A probe of micro space-time by the study of scalar fields

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Abstract

In this work, we try to find a way to describe the physical law of micro-world under the frame of a space-time theory. By introducing the scalar fields $\mathcal{D}(x)$ and $\zeta(x)$, we rewrite the action of conventional field theory and the Lagrangian describing the motion of the particle, where a modified space-time relation is obtained. To test the correctness of this attempt, we demonstrate the Klein-Gordon equation by the Hamilton-Jacobi method in four dimensional form and present a time delay for the ground state of hydrogen atom.

Keywords: general relativity; scalar field; variational principle

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I. INTRODUCTION

General relativity (GR) [1, 2] is a milestone in 20th century physics. It reveals the nature of gravity and greatly promotes the development of modern physics, especially the physics of black holes and cosmology, it also led to the emergence of various modified gravity theories, such as $f(R)$ gravity, $f(R,T)$ gravity, Lovelock's theorem, Brans–Dicke theory, Einstein–Gauss–Bonnet gravity, etc [3–17]. In the late 90's, the high redshift surveys of supernovae [18, 19] indicates the expansion of the universe is accelerating, this undoubtedly accelerated the birth of new theories of gravity. In past few decades, these modified gravity theories have made important progress in explaining the evolution of universe and in solving the problem of cosmological constant, inflation, or structure formation, etc. In this paper, we focus on the scalar-tensor field theory. As one of the famous modified theories, the scalar-tensor theory has attracted widespread attention. As we know, general relativity is a geometrical theory or a " tensor theory ", because the fundamental composition of this theory is a metric tensor field. Despite we should set the degrees of freedom in a theory as few as possible, it does not rule out the introduction of additional scalar fields. To some extend, scalar field was a common way to describe gravity before the advent of general relativity. In the original scalar tensor theory, the scalar field was related to a changing gravitational constant, thus the gravity can be adjusted. Gravity comes from two contributions: one is due to space-time geometry, and the other is due to a scalar field. Generally, the scalar field in a theory is often accompanied by profound physical essence. The sources of scalar fields are various, it could be the dilaton from string theory [20, 21], the scalar field in a brane world [22], or comes the size of compactified internal space, etc [23]. This feature of scalar field theory is very attractive because it provides a lot of freedom for proposing a new theory. In the development of modified gravity theories, the variational principle played an important role. Starting from the action amount, the equation of motion of the field or the particle can be easily obtained, the modification of a theory can be obtained by modifying its action, this method also provides an effective path for the birth of new theories.

Since the birth of general relativity, space-time has entered people's field of vision as a special research object, the background geometry is closely related to the distribution of matter, this indicates that space-time may have more interesting characteristics. In scalar-tensor field theory, the scalar field might be the source of these characteristics. Generally, GR is used in large-scale research, especially in cosmological scale, however, when it comes to micro-scale, there might be some problem. In cosmology, especially the fine-tuning problem [24, 25] indicate a gravity theory that can work at a micro scale is urgently needed, a modified field equation might be needed to describe the law of physics. Therefore, it is reasonable to add an additional scalar field to the action to alleviate the discomfort of conventional gravity theories in solving microscopic problems.

In this article, we try to use the methods of scalar-tensor theory and the concept of space-time to derive the law of physics in the microcosm. Generally, such an attempt is difficult. However, the introduction of the concept of space-time in GR, especially the view that space-time has a microstructure in some quantum gravity theories [26–35], provides a feasible way for this attempt. For technical details, we take a view that some microstructures of space-time would induce a scalar field $\mathcal{D}(x)$. Considering a particle moving in the background with such a scalar field, we propose the action of fields and the Lagrangian describing the motion of the particle. According to the action, one can get the relationship between the field $\mathcal{D}(x)$ and another scalar field $\zeta(x)$. With the Hamiltonian discussion, a modified space-time relation in general relativity is obtained. In order to test the practicability in describing the physical law in micro space-time, using the four dimensional Hamilton-Jacobi method, we demonstrate the Klein-Gordon equation [36–38]. Throughout this paper we use the signature $(-,+,+,+)$.

II. THE BASIC EQUATIONS

Considering a spin-independent particle moving in an arbitrary background, we choose the action of fields as

$$I = I_0 + \int \sqrt{-g}\lambda \left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \zeta \partial_\nu \zeta + \frac{1}{2} \left(\frac{mc}{\hbar} \right)^2 \mathcal{D} \zeta^2 \right] d^4 x, \tag{1}$$

and the Lagrangian describing the motion of the particle is

$$L = L_0 - \frac{1}{2} mc^2 \mathcal{D}. \tag{2}$$

Where I_0 can be the action of arbitrary theories we usually known, such as gauge theory [39–44] or various gravity theories. L_0 is the Lagrangian of the particle in arbitrary known theories and m is the proper mass of the particle. c is the speed of light and $\hbar$ is the reduced Planck constant (For convenience, we use natural units in which $c = \hbar = 1$ in the rest of this paper), λ is a constant with the dimension of length. The scalar function $\mathcal{D}(x)$ that we especially propose in this work form some microstructure of space-time represents a proper time field. Finally, $\zeta(x)$ is a scalar field with its direct explanation as

$$\zeta^2 = \eta, \tag{3}$$

here η is the proper space density of the particle caused by restricting the space of its existence, it can be also understood as the proper probability density because of the proportional relationship between two explanations. Thus it is reasonable to require the space density to satisfy the conservation law

$$\partial_\mu \left(\sqrt{-g} \eta u^\mu \right) = 0. \tag{4}$$

Similar to what we have done in conventional theories, we require that the space density of the existence be positive continuous, finite and single-valued at every point of space.

In order to further understand above formulas, we can write Eq. (1) and Eq. (2) more specifically. For example, considering a spin-independent particle moving in the electromagnetic field under Einstein gravity, with all elements the action and the Lagrangian take the form

$$I = \int \sqrt{-g}\left[\frac{1}{16\pi G}R - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \mathcal{L}_m + \lambda\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\zeta\partial_\nu\zeta + \frac{1}{2}m^2\mathcal{D}\zeta^2\right)\right]d^4x, \quad (5)$$

$$L\left(x^\mu, \dot{x}^\mu\right) = \frac{1}{2}mg_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + qA_\mu\dot{x}^\mu - \frac{1}{2}m\mathcal{D}. \quad (6)$$

Where R is the curvature scalar, the electromagnetic tensor $F_{\mu\nu}$ satisfies $F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu}$ and A_μ is the four-component vector potential. $\mathcal{L}_m$ is the matter Lagrangian density which provides the source of gravitational field and electromagnetic field. In Eq. (6), $\dot{x}^\mu = \frac{dx^\mu}{d\tau} = u^\mu$ is the four-velocity, τ is the proper time of the particle and q is the charge of the particle. For such a particle, the Hamiltonian is

$$H = \pi_\mu\dot{x}^\mu - L = \frac{1}{2}m\left(g_{\mu\nu}u^\mu u^\nu + \mathcal{D}\right), \quad (7)$$

where

$$\pi_\mu = \frac{\partial L}{\partial\dot{x}^\mu} = mu_\mu + qA_\mu \quad (8)$$

is the four canonical momentum vector. Then the conservation of the Hamilton Eq. (7) leads to

$$g_{\mu\nu}u^\mu u^\nu + \mathcal{D} = const. \quad (9)$$

When $\mathcal{D} = 0$, Eq. (9) should go back to the results of conventional gravity theory, that is

$$g_{\mu\nu}u^\mu u^\nu = -1, \quad (10)$$

thus we can determine the constant and re-express Eq. (9) as

$$g_{\mu\nu}u^{\mu}u^{\nu}+\mathcal{D}=-1. \tag{11}$$

To make a clear explanation to the above formula, let's rewrite it as:

$$g_{\mu\nu}dx^{\mu}dx^{\nu}=-d\bar{\tau}^{2}, \tag{12}$$

and

$$d\bar{\tau}=\sqrt{1+\mathcal{D}}d\tau. \tag{13}$$

Here $\bar{\tau}$ is the proper time of the local coordinate which moves together with the particle. Now we can see that the space-time relation of the coordinate in this work remains same to the convention of GR, but the proper time of the particle is not equal to the proper time of the coordinate because there is a proper time field, therefore, the comprehension of the four velocity and the proper space density of the existence of the particle also need to be justified by this way.

To get more understanding of the proper time field $\mathcal{D}(x)$, we can also pay attention to the equation of the scalar field ζ. Varying the action Eq. (5) with respect to ζ, we get

$$\partial_{\mu}\left(\sqrt{-g}g^{\mu\nu}\partial_{\nu}\zeta\right)+m^{2}\sqrt{-g}\mathcal{D}\zeta=0, \tag{14}$$

this equation shows that the relation between proper time field and the space density field, and it constitutes the basic equations of this work together with Eq. (4) and Eq. (11).

III. DEMONSTRATION AND VERIFICATION

In order to test the correctness of the basic equations in describing physical law at the microscopic scale, the Hamilton-Jacobi method [45] is chosen:

$$H\left(x^{\mu},\partial_{\mu}S\right)+\frac{\partial S}{\partial\tau}=0, \tag{15}$$

where S is the Hamilton-Jacobi function and π_μ has been substituted by $\partial_\mu S$. With

$$S = W\left(x^\mu, \pi_\mu\right) - a_h \tau, \tag{16}$$

one has

$$H\left(x^\mu, \partial_\mu W\right) = a_h = -\frac{1}{2} m. \tag{17}$$

The quantity W is Hamilton characteristic function and $\pi_\mu = \partial_\mu W$, now Eq. (11) can be rewritten as

$$g^{\mu\nu}\left(\partial_\mu W - qA_\mu\right)\left(\partial_\nu W - qA_\nu\right) + m^2 \mathcal{D} = -m^2, \tag{18}$$

and Eq. (4) can also get a new form in the same way:

$$\partial_\mu\left[\sqrt{-g}\zeta^2 g^{\mu\nu}\left(\partial_\nu W - qA_\nu\right)\right] = 0. \tag{19}$$

The equations Eq. (18), Eq. (19) and Eq. (14) can be the new basic formulas in describing the motion of a particle moving in the electromagnetic field under gravity.

To reach the task, we can introduce a new function ξ by defining:

$$W = \frac{\hbar}{i} \ln \xi, \tag{20}$$

substituting it into Eq. (18) and Eq. (19), then combining with Eq. (14), one can get

$$\frac{1}{\sqrt{-g}} \partial_\mu\left[\sqrt{-g} g^{\mu\nu}\left(\partial_\nu - iqA_\nu\right)\phi\right] - g^{\mu\nu} iqA_\mu\left(\partial_\nu - iqA_\nu\right)\phi - m^2\phi = 0, \tag{21}$$

this is the Klein-Gordon equation derived from this theory. Here $\phi = \xi\zeta$ is the wave-function of the particle.

As for the positive definiteness of the probability density, let us define $\gamma = \frac{dt}{d\tau}$, regarding to Eq. (4) and $\phi^*\phi = \eta$, then the equation of normalization can be written as

$$\int_\infty \sqrt{-g}\gamma\phi^*\phi d^3x = 1. \tag{22}$$

Now we can see that the wave-function in the Klein-Gordon equation has clearly got its explanation.

Using Eq. (3) and Eq. (14), the proper time field $\mathcal{D}(x)$ can be expressed as

$$\mathcal{D} = -\frac{1}{m^2}\left[\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu \ln\sqrt{\eta}\right) + \sqrt{-g}g^{\mu\nu}\partial_\mu \ln\sqrt{\eta}\partial_\nu \ln\sqrt{\eta}\right], \tag{23}$$

choosing the ground state of hydrogen atom as a example, the time delay can be obtained as

$$\langle\gamma\rangle = \left\langle\frac{dt}{d\tau}\right\rangle = \int_\infty \sqrt{1+\mathcal{D}}\eta d^3x \approx 1.000026638 \tag{24}$$

by computational method, where $\eta = \Psi^*\Psi$ and Ψ is the wave function of the ground state of hydrogen atom. This result might be a useful reference for experimental test.

IV. CONCLUSIONS AND OUTLOOK

In this paper, in order to probe the physical law in micro-world under the frame of space-time theory, we introduced the scalar fields $\mathcal{D}(x)$ and $\zeta(x)$, then a modified space-time relation is obtained. One can see that the proper time field $\mathcal{D}(x)$ relates to the probability density field. Also, using the Hamiltonian-Jacobi method, the Klein-Gordon equation was obtained, which shows this probe of micro space-time is effective. The time delay for the ground state of hydrogen atom is also calculated.

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CONFLICTS OF INTEREST

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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