multiplicative and derivative operators:*

$$\hat{M}_{RH\mathtt{s}}:=\bigg(-D_x^{-1}+D_y^{-1}+szD_y^{s-1}-\frac{g'(D_yyD_y)}{g(D_yyD_y)}\bigg)\frac{1}{f'(D_yyD_y)}, \tag{2.12}$$

$$\hat{M}_{SH\mathtt{s}}:=\bigg(y+2D_x^{-1}D_y+szD_y^{s-1}-\frac{g'(D_y)}{g(D_y)}\bigg)\frac{1}{f'(D_y)}, \tag{2.13}$$

or, equivalently

$$\hat{M}_{RH\mathtt{s}}:=\big(-D_x^{-1}+D_y^{-1}+szD_y^{s-1}\big)H'\big(H^{-1}(D_yyD_y)\big)+\frac{A'\big(H^{-1}(D_yyD_y)\big)}{A\big(H^{-1}(D_yyD_y)\big)}, \tag{2.14}$$

$$\hat{M}_{SH\mathtt{s}}:=\big(y+2D_x^{-1}D_y+szD_y^{s-1}\big)H'\big(H^{-1}(D_y)\big)+\frac{A'\big(H^{-1}(D_y)\big)}{A\big(H^{-1}(D_y)\big)} \tag{2.15}$$

and

$$\hat{P}_{RH\mathtt{s}}:=f(D_yyD_y), \tag{2.16}$$

$$\hat{P}_{SH\mathtt{s}}:=f(D_y), \tag{2.17}$$

or, equivalently

$$\hat{P}_{RH\mathtt{s}}:=H^{-1}(D_yyD_y), \tag{2.18}$$

$$\hat{P}_{SH\mathtt{s}}:=H^{-1}(D_y), \tag{2.19}$$

respectively.

Proof. Consider the following identity:

$$(D_yyD_y)\{C_0\big(-yf^{-1}(t)\big)\}=f^{-1}(t)\ C_0\big(-yf^{-1}(t)\big). \tag{2.20}$$

Since f^{-1} denotes the compositional inverse of the function f and $f(t)$ has an expansion (1.27) in powers of t, therefore we have

$$f(D_yyD_y)\{C_0\big(-yf^{-1}(t)\big)\}=t\ C_0\big(-yf^{-1}(t)\big). \tag{2.21}$$

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Differentiating equation (2.5) partially with respect to t and in view of relation (2.11), we find

$$\left(\left(\hat{M}_{RH}-\frac{g'\big(f^{-1}(t)\big)}{g\big(f^{-1}(t)\big)}\right)\frac{1}{f'\big(f^{-1}(t)\big)}\right)\frac{1}{g\big(f^{-1}(t)\big)}exp\big(\hat{M}_{RH}f^{-1}(t)\big)=\sum_{n=0}^{\infty}{}_{RH^{(s)}}\mathtt{s}_{n+1}(x,y,z)\frac{t^n}{n!}, \tag{2.22}$$

which on using monomiality principle equation (1.22) and definition (1.1) with $t=f^{-1}(t)$ gives

$$\left(\left(\hat{M}_{RH}-\frac{g'\big(f^{-1}(t)\big)}{g\big(f^{-1}(t)\big)}\right)\frac{1}{f'\big(f^{-1}(t)\big)}\right)\frac{1}{g\big(f^{-1}(t)\big)}exp\big(z(f^{-1}(t))^s\big)\ C_0\big(xf^{-1}(t)\big)$$
$$\times\ C_0\big(-yf^{-1}(t)\big)=\sum_{n=0}^{\infty}{}_{RH^{(s)}}\mathtt{s}_{n+1}(x,y,z)\frac{t^n}{n!}. \tag{2.23}$$

Since $g(t)$ is an invertible series and $f(t)$ is a delta series of t therefore $\frac{g'\left(f^{-1}(t)\right)}{g\left(f^{-1}(t)\right)}$ and $\frac{1}{f'\left(f^{-1}(t)\right)}$ possess power series expansions of $f^{-1}(t)$. Thus, in view of relation (2.20), the above equation becomes

$$\left(\left(\hat{M}_{RH}-\frac{g'\big(D_yyD_y\big)}{g\big(D_yyD_y\big)}\right)\frac{1}{f'\big(D_yyD_y\big)}\right)\left\{\frac{1}{g\big(f^{-1}(t)\big)}exp\big(z(f^{-1}(t))^s\big)\ C_0\big(xf^{-1}(t)\big)\right.$$
$$\left.\times\ C_0\big(-yf^{-1}(t)\big)\right\}=\sum_{n=0}^{\infty}{}_{RH^{(s)}}\mathtt{s}_{n+1}(x,y,z)\frac{t^n}{n!}, \tag{2.24}$$

which on using generating function (2.1) becomes

$$\left(\left(\hat{M}_{RH}-\frac{g'\big(D_yyD_y\big)}{g\big(D_yyD_y\big)}\right)\frac{1}{f'\big(D_yyD_y\big)}\right)\left\{\sum_{n=0}^{\infty}{}_{RH^{(s)}}\mathtt{s}_{n}(x,y,z)\frac{t^n}{n!}\right\}=\sum_{n=0}^{\infty}{}_{RH^{(s)}}\mathtt{s}_{n+1}(x,y,z)\frac{t^n}{n!}. \tag{2.25}$$

Adjusting the summation in the l.h.s. of the above equation and then equating the coefficients of like powers of t, we find

$$\left(\left(\hat{M}_{RH}-\frac{g'\big(D_yyD_y\big)}{g\big(D_yyD_y\big)}\right)\frac{1}{f'\big(D_yyD_y\big)}\right)\left\{{}_{RH^{(s)}}\mathtt{s}_{n}(x,y,z)\right\}={}_{RH^{(s)}}\mathtt{s}_{n+1}(x,y,z), \tag{2.26}$$

which, in view of equation (1.12) shows that the multiplicative operator for ${}_{RH^{(s)}}\mathtt{s}_{n}(x,y,z)$ is given as:

$$\hat{M}_{RH\mathtt{s}}=\left(\hat{M}_{RH}-\frac{g'\big(D_yyD_y\big)}{g\big(D_yyD_y\big)}\right)\frac{1}{f'\big(D_yyD_y\big)}. \tag{2.27}$$

Finally, using equation (1.8) in the r.h.s of above equation, we get assertion (2.12).
Next, consider the following identity

$$D_y\{exp\big(yf^{-1}(t)+z\big(f^{-1}(t)\big)^s\big)\}=f^{-1}(t)\ exp\big(yf^{-1}(t)+z\big(f^{-1}(t)\big)^s\big), \tag{2.28}$$

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and by making use of (1.10) and using a similar argument as in the above proof of (2.12), we establish the assertion (2.13).

Again, in view of identity (2.21), we have

$$f(D_yyD_y)\left\{\frac{1}{g\big(f^{-1}(t)\big)}exp\big(z(f^{-1}(t))^s\big)\ C_0\big(xf^{-1}(t)\big)\ C_0\big(-yf^{-1}(t)\big)\right\}$$
$$= t\ \frac{1}{g\big(f^{-1}(t)\big)}exp\big(z(f^{-1}(t))^s\big)\ C_0\big(xf^{-1}(t)\big)\ C_0\big(-yf^{-1}(t)\big), \quad (2.29)$$

which on using generating function (2.1) becomes

$$f(D_yyD_y)\left\{\sum_{n=0}^{\infty} {}_{R}H^{(s)}\mathbf{s}_n(x,y,z)\frac{t^n}{n!}\right\} = \sum_{n=1}^{\infty} {}_{R}H^{(s)}\mathbf{s}_{n-1}(x,y,z)\frac{t^n}{(n-1)!}. \quad (2.30)$$

Adjusting the summation in the l.h.s. of the above equation and then equating the coefficients of like powers of t, we get

$$f(D_yyD_y)\left\{{}_{R}H^{(s)}\mathbf{s}_n(x,y,z)\right\} = n\ {}_{R}H^{(s)}\mathbf{s}_{n-1}(x,y,z), \qquad n\geq 1, \quad (2.31)$$

which in view of equation (1.13) yields assertion (2.16). Similarly, we can obtain the assertion (2.17). Also, in view of relations (1.31) and (1.32), assertions (2.12), (2.13), (2.16) and (2.17) can be expressed equivalently as equations (2.14), (2.15), (2.18) and (2.19), respectively. □

In view of equation (1.20) and (1.21) and using equations (2.12), (2.13), (2.14) and (2.15), we deduce the following consequence of Theorem 2.2.

Corollary 2.1. *The Legendre-Gould Hopper based Sheffer polynomials* ${}_{R}H^{(s)}\mathbf{s}_n(x,y,z)$ *and* ${}_{S}H^{(s)}\mathbf{s}_n(x,y,z)$ *have the following explicit representations:*

$$ {}_{R}H^{(s)}\mathbf{s}_n(x,y,z) = \hat{M}_{RH\mathbf{s}}\{1\}, \quad (2.32)$$

$$ {}_{S}H^{(s)}\mathbf{s}_n(x,y,z) = \hat{M}_{SH\mathbf{s}}\{1\} \quad (2.33)$$

that is,

$$ {}_{R}H^{(s)}\mathbf{s}_n(x,y,z) = \left(\left(-D_x^{-1}+D_y^{-1}+szD_y^{s-1}-\frac{g'(D_yyD_y)}{g(D_yyD_y)}\right)\frac{1}{f'(D_yyD_y)}\right)^n\{1\}, \quad (2.34)$$

$$ {}_{S}H^{(s)}\mathbf{s}_n(x,y,z) = \left(\left(y+2D_x^{-1}D_y+szD_y^{s-1}-\frac{g'(D_y)}{g(D_y)}\right)\frac{1}{f'(D_y)}\right)^n\{1\}, \quad (2.35)$$

or, equivalently

$$ {}_{R}H^{(s)}\mathbf{s}_n(x,y,z) = \left(\big(-D_x^{-1}+D_y^{-1}+szD_y^{s-1}\big)H'\big(H^{-1}(D_yyD_y)\big)+\frac{A'\big(H^{-1}(D_yyD_y)\big)}{A\big(H^{-1}(D_yyD_y)\big)}\right)^n\{1\}, \quad (2.36)$$

$$ {}_{S}H^{(s)}\mathbf{s}_n(x,y,z) = \left(\big(y+2D_x^{-1}D_y+szD_y^{s-1}\big)H'\big(H^{-1}(D_y)\big)+\frac{A'\big(H^{-1}(D_y)\big)}{A\big(H^{-1}(D_y)\big)}\right)^n\{1\}, \quad (2.37)$$

respectively.

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Theorem 2.3. *The Legendre-Gould Hopper based Sheffer polynomials ${}_{RH^{(s)}}\mathtt{s}_n(x,y,z)$ and ${}_{SH^{(s)}}\mathtt{s}_n(x,y,z)$ are the solutions of the following differential equations:*

$$\left(\left(-D_x^{-1}+D_y^{-1}+szD_y^{s-1}-\frac{g'(D_yyD_y)}{g(D_yyD_y)}\right)\frac{f(D_yyD_y)}{f'(D_yyD_y)}-n\right){}_{RH^{(s)}}\mathtt{s}_n(x,y,z)=0, \tag{2.38}$$

or, equivalently

$$\Bigg(\bigg(\Big(-D_x^{-1}+D_y^{-1}+szD_y^{s-1}\Big)H'\big(H^{-1}(D_yyD_y)\big) +\frac{A'\big(H^{-1}(D_yyD_y)\big)}{A\big(H^{-1}(D_yyD_y)\big)}\bigg)H^{-1}(D_yyD_y)-n\Bigg){}_{RH^{(s)}}\mathtt{s}_n(x,y,z)=0 \tag{2.39}$$

and

$$\left(\left(y+2D_x^{-1}D_y+szD_y^{s-1}-\frac{g'(D_y)}{g(D_y)}\right)\frac{f(D_y)}{f'(D_y)}-n\right){}_{SH^{(s)}}\mathtt{s}_n(x,y,z)=0, \tag{2.40}$$

or, equivalently

$$\left(\left((y+2D_x^{-1}D_y+szD_y^{s-1})H'\big(H^{-1}(D_y)\big)+\frac{A'\big(H^{-1}(D_y)\big)}{A\big(H^{-1}(D_y)\big)}\right)H^{-1}(D_y)-n\right){}_{SH^{(s)}}\mathtt{s}_n(x,y,z)=0, \tag{2.41}$$

respectively.

Proof. Using equations (2.12) and (2.16) in the corresponding equation (1.16) for the Legendre-Gould Hopper based Sheffer polynomials ${}_{RH^{(s)}}\mathtt{s}_n(x,y,z)$, we get assertion (2.38). Also, using equations (2.14) and (2.15) in the corresponding equation (1.17) for the Legendre-Gould Hopper based Sheffer polynomials ${}_{SH^{(s)}}\mathtt{s}_n(x,y,z)$, we get assertion (2.40). Using similar argument, we can get assertion (2.39) and assertion (2.41). □

Remark 2.1. Since, for $g(t)=1$ (or $A(t)=1$), the Sheffer polynomials $\mathtt{s}_n(x)$ reduce to the associated Sheffer polynomials $\mathfrak{s}_n(x)$. Therefore, taking $g(t)=1$ (or $A(t)=1$) in the results obtained in Theorems 2.1-2.3 and denoting the resultant Legendre-Gould Hopper based associated Sheffer polynomials (LeGHASP) by ${}_{RH^{(s)}}\mathfrak{s}_n(x,y,z)$ and ${}_{SH^{(s)}}\mathfrak{s}_n(x,y,z)$, we deduce the following consequences of Theorems 2.1-2.3 :

Corollary 2.2. *The Legendre-Gould Hopper based associated Sheffer polynomials (LeGHASP) denoted by ${}_{RH^{(s)}}\mathfrak{s}_n(x,y,z)$ and ${}_{SH^{(s)}}\mathfrak{s}_n(x,y,z)$ are defined by the following generating functions*

$$exp\big(z(f^{-1}(t))^s\big)\ C_0\big(xf^{-1}(t)\big)\ C_0\big(-yf^{-1}(t)\big)=\sum_{n=0}^{\infty}{}_{RH^{(s)}}\mathfrak{s}_n(x,y,z)\frac{t^n}{n!}, \tag{2.42}$$

or, equivalently

$$exp\big(z(H(t))^s\big)\ C_0\big(xH(t)\big)\ C_0\big(-yH(t)\big)=\sum_{n=0}^{\infty}{}_{RH^{(s)}}\mathfrak{s}_n(x,y,z)\frac{t^n}{n!} \tag{2.43}$$

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and

$$exp\big(yf^{-1}(t)+z(f^{-1}(t))^s\big)\ C_0\big(-x(f^{-1}(t))^2\big)=\sum_{n=0}^{\infty}{}_{SH^{(s)}}\mathfrak{s}_n(x,y,z)\frac{t^n}{n!}, \tag{2.44}$$

or, equivalently

$$exp\big(yH(t)+z(H(t))^s\big)\ C_0\big(-x(H(t))^2\big)=\sum_{n=0}^{\infty}{}_{SH^{(s)}}\mathfrak{s}_n(x,y,z)\frac{t^n}{n!}, \tag{2.45}$$

respectively.

Corollary 2.3. *The Legendre-Gould Hopper based associated Sheffer polynomials* ${}_{RH^{(s)}}\mathfrak{s}_n(x,y,z)$ *and* ${}_{SH^{(s)}}\mathfrak{s}_n(x,y,z)$ *are quasi-monomial with respect to the following multiplicative and derivative operators:*

$$\hat{M}_{RH\mathfrak{s}}:=\Big(-D_x^{-1}+D_y^{-1}+szD_y^{s-1}\Big)\frac{1}{f'(D_yyD_y)}, \tag{2.46}$$

$$\hat{M}_{SH\mathfrak{s}}:=\Big(y+2D_x^{-1}D_y+szD_y^{s-1}\Big)\frac{1}{f'(D_y)}, \tag{2.47}$$

or, equivalently

$$\hat{M}_{RH\mathfrak{s}}:=\big(-D_x^{-1}+D_y^{-1}+szD_y^{s-1}\big)H'\big(H^{-1}(D_yyD_y)\big), \tag{2.48}$$

$$\hat{M}_{SH\mathfrak{s}}:=\big(y+2D_x^{-1}D_y+szD_y^{s-1}\big)H'\big(H^{-1}(D_y)\big) \tag{2.49}$$

and

$$\hat{P}_{RH\mathfrak{s}}:=f(D_yyD_y), \tag{2.50}$$

$$\hat{P}_{SH\mathfrak{s}}:=f(D_y), \tag{2.51}$$

or, equivalently

$$\hat{P}_{RH\mathfrak{s}}:=H^{-1}(D_yyD_y), \tag{2.52}$$

$$\hat{P}_{SH\mathfrak{s}}:=H^{-1}(D_y), \tag{2.53}$$

respectively.

Corollary 2.4. *The Legendre-Gould Hopper based associated Sheffer polynomials* ${}_{RH^{(s)}}\mathfrak{s}_n(x,y,z)$ *and* ${}_{SH^{(s)}}\mathfrak{s}_n(x,y,z)$ *are the solutions of the following differential equations:*

$$\Big(\big(-D_x^{-1}+D_y^{-1}+szD_y^{s-1}\big)\frac{f(D_yyD_y)}{f'(D_yyD_y)}-n\Big)\,{}_{RH^{(s)}}\mathfrak{s}_n(x,y,z)=0, \tag{2.54}$$

or, equivalently

$$\Big(\Big(\Big(-D_x^{-1}+D_y^{-1}+szD_y^{s-1}\Big)H'\big(H^{-1}(D_yyD_y)\big)\Big)H^{-1}(D_yyD_y)-n\Big)\,{}_{RH^{(s)}}\mathfrak{s}_n(x,y,z)=0 \tag{2.55}$$

and

$$\Big(\big(y+2D_x^{-1}D_y+szD_y^{s-1}\big)\frac{f(D_y)}{f'(D_y)}-n\Big)\,{}_{SH^{(s)}}\mathfrak{s}_n(x,y,z)=0, \tag{2.56}$$

or, equivalently

$$\Big(\Big(\big(y+2D_x^{-1}D_y+szD_y^{s-1}\big)H'\big(H^{-1}(D_y)\big)\Big)H^{-1}(D_y)-n\Big)\,{}_{SH^{(s)}}\mathfrak{s}_n(x,y,z)=0, \tag{2.57}$$

respectively.

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We have mentioned special cases of the LeGHP ${}_SH_n^{(s)}(x,y,z)$ and $\frac{{}_RH_n^{(s)}(x,y,z)}{n!}$ in Table 1. Now, for the same choice of the variables and indices the LeGHSP ${}_{RH^{(s)}}\mathsf{s}_n(x,y,z)$ and ${}_{SH^{(s)}}\mathsf{s}_n(x,y,z)$ reduce to the corresponding special cases. We mention these known and new special polynomials related to the Sheffer sequences in Table 4.

Table 4. Special cases of LeGHSP ${}_{RH^{(s)}}\mathsf{s}_n(x,y,z)$ and ${}_{SH^{(s)}}\mathsf{s}_n(x,y,z)$

S. No.	Values of the indices and variables	Relation between LeGHSP ${}_{RH^{(s)}}\mathsf{s}_n(x,y,z)$, ${}_{SH^{(s)}}\mathsf{s}_n(x,y,z)$ and its special case	Name of the special polynomials
I.	$x=0$	${}_{SH^{(s)}}\mathsf{s}_n(0,y,z) = {}_{H^{(s)}}\mathsf{s}_n(y,z)$	Gould-Hopper based Sheffer polynomials (GHSP)
II.	$z=0$	${}_{SH^{(s)}}\mathsf{s}_n(x,y,0) = {}_{2L}\mathsf{s}_n(x,y)$	2-Variable Legendre based Sheffer polynomials (2VLeSP)
III.	i. $s=m;\ x=0,$ $y\to -D_x^{-1},\ z\to y$ ii.$s=m;\ y=0,\ z\to y$	${}_{SH^{(m)}}\mathsf{s}_n(\frac{-x}{t},0,y) = {}_{[m]L}\mathsf{s}_n(x,y)$ ${}_{RH^{(m)}}\mathsf{s}_n(x,0,y) = {}_{[m]L}\mathsf{s}_n(x,y)$	2-Variable Generalized Laguerre Type based Sheffer polynomials (2VGLTSP)
IV.	$s=m-1;\ x=0,$ $y\to x,\ z\to y$	${}_{SH^{(m-1)}}\mathsf{s}_n(0,x,y) = {}_{U^{(m)}}\mathsf{s}_n(x,y)$	Generalized Chebyshev based Sheffer polynomials (GCSP)
V.	i.$s=1, x=0,\ z=0$ ii.$s=1;\ y=0,\ z\to y$	${}_{SH^{(1)}}\mathsf{s}_n(0,y,-D_x^{-1}) = {}_L\mathsf{s}_n(x,y)$ ${}_{RH^{(1)}}\mathsf{s}_n(x,o,y) = {}_L\mathsf{s}_n(x,y)$	2-Variable Laguerre based Sheffer polynomials (2VLSP)
VI.	$z=0$	${}_{RH^{(s)}}\mathsf{s}_n(x,y,0) = {}_R\mathsf{s}_n(x,y)$	2-Variable Legendre based Sheffer polynomials (2VLeSP)
VII.	$x=0,\ y\to x,$ $z\to yD_yy$	${}_{SH^{(s)}}\mathsf{s}_n(0,x,yD_yy) = {}_{e^{(s)}}\mathsf{s}_n(x,y)$	2-Variable truncated based Sheffer polynomials of order s (2VTSP)
VIII.	$s=2;\ x=0$	${}_{SH^{(2)}}\mathsf{s}_n(0,y,z) = {}_H\mathsf{s}_n(y,z)$	2-Variable Hermite-Kamp de Fériet é based Sheffer polynomials (2VHKFSP)
IX.	i. $s=2; x=0,$ $y\to D_x^{-1}, z\to y$ ii.$s=2;\ x=0,$ $y\to x, z\to y$	${}_{SH^{(2)}}\mathsf{s}_n(0,D_x^{-1},y) = {}_G\mathsf{s}_n(x,y)$ ${}_{RH^{(2)}}\mathsf{s}_n(0,x,y) = {}_G\mathsf{s}_n(x,y)$	Hermite Type based Sheffer polynomials (HTSP)
X.	i.$x\to(\frac{x^2-1}{4}),$ $y\to x,\ z=0$ ii.$s=1; x\to(\frac{1-x}{2}),$ $y\to(\frac{1+x}{2}),\ z=0$	${}_{SH^{(s)}}\mathsf{s}_n(\frac{x^2-1}{4},x,0) = {}_P\mathsf{s}_n(x)$ ${}_{RH^{(1)}}\mathsf{s}_n(\frac{1-x}{2},\frac{1+x}{2},0) = {}_P\mathsf{s}_n(x)$	Legendre based Sheffer polynomials (LeSP)
XI.	$s=3;,\ x\to zD_z\ z,$ $y\to x, z\to y$	${}_{SH^{(3)}}\mathsf{s}_n(zD_z\ z,x,y) = {}_{H^{(3,2)}}\mathsf{s}_n(x,y,z)$	Bell-Type based Sheffer polynomials (BTSP)

Remark 2.2. In view of the special cases mentioned in Table 4, the results for the special polynomials related to the Sheffer sequences can be obtained.

Next, we derive certain operational representations for the LeGHSP ${}_{RH^{(s)}}\mathsf{s}_n(x,y,z)$ and ${}_{SH^{(s)}}\mathsf{s}_n(x,y,z)$.

3. Operational and integral representations

To establish the operational representation for the LeGHSP ${}_{RH^{(s)}}\mathsf{s}_n(x,y,z)$ and ${}_{SH^{(s)}}\mathsf{s}_n(x,y,z)$, we prove the following results:

Theorem 3.1. *The following operational representation between the LeGHSP* ${}_{RH^{(s)}}\mathsf{s}_n(x,y,z)$, ${}_{SH^{(s)}}\mathsf{s}_n(x,y,z)$ *and the Sheffer polynomials* $\mathsf{s}_n(x)$ *hold true:*

$$ {}_{RH^{(s)}}\mathsf{s}_n(x,y,z) = exp\left(z\frac{\partial^s}{\partial y^s}\right)\mathsf{s}_n\big(-D_x^{-1}+D_y^{-1}\big) \tag{3.1}$$

and

$$ {}_{SH^{(s)}}\mathsf{s}_n(x,y,z) = exp\left(D_x^{-1}\frac{\partial^2}{\partial y^2}+z\frac{\partial^s}{\partial y^s}\right)\mathsf{s}_n(y), \tag{3.2}$$

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respectively.

Proof. In view of equation (2.11), the proof of (3.1) is direct use of identity (2.6) and similarly the proof of (3.2) can be obtained. □

Theorem 3.2. *The following operational representation between the LeGHSP* ${}_{RH^{(s)}}\mathtt{s}_n(x,y,z)$ *and the 2VLeSP* ${}_{R}\mathtt{s}_n(x,y)$ *holds true:*

$$ {}_{RH^{(s)}}\mathtt{s}_n(x,y,z) = exp\bigg((-1)^s z\frac{\partial^s}{\partial D_x^{-s}}\bigg){}_{R}\mathtt{s}_n(x,y), \tag{3.3} $$

or, equivalently

$$ {}_{RH^{(s)}}\mathtt{s}_n(x,y,z) = exp\bigg(z\frac{\partial^s}{\partial D_y^{-s}}\bigg){}_{R}\mathtt{s}_n(x,y). \tag{3.4} $$

Proof. From equation (2.2), we have

$$ (-1)^s\frac{\partial^s}{\partial D_x^{-s}}\ {}_{RH^{(s)}}\mathtt{s}_n(x,y,z) = \frac{\partial}{\partial z}\ {}_{RH^{(s)}}A_n(x,y,z). \tag{3.5} $$

Since, in view of Table 1(VI), we have

$$ {}_{R}H_n^{(s)}(x,y,0) = R_n(x,y). \tag{3.6} $$

Consequently, from Table 4(VI), we have

$$ {}_{RH^{(s)}}\mathtt{s}_n(x,y,0) = {}_{R}\mathtt{s}_n(x,y). \tag{3.7} $$

Now, solving equation (3.5) subject to initial condition (3.7), we get assertion (3.3). Again using a similar argument as in the proof of (3.3), we establish the assertion (3.4). □

Theorem 3.3. *The following operational representation between the LeGHSP* ${}_{SH^{(s)}}\mathtt{s}_n(x,y,z)$ *and the 2VLeSP* ${}_{2L}\mathtt{s}_n(x,y)$ *holds true:*

$$ {}_{SH^{(s)}}\mathtt{s}_n(x,y,z) = exp\bigg(z\frac{\partial^s}{\partial y^s}\bigg){}_{2L}\mathtt{s}_n(x,y). \tag{3.8} $$

Proof. Using a similar argument as in the proof of Theorem 3.2, we establish the assertion (3.8) of Theorem 3.3. □

Theorem 3.4. *The following operational representation between the LeGHSP* ${}_{RH^{(s)}}\mathtt{s}_n(x,y,z)$ *and the 2VGLTSP* ${}_{[m]L}\mathtt{s}_n(x,y)$ *hold true:*

$$ {}_{RH^{(m)}}\mathtt{s}_n(x,z,y) = exp\bigg(-D_z^{-1}\frac{\partial}{\partial D_x^{-1}}\bigg){}_{[m]L}\mathtt{s}_n(x,y). \tag{3.9} $$

Proof. From equations (1.5) and (2.2), we have

$$ \frac{\partial}{\partial D_z^{-1}}\ {}_{RH^{(m)}}\mathtt{s}_n(x,z,y) = -\frac{\partial}{\partial D_x^{-1}}\ {}_{RH^{(m)}}\mathtt{s}_n(x,z,y). \tag{3.10} $$

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Since, in view of Table 1(III), we have

$$ {}_RH_n^{(m)}(x,0,y) = {}_{[m]}L_n(x,y). \tag{3.11} $$

Consequently, from Table 4(III), we have

$$ {}_{RH^{(m)}}\mathtt{s}_n(x,0,y) = {}_{[m]L}\mathtt{s}_n(x,y). \tag{3.12} $$

Solving equation (3.10) subject to initial condition (3.12), we get assertion (3.9). □

Theorem 3.5. *The following operational representation between the LeGHSP* ${}_{sH^{(s)}}\mathtt{s}_n(x,y,z)$ *and the GHSP* ${}_{H^{(s)}}\mathtt{s}_n(y,z)$ *hold true:*

$$ {}_{sH^{(s)}}\mathtt{s}_n(x,y,z) = exp\left(D_x^{-1}\frac{\partial^2}{\partial y^2}\right){}_{H^{(s)}}\mathtt{s}_n(y,z) \tag{3.13} $$

Proof. From equations (1.5) and (2.4), we have

$$ \frac{\partial^2}{\partial y^2}\,{}_{sH^{(s)}}\mathtt{s}_n(x,z,y) = \frac{\partial}{\partial D_x^{-1}}\,{}_{sH^{(s)}}\mathtt{s}_n(x,z,y). \tag{3.14} $$

Since, in view of Table 1(I), we have

$$ {}_sH_n^{(s)}(0,y,z) = H_n^{(s)}(y,z). \tag{3.15} $$

Consequently, from Table 4(I), we have

$$ {}_{sH^{(s)}}\mathtt{s}_n(0,y,z) = {}_{H^{(s)}}\mathtt{s}_n(y,z). \tag{3.16} $$

Solving equation (3.14) subject to initial condition (3.16), we get assertion (3.13). □

Next, we prove the integral for the LeGHSP ${}_{RH^{(s)}}\mathtt{s}_n(x,y,z)$ and ${}_{sH^{(s)}}\mathtt{s}_n(x,y,z)$ in the form of following theorem:

Theorem 3.6. *The following integral representations for the LeGHSP* ${}_{RH^{(s)}}\mathtt{s}_n(x,y,z)$ *and* ${}_{sH^{(s)}}\mathtt{s}_n(x,y,z)$ *hold true:*

$$ {}_{RH^{(s)}}\mathtt{s}_n(x,y,z) = \int_0^\infty e^{-v}\,{}_{RH^{(s)}}\mathtt{s}_n\left(x,y,vD_z^{-1}\right)\,dv \tag{3.17} $$

and

$$ {}_{sH^{(s)}}\mathtt{s}_n(x,y,z) = \int_0^\infty e^{-v}\,{}_{sH^{(s)}}\mathtt{s}_n\left(x,y,vD_z^{-1}\right)\,dv, \tag{3.18} $$

respectively.

Proof. Using equation (1.1) in the l.h.s. of relation (2.1), we get

$$ \frac{1}{g(f^{-1}(t))}\sum_{n=0}^{\infty}\frac{{}_RH_n^{(s)}(x,y,z)}{n!}\frac{\left(f^{-1}(t)\right)^n}{n!} = \sum_{n=0}^{\infty}{}_{RH^{(s)}}\mathtt{s}_n(x,y,z)\frac{t^n}{n!}. \tag{3.19} $$

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Next, on using the integral representation of LeGHP $\frac{{}_RH_n^{(s)}(x,y,z)}{n!}$ [30]:

$$ {}_RH_n^{(s)}(x,y,z) = \int_0^\infty e^{-v}\,{}_RH_n^{(s)}\big(x,y,vD_z^{-1}\big)\,dv \tag{3.20} $$

in the l.h.s of equation (3.19) and interchanging the sides, we have

$$ \sum_{n=0}^{\infty} {}_{RH^{(s)}}\mathfrak{s}_n(x,y,z)\frac{t^n}{n!} = \frac{1}{g(f^{-1}(t))}\sum_{n=0}^{\infty}\left(\int_0^\infty e^{-v}\,{}_RH_n^{(s)}\big(x,y,vD_z^{-1}\big)\,dv\right)\frac{\big(f^{-1}(t)\big)^n}{n!n!}, \tag{3.21} $$

or, equivalently

$$ \sum_{n=0}^{\infty} {}_{RH^{(s)}}\mathfrak{s}_n(x,y,z)\frac{t^n}{n!} = \int_0^\infty e^{-v}\sum_{n=0}^{\infty}\left(\frac{1}{g(f^{-1}(t))}\frac{{}_RH_n^{(s)}\big(x,y,vD_z^{-1}\big)}{n!}\frac{\big(f^{-1}(t)\big)^n}{n!}\right)dv. \tag{3.22} $$

Again, using equation (3.19) in the r.h.s of the above equation, we find

$$ \sum_{n=0}^{\infty} {}_{RH^{(s)}}\mathfrak{s}_n(x,y,z)\frac{t^n}{n!} = \int_0^\infty e^{-v}\sum_{n=0}^{\infty}\left({}_{RH^{(s)}}\mathfrak{s}_n\big(x,y,vD_z^{-1}\big)\frac{t^n}{n!}\right)dv, \tag{3.23} $$

or, equivalently

$$ \sum_{n=0}^{\infty} {}_{RH^{(s)}}\mathfrak{s}_n(x,y,z)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\int_0^\infty e^{-v}\,{}_{RH^{(s)}}\mathfrak{s}_n\big(x,y,vD_z^{-1}\big)\,dv\right)\frac{t^n}{n!}. \tag{3.24} $$

Finally, equating the coefficient of like powers of t in both sides of the above equation, we get assertion (3.17). Similarly, we can get assertion (3.18) □

Corollary 3.1. *The following operational representation between the LeGHASP* ${}_{RH^{(s)}}\mathfrak{s}_n(x,y,z)$, ${}_{SH^{(s)}}\mathfrak{s}_n(x,y,z)$ *and the associated Sheffer polynomials* $\mathfrak{s}_n(x)$ *hold true:*

$$ {}_{RH^{(s)}}\mathfrak{s}_n(x,y,z) = exp\left(z\frac{\partial^s}{\partial y^s}\right)\mathfrak{s}_n\big(-D_x^{-1}+D_y^{-1}\big) \tag{3.25} $$

and

$$ {}_{SH^{(s)}}\mathfrak{s}_n(x,y,z) = exp\left(D_x^{-1}\frac{\partial^2}{\partial y^2}+z\frac{\partial^s}{\partial y^s}\right)\mathfrak{s}_n(y), \tag{3.26} $$

respectively.

Corollary 3.2. *The following operational representation between the LeGHASP* ${}_{RH^{(s)}}\mathfrak{s}_n(x,y,z)$ *and the 2VLeASP* ${}_R\mathfrak{s}_n(x,y)$ *holds true:*

$$ {}_{RH^{(s)}}\mathfrak{s}_n(x,y,z) = exp\left((-1)^s z\frac{\partial^s}{\partial D_x^{-s}}\right){}_R\mathfrak{s}_n(x,y), \tag{3.27} $$

or, equivalently

$$ {}_{RH^{(s)}}\mathfrak{s}_n(x,y,z) = exp\left(z\frac{\partial^s}{\partial D_y^{-s}}\right){}_R\mathfrak{s}_n(x,y). \tag{3.28} $$

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Corollary 3.3. *The following operational representation between the LeGHASP* ${}_{S}H^{(s)}\mathfrak{s}_n(x,y,z)$ *and the 2VLeTASP* ${}_{2}L\mathfrak{s}_n(x,y)$ *holds true:*

$$ {}_{S}H^{(s)}\mathfrak{s}_n(x,y,z) = exp\Big(z\frac{\partial^s}{\partial y^s}\Big)\, {}_{2}L\mathfrak{s}_n(x,y). \tag{3.29}$$

Corollary 3.4. *The following operational representation between the LeGHASP* ${}_{R}H^{(s)}\mathfrak{s}_n(x,y,z)$ *and the 2VGLTASP* ${}_{[m]}L\mathfrak{s}_n(x,y)$ *hold true:*

$$ {}_{R}H^{(m)}\mathfrak{s}_n(x,z,y) = exp\Big(-D_z^{-1}\frac{\partial}{\partial D_x^{-1}}\Big)\, {}_{[m]}L\mathfrak{s}_n(x,y). \tag{3.30}$$

Corollary 3.5. *The following operational representation between the LeGHASP* ${}_{S}H^{(s)}\mathfrak{s}_n(x,y,z)$ *and the GHASP* ${}_{H^{(s)}}\mathfrak{s}_n(y,z)$ *hold true:*

$$ {}_{S}H^{(s)}\mathfrak{s}_n(x,y,z) = exp\Big(D_x^{-1}\frac{\partial^2}{\partial y^2}\Big)\, {}_{H^{(s)}}\mathfrak{s}_n(y,z) \tag{3.31}$$

Corollary 3.6. *The following integral representations for the LeGHASP* ${}_{R}H^{(s)}\mathfrak{s}_n(x,y,z)$ *and* ${}_{S}H^{(s)}\mathfrak{s}_n(x,y,z)$ *hold true:*

$$ {}_{R}H^{(s)}\mathfrak{s}_n(x,y,z) = \int_0^\infty e^{-v}\, {}_{R}H^{(s)}\mathfrak{s}_n\big(x,y,vD_z^{-1}\big)\, dv \tag{3.32}$$

and

$$ {}_{S}H^{(s)}\mathfrak{s}_n(x,y,z) = \int_0^\infty e^{-v}\, {}_{S}H^{(s)}\mathfrak{s}_n\big(x,y,vD_z^{-1}\big)\, dv, \tag{3.33}$$

respectively.

4. Examples

The Sheffer polynomials have been studied because of their remarkable applications not only in different branches of mathematics but also in physics. The Sheffer and associated Sheffer class contains a number of important special polynomials. In this section, some results for the corresponding members of the Legendre-Gould Hopper based Sheffer polynomial families are obtained. We consider the following examples:

Example 4.1: Since, for $A(t) = e^{-t^r}$ and $H(t) = vt$, the Sheffer polynomial $\mathfrak{s}_n(x)$ becomes the generalized Hermite polynomial (GHP) $H_{n,r,v}(x)$ (Table 2(I)). Therefore, for the same choice of $A(t)$ and $H(t)$, the LeGHSP reduce to the Legendre-Gould Hopper based generalized Hermite polynomials (LeGHGHP) ${}_{R}H^{(s)}H_{n,r,v}(x,y,z)$ and ${}_{S}H^{(s)}H_{n,r,v}(x,y,z)$.

Thus, In view of equations (2.2), (2.14), (2.18), (2.39), (3.1),(3.3), (3.4), (3.9) and (3.17), we get the following results for the LeGHGHP ${}_{R}H^{(s)}H_{n,r,v}(x,y,z)$:

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Table 5. Results for the LeGHGHP ${}_{R}H^{(s)}H_{n,r,v}(x,y,z)$

I.	Generating functions	$e^{-t^r} exp\big(z(vt)^s\big)\ C_0\big(xvt)\big)\ C_0\big(-yvt\big) = \sum_{n=0}^{\infty} {}_{R}H^{(s)}H_{n,r,v}(x,y,z)\frac{t^n}{n!}$
II.	Multiplicative and derivative operators	$\hat{M} := v(-D_x^{-1} + D_y^{-1} + szD_y^{s-1}) - r(\frac{1}{v}D_yyD_y)^{r-1}, \quad \hat{P} := \frac{1}{v}D_yyD_y$
III.	Differential equations	$\left(\left(\left(-D_x^{-1} + D_y^{-1} + szD_y^{s-1}\right)(D_yyD_y) - r\left(\frac{D_yyD_y}{v}\right)^r\right) - n\right) {}_{R}H^{(s)}H_{n,r,v}(x,y,z) = 0$
IV.	Operational representations	${}_{R}H^{(s)}H_{n,r,v}(x,y,z) = exp\left(z\frac{\partial^s}{\partial y^s}\right)H_{n,r,v}(-D_x^{-1} + D_y^{-1})$, ${}_{R}H^{(s)}H_{n,r,v}(x,y,z) = exp\left((-1)^s z\frac{\partial^s}{\partial D_x^{-s}}\right){}_{R}H_{n,r,v}(x,y)$, ${}_{R}H^{(s)}H_{n,r,v}(x,y,z) = exp\left(z\frac{\partial^s}{\partial D_y^{-s}}\right){}_{R}H_{n,r,v}(x,y)$, ${}_{R}H^{(s)}H_{n,r,v}(x,y,z) = exp\left(-D_z^{-1}\frac{\partial}{\partial D_x^{-1}}\right){}_{[m]L}H_{n,r,v}(x,y)$, ${}_{R}H^{(s)}H_{n,r,v}(x,y,z) = \int_0^\infty e^{-v}\ {}_{R}H^{(s)}H_{n,r,v}(x,y,vD_z^{-1})\ dv.$

Also, In view of equations (2.4), (2.15), (2.19), (2.41), (3.2),(3.8), (3.13) and (3.18), we get the following results for the LeGHGHP ${}_{S}H^{(s)}H_{n,r,v}(x,y,z)$:

Table 6. Results for the LeGHGHP ${}_{S}H^{(s)}H_{n,r,v}(x,y,z)$

I.	Generating functions	$e^{-t^r} exp\big(yvt + z(vt)^s\big)\ C_0\big(-x(vt)^2\big) = \sum_{n=0}^{\infty} {}_{S}H^{(s)}H_{n,r,v}(x,y,z)\frac{t^n}{n!}$
II.	Multiplicative and derivative operators	$\hat{M} := v(y + 2D_x^{-1}D_y + szD_y^{s-1}) - r(\frac{1}{v}D_y)^{r-1}, \quad \hat{P} := \frac{1}{v}D_y$
III.	Differential equations	$\left(\left(\left(y + 2D_x^{-1}D_y + szD_y^{s-1}\right)D_y - r\left(\frac{D_y}{v}\right)^r\right) - n\right) {}_{S}H^{(s)}H_{n,r,v}(x,y,z) = 0$
IV.	Operational representations	${}_{S}H^{(s)}H_{n,r,v}(x,y,z) = exp\left(D_x^{-1}\frac{\partial^2}{\partial y^2} + z\frac{\partial^s}{\partial y^s}\right)H_{n,r,v}(y)$, ${}_{S}H^{(s)}H_{n,r,v}(x,y,z) = exp\left(z\frac{\partial^s}{\partial y^s}\right){}_{2L}H_{n,r,v}(x,y)$, ${}_{S}H^{(s)}H_{n,r,v}(x,y,z) = exp\left(D_x^{-1}\frac{\partial^2}{\partial y^2}\right){}_{H^{(s)}}H_{n,r,v}(y,z)$, ${}_{S}H^{(s)}H_{n,r,v}(x,y,z) = \int_0^\infty e^{-v}\ {}_{S}H^{(s)}H_{n,r,v}(x,y,vD_z^{-1})\ dv.$

Example 4.2: Since, for $A(t) = (1-t)^{-\nu-1}$ and $H(t) = \frac{t}{t-1}$, the Sheffer polynomial $\mathtt{s}_n(x)$ becomes the generalized Laguerre polynomial (GHP) $n!L_n^{(\nu)}(x)$ (Table 2(II)). Therefore, for the same choice of $A(t)$ and $H(t)$, the LeGHSP reduce to the Legendre-Gould Hopper based generalized Laguerre polynomials (LeGHGLP) $n!\ {}_{R}H^{(s)}L_n^{(\nu)}(x,y,z)$ and $n!\ {}_{S}H^{(s)}L_n^{(\nu)}(x,y,z)$.

Thus, In view of equations (2.2), (2.14), (2.18), (2.39), (3.1),(3.3), (3.4), (3.9) and (3.17), we get the following results for the LeGHGLP $n!\ {}_{R}H^{(s)}L_n^{(\nu)}(x,y,z)$:

Table 7. Results for the LeGHGLP $n!\ {}_{R}H^{(s)}L_n^{(\nu)}(x,y,z)$

I.	Generating functions	$(1-t)^{-\nu-1}exp\left(z\left(\frac{t}{t-1}\right)^s\right)\ C_0\left(\frac{xt}{t-1}\right)\ C_0\left(-\frac{yt}{t-1}\right) = \sum_{n=0}^{\infty} {}_{R}H^{(s)}L_n^{(\nu)}(x,y,z)\frac{t^n}{n!}$
II.	Multiplicative and derivative operators	$\hat{M} := -\left(-D_x^{-1} + D_y^{-1} + szD_y^{s-1}\right)(D_yyD_y - 1)^2 + (\nu+1)(1 - D_yyD_y), \quad \hat{P} := \frac{D_yyD_y}{D_yyD_y - 1}$
III.	Differential equations	$\left(\left(\left(-D_x^{-1} + D_y^{-1} + szD_y^{s-1}\right)\left((D_yyD_y)^2 - D_yyD_y\right) - (\nu+1)(D_yyD_y)\right) - n\right) {}_{R}H^{(s)}L_n^{(\nu)}(x,y,z) = 0$
IV.	Operational representations	${}_{R}H^{(s)}L_n^{(\nu)}(x,y,z) = exp\left(z\frac{\partial^s}{\partial y^s}\right)L_n^{(\nu)}(-D_x^{-1} + D_y^{-1})$, ${}_{R}H^{(s)}L_n^{(\nu)}(x,y,z) = exp\left((-1)^s z\frac{\partial^s}{\partial D_x^{-s}}\right){}_{R}L_n^{(\nu)}(x,y)$, ${}_{R}H^{(s)}L_n^{\nu}(x,y,z) = exp\left(z\frac{\partial^s}{\partial D_y^{-s}}\right){}_{R}L_n^{(\nu)}(x,y)$, ${}_{R}H^{(s)}L_n^{\nu}(x,y,z) = exp\left(-D_z^{-1}\frac{\partial}{\partial D_x^{-1}}\right){}_{[m]L}L_n^{(\nu)}(x,y)$, ${}_{R}H^{(s)}L_n^{(\nu)}(x,y,z) = \int_0^\infty e^{-v}\ {}_{R}H^{(s)}L_n^{(\nu)}(x,y,vD_z^{-1})\ dv.$

Also, In view of equations (2.4), (2.15), (2.19), (2.41), (3.2),(3.8), (3.13) and (3.18), we get the following results for the LeGHGLP ${}_{S}H^{(s)}L_n^{(\nu)}(x,y,z)$:

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Table 8. Results for the LeGHGLP ${}_{S}H^{(s)}L_n^{(\nu)}(x,y,z)$

I.	Generating functions	$(1-t)^{-\nu-1}exp\Big(\frac{yt}{t-1}+z\Big(\frac{t}{t-1}\Big)^s\Big)\ C_0\Big(-x\Big(\frac{t}{t-1}\Big)^2\Big)=\sum_{n=0}^{\infty}{}_{S}H^{(s)}L_n^{(\nu)}(x,y,z)\frac{t^n}{n!}$
II.	Multiplicative and derivative operators	$\hat{M}:=-\Big(y+2D_x^{-1}D_y+szD_y^{s-1}\Big)(D_y-1)^2+(\nu+1)(1-D_y),\quad \hat{P}:=\frac{D_y}{D_y-1}$
III.	Differential equations	$\Big(\Big(\Big(y+2D_x^{-1}D_y+szD_y^{s-1}\Big)\Big(D_y^2-D_y\Big)-(\nu+1)D_y\Big)-n\Big)\ {}_{S}H^{(s)}L_n^{(\nu)}(x,y,z)=0$
IV.	Operational representations	${}_{S}H^{(s)}L_n^{(\nu)}(x,y,z)=exp\Big(D_x^{-1}\frac{\partial^2}{\partial y^2}+z\frac{\partial^s}{\partial y^s}\Big)L_n^{(\nu)}(y)$, ${}_{S}H^{(s)}L_n^{(\nu)}(x,y,z)=exp\Big(z\frac{\partial^s}{\partial y^s}\Big)\ {}_{2L}L_n^{(\nu)}(x,y)$, ${}_{S}H^{(s)}L_n^{(\nu)}(x,y,z)=exp\Big(D_x^{-1}\frac{\partial^2}{\partial y^2}\Big)\ {}_{H^{(s)}}L_n^{(\nu)}(y,z)$, ${}_{S}H^{(s)}L_n^{(\nu)}(x,y,z)=\int_0^\infty e^{-v}\ {}_{S}H^{(s)}L_n^{(\nu)}(x,y,vD_z^{-1})\ dv$.

Example 4.3: Since, for $A(t)=\frac{t}{1-t}$ and $H(t)=ln\big(\frac{1+t}{1-t}\big)$, the Sheffer polynomial $\mathfrak{s}_n(x)$ becomes the Pidduck polynomial (PP) $P_n(x)$ (Table 2(III)). Therefore, for the same choice of $A(t)$ and $H(t)$, the LeGHSP reduce to the Legendre-Gould Hopper based Pidduck polynomials (LeGHPP) ${}_{R}H^{(s)}P_n(x,y,z)$ and ${}_{S}H^{(s)}P_n(x,y,z)$.

Thus, In view of equations (2.2), (2.14), (2.18), (2.39), (3.1),(3.3), (3.4), (3.9) and (3.17), we get the following results for the LeGHPP ${}_{R}H^{(s)}P_n(x,y,z)$:

Table 9. Results for the LeGHPP ${}_{R}H^{(s)}P_n(x,y,z)$

I.	Generating functions	$\frac{t}{1-t}exp\Big(z\Big(ln\big(\frac{1+t}{1-t}\big)\Big)^s\Big)\ C_0\Big(x\ ln\big(\frac{1+t}{1-t}\big)\Big)\ C_0\Big(-y\ ln\big(\frac{1+t}{1-t}\big)\Big)=\sum_{n=0}^{\infty}{}_{R}H^{(s)}P_n(x,y,z)\frac{t^n}{n!}$
II.	Multiplicative and derivative operators	$\hat{M}:=\Big(-D_x^{-1}+D_y^{-1}+szD_y^{s-1}\Big)\Big(\frac{(e^{D_yyD_y}+1)^2}{2e^{D_yyD_y}}\Big)+\frac{(e^{D_yyD_y}+1)^2}{2(e^{D_yyD_y}-1)},\quad \hat{P}:=\frac{e^{D_yyD_y}-1}{e^{D_yyD_y}+1}$
III.	Differential equations	$\Big(\Big(\Big(-D_x^{-1}+D_y^{-1}+szD_y^{s-1}\Big)\Big(\frac{e^{2D_yyD_y}-1}{2e^{D_yyD_y}}\Big)+\frac{1}{2}(e^{D_yyD_y}+1)\Big)-n\Big)\ {}_{R}H^{(s)}P_n(x,y,z)=0$
IV.	Operational representations	${}_{R}H^{(s)}P_n(x,y,z)=exp\Big(z\frac{\partial^s}{\partial y^s}\Big)P_n(-D_x^{-1}+D_y^{-1})$, ${}_{R}H^{(s)}P_n(x,y,z)=exp\Big((-1)^sz\frac{\partial^s}{\partial D_x^{-s}}\Big)\ {}_{R}P_n(x,y)$, ${}_{R}H^{(s)}P_n(x,y,z)=exp\Big(z\frac{\partial^s}{\partial D_y^{-s}}\Big)\ {}_{R}P_n(x,y)$, ${}_{R}H^{(s)}P_n(x,y,z)=exp\Big(-D_z^{-1}\frac{\partial}{\partial D_x^{-1}}\Big)\ {}_{[m]L}P_n(x,y)$, ${}_{R}H^{(s)}P_n(x,y,z)=\int_0^\infty e^{-v}\ {}_{R}H^{(s)}P_n(x,y,vD_z^{-1})\ dv$.

Also, In view of equations (2.4), (2.15), (2.19), (2.41), (3.2),(3.8), (3.13) and (3.18), we get the following results for the LeGHPP ${}_{S}H^{(s)}P_n(x,y,z)$:

Table 10. Results for the LeGHPP ${}_{S}H^{(s)}P_n(x,y,z)$

I.	Generating functions	$\frac{t}{1-t}exp\Big(y\ ln\big(\frac{1+t}{1-t}\big)+z\Big(ln\big(\frac{1+t}{1-t}\big)\Big)^s\Big)\ C_0\Big(-x\ \Big(ln\big(\frac{1+t}{1-t}\big)\Big)^2\Big)=\sum_{n=0}^{\infty}{}_{S}H^{(s)}P_n(x,y,z)\frac{t^n}{n!}$
II.	Multiplicative and derivative operators	$\hat{M}:=-\Big(y+2D_x^{-1}D_y+szD_y^{s-1}\Big)\Big(\frac{(e^{D_y}+1)^2}{2e^{D_y}}\Big)+\frac{(e^{D_y}+1)^2}{2(e^{D_y}-1)},\quad \hat{P}:=\frac{e^{D_y}-1}{e^{D_y}+1}$
III.	Differential equations	$\Big(\Big(\Big(y+2D_x^{-1}D_y+szD_y^{s-1}\Big)\Big(\frac{e^{2D_y}-1}{2e^{D_y}}\Big)+\frac{1}{2}(e^{D_y}+1)\Big)-n\Big)\ {}_{S}H^{(s)}P_n(x,y,z)=0$
IV.	Operational representations	${}_{S}H^{(s)}P_n(x,y,z)=exp\Big(D_x^{-1}\frac{\partial^2}{\partial y^2}+z\frac{\partial^s}{\partial y^s}\Big)P_n(y)$, ${}_{S}H^{(s)}P_n(x,y,z)=exp\Big(z\frac{\partial^s}{\partial y^s}\Big)\ {}_{2L}P_n(x,y)$, ${}_{S}H^{(s)}P_n(x,y,z)=exp\Big(D_x^{-1}\frac{\partial^2}{\partial y^2}\Big)\ {}_{H^{(s)}}P_n(y,z)$, ${}_{S}H^{(s)}P_n(x,y,z)=\int_0^\infty e^{-v}\ {}_{S}H^{(s)}P_n(x,y,vD_z^{-1})\ dv$.

Example 4.4: Since, for $H(t)=ln\big(\frac{1+t}{1-t}\big)$, the associated Sheffer polynomial $\mathfrak{s}_n(x)$ becomes the Mittag-Leffler polynomial (MLP) $M_n(x)$ (Table 3(I)). Therefore, for the same choice of $H(t)$, the LeGHASP reduce to the Legendre-Gould Hopper based Mittag-Leffler polynomials (LeGHMLP) ${}_{R}H^{(s)}M_n(x,y,z)$ and ${}_{S}H^{(s)}M_n(x,y,z)$.

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Thus, in view of equations (2.43), (2.48), (2.52), (2.55), (3.25),(3.27), (3.28), (3.30) and (3.32), we get the following results for the LeGHMLP ${}_{R}H^{(s)}M_n(x,y,z)$:

Table 11. Results for the LeGHMLP ${}_{R}H^{(s)}M_n(x,y,z)$

I.	Generating functions	$exp\Big(z\Big(ln(\frac{1+t}{1-t})\Big)^s\Big)\ C_0\Big(x\ ln(\frac{1+t}{1-t})\Big)\ C_0\Big(-y\ ln(\frac{1+t}{1-t})\Big)=\sum_{n=0}^{\infty}{}_{R}H^{(s)}M_n(x,y,z)\frac{t^n}{n!}$
II.	Multiplicative and derivative operators	$\hat{M}:=\Big(-D_x^{-1}+D_y^{-1}+szD_y^{s-1}\Big)\Big(\frac{(e^{D_y y D_y}+1)^2}{2e^{D_y y D_y}}\Big),\quad \hat{P}:=\frac{e^{D_y y D_y}-1}{e^{D_y y D_y}+1}$
III.	Differential equations	$\Big(\Big(-D_x^{-1}+D_y^{-1}+szD_y^{s-1}\Big)\Big(\frac{e^{2D_y y D_y}-1}{2e^{D_y y D_y}}\Big)-n\Big){}_{R}H^{(s)}M_n(x,y,z)=0$
IV.	Operational representations	${}_{R}H^{(s)}M_n(x,y,z)=exp\Big(z\frac{\partial^s}{\partial y^s}\Big)M_n(-D_x^{-1}+D_y^{-1}),\ {}_{R}H^{(s)}M_n(x,y,z)=exp\Big((-1)^s z\frac{\partial^s}{\partial D_x^{-s}}\Big){}_{R}M_n(x,y),$ ${}_{R}H^{(s)}M_n(x,y,z)=exp\Big(z\frac{\partial^s}{\partial D_y^{-s}}\Big){}_{R}M_n(x,y),\ {}_{R}H^{(s)}M_n(x,y,z)=exp\Big(-D_z^{-1}\frac{\partial}{\partial D_x^{-1}}\Big)_{[m]}{}_{L}M_n(x,y),$ ${}_{R}H^{(s)}M_n(x,y,z)=\int_0^\infty e^{-v}\,{}_{R}H^{(s)}M_n(x,y,vD_z^{-1})\,dv.$

Also, in view of equations (2.45), (2.49), (2.53), (2.57), (3.26),(3.29), (3.31) and (3.33), we get the following results for the LeGHMLP ${}_{S}H^{(s)}M_n(x,y,z)$:

Table 12. Results for the LeGHMLP ${}_{S}H^{(s)}M_n(x,y,z)$

I.	Generating functions	$exp\Big(y\ ln(\frac{1+t}{1-t})+z\Big(ln(\frac{1+t}{1-t})\Big)^s\Big)\ C_0\Big(-x\ \Big(ln(\frac{1+t}{1-t})\Big)^2\Big)=\sum_{n=0}^{\infty}{}_{S}H^{(s)}M_n(x,y,z)\frac{t^n}{n!}$
II.	Multiplicative and derivative operators	$\hat{M}:=-\Big(y+2D_x^{-1}D_y+szD_y^{s-1}\Big)\Big(\frac{(e^{D_y}+1)^2}{2e^{D_y}}\Big),\quad \hat{P}:=\frac{e^{D_y}-1}{e^{D_y}+1}$
III.	Differential equations	$\Big(\Big(y+2D_x^{-1}D_y+szD_y^{s-1}\Big)\Big(\frac{e^{2D_y}-1}{2e^{D_y}}\Big)-n\Big){}_{S}H^{(s)}M_n(x,y,z)=0$
IV.	Operational representations	${}_{S}H^{(s)}M_n(x,y,z)=exp\Big(D_x^{-1}\frac{\partial^2}{\partial y^2}+z\frac{\partial^s}{\partial y^s}\Big)M_n(y),$ ${}_{S}H^{(s)}M_n(x,y,z)=exp\Big(z\frac{\partial^s}{\partial y^s}\Big){}_{2L}M_n(x,y),$ ${}_{S}H^{(s)}M_n(x,y,z)=exp\Big(D_x^{-1}\frac{\partial^2}{\partial y^2}\Big){}_{H^{(s)}}M_n(y,z),$ ${}_{S}H^{(s)}M_n(x,y,z)=\int_0^\infty e^{-v}\,{}_{S}H^{(s)}M_n(x,y,vD_z^{-1})\,dv.$

Example 4.5: Since, for $H(t)=e^t-1$, the associated Sheffer polynomial $\mathfrak{s}_n(x)$ becomes the Exponential polynomial (EP) $\varphi_n(x)$ (Table 3(II)). Therefore, for the same choice of $H(t)$, the LeGHASP reduce to the Legendre-Gould Hopper based Exponential polynomials (LeGHEP) ${}_{R}H^{(s)}\varphi_n(x,y,z)$ and ${}_{S}H^{(s)}\varphi_n(x,y,z)$.

Thus, in view of equations (2.43), (2.48), (2.52), (2.55), (3.25),(3.27), (3.28), (3.30) and (3.32), we get the following results for the LeGHEP ${}_{R}H^{(s)}\varphi_n(x,y,z)$:

Table 13. Results for the LeGHEP ${}_{R}H^{(s)}\varphi_n(x,y,z)$

I.	Generating functions	$exp(z(e^t-1)^s)\ C_0(x\ (e^t-1))\ C_0(-y\ (e^t-1))=\sum_{n=0}^{\infty}{}_{R}H^{(s)}\varphi_n(x,y,z)\frac{t^n}{n!}$
II.	Multiplicative and derivative operators	$\hat{M}:=\Big(-D_x^{-1}+D_y^{-1}+szD_y^{s-1}\Big)(1+D_y y D_y),\quad \hat{P}:=ln(1+D_y y D_y)$
III.	Differential equations	$\Big(\Big(-D_x^{-1}+D_y^{-1}+szD_y^{s-1}\Big)(1+D_y y D_y)\ ln(1+D_y y D_y)-n\Big){}_{R}H^{(s)}\varphi_n(x,y,z)=0$
IV.	Operational representations	${}_{R}H^{(s)}\varphi_n(x,y,z)=exp\Big(z\frac{\partial^s}{\partial y^s}\Big)\varphi_n(-D_x^{-1}+D_y^{-1}),\ {}_{R}H^{(s)}\varphi_n(x,y,z)=exp\Big((-1)^s z\frac{\partial^s}{\partial D_x^{-s}}\Big){}_{R}\varphi_n(x,y),$ ${}_{R}H^{(s)}\varphi_n(x,y,z)=exp\Big(z\frac{\partial^s}{\partial D_y^{-s}}\Big){}_{R}\varphi_n(x,y),\ {}_{R}H^{(s)}\varphi_n(x,y,z)=exp\Big(-D_z^{-1}\frac{\partial}{\partial D_x^{-1}}\Big)_{[m]}{}_{L}\varphi_n(x,y),$ ${}_{R}H^{(s)}\varphi_n(x,y,z)=\int_0^\infty e^{-v}\,{}_{R}H^{(s)}\varphi_n(x,y,vD_z^{-1})\,dv.$

Also, in view of equations (2.45), (2.49), (2.53), (2.57), (3.26),(3.29), (3.31) and (3.33), we get the following results for the LeGHEP ${}_{S}H^{(s)}\varphi_n(x,y,z)$:

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Table 14. Results for the LeGHEP ${}_{S}H^{(s)}\varphi_n(x,y,z)$

I.	Generating functions	$exp(y\ (e^t-1)+z(e^t-1)^s)\ C_0\Big(-x\ (e^t-1)^2\Big)=\sum\limits_{n=0}^{\infty}{}_{S}H^{(s)}\varphi_n(x,y,z)\frac{t^n}{n!}$
II.	Multiplicative and derivative operators	$\hat{M}:=-\Big(y+2D_x^{-1}D_y+szD_y^{s-1}\Big)(1+D_y),\quad \hat{P}:=ln(1+D_y)$
III.	Differential equations	$\Big(\Big(y+2D_x^{-1}D_y+szD_y^{s-1}\Big)(1+D_y)\ ln(1+D_y)-n\Big){}_{S}H^{(s)}\varphi_n(x,y,z)=0$
IV.	Operational representations	${}_{S}H^{(s)}\varphi_n(x,y,z)=exp\Big(D_x^{-1}\frac{\partial^2}{\partial y^2}+z\frac{\partial^s}{\partial y^s}\Big)\varphi_n(y),\ {}_{S}H^{(s)}\varphi_n(x,y,z)=exp\Big(z\frac{\partial^s}{\partial y^s}\Big){}_{2L}\varphi_n(x,y),$ ${}_{S}H^{(s)}\varphi_n(x,y,z)=exp\Big(D_x^{-1}\frac{\partial^2}{\partial y^2}\Big){}_{H^{(s)}}\varphi_n(y,z),\ {}_{S}H^{(s)}\varphi_n(x,y,z)=\int_0^\infty e^{-v}\ {}_{S}H^{(s)}\varphi_n(x,y,vD_z^{-1})\ dv.$

Similarly, for the other members of the Sheffer and associated families (see Tables 2 and 3), there exist new special polynomials belonging to the LeGHSP and LeGHASP families respectively. The generating functions and other properties of these special polynomials can be obtained from the results derived in the second and third sections.

5. Concluding Remarks

A determinantal definition for the classical Bernoulli polynomials introduced by Costabile et.al. [10] has given a new approach to Bernoulli polynomials which was further extended to provide the determinantal definition of the Appell polynomials [11]. Recently, the determinantal definition of Appell sequences is extended to Sheffer sequences by using the theory of Riordan arrays [29].

The determinantal approach considered in [10, 11, 29] provides motivation to consider the determinantal form of the new families of special polynomials. In this section, we give the determinantal definitions for the Legendre-Gould Hopper based Sheffer polynomials ${}_{R}H^{(s)}\mathtt{s}_n(x,y,z)$ and ${}_{S}H^{(s)}\mathtt{s}_n(x,y,z)$ as:

Definition 5.1. The Legendre-Gould Hopper based Sheffer polynomials ${}_{S}H^{(s)}\mathtt{s}_n(x,y,z)$ of degree n are defined by

$$ {}_{S}H^{(s)}\mathtt{s}_0(x,y,z)=\frac{1}{\alpha_{0,0}}, \tag{5.1}$$

$$ {}_{S}H^{(s)}\mathtt{s}_n(x,y,z)=\frac{(-1)^n}{\alpha_{0,0}\alpha_{1,1}\ldots\alpha_{n,n}}\begin{vmatrix} 1 & {}_{S}H_1^{(s)}(x,y,z) & {}_{S}H_2^{(s)}(x,y,z) & \ldots & {}_{S}H_{n-1}^{(s)}(x,y,z) & {}_{S}H_n^{(s)}(x,y,z) \\ \alpha_{0,0} & \alpha_{1,0} & \alpha_{2,0} & \ldots & \alpha_{n-1,0} & \alpha_{n,0} \\ 0 & \alpha_{1,1} & \alpha_{2,1} & \ldots & \alpha_{n-1,1} & \alpha_{n,1} \\ 0 & 0 & \alpha_{2,2} & \ldots & \alpha_{n-1,2} & \alpha_{n,1} \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ 0 & 0 & 0 & \ldots & \alpha_{n-1,n-1} & \alpha_{n,n-1} \end{vmatrix}, $$

$$ =\frac{(-1)^n}{\alpha_{0,0}\alpha_{1,1}\ldots\alpha_{n,n}}\det\begin{pmatrix}\mathtt{H}_{n+1}(x,y,z)\\ \mathtt{S}_{n\times(n+1)}\end{pmatrix}, \tag{5.2}$$

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where $\mathtt{H}_{n+1}(x,y,z) = \big(1, {}_SH_1^{(s)}(x,y,z), {}_SH_2^{(s)}(x,y,z), ..., {}_SH_n^{(s)}(x,y,z)\big)$ and ${}_SH_n^{(s)}(x,y,z)(n = 0,1,2,...,)$ are the Legendre-Gould Hopper polynomials defined by equation (1.2), $\mathtt{S}_{n\times(n+1)} = (a_{j-1,i-1})1 \leq i \leq n, 1 \leq j \leq n+1$ and $\alpha_{n,k}$ is the (n,k) entry of the Riordan array $(g(t), f(t))$.

Definition 5.1. The Legendre-Gould Hopper based Sheffer polynomials ${}_{RH^{(s)}}\mathtt{s}_n(x,y,z)$ of degree n are defined by

$$
{}_{RH^{(s)}}\mathtt{s}_0(x,y,z) = \frac{1}{\alpha_{0,0}}, \tag{5.3}
$$

$$
{}_{RH^{(s)}}\mathtt{s}_n(x,y,z) = \frac{(-1)^n}{\alpha_{0,0}\alpha_{1,1}...\alpha_{n,n}}
\begin{vmatrix}
1 & \frac{{}_RH_1^{(s)}(x,y,z)}{1!} & \frac{{}_RH_2^{(s)}(x,y,z)}{2!} & \cdots & \frac{{}_RH_{n-1}^{(s)}(x,y,z)}{(n-1)!} & \frac{{}_RH_n^{(s)}(x,y,z)}{n!} \\
\alpha_{0,0} & \alpha_{1,0} & \alpha_{2,0} & \cdots & \alpha_{n-1,0} & \alpha_{n,0} \\
0 & \alpha_{1,1} & \alpha_{2,1} & \cdots & \alpha_{n-1,1} & \alpha_{n,1} \\
0 & 0 & \alpha_{2,2} & \cdots & \alpha_{n-1,2} & \alpha_{n,1} \\
. & . & . & \cdots & . & . \\
. & . & . & \cdots & . & . \\
0 & 0 & 0 & \cdots & \alpha_{n-1,n-1} & \alpha_{n,n-1}
\end{vmatrix},
$$

$$
= \frac{(-1)^n}{\alpha_{0,0}\alpha_{1,1}...\alpha_{n,n}} \det\begin{pmatrix} H_{n+1}(x,y,z) \\ \mathtt{S}_{n\times(n+1)} \end{pmatrix}, \tag{5.4}
$$

where $H_{n+1}(x,y,z) = \left(1, \frac{{}_RH_1^{(s)}(x,y,z)}{1!}, \frac{{}_RH_2^{(s)}(x,y,z)}{2!}, ..., \frac{{}_RH_n^{(s)}(x,y,z)}{n!}\right)$ and $\frac{{}_RH_n^{(s)}(x,y,z)}{n!}(n = 0,1,2,...,)$ are the Legendre-Gould Hopper polynomials defined by equation (1.1), $\mathtt{S}_{n\times(n+1)} = (\alpha_{j-1,i-1})1 \leq i \leq n, 1 \leq j \leq n+1$ and $a_{n,k}$ is the (n,k) entry of the Riordan array $(g(t), f(t))$.

Further, we consider an example of mixed families of special polynomials belonging to Legendre-Gould Hopper based Sheffer polynomials.

In [29], it has been shown that for $\alpha_{n,k} = (-1)^k \frac{n!}{k!}\binom{n+\nu}{n-k}$ the determinantal definition of the Sheffer polynomials reduces to the determinantal definition of the Laguerre polynomials of order ν, $L_n^{(\nu)}(x)$. Therefore, on taking $\mathtt{s}_n(x) = L_n^{(\nu)}(x)$ and $\alpha_{n,k} = (-1)^k \frac{n!}{k!}\binom{n+\nu}{n-k}$ in equations (5.3) and (5.4), the determinantal definition of Legendre-Gould Hopper based generalized Laguerre polynomials $n!\ {}_{RH^{(s)}}L_n^{(\nu)}(x,y,z)$ is given as:

Definition 5.1. The Legendre-Gould Hopper based generalized Laguerre polynomials ${}_{RH^{(s)}}L_n^{(\nu)}(x,y,z)$ of degree n are defined by

$$
{}_{RH^{(s)}}L_0^{(\nu)}(x,y,z) = 1, \tag{5.5}
$$

$$
{}_{R}H^{(s)}L_n^{(\nu)}(x,y,z) = \frac{(-1)^n}{\alpha_{0,0}\alpha_{1,1}...\alpha_{n,n}}
\begin{vmatrix}
1 & \frac{{}_RH_1^{(s)}(x,y,z)}{1!} & \frac{{}_RH_2^{(s)}(x,y,z)}{2!} & ... & \frac{{}_RH_{n-1}^{(s)}(x,y,z)}{(n-1)!} & \frac{{}_RH_n^{(s)}(x,y,z)}{n!} \\
1 & \nu+1 & (\nu+2)_2 & ... & (\nu+n-1)_{n-1} & (\nu+n)_n \\
0 & -1 & -2(\nu+2) & ... & -(n-1)(\nu+n-1)_{n-2} & -n(\nu+n)_{n-1} \\
0 & 0 & 1 & ... & \frac{(n-1)(n-2)}{2}(\nu+n-1)_{n-3} & \frac{n(n-1)}{2}(\nu+n)_{n-2} \\
. & . & . & ... & . & . \\
. & . & . & ... & . & . \\
0 & 0 & 0 & ... & (-1)^{n-1} & (-1)^{n-1}n(\nu+n)
\end{vmatrix},
$$

$$
= (-1)^{\frac{n(n+3)}{2}} \det \begin{pmatrix} H_{n+1}(x,y,z) \\ \mathsf{S}_{n\times(n+1)} \end{pmatrix}, \tag{5.6}
$$

where $H_{n+1}(x,y,z) = \left(1, \frac{{}_RH_1^{(s)}(x,y,z)}{1!}, \frac{{}_RH_2^{(s)}(x,y,z)}{2!}, ..., \frac{{}_RH_n^{(s)}(x,y,z)}{n!}\right)$ and $\frac{{}_RH_n^{(s)}(x,y,z)}{n!}(n = 0,1,2,...,)$ are the Legendre-Gould Hopper polynomials defined by equation (1.1), $\mathsf{S}_{n\times(n+1)} = (\alpha_{j-1,i-1})1 \leq i \leq n, 1 \leq j \leq n+1$ and $a_{n,k}$ is the (n,k) entry of the Riordan array $(g(t), f(t))$.

Also, by giving suitable values to the variables and indices we can find the determinantal definitions of the other members belonging to the Legendre-Gould Hopper based Sheffer family. We conclude this paper with a further remark concerning the possibility of extending the method to other generalized special polynomials.

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