

Some notes on the parametric Gevrey asymptotics in two complex time variables through truncated Laplace transforms

Guoting Chen^{*}, Alberto Lastra[†] and Stéphane Malek[‡]

chenguoting@hit.edu.cn alberto.lastra@uah.es

Stephane.Malek@math.univ-lille1.fr

Abstract

This paper is a slightly modified, abridged version of a previous work “Parametric Gevrey asymptotics in two complex time variables through truncated Laplace transforms” motivated by our contribution in the conference “Formal and Analytic Solutions of Diff. (differential, partial differential, difference, q-difference, q-difference-differential) Equations on the Internet” (FASnet20). It aims to clarify and give further detail at some crucial points concerning the asymptotic behavior of the solutions of the problems studied in that work.

Key words: Asymptotic expansion, Borel-Laplace transform, Fourier transform, initial value problem, formal power series, linear partial differential equation, singular perturbation. 2010 MSC: 35C10, 35C20.

1 Introduction

The main aim of this revision is to clarify some points and give answer to some questions on a recent work on truncated Laplace transform [1] which was presented at the conference “Formal and Analytic Solutions of Diff. (differential, partial differential, difference, q-difference, q-difference-differential,...) Equations on the Internet” (FASnet20), held virtually during the last week of June, 2020. Moreover, we motivate future problems in this direction.

All technical difficulties have been simplified, whose proof can be found in detail in our work [1], in order to focus on the mentioned details. In this revision we highlight two aspects that have not been detailed in that paper. Namely, we utterly explain the necessity to use a truncated Laplace transform instead of a complete one in order to extract asymptotic information from our constructed solutions. Furthermore, we provide a more geometric description (with the help of enlightening drawings) of a technical part needed for the study of the difference of consecutive solutions in the framework of the Ramis-Sibuya approach.

The motivation on the use of truncated Laplace transform leans on previous recent works in which truncated Laplace transform has been applied in different settings, namely in the study of sharp lower estimates [2] and its numerical properties [3] and [4].

^{*}School of Science, Harbin Institute of Technology, 518055 Shenzhen, China

[†]University of Alcalá, Departamento de Física y Matemáticas, Ap. de Correos 20, E-28871 Alcalá de Henares (Madrid), Spain

[‡]University of Lille, Laboratoire Paul Painlevé, 59655 Villeneuve d’Ascq cedex, France

Truncated Laplace transform appears in the classical theory of asymptotic expansions of complex functions at the time of constructing a function with prescribed Gevrey asymptotic behavior at the origin [5, 6] and also in the study of singularities of canard solutions to singularly perturbed equations [7]. Further applications can be found in many other recent publications [1].

In the next section, Section 2, we recall the main problem under study, and the main steps to provide analytic solutions to the problem, as well as formal solutions which are related by means of certain asymptotic expansions. Further remarks on some steps are also shown. In Section 3, we give further detail on the geometric aspects on the difference of two consecutive solutions, and describe further properties and future research.

2 Review of the main results

This section is devoted to review the main results in the work “Parametric Gevrey asymptotics in two complex time variables through truncated Laplace transforms” [1], stated without proof. We have also decided not to enter into many details and refer to the precise expression or result in the original text.

We focus our attention on the next initial value problem which involves two complex time variables t_1, t_2 and a small complex parameter ϵ ,

$$(1) \quad Q(\partial_z)u(t_1, t_2, z, \epsilon) = \epsilon^{\Delta_{D_1 D_2}} (t_1^{k_1+1} \partial_{t_1})^{\delta_{D_1}} (t_2^{k_2+1} \partial_{t_2})^{\tilde{\delta}_{D_2}} R_{D_1 D_2}(\partial_z)u(t_1, t_2, z, \epsilon) \\ + \sum_{\substack{1 \leq \ell_1 \leq D_1 - 1 \\ 1 \leq \ell_2 \leq D_2 - 1}} \epsilon^{\Delta_{\ell_1 \ell_2}} (t_1^{k_1+1} \partial_{t_1})^{\delta_{\ell_1}} t_2^{d_{\ell_2}} \partial_{t_2}^{\tilde{\delta}_{\ell_2}} c_{\ell_1 \ell_2}(z, \epsilon) R_{\ell_1 \ell_2}(\partial_z)u(t_1, t_2, z, \epsilon) + f(t_1, t_2, z, \epsilon),$$

under given initial conditions $u(0, t_2, z, \epsilon) \equiv u(t_1, 0, z, \epsilon) \equiv 0$.

Here, we assume that $Q, R_{D_1 D_2}, R_{\ell_1 \ell_2}$ are polynomials, $k_1, k_2 \geq 1$ are integers, the coefficients $c_{\ell_1 \ell_2}(z, \epsilon)$ are bounded holomorphic functions on some horizontal strip

$$H_\beta = \{z \in \mathbb{C} : |\operatorname{Im}(z)| < \beta\},$$

for some $\beta > 0$, with respect to z , and holomorphic with respect to ϵ on a disc $D(0, \epsilon_0)$, $\epsilon_0 > 0$.

The forcing term $f(t_1, t_2, z, \epsilon)$ is a holomorphic function in t_1, t_2 on $\mathbb{C}^* \times D(0, h')$, for some radius $h' > 0$, bounded holomorphic with respect to z on H_β and on any given open sector \mathcal{E} centered at 0, $\mathcal{E} \subseteq D(0, \epsilon_0)$ for some $\epsilon_0 > 0$, with respect to the perturbation parameter ϵ .

Our goal is the construction of holomorphic solutions $u(t_1, t_2, z, \epsilon)$ of (1) where t_1, t_2, ϵ are located on sectors in \mathbb{C} , together with the analysis of their asymptotic expansions with respect to ϵ ,

$$u(t_1, t_2, z, \epsilon) \sim_{\epsilon \rightarrow 0} \hat{u}(t_1, t_2, z, \epsilon) = \sum_{n \geq 0} u_n(t_1, t_2, z) \epsilon^n.$$

We search for solutions as a double Laplace and Fourier transform

$$(2) \quad u(t_1, t_2, z, \epsilon) = \frac{k_1 k_2}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \int_{L_{d_1}} \int_{L_{d_2}} \omega(u_1, u_2, m, \epsilon) \\ \times \exp\left(-\left(\frac{u_1}{\epsilon t_1}\right)^{k_1} - \left(\frac{u_2}{\epsilon t_2}\right)^{k_2}\right) e^{izm} \frac{du_2}{u_2} \frac{du_1}{u_1} dm,$$

along half-lines $L_{d_j} = [0, \infty)e^{\sqrt{-1}d_j}$ for suitable directions $d_j \in \mathbb{R}$, $j = 1, 2$.

This approach has been successfully applied in two previous works [8, 9] for other singularly perturbed families of PDEs with two complex time variables that can be expressed (in the linear setting) in the form

$$Q(\partial_z)\partial_{t_1}\partial_{t_2}y(t_1, t_2, z, \epsilon) = \mathcal{P}_1(\epsilon, t_1, t_2, \partial_{t_1}, \partial_{t_2}, \partial_z)y(t_1, t_2, z, \epsilon) + f(t_1, t_2, z, \epsilon),$$

where the differential operators with polynomial coefficients

$$\mathcal{P}_2(\epsilon, t_1, t_2, \partial_{t_1}, \partial_{t_2}, \partial_z) := Q(\partial_z)\partial_{t_1}\partial_{t_2} - L_1(\epsilon, t_1, t_2, \partial_{t_1}, \partial_{t_2}, \partial_z)$$

for L_1 being an operator which comprises the leading terms of \mathcal{P}_1 :

- can be factorized in a special manner as

$$\mathcal{P}_2 = \mathcal{P}_{2.1}(\epsilon, t_1, \partial_{t_1}, \partial_z) \cdot \mathcal{P}_{2.2}(\epsilon, t_2, \partial_{t_2}, \partial_z)$$

with factors that only depend on one time variable. In this case, the ‘‘Borel map’’ $\omega(u_1, u_2, m, \epsilon)$ is defined w.r.t. its first two variables on domains of the form $(S_{d_1} \cup D(0, \rho_1)) \times (S_{d_2} \cup D(0, \rho_2))$, where S_{d_j} stands for an infinite sector with vertex at the origin and bisecting direction d_j for $j = 1, 2$, and $\rho_j > 0$ is small enough [8].

- can not be factorized in the previous manner, and are of a special shape. The related domains for the ‘‘Borel map’’ $(u_1, u_2) \mapsto \omega(u_1, u_2, m, \epsilon)$ are of the form $S_{d_1} \times (S_{d_2} \cup D(0, \rho_2))$ or $(S_{d_1} \cup D(0, \rho_1)) \times S_{d_2}$, together with a polydisc at the origin [9].

In the study of (1), none of the solutions provided for the two previous problems hold and $\omega(u_1, u_2, m, \epsilon)$ in (2) can only be defined on products of unbounded sectors $S_{d_1} \times S_{d_2}$ with respect to (u_1, u_2) . Therefore, the actual solution $u(t_1, t_2, z, \epsilon)$ can be built up, whereas no asymptotic features with respect to ϵ can be obtained.

Indeed, $u(t_1, t_2, z, \epsilon)$ solves (1) provided that $\omega(u_1, u_2, m, \epsilon)$ solves a convolution equation of the form

$$(3) \quad P_m(u_1, u_2)\omega(u_1, u_2, m, \epsilon) = \text{convolution terms in } \omega(u_1, u_2, m, \epsilon) + \text{entire forcing term},$$

where

$$(4) \quad P_m(u_1, u_2) = Q(im) - k_1^{\delta_{D_1}} k_2^{\delta_{D_2}} u_1^{k_1 \delta_{D_1}} u_2^{k_2 \delta_{D_2}} R_{D_1 D_2}(im),$$

whose precise shape is detailed in our work [1].

Under the assumption that the quotient $Q(im)/R_{D_1 D_2}(im)$ remains inside certain sectorial annulus, then, for every fixed $\rho_0 > 0$, the map $u_1 \mapsto P_m(u_1, u_2)$ has $k_1 \delta_{D_1}$ complex roots in $D(0, \rho_0)$, provided that $u_2 \in S_{d_2}$ with large enough $|u_2|$.

The observation above on ω follows from the fact that in solving (3), one needs to invert $P_m(u_1, u_2)$.

As a result, we need to follow another approach in order to analyze the asymptotic expansions with respect to ϵ . Our idea consists on the next construction: instead of a solution expressed as a double Laplace transform, we search for a genuine solution in the form of a Fourier, truncated Laplace and Laplace transform, namely

$$(5) \quad u(t_1, t_2, z, \epsilon) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \int_{L_{1,\epsilon}} \int_{L_{d_2}} \omega(u_1, u_2, m, \epsilon) \times \exp\left(-\left(\frac{u_1}{\epsilon t_1}\right)^{k_1} - \left(\frac{u_2}{\epsilon t_2}\right)^{k_2}\right) e^{izm} \frac{du_2}{u_2} \frac{du_1}{u_1} dm,$$

where $L_{d_2} = [0, \infty)e^{\sqrt{-1}d_2}$ is such that $d_2 \in \mathbb{R}$ is a suitable direction, and $L_{1,\epsilon}$ stands for a segment of the form $\left[0, \left(C_1/\epsilon^{\lambda k_2 \tilde{\delta}_{D_2}}\right) e^{\sqrt{-1}\theta_1}\right]$, for some $C_1, \lambda > 0$ with $\lambda < (k_1 \delta_{D_1})^{-1}$, and an appropriate angle $\theta_1 \in \mathbb{R}$.

The first important feature is that the solution (5) remains close, as $\epsilon \rightarrow 0$, to a double Laplace transform in both time variables as mentioned earlier, since $C_1/\epsilon^{\lambda k_2 \tilde{\delta}_{D_2}} \rightarrow \infty$ as $\epsilon \rightarrow 0$. The second important property is that the asymptotic expansions relatively to ϵ can be reached out for this solution. Accordingly, we impose that the forcing term shares the same shape as the solution, namely

$$(6) \quad f(t_1, t_2, z, \epsilon) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \int_{L_{1,\epsilon}} \int_{L_{d_2}} \psi(u_1, u_2, m, \epsilon) \\ \times \exp\left(-\left(\frac{u_1}{\epsilon t_1}\right)^{k_1} - \left(\frac{u_2}{\epsilon t_2}\right)^{k_2}\right) e^{izm} \frac{du_2}{u_2} \frac{du_1}{u_1} dm,$$

where ψ is a polynomial in u_1 , entire in u_2 with at most exponential growth of order k_2 , continuous with respect to m with exponential decay on \mathbb{R} and holomorphic with respect to ϵ on $D(0, \epsilon_0)$. The resulting function $f(t_1, t_2, z, \epsilon)$ is holomorphic on $\mathbb{C}^* \times D(0, h') \times H_\beta \times \mathcal{E}$, for any sector $\mathcal{E} \subseteq D(0, \epsilon_0)$ centered at the origin. From the fact that f approaches a double Laplace transform in t_1, t_2 as $\epsilon \rightarrow 0$, observe that f remains close to a polynomial in t_1 (on some sector) as $\epsilon \rightarrow 0$, $\epsilon \in \mathcal{E}$.

It is worth noticing that the approach proposed in this situation differs from that previously worked out [10] which concerns a subclass of (1) where the differential operators in t_2 belong to a less general class of operators of the form $(t_2^{k_2+1} \partial_{t_2})^{\tilde{\delta}_{\ell_2}}$, and the solutions are built up as Fourier and single Laplace transform along appropriate half-lines, and with a special kernel.

The precise set of conditions we impose to (1) are the following:

$$(7) \quad \Delta_{D_1 D_2} = k_1 \delta_{D_1} + k_2 \tilde{\delta}_{D_2}, \text{ and for all } 1 \leq \ell_j \leq D_j - 1, \quad j = 1, 2,$$

$$\Delta_{\ell_1 \ell_2} > k_1 \delta_{\ell_1} + \frac{k_2 \tilde{\delta}_{D_2} \delta_{\ell_1}}{\delta_{D_1}}, \quad d_{\ell_2} > \tilde{\delta}_{\ell_2} (k_2 + 1), \quad \tilde{\delta}_{D_2} \delta_{\ell_1} \geq \delta_{D_1} (\tilde{\delta}_{\ell_2} + 1/k_2).$$

The quotient $Q(im)/R_{D_1 D_2}(im)$ remains inside a fixed unbounded sector $S_{Q, R_{D_1 D_2}}$ with positive distance to the origin. The first main result concerns the construction of actual holomorphic solutions to (1).

Theorem 1 (First statement of Theorem 1 [1]) *There exist:*

- (a) a finite set of bounded sectors covering a punctured disc at the origin, $\{\mathcal{E}_p\}_{0 \leq p \leq \iota-1}$, with $\mathcal{E}_p \subseteq D(0, \epsilon_0)$,
- (b) a set of directions $d_{2,p} \in \mathbb{R}$, with $0 \leq p \leq \iota - 1$,
- (c) A pair of bounded sectors $\mathcal{T}_1, \mathcal{T}_2$,

such that a holomorphic solution

$$(8) \quad u_p(t_1, t_2, z, \epsilon) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \int_{L_{1,p,\epsilon}} \int_{L_{d_{2,p}}} \omega_p(u_1, u_2, m, \epsilon) \\ \times \exp\left(-\left(\frac{u_1}{\epsilon t_1}\right)^{k_1} - \left(\frac{u_2}{\epsilon t_2}\right)^{k_2}\right) e^{izm} \frac{du_2}{u_2} \frac{du_1}{u_1} dm,$$

of (1) is defined on $\mathcal{T}_1 \times \mathcal{T}_2 \times H_\beta \times \mathcal{E}_p$, for all $0 \leq p \leq \iota - 1$.

For all $0 \leq p \leq \iota - 1$, the paths of integration $L_{1,p,\epsilon}$ depend on $\epsilon \in \mathcal{E}_p$, and represent the segment $[0, (C_1/\epsilon^{\lambda k_2 \tilde{\delta}_{D_2}})e^{\sqrt{-1}\theta_{1,p}}]$, for some $\theta_{1,p} \in \mathbb{R}$, and $L_{d_2,p} = [0, \infty)e^{\sqrt{-1}d_{2,p}}$. The Borel function $\omega_p(u_1, u_2, m, \epsilon)$ turns out to be continuous with respect to m on \mathbb{R} with exponential decay at infinity, and holomorphic with exponential growth with respect to (u_1, u_2) on domains which depend on ϵ given by:

- The polydisc $D(0, r_1(\epsilon)) \times D(0, r_2(\epsilon))$, where $r_1(\epsilon) = C_1/|\epsilon|^{\lambda k_2 \tilde{\delta}_{D_2}}$ and $r_2(\epsilon) = \frac{1}{2}|\epsilon|^{\lambda k_1 \delta_{D_1}}$.
- a product $S_{\theta_{1,p}, r_1(\epsilon)} \times (S_{d_2,p} \cup D(0, r_2(\epsilon)))$, where $S_{\theta_{1,p}, r_1(\epsilon)}$ is a sector centered at 0 with small aperture, bisecting direction $\theta_{1,p} - \lambda k_2 \tilde{\delta}_{D_2} \arg(\epsilon)$, and radius $r_1(\epsilon)$; $S_{d_2,p}$ is an unbounded sector centered at 0, with small opening and bisecting direction $d_{2,p}$.

The previous domains appear in the resolution of the related convolution problem for ω_p since $P_m(u_1, u_2)$ is invertible on these domains with appropriate lower bounds.

Our second main result deals with the asymptotic expansions of u_p relatively to ϵ .

Theorem 2 (Theorem 3 [1]) *Let*

$$(9) \quad \alpha = \min\{k_2(1 - \lambda k_1 \delta_{D_1}), k_1(1 + \lambda k_2 \tilde{\delta}_{D_2})\}.$$

There exists a formal power series

$$\hat{u}(t_1, t_2, z, \epsilon) = \sum_{m \geq 0} H_m(t_1, t_2, z) \frac{\epsilon^m}{m!},$$

where $H_m(t_1, t_2, z)$ are bounded holomorphic functions on $\mathcal{T}_1 \times \mathcal{T}_2 \times H_\beta$ for all $m \geq 0$, which solves (1) and is the common asymptotic expansion of Gevrey order $1/\alpha$ with respect to ϵ on \mathcal{E}_p of $u_p(t_1, t_2, z, \epsilon)$ for all $0 \leq p \leq \iota - 1$, i.e.

$$\sup_{(t_1, t_2, z) \in \mathcal{T}_1 \times \mathcal{T}_2 \times H_\beta} \left| u_p(t_1, t_2, z, \epsilon) - \sum_{m=0}^{N-1} H_m(t_1, t_2, z) \frac{\epsilon^m}{m!} \right| \leq CM^N \Gamma\left(1 + \frac{N}{\alpha}\right) |\epsilon|^N,$$

for all $\epsilon \in \mathcal{E}_p$, $N \geq 1$ and $0 \leq p \leq \iota - 1$, for well chosen constants $C, M > 0$.

This result leans on the application of the cohomological approach given by Ramis-Sibuya theorem and appropriate bounds on the difference of two consecutive solutions of (1) stated in the second statement of Theorem 1 [1], and described in the next section.

3 Further comments and open problems

It turns out that the location of the complex singularities of the partial Borel map $u_2 \mapsto \omega_p(u_1, u_2, m, \epsilon)$ merging at 0 as ϵ tends to 0 (due to our construction by means of a parameter depending truncated Laplace transform) has a direct effect on the Gevrey order $1/\alpha$ of the asymptotic expansions of our solutions.

The first occurrence of such an interplay between the accumulating singularities in the Borel plane at the origin and the asymptotics in the physical space appears in the important work [11] by B. Braaksma and L. Stolovitch where normal forms of singular holomorphic vector fields are studied. Later on, the authors and their colleagues have observed related phenomena in the study of

- PDEs with Fuchsian and irregular singularities in [12], singularly perturbed problems related to parametric summability in [13] and parametric multisummability in [14], [15],
- q-difference-differential equations with Fuchsian and irregular singularities in [16], [17], singularly perturbed Cauchy problems related to parametric q-Gevrey asymptotics in [18] and initial value problems mixing both Gevrey and q-Gevrey expansions in [19].

The Gevrey asymptotic expansion appearing in Theorem 2 is explained in terms of the difference of two consecutive solutions (in the sense that the solutions are related to consecutive sectors in $(\mathcal{E}_p)_{0 \leq p \leq \iota-1}$) of (1) on the intersection of their domains. More precisely for all $0 \leq p \leq \iota - 1$, identifying the indices ι and 0, we have shown the existence of two constants $K, M > 0$ with

$$\sup_{(t_1, t_2, z) \in \mathcal{T}_1 \times \mathcal{T}_2 \times H_\beta} |u_{p+1}(t_1, t_2, z, \epsilon) - u_p(t_1, t_2, z, \epsilon)| \leq K \exp\left(-\frac{M}{|\epsilon|^\alpha}\right),$$

for all $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$, and $0 \leq p \leq \iota - 1$. In order to provide these bounds, we use deformations of the integration paths involved in the solution (5). More precisely, the following three cases occur:

Case 1: One can choose $L_{1,p,\epsilon} \equiv L_{1,p+1,\epsilon}$, i.e. $\theta_{1,p} = \theta_{1,p+1}$ and $L_{d_2,p}$ differs from $L_{d_2,p+1}$. Then, the path

$$L_{1,p,\epsilon} \times L_{d_2,p+1} - L_{1,p,\epsilon} \times L_{d_2,p} = L_{1,p,\epsilon} \times (L_{d_2,p+1} - L_{d_2,p}),$$

represented in Figure 1 is deformed into the path displayed in Figure 2. We notice that this deformation can be performed since the function $(u_1, u_2) \mapsto \omega_j(u_1, u_2, m, \epsilon)$, $j = p, p+1$ is holomorphic on $D(0, r_1(\epsilon)) \times D(0, r_2(\epsilon))$. The integration along this new configuration gives rise to bounds of exponential decay at the origin of order $k_2(1 - \lambda k_1 \delta_{D_1})$ with respect to the perturbation parameter, in the intersection of the corresponding sectors of the difference of the solutions, uniformly with respect to the other variables.

Case 2: One can choose $L_{d_2,p} \equiv L_{d_2,p+1}$, but $L_{1,p,\epsilon}$ differs from $L_{1,p+1,\epsilon}$. Then, the path

$$L_{1,p+1,\epsilon} \times L_{d_2,p} - L_{1,p,\epsilon} \times L_{d_2,p}$$

is split into three pieces, and deformed to the concatenation of the following paths:

Piece 2.1: $L_{1,p+1,\epsilon} \times L_{d_2,p,r_2(\epsilon)}$ (see Figure 3)

Piece 2.2: $-L_{1,p,\epsilon} \times L_{d_2,p,r_2(\epsilon)}$ (see Figure 4)

Piece 2.3: $(L_{1,p+1,\epsilon} - L_{1,p,\epsilon}) \times L_{r_2(\epsilon), d_2, p}$ (see Figure 5).

Notice that the deformation involved in this piece can be performed since the map $(u_1, u_2) \mapsto \omega_j(u_1, u_2, m, \epsilon)$, $j = p, p+1$ is holomorphic on $D(0, r_1(\epsilon)) \times D(0, r_2(\epsilon))$. The deformation path is displayed in Figure 6. As in the first case, the integration along this new configuration in the pieces 2.1 and 2.2 gives rise to bounds of exponential decay at the origin of order $k_2(1 - \lambda k_1 \delta_{D_1})$ with respect to the perturbation parameter, in the intersection of the corresponding sectors of the difference of the solutions, uniformly with respect to the other variables. On the other hand, the integration along the arrangement in the piece 2.3 provides exponential decay at the origin of order $k_1(1 + \lambda k_2 \delta_{D_2})$.

Case 3: Assume that $L_{d_2,p}$ does not coincide with $L_{d_2,p+1}$, and $L_{1,p,\epsilon}$ differs from $L_{1,p+1,\epsilon}$. In this case, the path

$$L_{1,p+1,\epsilon} \times L_{d_2,p+1} - L_{1,p,\epsilon} \times L_{d_2,p}$$

is split into four pieces, and deformed as follows:

Piece 3.1: $L_{1,p+1,\epsilon} \times L_{r_2(\epsilon), d_2, p+1}$ (see Figure 7)

Piece 3.2: $L_{1,p+1,\epsilon} \times L_{d_2,p+1,r_2(\epsilon)}$ (see Figure 8)

Piece 3.3: $-L_{1,p,\epsilon} \times L_{r_2(\epsilon),d_2,p}$ (see Figure 9)

Piece 3.4: $-L_{1,p,\epsilon} \times L_{d_2,p,r_2(\epsilon)}$ (see Figure 10)

The piece 3.3 is deformed into two further block, namely $-L_{1,p,\epsilon} \times C_{r_2(\epsilon)}$ and $-L_{1,p,\epsilon} \times L_{r_2(\epsilon),d_2,p+1}$, say piece 3.3 (1) and piece 3.3 (2), represented in Figure 11 and Figure 12, respectively.

Finally, the piece 3.1 together with the piece 3.3 (2) can be deformed into $C_{r_1(\epsilon)} \times L_{r_2(\epsilon),d_2,p+1}$, shown in Figure 13.

Notice that the deformation from the path related to the piece 3.3, into piece 3.3 (1) and piece 3.3 (2); and the piece 3.1 together with the piece 3.3 (2) can be performed since the map $(u_1, u_2) \mapsto \omega_j(u_1, u_2, m, \epsilon)$, $j = p, p + 1$ is holomorphic on the polydisc $D(0, r_1(\epsilon)) \times D(0, r_2(\epsilon))$. As a result, the integration along the concatenation of the pieces 3.2, 3.4 and 3.3 (1) leads to bounds of exponential decay at 0 with respect to the perturbation parameter in the intersection of the corresponding sectors, uniformly with respect to the other variables, of order $k_2(1 - \lambda k_1 \delta_{D_1})$, and the integration along the deformed piece drawn in Figure 13 gives rise to bounds of exponential decay of order $k_1(1 + \lambda k_2 \tilde{\delta}_{D_2})$.

Since the truncated Laplace transform does not behave properly under products (unlike the complete one), one major challenging problem would be to generalize our statement to the case of nonlinear equations. Of course, the construction of solutions by means of double complete Laplace transforms remains possible in that extended situation. But the extraction of asymptotic information out of the solutions, which is the core of our study, remains an unsolved question left for future research.

Funding: The second and third authors are partially supported by the project MTM2016-77642-C2-1-P of Ministerio de Economía y Competitividad, Spain; the second author is partially supported by Dirección General de Investigación e Innovación, Consejería de Educación e Investigación of Comunidad de Madrid (Spain), and Universidad de Alcalá under grant CM/JIN/2019-010.

References

- [1] G. Chen, A. Lastra, S. Malek, *Parametric Gevrey asymptotics in two complex time variables through truncated Laplace transforms*. Adv. Difference Equ. 2020, Paper No. 307, 31 pp.
- [2] R. Lederman, S. Steinerberger, *Lower bounds for truncated Fourier and Laplace transforms*. Integral Equations Operator Theory 87 (2017), no. 4, 529–543.
- [3] R. Lederman, V. Rokhlin, *On the analytical and numerical properties of the truncated Laplace transform. Part II*. SIAM J. Numer. Anal. 54 (2016), no. 2, 665–687.
- [4] R. Lederman, V. Rokhlin, *On the analytical and numerical properties of the truncated Laplace transform I*. SIAM J. Numer. Anal. 53 (2015), no. 3, 1214–1235.
- [5] W. Balsler, *Formal power series and linear systems of meromorphic ordinary differential equations*. Universitext. Springer-Verlag, New York, 2000. xviii+299 pp.
- [6] M. Loday-Richaud, *Divergent series, summability and resurgence. II. Simple and multiple summability. With prefaces by Jean-Pierre Ramis, ric Delabaere, Claude Mitschi and David Sauzin*. Lecture Notes in Mathematics, 2154. Springer, Cham, 2016. xxiii+272 pp.

- [7] P. D'Escurac, *The Borel transform of canard values and its singularities*. Formal and analytic solutions of diff. equations, 149175, Springer Proc. Math. Stat., 256, Springer, Cham, 2018.
- [8] A. Lastra, S. Malek, *On parametric Gevrey asymptotics for some nonlinear initial value problems in symmetric complex time variables*, Asymptotic Analysis vol. 118, no. 1-2, 49–79, 2020.
- [9] A. Lastra, S. Malek, *On parametric Gevrey asymptotics for some initial value problems in two asymmetric complex time variables*. Results Math. 73 (2018), no. 4, Paper No. 155, 46 pp.
- [10] A. Lastra, S. Malek, *Boundary layer expansions for initial value problems with two complex time variables*. Adv. Difference Equ. 2020, Paper No. 20, 24 pp.
- [11] B. Braaksma, L. Stolovitch, *Small divisors and large multipliers*. Ann. Inst. Fourier (Grenoble) 57 (2007), no. 2, 603–628.
- [12] S. Malek, *On Gevrey functional solutions of partial differential equations with Fuchsian and irregular singularities*. J. Dyn. Control Syst. 15 (2009), no. 2, 277–305.
- [13] A. Lastra, S. Malek, J. Sanz, *On Gevrey solutions of threefold singular nonlinear partial differential equations*. J. Differential Equations 255 (2013), no. 10, 3205–3232.
- [14] A. Lastra, S. Malek, *Multi-level Gevrey solutions of singularly perturbed linear partial differential equations*. Adv. Differential Equations 21 (2016), no. 7-8, 767–800.
- [15] A. Lastra, S. Malek, *Some notes on the multi-level Gevrey solutions of singularly perturbed linear partial differential equations*. Analytic, algebraic and geometric aspects of differential equations, 415–427, Trends Math., Birkhäuser/Springer, Cham, 2017.
- [16] A. Lastra, S. Malek, J. Sanz, *On q -asymptotics for linear q -difference-differential equations with Fuchsian and irregular singularities*. J. Differential Equations 252 (2012), no. 10, 5185–5216.
- [17] A. Lastra, S. Malek, J. Sanz, *On q -asymptotics for q -difference-differential equations with Fuchsian and irregular singularities*. Formal and analytic solutions of differential and difference equations, 73–90, Banach Center Publ., 97, Polish Acad. Sci. Inst. Math., Warsaw, 2012.
- [18] A. Lastra, S. Malek, *On parametric Gevrey asymptotics for singularly perturbed partial differential equations with delays*. Abstr. Appl. Anal. 2013, Art. ID 723040, 18 pp.
- [19] A. Lastra, S. Malek, *On multiscale Gevrey and q -Gevrey asymptotics for some linear q -difference differential initial value Cauchy problems*. J. Difference Equ. Appl. 23 (2017), no. 8, 1397–1457.

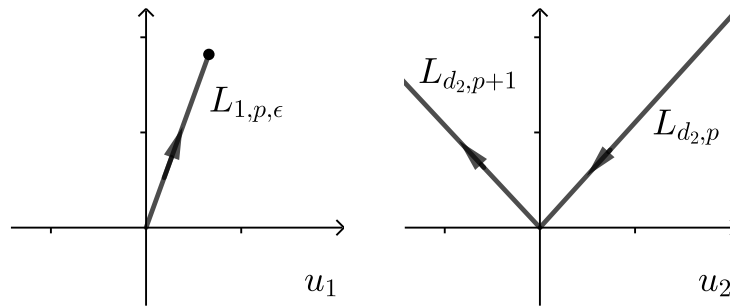


Figure 1: Integration path. Case 1

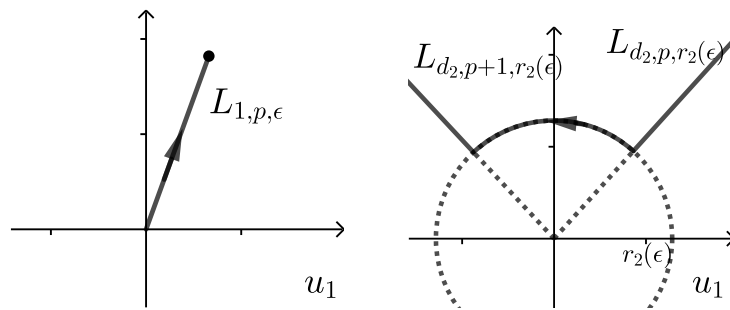


Figure 2: Deformation of the integration path. Case 1

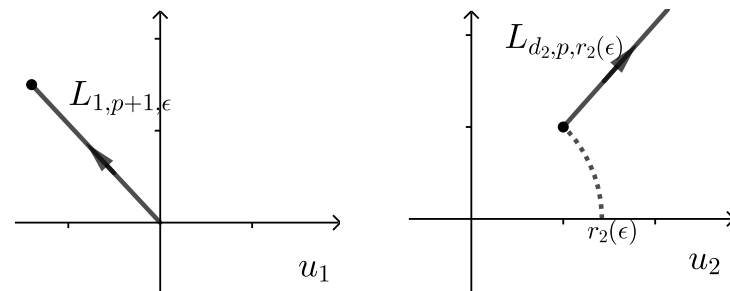


Figure 3: Integration path. Case 2. Piece 2.1

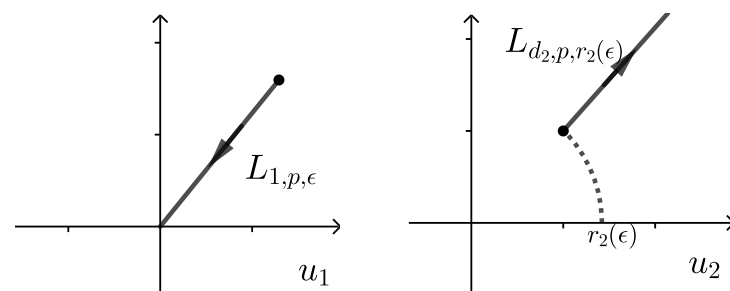


Figure 4: Integration path. Case 2. Piece 2.2

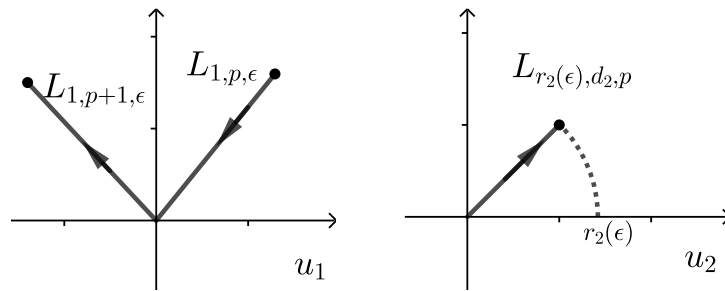


Figure 5: Integration path. Case 2. Piece 2.3

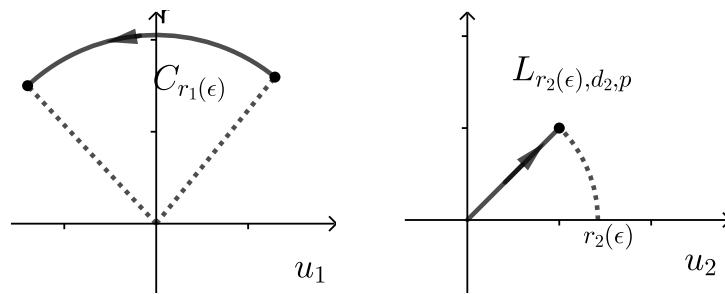


Figure 6: Deformation of the integration path. Case 2. Piece 2.3

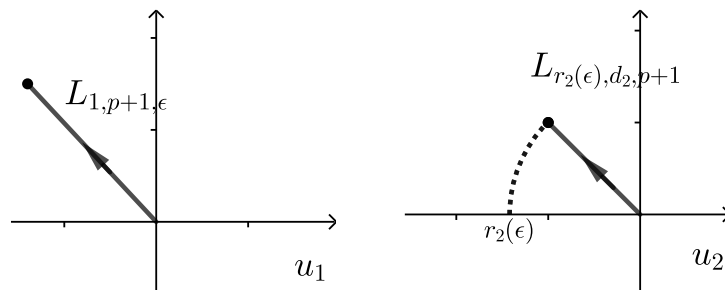


Figure 7: Integration path. Case 3. Piece 3.1

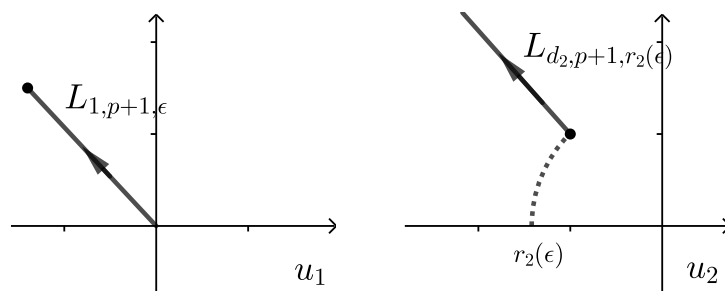


Figure 8: Integration path. Case 3. Piece 3.2

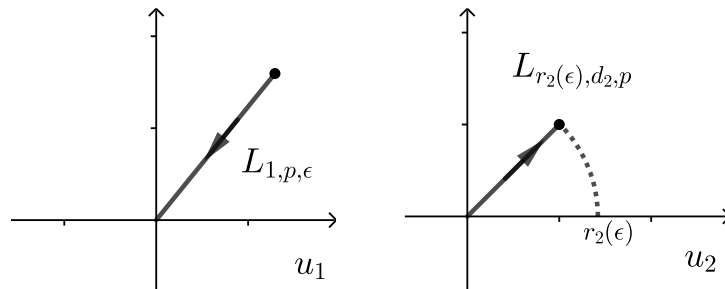


Figure 9: Integration path. Case 3. Piece 3.3

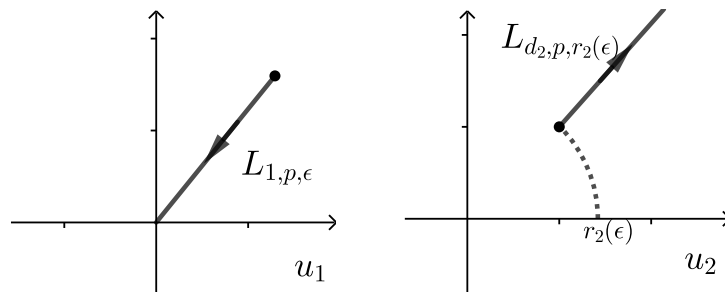


Figure 10: Integration path. Case 3. Piece 3.4

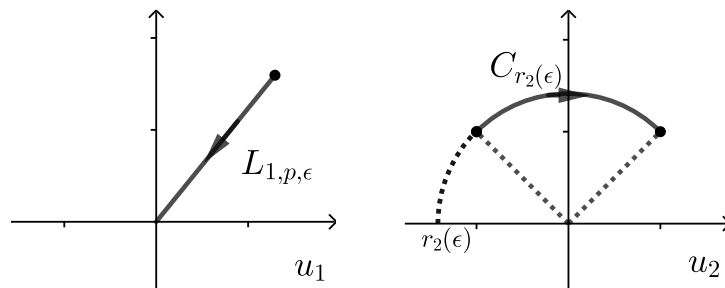


Figure 11: Deformation of the integration path. Case 3. Piece 3.3 (1)

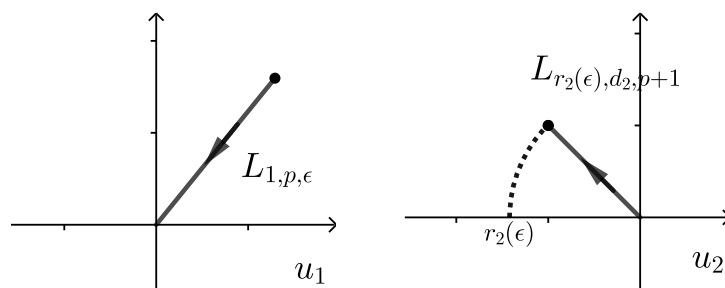


Figure 12: Deformation of the integration path. Case 3. Piece 3.3 (2)

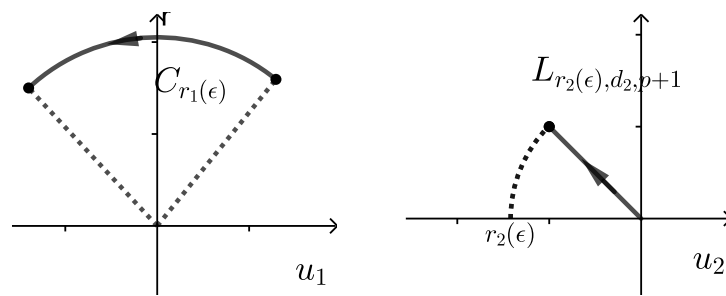


Figure 13: Deformation of the integration path. Case 3. Piece 3.1 together with Piece 3.3 (2)