Maximal order of NG-Transformation group

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Abstract: In this paper, we consider the problem that the maximal order consider the groups that consisting of transformations we called NG-Transformation on a nonempty set *A* has no bijection as its element. We find the order of these groups not greater that (*n*-1)!. In addition, we will prove our result by showing that any kind of NG-group in the theorem be isomorphic to a permutation group on a quotient set of *A* with respect to an equivalence relation on *A*.

Keywords: NG-group, permutation group, permutation group,χ-subgroup.

Introduction: We consider the problem that the maximal order of a group consisting of transformations on a nonempty set *A* and the group has no bijection as its element. Recall a permutation group on *A* is a group consisting of bijections from *A* to *A* with respect to compositions of mappings. It is well known that any permutation group on a set *A* with cardinality n has order not greater that *n*!. There are some authors, [15],[14], problem 1.4 in [6], considering groups which consists of non-bijective transformations on *A* where the binary operation is the composition of mappings. Our first result is on the orders of such groups.

Theorem1.1. *let A be a set with cardinality n. Suppose NG be groups consisting of non bijective transformations on A, where the binary operation on NG is the composition of transformation. Then the order of NG is not greater than (n-1)! and there are such groups having order(n-1)!.*

We will prove Theorem1.1 by showing that any kind of group in the theorem be isomorphic to a permutation group on a quotient set of *A* with respect to an equivalence relation on *A*.

Definition 1.1. A class of group χ is called *an SHP-class* if it is closed under taking subgroups, homomorphic images and products of normal subgroups. The latter condition mean that if *U* and *V* are normal in *G* and both *U* and *V* lie in χ, then $UV \in \chi$. If a group *G* belong to χ, we will say *G* is an χ-group.

Remark 1.2. If χ is an SHP-class and $U,V \lhd G$ are such that *G/U* and *G/V* are χ-groups, then $G/(U \cap V)$ is isomorphic to a subgroup of the χ-group $(U/G) \rtimes (G/V)$, and thus $G/(U \cap V)$ is an χ-group. It follows that given a finite group *G*, there exists a unique smallest normal subgroup *N* such that $G/N \in \chi$, and we write $N = G\chi$.

The following theorem was found in [14]; see also lemma 2.32 in [3].

Theorem 1.2 *let χ be an SHP-class, and suppose G=UV, where U and V are subnormal in G. then $G\chi = U\chi\,V\chi$.*

We can take the SHP-class to the class of p-groups, the class of nilpotent groups, etc. theorem1.2 will imply Lemma 9.15, problem 9B.5, Corollary 9.27, problem 9C.2, as corollaries.

Remark 1.3. It was noted in Sec.4 of [8] that if we replace the condition that χ is an SHP-class by some weaker condition that the class χ is such that whose composition factors all lie in some given set of simple groups then theorem 1.2 will fail in this case.

Definition 1.2. let χ be an *SHP-class* and *G* be a finite group. We denoted the maximal normal χ-subgroup of *G* by $G\chi$.

We will consider the question that if *G=UV* with *U,V* subnormal in *G* then it holds that $G\chi = U\chi V\chi$ or not. If *p* is a prime and take the SHP-class χ to be the class of all finite *p*-group, then for any finite group $G\chi$ will be $O_p(G)$ and we have the following theorem, which is inspired by P.Iin,[12].

Theorem1.3. *Let p and q be two primes such that $q\equiv1(mod\ p)$. Let $N=G_q$ be cyclic group of order q and $H=<x>\times<y>$ an elementary abelian group of order p^2. Let $<x>$ act on N faithfully and $<y>$ act on N trivially. Set $G= N\rtimes H$ to the semidirect product of N and H. Let $U=N<x>$ and $V=<xy>$. Then*

(1) *U,V are both subnormal in G.*

(2) *$O_p(G)=<y>$ and$O_p(U)=O_p(V)$=1. In particular, $Op(G)\neq O_p(U)O_p(V)$.*

Preliminary: In this section, we review some basics concepts of the finite group theory that are assumed in our paper. The detailed here in lots of abstract algebra and finite group theory books for more including all treating of some of this material for example, in [11],[6],and,[9] would be good supplementary sources for the theory needed here.

Definition 2.1. A binary relation ~ in *A* is called *an equivalence relation* on *A. If it satisfy the following three conditions:*
  (i)   *a~a* for any *a* $\in A$;
  (ii)   for any *a,b*$\in$ *A*, if *a~b* then *b~a* ;
  (iii)   for any *a,b,c*$\in A$, if *a~b* and *b~c* then *a~c*.

For the set A, where use A^A to dented the set of all its transforms. For any $f \in A^A$, we use Im(f) to denoted the image of f. Also, Z and $Z_>$ will respective dented the set of integers and positive integers.

Definition 2.2. Let ~ be an equivalence relation on *A*. For an element $a \in A$, we call $\{x \in A \mid x \sim\}$ the equivalence class of a determined by ~, which is denoted by$[a]_\sim$. And $A/\sim = \{[a]_\sim \mid a \in A\}$ is called the quotient set of *S* relative to the equivalence relation ~.

Lemma2.3.*([14],theorem 1) For any $f \in G$ and the e the identity element of G, $\sim_e = \sim_f$.*
Proof. For any $a \in A$, our goal is to show that $[a]_f = [a]_e$. On one hand, if $x \in [a]_f$, i.e. $f(x) = f(a)$. Since *G* is a group with identity element *e*, there is a transformation $f' \in G$ such that $f'f = e = f'f$. Therefore, $e(x) = f'(f(x)) = f'(f(a)) = e(a)$, Which yields that $x \in [a]_e$. On the other hand, if $y \in [a]_e$ i.e. $e(a)e(y)$. Hence, $f(a) = (fe)(a) = f(e(y)) = (fe)(y) = f(y)$, which implies $y \in \in [a]_f$. It follows that $[a]_e = [a]_f$ for any $a \in A$, as wanted.

Remark 2.1. For Lemma 2.4, we see that $\sim_f = \sim_g$ for any element $f, g \in G$ .The following Theorem is revised version of Theorem 2,[14].

Theorem 2.5. *Let f be an element in A^A and $\hat{f}$ be the induced transformation of f on $A/\sim_f$, i.e $\hat{f}$: $A/\sim_f \longrightarrow A/\sim_f$, $[x]_f \mapsto [f(x)]_f$. Then the following hold:*
(i)   The exists a groups $G \subseteq A^A$ containing *f* as the identity element *iff $f^2 = f$*.
(ii)   There is a groups $G \subseteq A^A$ containing *f* as the identity element *iff $\hat{f}$ is abijective on $A/\sim_f$.*

The following two corollaries are from [14] and we make some corrections to the original proofs. Actually, we adopt the restriction of finiteness on *A* in the first corollary from the original one. And we used the finiteness on *A* in the second corollary; the original one did not use it.

Corollary 2.6. *Let f be an element in A^A. Then $f^2 = f$ iff the induced mapping $\hat{f}$ on $A/\sim_f$ is the identity element.*
Proof. On one hand, suppose that $f^2 = f$. Then for any $[x]_f \in A/\sim_f$, as$f(x) = f(f(x))$ we see that $[x]_f = [f(x)]_f$. It follows that $\hat{f} ([x]_f) = [f(x)]_f = [x]_f$; which implies that $\hat{f}$ is the identity mapping on $A/\sim_f$. On the other hand, assume that $\hat{f}$ is the identity mapping on $A/\sim_f$. Then for any $[x]_f \in A/\sim_f$, the condition that $\hat{f} ([x]_f) = [x]_f$ will imply that $[f(x)]_f = [x]_f$ and hence $f(f(x)) = f(x)$. It follows that $f^2 = f$ as required.

Corollary 2.7. *Suppose that A is a finite set and f is an element in A^A. Then there is a group $G \subseteq A^A$ containing f as an element iff Im(f) = Im(f^2).*
Proof. On one hand, suppose that there is a group $G \subseteq A^A$ containing *f* as an element. Let *e* be the identity element of *G*. Then by Theorem 2.5, the induced mapping $\hat{f}$ is a bijection on $A/\sim_f$. In particular, $\hat{f}$ is surjective and thus for any $x \in A$, there is *a* $[y]_f \in A/\sim_f$ such that $\hat{f} ([y]_f) = [x]_f = [f(y)]_f$; which yields that $f(x) = f(f(y)) = (f^2)(y)$. As

a result, $\text{Im}(f) \subseteq \text{Im}(f^2)$ and thus $\text{Im}(f) = \text{Im}(f^2)$. On the other hand, suppose that $\text{Im}(f) = \text{Im}(f^2)$. Thus, for any $f(x) \in \text{Im}(f)$ there is a $y \in A$ such that $f(x) = f(f(y))$ and hence $\hat{f}([y]f) = [x]f$; which implies that $\hat{f}$ is surjective on $A/\sim_f$. Note that we are assuming that A is finite and so is $A/\sim_f$. We have that the induced mapping $\hat{f}$ is bijective. By Theorem 2. 5, the assertion follows.

Remark 2. 2. Let $G \subseteq A^A$ be a group. We have seen, in Remark 2.1, that $\sim_f = \sim_g$ for any elements in G and we will denote the common equivalence relation by $\sim$. Also, by Theorem 2.5, each element $f \in G$ will induce a bijection $\hat{f}$ on $A/\sim$**.**

The following theorem is crucial since it turns a group $G \subseteq A^A$ into a permutation group.

Theorem 2.8. *Let $G \subseteq A^A$ be a group. Set $\hat{G} = \{\hat{f} \mid f \in G\}$; then $\hat{G}$ is a permutation group on $A/\sim$ and $\rho: G \longrightarrow \hat{G}, f \mapsto \hat{f},$ is an isomorphism.*

Proof. For any $f, g \in G$ and any $[a] \in A/\sim$, we have $\rho(fg)([a]) = [(fg)(a)] = [f(g(a))] = \rho(f)([g(a)]) = (\rho(f) \ \rho(g))([a])$; which implies that $\rho(fg) = \rho(f) \ \rho(g)$ and thus ρ is a homomorphism. By the definition of $\hat{G}$, it is obvious that ρ is surjective. Now suppose that $\rho(f) = \rho(g)$ for two elements $f, g \in G$, i.e.$[f(a)] = [g(a)]$, $\forall a \in A$: Let e be the identity element of G, then we have $[f(a)]_e = [g(a)]_e$; $\forall a \in A$. It follows that $e(f(a)) = e(g(a))$; $\forall a \in A$ .Hence,
$f(a) = (ef)(a) = e(f(a)) = e(g(a)) = g(a)$, $\forall a \in A$, and therefore $f = g$. We conclude that ρ is injective. As a consequence, ρ is an isomorphism.

Definition 2.3. A subgroup H of a group G is called *characteristic* in G, denoted H char G, if every automorphism of G maps H to itself, that is $\rho(H) = H$ for all $\rho \in \text{Aut}(G)$.
Remark 2.3. If H is characteristic in G in K and K is characteristic in G, then H is characteristic in G.

Let G be a finite group. We have the following two lemmas. They are from Section 2 of [8].
Lemma 2.9. *Suppose that χ is an SHP-class.*

(a) *Let $\leq G$ be a subgroup. Then $H^\chi \leq G^\chi$.*
(b) *Let $N \triangleleft G$ be a normal subgroup of G and write $\bar{G} = G/N$. then $\bar{G}^\chi = \overline{\bar{G}^\chi}$.*
(c) *G^χ is characteristic in G.*
(d) *$O_\chi (G)$ is characteristic in G.*

The following lemma is a generalization of Problem 2A.1 in[7].
Lemma 2.10. *Let A and B be two subnormal χ-subgroups of G. Then the subgroup $\langle A,B \rangle$ generated by A and B are χ-subgroup of G.*
Proof. Let A be a subnormal χ-subgroup of G. We use induction on the subnormal depth r, $A \subseteq O_\chi$ of A in G to show that if $r = 1$, then $A \triangleleft$ and thus $A \subseteq O_\chi (G)$ since $O_\chi(G)$ is the largest normal χ-subgroup of G. Suppose $r > 1$ and the containment holds for

r-1. Let $A_1 = A \lhd \ldots \lhd H_{r-1} \lhd H_r = G$ be a subnormal series from A to G: Then $A \subseteq O_\chi(G)$ by indctive hypothesis. Since $O_\chi(G)$ char H_{r-1} and $H_{r-1} \lhd G$; $O_\chi(G) \lhd G$ and then $O_\chi(G)$ $(H_{r-1}) \subseteq O_\chi(G)$. We conclude that $A \subseteq O_\chi(G)$. In general, for any two subnormal χ-subgroups A and B, $A,B \subseteq O_\chi(G)$ and thus $<A,B> \subseteq O_\chi(G)$ as wanted.

3. Proofs of Main Results

Now let A be a set having n letters written as $\{1, 2, \ldots, n\}$. We have the following theorem, which is Theorem 1.1.

Theorem 3.1. *Let A be a set with cardinality n with n≥3. Suppose NG is a group consisting of non-bijective transformations on A, where the binary operation on NG is the composition of transformations. Then the order of NG is not greater than (n-1)! and there are such groups having order (n-1)!:*

Proof. Let NG be a group consisting of non-bijective transformations on A. By Remark 2.1, we know that $\sim_f = \sim_g$ for any element $f, g \in NG$ and we denote the common equivalence relation by $\sim$. Note that NG is a group consisting of non-bijective transformations, then we see that the equivalence relation is not the equality relation $=$ on A. Thus, we have that the quotient set $A/\sim$ has order less than n-1. Additionally, NG is isomorphic to a permutation group on $A/\sim$ by Theorem 2.8. It follows that the order of NG is less than $(n$-1)! as any permutation group on $A/\sim$ has order less than $(n$-1)!.

Note that In defining a permutation s on the set $\{1, 3, \ldots, n\}$, there are n-1 choices for $\rho(1)$, n-2 choices of $\rho(3) \neq \rho(1)$, n- 2 choices of $\rho(4)$ ($\neq \rho(1)$, $\rho(3)$), etc., i.e. totally $(n$-1)$(n$-2)1 = $(n$-1)!.

Theorem 3.2. *Let χ be an SHP-calss, and suppose G =UV; where U and V are subnormal in G. Then $G^\chi = U^\chi V^\chi$.*

Proof. We use induction of the subnormal depth of U in G to prove the result. First, if the subnormal depth of U in G is one, i.e. $U \lhd G$. Since U^χ is characteristic in U and U is normal in G we see that U^χ is normal in G.

Let $\bar{G} = G/U^\chi$. By the hypothesis, $\bar{G} = \bar{U}$ $\bar{V}$ where $\bar{U} = U/U^\chi$, $\bar{V} = V U^\chi/U^\chi$. Thus, $\bar{U}$ is a normal χ-group of $\bar{G}$ and $\bar{V}$ is subnormal in $\bar{G}$. By Lemma 2.10, we have $\bar{G}^\chi = \bar{V}^\chi$. By Lemma 2.9 (b), $\bar{G}^\chi = \bar{G}^{\bar{\chi}}$, $\bar{V}^\chi = \bar{V}^{\bar{\chi}} = \bar{U}^{\bar{\chi}} \bar{V}^{\bar{\chi}}$. By correspondence theorem, we have $G^\chi = U^\chi V^\chi$; as required. Now suppose that the subnormal depth of U in G is r with $r > 1$: Let $U_1 = U \lhd \ldots \lhd U_r \lhd G$ be a subnormal series from U to G with length r. By Dedekind's lemma, $U_r = U(V \cap U_r)$. As both U and $V \cap U_r$ are subnormal in U_r and U has subnormal depth r- 1 in U_r, we obtain that

$(U_r)^\chi = U^\chi (V \cap U_r)^\chi$ by inductive hypothesis. Also, $G = U_r V$ with U_r normal in G and V subnormal in G, and hence $G^\chi = (U_r)^\chi V^\chi$ by the first paragraph of the proof. It follows that $G^\chi = (U_r)^\chi V^\chi = U^\chi (V \cap U_r)^\chi V^\chi = U^\chi V^\chi$, because $(V \cap U_r)^\chi \subseteq V^\chi$ by Lemma 2.9 (a).

Theorem 3.3. *Let p and q be two primes such that q≡ 1(modp). Let N = C_q be cyclic group of order q and H = $<x>$ an elementary abelian group of order p^2. Let $<x>$ act on N faithfully and $<y>$ act on N trivially. Set G = N ⋊ H to be the semidirect product of N and H. Let U = N$<x>$ and V = N$<xy>$. Then U, V are both subnormal in G and G = UV .*

    (i)   *$O_p(G) = <y>$ and $O_p(U) = O_p(V) = 1$. In particular, $O_p(G) \neq O_p(U) O_p(V)$.*

Proof. Since N is normal in G and the quotient group $G=N/H$ is abelian, we deduce that the derived subgroup G' is contained in N. It follows that both U and V contain G' as a subgroup, which implies that U and V are normal in G. Obviously, $G = UV$. Assertion (i) holds. Note that the Sylow p-subgroup of G is not normal since $<x>$ act on N faithfully and hence $O_p(G)$ has order less than p^2. However, as $<y>$ act on N trivially, N normalizes $<y>$ which yields that $<y>$ is a normal p-subgroup of G. It is easy to see that $O_p(G) = <y>$. Both $<x>$ and $<xy>$ act faithfully on N, which yields that $O_p(U) = Op(V) = 1$; as wanted.

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