 doi:10.20944/preprints202010.0142.v1

Dynamics of System of Higher Order Difference Equations with Quadratic Terms

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Abstract

This paper aims to investigate the global asymptotic stability of following system of higher order difference equations with quadratic terms:

$$x_{n+1} = A + B\frac{y_n}{y_{n-m}^2}, y_{n+1} = A + B\frac{x_n}{x_{n-m}^2}$$

where A and B are positive numbers and the initial values are positive numbers. We also study the rate of convergence and oscillation behaviour of the solutions of related system.

Keywords: Difference equations; global asymptotic stability; boundedness; rate of convergence; oscillation

1 Introduction

Nowadays, investigation of dynamic behaviour of recursive sequences (also called difference equations) have attracted significant interest. This attention concern with applications to different fields of sciences. Applied sciences need many mathematical models. Mathematical models easily can be create from difference equations or their systems. Although the appearance of the difference equations is simple, the solutions of these equations are difficult to understand and are different from each other. Particularly since the solutions of the higher order difference equations differ in every order, it becomes difficult to understand the behavior of their solutions. Especially researchers have studied the global asymptotic stability, boundedness and oscillatory behavior of system of difference equations. There are many examples related to difference equations or systems. Therefore, studies on difference equations are increasing day by day and will continue to increase. There are many articles and books in the literature on the theory of difference equations and systems, see [1]-[26]. Additionally, there are many papers related to our study as follows:

In [6], Camouzis et al investigated the dynamics of solutions of following system of difference equations

$$x_{n+1} = 1 + \frac{x_n}{y_{n-m}}, y_{n+1} = 1 + \frac{y_n}{x_{n-m}}.$$

They especially studied the global asymptotic behavior of solutions of the related system.

In [19], Papaschinopoulos et al dealt with the oscillation, the bounded solutions, and the

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global asymptotic stability of the system of difference equations

$$x_{n+1} = A + \frac{y_n}{x_{n-q}}, y_{n+1} = A + \frac{x_n}{y_{n-q}}.$$

In [26], Zhang et al handled the boundedness, persistence and global asymptotic stability of solutions of system of difference equations

$$x_{n+1} = A + \frac{y_{n-m}}{x_n}, y_{n+1} = A + \frac{x_{n-m}}{y_n}.$$

In [25], Zhang et al studied the dynamics of the following symmetrical system of rational difference equation

$$x_{n+1} = A + \frac{y_{n-k}}{y_n}, y_{n+1} = A + \frac{x_{n-k}}{x_n}.$$

Additionally, in [10], Gümüş dealt with the global asymptotic stability and rate of convergence of related system.

In [3], Beso et al investigated the boundedness and global asymptotic stability of solutions of following difference equation

$$x_{n+1} = \gamma + \delta \frac{x_n}{x_{n-1}^2}.$$

In [23], Taşdemir investigated the oscillation, global asymptotic stability and rate of convergence of solutions of following system of second order difference equations with quadratic terms

$$x_{n+1} = A + B\frac{y_n}{y_{n-1}^2}, y_{n+1} = A + B\frac{x_n}{x_{n-1}^2}.$$

In this paper, we study the system of higher order difference equations with quadratic terms

$$x_{n+1} = A + B\frac{y_n}{y_{n-m}^2}, y_{n+1} = A + B\frac{x_n}{x_{n-m}^2} \tag{1}$$

where A and B are positive numbers and the initial values are positive numbers and $m \in \{2, 3, \cdots\}$. We especially investigate the stability, global behaviour and rate of convergence of solutions of system (1). We also study the oscillation behaviour of solutions of related system.

Now, we give three theorems which are used during this study.

Theorem 1 (Linearized Stability Theorem [16]**, p.11)** *Assume that*

$$X_{n+1} = F(X_n), n = 0, 1, \cdots,$$

is a system of difference equations such that $\bar{X}$ is a fixed point of F.

(i) *If all eigenvalues of the Jacobian matrix B about $\bar{X}$ lie inside the open unit disk $|\lambda| < 1$, that is, if all of them have absolute value less than one, then $\bar{X}$ is locally asymptotically stable.*

(ii) *If at least one of them has a modulus greater than one, then $\bar{X}$ is unstable.*

Theorem 2 ([4]**)** *Let $n \in N_{n_0}^+$ and $g(n, u, v)$ be a decreasing function in u and v for any* *fixed n. Suppose that for $n \geq n_0$, the inqualities*

$$\begin{aligned} y_{n+1} &\leq g\left(n, y_n, y_{n-1}\right) \\ u_{n+1} &\geq g\left(n, y_n, y_{n-1}\right) \end{aligned}$$

hold. Then

$$y_{n_0-1} \leq u_{n_0-1}, y_{n_0} \leq u_{n_0}$$

implies that

$$y_n \leq u_n, n \geq n_0.$$

Let us consider the following system of difference equations:

$$E_{n+1} = \left(A + B\left(n\right)\right) E_n, \tag{2}$$

where E_n is a $k-$dimensional vector, $A \in C^{k \times k}$ is a constant matrix, and $B : \mathbb{Z}^+ \to C^{k \times k}$ is a matrix function satisfying

$$\|B(n)\| \to 0, \tag{3}$$

as $n \to \infty$, where $\|\cdot\|$ denotes any matrix norm that is associated with the vector norm

$$\|(x, y)\| = \sqrt{x^2 + y^2}.$$

Theorem 3 (Perron's Theorem, [20]**)** *Assume that condition (3) holds. If E_n is a solution of (2), then either $E_n = 0$ for all $n \to \infty$, or*

$$\lim_{n \to \infty} \sqrt[n]{\|E_n\|},$$

or

$$\lim_{n \to \infty} \frac{\|E_{n+1}\|}{\|E_n\|},$$

exists and is equal to modulus of one of the eigenvalues of matrix A.

2 Main Results

In this here, we firstly apply the following the change of the variables for system (1):

$$t_n = \frac{x_n}{A}, z_n = \frac{y_n}{A}.$$

Hence we obtain the new system as:

$$t_{n+1} = 1 + p\frac{z_n}{z_{n-m}^2}, z_{n+1} = 1 + p\frac{t_n}{t_{n-m}^2} \tag{4}$$

where $p = \frac{B}{A^2} > 0$. From here on, we consider the system (4).

Now we examine the equilibrium point of system (4).

Lemma 4 *Let $p > 0$. The system (4) has a unique positive equilibrium point such as*

$$(\bar{t}, \bar{z}) = \left(\frac{1 + \sqrt{1 + 4p}}{2}, \frac{1 + \sqrt{1 + 4p}}{2}\right).$$

2.1 Boundedness of system (4)

This section, we determine the boundedness character of system (4).

Lemma 5 *Assume that $\{(t_n, z_n)\}$ be a positive solution of system (4) and $p > 0$. Then $t_n > 1$ and $z_n > 1$ for $n \geq 1$.*

Proof. Let $\{(t_n, z_n)\}$ be a positive solution of system (4). Hence we obtain from system (4):

$$\begin{aligned} t_1 &= 1 + p\frac{z_0}{z_{-m}^2} > 1, \\ z_1 &= 1 + p\frac{t_0}{t_{-m}^2} > 1. \end{aligned}$$

Thus, we have the followings by induction

$$\begin{aligned} t_{n+1} &= 1 + p\frac{z_n}{z_{n-m}^2} > 1, \\ z_{n+1} &= 1 + p\frac{t_n}{t_{n-m}^2} > 1. \end{aligned}$$

Therefore, the system (4) has a bound from below as desired. ■

Theorem 6 *Suppose that $0 < p < 1$. Therefore every solutions of system (4) have a bound from above.*

Proof. Now we know from Lemma 5 that $t_n > 1$ and $z_n > 1$ for $n \geq 1$ and $p > 0$. Otherwise, every solutions of system (4) satisfies

$$t_{n+1} < 1 + p + p^2 t_{n-1}, n \geq 1. \tag{5}$$

We consider Theorem 2. According to this, $t_n \leq u_n$, $n = 0, 1, ...$, where $\{u_n\}$ satisfy

$$u_{n+1} = 1 + p + p^2 u_{n-1}, n \geq 1, \tag{6}$$

such that

$$u_s = t_s, u_{s+1} = t_{s+1}, s \in \{-1, 0, 1, ...\}, n \geq s.$$

Therefore the solution u_n of the difference equation (6) is

$$u_n = \frac{1}{1-p} + p^n C_1 + (-p)^n C_2. \tag{7}$$

Moreover, we obtain the relations (5) and (7) imply that

$$t_{n+1} - u_{n+1} \leq p^2 (t_{n-1} - u_{n-1}), n > s, p \in (0, 1).$$

Thus we get

$$t_n \leq u_n, n > s. \tag{8}$$

Hence, we obtain from (7), (8) and Lemma 5,

$$1 < t_n \leq \frac{1}{1-p} + p^n C_1 + (-p)^n C_2 = M_1,$$

where

$$\begin{aligned} C_1 &= \frac{1}{2p}\left(pt_0 + t_1 - \frac{1+p}{1-p}\right), \\ C_2 &= \frac{1}{2p}\left(pt_0 - t_1 + 1\right). \end{aligned}$$

Likewise we have that

$$1 < z_n \leq \frac{1}{1-p} + p^n C_3 + (-p)^n C_4 = M_2,$$

where

$$\begin{aligned} C_3 &= \frac{1}{2p}\left(pz_0 + z_1 - \frac{1+p}{1-p}\right), \\ C_4 &= \frac{1}{2p}\left(pz_0 - z_1 + 1\right). \end{aligned}$$

■

2.2 Stability of System

Now, we firstly consider a transformation such as:

$$(t_n, t_{n-1}, \cdots, t_{n-m}, z_n, z_{n-1}, \cdots, z_{n-m}) \to (f, f_1, \cdots, f_m, g, g_1, \cdots, g_m)$$

where $f = 1 + p\frac{z_n}{z_{n-m}^2}, f_1 = t_n, \cdots f_m = t_{n-m}, g = 1 + p\frac{t_n}{t_{n-m}^2}, g_1 = z_n, \cdots, g_m = z_{n-m}$. Hence we obtain the following the jacobian matrix about equilibrium point $(\bar{t}, \bar{z})$:

$$B(\bar{t}, \bar{z}) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \frac{p}{\bar{z}^2} & 0 & \cdots & 0 & \frac{-2p}{\bar{z}^2} \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{p}{\bar{t}^2} & 0 & \cdots & 0 & \frac{-2p}{\bar{t}^2} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}_{(2m+2)\times(2m+2)}.$$

Theorem 7 *The positive equilibrium point $(\bar{t}, \bar{z})$ of system (4) is locally asymptotically stable.*

Proof. The linearized system of system (4) about the positive equilibrium point is given by

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$X_{N+1} = B\left(\bar{t}, \bar{z}\right) X_N$, where

$$X_N = \begin{pmatrix} t_n \\ \vdots \\ t_{n-m} \\ z_n \\ \vdots \\ z_{n-m} \end{pmatrix},$$

$$E = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & c & 0 & \cdots & 0 & -2c \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ c & 0 & \cdots & 0 & -2c & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}_{(2m+2)\times(2m+2)}$$

and $c = \frac{p}{\bar{z}^2} = \frac{p}{\bar{t}^2}$.

Let $\lambda_1, \lambda_2, \cdots, \lambda_{2m+2}$ denote the $2m+2$ eigenvalues of Matrix E. Let $D = diag\left(d_1, d_2, , d_{2m+2}\right)$ be a diagonal matrix such that $d_1 = d_{m+2} = 1, d_k = d_{m+1+k} = 1 - k\varepsilon, 1 \leq k \leq m$ and

$$0 < \varepsilon < \min\left\{\frac{1}{m+1}, \frac{1-3c}{m+1}\right\}.$$

Clearly, D is invertible matrix. In computing matrix DED^{-1}, we get that

$$DED^{-1} =$$

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & cd_1d_{m+2}^{-1} & 0 & \cdots & 0 & -2cd_1d_{2m}^{-1} \\ d_2d_1^{-1} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & d_{m+1}d_m^{-1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ cd_{m+2}d_1^{-1} & 0 & \cdots & 0 & -2cd_{m+2}d_{m+1}^{-1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & d_{m+3}d_{m+2}^{-1} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & d_{2m+2}d_{2m+1}^{-1} & 0 \end{pmatrix}$$

From the following two inequalities

$$\begin{aligned} 1 &= d_1 > d_2 > \cdots > d_m > d_{m+1} > 0, \\ 1 &= d_{m+2} > d_{m+3} > \cdots > d_{2m+1} > d_{2m+2} > 0 \end{aligned}$$

it implies that

$$\begin{aligned} d_2d_1^{-1} &< 1, d_3d_2^{-1} < 1, \cdots, d_{m+1}d_m^{-1} < 1, \\ d_{m+3}d_{m+2}^{-1} &< 1, d_{m+4}d_{m+3}^{-1} < 1, \cdots, d_{2m+2}d_{2m+1}^{-1} < 1. \end{aligned}$$

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Moreover we have that

$$\begin{aligned}
cd_1 d_{m+2}^{-1} + 2cd_1 d_{2m+2}^{-1} &= c\left(1 + \frac{2}{1-(m+1)\varepsilon}\right) < 1, \\
cd_{m+2} d_1^{-1} + 2cd_{m+2} d_{m+1}^{-1} &= c\left(1 + \frac{2}{1-(m+1)\varepsilon}\right) < 1.
\end{aligned}$$

It is a well-known fact that E has the same eigenvalues as DED^{-1}. Thus we get

$$\begin{aligned}
\max_{1\le k\le 2m+2} |\lambda_k| &= \left\| DED^{-1} \right\| \\
&= \max \left\{ \begin{array}{c} d_2 d_1^{-1}, \cdots, d_{m+1} d_m^{-1}, \\ d_{m+3} d_{m+2}^{-1}, \cdots, d_{2m+2} d_{2m+1}^{-1}, \\ cd_1 d_{m+2}^{-1} + 2cd_1 d_{2m+2}^{-1}, \\ cd_{m+2} d_1^{-1} + 2cd_{m+2} d_{m+1}^{-1} \end{array} \right\} \\
&< 1.
\end{aligned}$$

So, the positive equilibrium point $(\bar{t}, \bar{z})$ of system (4) is locally asymptotically stable. ■

Theorem 8 *Suppose that* $0 < p < \frac{1}{2}$. *Then the positive equilibrium point of system (4) is globally asymptotically stable.*

Proof. From Theorem 6, we know that

$$\begin{aligned}
1 &< l_1 = \lim_{n\to\infty} t_n \le M_1, \\
1 &< l_2 = \lim_{n\to\infty} z_n \le M_2, \\
1 &< L_1 = \lim_{n\to\infty} t_n \le M_1, \\
1 &< L_2 = \lim_{n\to\infty} z_n \le M_2.
\end{aligned}$$

Thus we have the followings for system (4)

$$\begin{aligned}
L_1 &\le 1 + p\frac{L_2}{l_2^2}, l_1 \ge 1 + p\frac{l_2}{L_2^2}, \\
L_2 &\le 1 + p\frac{L_1}{l_1^2}, l_2 \ge 1 + p\frac{l_1}{L_1^2}.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
L_1 + p\frac{l_1}{L_1} &\le L_1 l_2 \le l_2 + p\frac{L_2}{l_2}, \\
L_2 + p\frac{l_2}{L_2} &\le L_2 l_1 \le l_1 + p\frac{L_1}{l_1}.
\end{aligned}$$

 doi:10.20944/preprints202010.0142.v1

Hence we get that

$$\begin{aligned} L_1 + p\frac{l_1}{L_1} + L_2 + p\frac{l_2}{L_2} &\leq l_2 + p\frac{L_2}{l_2} + l_1 + p\frac{L_1}{l_1}, \\ L_1 + p\frac{l_1}{L_1} + L_2 + p\frac{l_2}{L_2} - l_2 - p\frac{L_2}{l_2} - l_1 - p\frac{L_1}{l_1} &\leq 0, \\ (L_1 - l_1)\left(1 - p\left(\frac{1}{l_1} + \frac{1}{L_1}\right)\right) + (L_2 - l_2)\left(1 - p\left(\frac{1}{l_2} + \frac{1}{L_2}\right)\right) &\leq 0. \end{aligned}$$

Now if $p \in \left(0, \frac{1}{2}\right)$ then

$$\begin{aligned} 1 - p\left(\frac{1}{l_1} + \frac{1}{L_1}\right) &> 0, \\ 1 - p\left(\frac{1}{l_2} + \frac{1}{L_2}\right) &> 0. \end{aligned}$$

Therefore, we obtain that

$$L_1 - l_1 = 0, L_2 - l_2 = 0.$$

So, $L_1 = l_1$ and $L_2 = l_2$. ■

2.3 Rate of convergence of system (4)

This section, we investigate the rate of convergence of system (4).

Theorem 9 *Assume that $0 < p < \frac{1}{2}$ and $\{(t_n, z_n)\}$ be a solution of the system (4) such that $\lim_{n\to\infty} t_n = \bar{t}$ and $\lim_{n\to\infty} z_n = \bar{z}$. Then the error vector*

$$E_n = \begin{pmatrix} e_n^1 \\ \vdots \\ e_{n-m}^1 \\ e_n^2 \\ \vdots \\ e_{n-m}^2 \end{pmatrix} = \begin{pmatrix} t_n - \bar{t} \\ \vdots \\ t_{n-m} - \bar{t} \\ z_n - \bar{z} \\ \vdots \\ z_{n-m} - \bar{z} \end{pmatrix}$$

of every solution of system (4) satisfies both of the following asymptotic relations:

$$\begin{aligned} \lim_{n\to\infty} \sqrt[n]{\|E_n\|} &= \left|\lambda F_J(\bar{t}, \bar{z})\right|, \\ \lim_{n\to\infty} \frac{\|E_{n+1}\|}{\|E_n\|} &= \left|\lambda F_J(\bar{t}, \bar{z})\right|, \end{aligned}$$

where $\lambda F_J(\bar{t}, \bar{z})$ are the characteristic roots of the Jacobian matrix $F_J(\bar{t}, \bar{z})$.

Proof. We firstly set to find the error terms,

$$\begin{aligned} t_{n+1}-\bar{t} &= \sum_{i=0}^{k} A_i\left(t_{n-i}-\bar{t}\right)+\sum_{i=0}^{k} B_i\left(z_{n-i}-\bar{z}\right), \\ z_{n+1}-\bar{z} &= \sum_{i=0}^{k} C_i\left(t_{n-i}-\bar{t}\right)+\sum_{i=0}^{k} D_i\left(z_{n-i}-\bar{z}\right), \end{aligned}$$

and $e_n^1=t_n-\bar{t}, e_n^2=z_n-\bar{z}$. Hence we obtain

$$\begin{aligned} e_{n+1}^1 &= \sum_{i=0}^{k} A_i e_{n-i}^1+\sum_{i=0}^{k} B_i e_{n-i}^2, \\ e_{n+1}^2 &= \sum_{i=0}^{k} C_i e_{n-i}^1+\sum_{i=0}^{k} D_i e_{n-i}^2, \end{aligned}$$

where $A_i=0$ and $D_i=0$ for $i=0,1,\cdots,m$,

$$\begin{aligned} B_0 &= \frac{p}{z_{n-m}^2}, B_i=0, i\in\{1,2,\cdots,m-1\}, B_m=\frac{-p\left(\bar{z}+z_{n-m}\right)}{\bar{z}z_{n-m}^2}, \\ C_0 &= \frac{p}{t_{n-m}^2}, C_i=0, i\in\{1,2,\cdots,m-1\}, C_m=\frac{-p\left(\bar{t}+t_{n-m}\right)}{\bar{t}t_{n-m}^2}. \end{aligned}$$

Taking the limits, we have $\lim_{n\to\infty} A_i=\lim_{n\to\infty} D_i=0$ for $i\in\{0,1,\cdots,m\}$ and $\lim_{n\to\infty} B_i=\lim_{n\to\infty} C_i=0$ for $i\in\{1,\cdots,m-1\}$. Moreover we obtain that

$$\begin{aligned} \lim_{n\to\infty} B_0 &= \frac{p}{\bar{z}^2}, \lim_{n\to\infty} B_m=\frac{-2p}{\bar{z}^2}, \\ \lim_{n\to\infty} C_0 &= \frac{p}{\bar{t}^2}, \lim_{n\to\infty} C_m=\frac{-2p}{\bar{t}^2}. \end{aligned}$$

Hence

$$\begin{aligned} B_0 &= \frac{p}{\bar{z}^2}+a_n, B_m=\frac{-2p}{\bar{z}^2}+b_n, \\ C_0 &= \frac{p}{\bar{t}^2}+c_n, C_m=\frac{-2p}{\bar{t}^2}+d_n, \end{aligned}$$

where $a_n\to 0, b_n\to 0, c_n\to 0, d_n\to 0$ as $n\to\infty$. Hence, we get the system of the form (2)

$$E_{n+1}=\left(A+B\left(n\right)\right)E_n$$

where

$$A=\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \frac{p}{\bar{z}^2} & 0 & \cdots & 0 & \frac{-2p}{\bar{z}^2} \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{p}{\bar{t}^2} & 0 & \cdots & 0 & \frac{-2p}{\bar{t}^2} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

$$B(n) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & a_n & 0 & \cdots & 0 & b_n \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ c_n & 0 & \cdots & 0 & d_n & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

and $\|B(n)\| \to 0$ as $n \to \infty$. Therefore, we can write the limiting system of error terms about the equilibrium point $(\bar{t}, \bar{z})$ as follows:

$$\begin{pmatrix} e^1_{n+1} \\ e^1_n \\ \vdots \\ e^1_{n-m+1} \\ e^2_{n+1} \\ e^2_n \\ \vdots \\ e^2_{n-m+1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \frac{p}{\bar{z}^2} & 0 & \cdots & 0 & \frac{-2p}{\bar{z}^2} \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{p}{\bar{t}^2} & 0 & \cdots & 0 & \frac{-2p}{\bar{t}^2} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \begin{pmatrix} e^1_n \\ e^1_{n-1} \\ \vdots \\ e^1_{n-m} \\ e^2_n \\ e^2_{n-1} \\ \vdots \\ e^2_{n-m} \end{pmatrix},$$

which is same as linearized system of system (4) about equilibrium point $(\bar{t}, \bar{z})$. ■

2.4 Oscillatory of solutions of system (4)

We now study the oscillatory of solutions of system(4).

Theorem 10 *Suppose that $\{(t_n, z_n)\}$ be a positive solution of system (4) and $p > 0$. Then, either $\{(t_n, z_n)\}$ solution of system (4) has a single semicycle or $\{(t_n, z_n)\}$ solution of system (4) has semicycles with at most m terms.*

Proof. Let $\{(t_n, z_n)\}$ solution of system (4) has at least two semicycles. Hence there exists $N \geq 0$ such that either

$$t_N, z_{N+1} < \bar{t} = \bar{z} < t_{N+1}, z_N$$

or

$$t_{N+1}, z_N < \bar{t} = \bar{z} < t_N, z_{N+1}.$$

Firstly we assume that $t_N, z_{N+1} < \bar{t} = \bar{z} < t_{N+1}, z_N$. Moreover, we suppose that positive semicycle have m terms and it begins with the term (t_{N+1}, z_{N+1}). Thus we obtain the followings

$$\begin{aligned} t_N &< \bar{t} = \bar{z} < t_{N+m}, \\ z_{N+m} &< \bar{t} = \bar{z} < z_N. \end{aligned}$$

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From this we get

$$\begin{aligned} t_{N+m+1} &= 1+p\frac{z_{N+m}}{z_N^2} < \bar{t} = \bar{z}, \\ z_{N+m+1} &= 1+p\frac{t_{N+m}}{t_N^2} > \bar{t} = \bar{z}. \end{aligned}$$

■

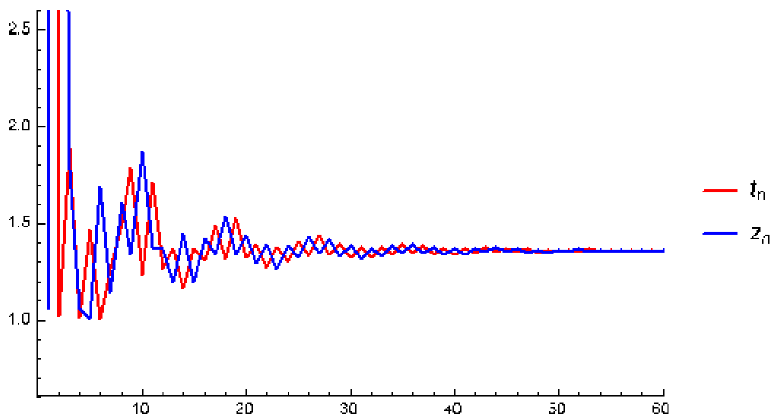
3 Numerical examles

In this here, we provide two examples to verify our theoretical results.

Example 1 *We handle system (4) with* $m = 3$ *and* $p = 0.49$. *Therefore we obtain the following fourth order system of difference equations*

$$t_{n+1} = 1 + 0.49\frac{z_n}{z_{n-3}^2}, z_{n+1} = 1 + 0.49\frac{t_n}{t_{n-3}^2}. \tag{9}$$

Now we consider this system with the initial values $t_{-3} = 6, t_{-2} = 1, t_{-1} = 0.8, t_0 = 4, z_{-3} = 0.4, z_{-2} = 5, z_{-1} = 3$ *and* $z_0 = 10$. *Thus positive equilibrium point* $(\bar{t}, \bar{z}) = (1.36, 1.36)$ *of system (9) is globally asymptotically stable. Figure 1 verifies to our theoretical results.*



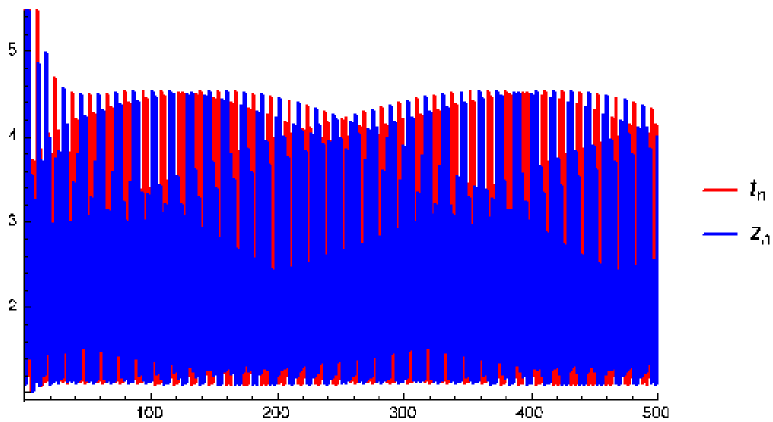
1. Plot of system (4) with $m = 3$ and $p = 0.49$.

Example 2 *We take system (4) with* $m = 2$ *and* $p = 1.5$. *Hence we have the following third order system of difference equations*

$$t_{n+1} = 1 + 1.5\frac{z_n}{z_{n-2}^2}, z_{n+1} = 1 + 1.5\frac{t_n}{t_{n-2}^2}. \tag{10}$$

 doi:10.20944/preprints202010.0142.v1

We also consider this system with the initial values $t_{-2} = 3, t_{-1} = 4, t_0 = 0.6, z_{-2} = 2, z_{-1} = 0.4$ *and* $z_0 = 3$. *Thus the solutions of system* (10) *oscillate about positive equilibrium point* $(\bar{t}, \bar{z}) = (1.82, 1.82)$. *Figure 2 verifies our theoretical results.*



2. Plot of system (4) with $m = 2$ and $p = 1.5$.

4 Conclusions and an Open Problem

In this study, we analysis the dynamics of system (4) of higher order difference equations with quadratic terms. We first examine the equilibrium point of related system. Then we investigate the existence of bounded solutions of system (4). We also study the stability analysis of solutions of system (4). As a result of this, we obtain that positive equilibrium point of system (4) is globally asymptotically stable when $0 < p < \frac{1}{2}$. Additionally we investigate the rate of convergence of system (4). Moreover we have an oscillatory result such that the solution of system (4) has a single semicycle or the solution of system (4) has semicycles with at most m terms. In addition we give two numerical examles to verify our theoretical results.

Open Problem: Investigate the dynamics of following non-symmetric system of difference equations

$$x_{n+1} = A + B\frac{y_n}{y_{n-m}^2}, y_{n+1} = C + D\frac{x_n}{x_{n-m}^2}, \ n = 0, 1, ...,$$

where the initial values are real numbers and both $A \neq B$ and $C \neq D$ are real numbers.

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