

## A novel kind of the type 2 poly-Fubini polynomials and numbers

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**Abstract.** Motivation by the definition of the type 2 poly-Bernoulli polynomials introduced by Kim-Kim [9], in the present paper, we consider a new class of new generating function for the Fubini polynomials, called the type 2 poly-Fubini polynomials by means of the polyexponential function. Then, we derive some useful relations and properties. We show that the type 2 poly-Fubini polynomials equal a linear combination of the classical of the Fubini polynomials and Stirling numbers of the first kind. In a special case, we give a relation between the type 2 poly-Fubini polynomials and Bernoulli polynomials of order  $r$ . Moreover, inspired by the definition of the unipoly-Bernoulli polynomials introduced by Kim-Kim [9]. We introduce the type 2 unipoly-Fubini polynomials by means of unipoly function and give multifarious properties including derivative and integral properties. Furthermore, we provide a correlation between the unipoly-Fubini polynomials and the classical Fubini polynomials.

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### 1. Introduction

Special polynomials have their origin in the solution of the differential equations (or partial differential equations) under some conditions. Special polynomials can be defined in a various ways such as by generating functions, by recurrence relations, by  $p$ -adic integrals in the sense of fermionic and bosonic, by degenerate versions, etc.

Kim-Kim have introduced polyexponential function in [9] and its degenerate version in [19-22]. By making use of aforementioned function, they have introduced a new class of some numbers and polynomials. This idea provides a powerful tool in order to define special numbers and polynomials by making use of polyexponential function. One may see that the notion of polyexponential function form a special class of polynomials because of their great applicability. The importance of these polynomials would be to find applications in analytic number theory, applications in classical analysis and statistics.

The classical Bernoulli  $B_n(x)$ , Euler  $E_n(x)$  and Genocchi  $G_n(x)$  polynomial are defined by means of the following generating function as follows

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, |t| < 2\pi, \frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, |t| < \pi,$$

and

$$\frac{2t}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}, |t| < \pi, \text{ (see [4, 11, 14])} \quad (1.1)$$

respectively.

Kargin [19] defined the two-variable Fubini polynomials by the following generating function

$$\frac{e^{xt}}{1 - y(e^t - 1)} = \sum_{n=0}^{\infty} F_n(x, y) \frac{t^n}{n!}, \text{ (see [3, 7, 16, 17, 18])}. \quad (1.2)$$

When  $x = 0$  in (1.2), the two variable Fubini polynomials  $F_n(x, y)$  reduce to the usual Fubini polynomials given by

$$\frac{1}{1 - y(e^t - 1)} = \sum_{n=0}^{\infty} F_n(y) \frac{t^n}{n!}, \text{ (see [1, 2, 5])}. \quad (1.3)$$

It is easy to see that

$$F_n\left(x, -\frac{1}{2}\right) = E_n(x), F_n\left(-\frac{1}{2}\right) = E_n \quad (1.4)$$

and

$$F_n(y) = \sum_{k=0}^n S_2(n, k) k! y^k.$$

For  $y = 1$  in (1.3), we get the familiar Fubini numbers  $F_n(1) = F_n$  as follows

$$\frac{1}{2 - e^t} = \sum_{n=0}^{\infty} F_n \frac{t^n}{n!}, \text{ (see [1, 2])}. \quad (1.5)$$

For more information about the applications of the usual Fubini polynomials and numbers (see [24, 25, 28, 29, 30]).

Kim-Kim [9] introduced the polyexponential function, as an inverse to the polylogarithm function to be

$$\text{Ei}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)! n^k}, (k \in \mathbb{Z}). \quad (1.6)$$

For  $k = 1$ , (1.6) gives

$$\text{Ei}_1(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!} = e^x - 1. \quad (1.7)$$

For  $k \in \mathbb{Z}$ , Kim-Kim considered the type 2 poly-Bernoulli polynomials are defined by means of the following generating function

$$\frac{\text{Ei}_k(\log(1+t))}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!}, \text{ (see [9])}. \quad (1.8)$$

In the special case,  $x = 0$ ,  $B_n^{(k)} = B_n^{(k)}(0)$  are called the type 2 poly-Bernoulli numbers. Note that  $B_n^{(1)}(x) = B_n(x)$  are called the ordinary Bernoulli polynomials.

For  $k \in \mathbb{Z}$ , the polylogarithm function is defined by

$$\text{Li}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k}, (|x| < 1), \text{ (see, [26])}. \quad (1.9)$$

Note that

$$\text{Li}_1(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = -\log(1-x). \quad (1.10)$$

In (2020), Lee et al. [27] introduced the type 2 poly-Euler polynomials are given by

$$\frac{\text{Ei}_k(1+2t)}{t(e^t+1)}e^{xt} = \sum_{n=0}^{\infty} E_n^{(k)}(x) \frac{t^n}{n!}. \quad (1.11)$$

In the case when  $x = 0$ ,  $E_n^{(k)} = E_n^{(k)}(0)$  are called the type 2 poly-Euler numbers.

The Daehee polynomials are defined by

$$\frac{\log(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!}, \text{ (see [6, 15])}. \quad (1.12)$$

When  $x = 0$ ,  $D_n = D_n(0)$  are called the Daehee numbers.

For  $n \geq 0$ , the Stirling numbers of the first kind are defined by

$$(x)_n = \sum_{l=0}^n S_1(n, l) x^l, \text{ (see [4, 9])}, \quad (1.13)$$

where  $(x)_0 = 1$ , and  $(x)_n = x(x-1) \cdots (x-n+1)$ , ( $n \geq 1$ ). From (1.13), it is easily to see that

$$\frac{1}{k!}(\log(1+t))^k = \sum_{n=k}^{\infty} S_1(n, k) \frac{t^n}{n!}, \text{ (} k \geq 0 \text{), (see [4, 9])}. \quad (1.14)$$

In the inverse expression to (1.13), the Stirling numbers of the second kind are defined by

$$x^n = \sum_{l=0}^n S_2(n, l) (x)_l, \text{ (see [10, 12, 13])}. \quad (1.15)$$

From (1.15), it is easily to see that

$$\frac{1}{k!}(e^t-1)^k = \sum_{n=k}^{\infty} S_2(n, k) \frac{t^n}{n!}, \text{ (see [1-30])}. \quad (1.16)$$

In this paper, we construct the poly-Fubini polynomials and numbers, called the type 2 poly-Fubini polynomials and numbers by using the polyexponential function and derive several properties on the poly-Fubini polynomials and numbers. In the final section, we define type 2 unipoly-Fubini polynomials by means of unipoly function and derive explicit expressions.

## 2. Type 2 poly-Fubini polynomials and numbers

In this section, we define the type 2 poly-Fubini polynomials using the polyexponential functions and represent the usual Fubini polynomials (more precisely, the values of Fubini polynomials at 1) when  $k = 1$ . At the same time, we give explicit expressions and identities involving those polynomials.

For  $k \in \mathbb{Z}$ , the type 2 poly-Fubini polynomials are defined via the following exponential generating function (in a suitable neighborhood of  $t = 0$ ) including the polyexponential function as given below:

$$\frac{\text{Ei}_k(\log(1+t))}{t(1-y(e^t-1))}e^{xt} = \sum_{n=0}^{\infty} F_n^{(k)}(x; y) \frac{t^n}{n!}. \quad (2.1)$$

At the point  $x = 0$  and  $y = 1$ ,  $F_n^{(k)} = F_n^{(k)}(0; 1)$  are called the type 2 poly-Fubini numbers.

For  $k = 1$ , by using (1.6) and (2.1), we see that

$$\frac{\text{Ei}_1(\log(1+t))}{t(1-y(e^t-1))} e^{xt} = \frac{1}{1-y(e^t-1)} e^{xt} = \sum_{n=0}^{\infty} F_n(x) \frac{t^n}{n!}, \quad (2.2)$$

where  $F_n(x)$  are called the Fubini polynomials (see Eq. (1.3)).

By (2.1) and (2.2), we get

$$F_n^{(1)}(x) = F_n(x), (n \geq 0).$$

Using (1.6), we see that

$$\begin{aligned} \text{Ei}_k(\log(1+t)) &= \sum_{m=1}^{\infty} \frac{(\log(1+t))^m}{(m-1)!m^k} \\ &= \sum_{m=0}^{\infty} \frac{(\log(1+t))^{m+1}}{m!(m+1)^k} \\ &= \sum_{m=0}^{\infty} \frac{1}{(m+1)^{k-1}} \frac{1}{(m+1)!} (\log(1+t))^{m+1} \\ &= \sum_{m=0}^{\infty} \frac{1}{(m+1)^{k-1}} \sum_{l=m+1}^{\infty} S_1(l, m+1) \frac{t^l}{l!}. \end{aligned} \quad (2.3)$$

From (2.3) and (2.1), we have

$$\begin{aligned} \frac{e^{xt}}{t(1-y(e^t-1))} \text{Ei}_k(\log(1+t)) &= \frac{1}{1-y(e^t-1)} e^{xt} \sum_{m=0}^{\infty} \frac{1}{(m+1)^{k-1}} \sum_{l=m}^{\infty} \frac{S_1(l+1, m+1)}{l+1} \frac{t^l}{l!} \\ &= \sum_{n=0}^{\infty} F_n(x; y) \frac{t^n}{n!} \sum_{m=0}^{\infty} \frac{1}{(m+1)^{k-1}} \sum_{l=m}^{\infty} \frac{S_1(l+1, m+1)}{l+1} \frac{t^l}{l!} \\ L.H.S &= \sum_{n=0}^{\infty} \left( \sum_{l=0}^n \binom{n}{l} \sum_{m=0}^l \frac{1}{(m+1)^{k-1}} \frac{S_1(l+1, m+1)(1-u)^l}{l+1} F_{n-l}(x; y) \right) \frac{t^n}{n!}. \end{aligned} \quad (2.4)$$

Therefore, by (2.3) and (2.4), we obtain the following theorem.

**Theorem 2.1.** For  $k \in \mathbb{Z}$  and  $n \geq 0$ , we have

$$F_n^{(k)}(x; y) = \sum_{l=0}^n \binom{n}{l} \sum_{m=0}^l \frac{1}{(m+1)^{k-1}} \frac{S_1(l+1, m+1)(1-u)^l}{l+1} F_{n-l}(x; y).$$

**Corollary 2.1.** For  $n \geq 0$ , we have

$$F_n(x; y) = \sum_{l=0}^n \binom{n}{l} \sum_{m=0}^l \frac{S_1(l+1, m+1)}{l+1} F_{n-l}(x; y).$$

Moreover,

$$\sum_{l=1}^n \binom{n}{l} \sum_{m=0}^l \frac{S_1(l+1, m+1)2^l}{l+1} F_{n-l}(x; y) = 0, (n \geq 1).$$

It is well-known from ([8, 11]) that

$$\left(\frac{t}{\log(1+t)}\right)^r (1+t)^{x-1} = \sum_{n=0}^{\infty} B_n^{(n-r+1)}(x) \frac{t^n}{n!}, (r \in \mathbb{C}), \quad (2.5)$$

where  $B_n^{(r)}(x)$  are called the higher-order Bernoulli polynomials which are given by the generating function

$$\left(\frac{t}{e^t - 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}.$$

Using (1.6), we have

$$\begin{aligned} \frac{d}{dx} \text{Ei}_k(\log(1+x)) &= \frac{d}{dx} \sum_{n=1}^{\infty} \frac{(\log(1+x))^n}{(n-1)!n^k} \\ &= \frac{1}{1+x} \sum_{n=1}^{\infty} \frac{n(\log(1+x))^{n-1}}{(n-1)!n^k} \\ &= \frac{1}{(1+x)\log(1+x)} \sum_{n=1}^{\infty} \frac{(\log(1+x))^n}{(n-1)!n^{k-1}} \\ &= \frac{1}{(1+x)\log(1+x)} \text{Ei}_{k-1}(\log(1+x)). \end{aligned} \quad (2.6)$$

From (2.6),  $k \geq 1$ , we have

$$\begin{aligned} \text{Ei}_k(\log(1+x)) &= \int_0^x \frac{1}{(1+t)\log(1+t)} \\ &\times \underbrace{\int_0^t \frac{1}{(1+t)\log(1+t)} \cdots \int_0^t \frac{t}{(1+t)\log(1+t)} dt dt \cdots dt}_{k-2\text{-times}}. \end{aligned} \quad (2.7)$$

$$= x \sum_{m=0}^{\infty} \sum_{m_1+\cdots+m_{k-1}=m} \binom{m}{m_1, \dots, m_{k-1}} \frac{B_{m_1}^{(m_1)}}{m_1+1} \frac{B_{m_2}^{(m_2)}}{m_1+m_2+1} \cdots \frac{B_{m_{k-1}}^{(m_{k-1})}}{m_1+\cdots+m_{k-1}+1} \frac{x^m}{m!}$$

From (2.7), we observe that

$$\begin{aligned} \sum_{n=0}^{\infty} F_n^{(k)}(y) \frac{x^n}{n!} &= \frac{1}{x(1-y(e^x-1))} \text{Ei}_k(\log(1+x)) \\ &= \frac{x}{x(1-y(e^x-1))} \\ &\times \sum_{m=0}^{\infty} \sum_{m_1+\cdots+m_{k-1}=m} \binom{m}{m_1, \dots, m_{k-1}} \frac{B_{m_1}^{(m_1)}}{m_1+1} \frac{B_{m_2}^{(m_2)}}{m_1+m_2+1} \cdots \frac{B_{m_{k-1}}^{(m_{k-1})}}{m_1+\cdots+m_{k-1}+1} \frac{x^m}{m!} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^n \binom{n}{m} \sum_{m=0}^{\infty} \\ &\times \sum_{m_1+\cdots+m_{k-1}=m} \binom{m}{m_1, \dots, m_{k-1}} \frac{B_{m_1}^{(m_1)}}{m_1+1} \frac{B_{m_2}^{(m_2)}}{m_1+m_2+1} \cdots \frac{B_{m_{k-1}}^{(m_{k-1})}}{m_1+\cdots+m_{k-1}+1} F_{n-m}(y) \frac{x^n}{n!}. \end{aligned} \quad (2.8)$$

Therefore, by (2.7) and (2.8), we get the following theorem.

**Theorem 2.2.** For  $n \geq 0$ , we have

$$F_n^{(k)}(y) = \sum_{m=0}^n \binom{n}{m} \sum_{m=0}^{\infty} \times \sum_{m_1+\dots+m_{k-1}=m} \binom{m}{m_1, \dots, m_{k-1}} \frac{B_{m_1}^{(m_1)}}{m_1+1} \frac{B_{m_2}^{(m_2)}}{m_1+m_2+1} \cdots \frac{B_{m_{k-1}}^{(m_{k-1})}}{m_1+\dots+m_{k-1}+1} F_{n-m}(y).$$

**Corollary 2.2.** For  $n \geq 0$ , we have

$$F_n^{(2)}(y) = \sum_{m=0}^n \binom{n}{m} \frac{B_m^{(m)}}{m+1} F_{n-l}.$$

Let  $s \in \mathbb{C}$  and  $k \in \mathbb{Z}$  with  $k \geq 1$ . We consider the function  $\eta_k$  by the representation of improper integral as follows

$$\eta_k(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{t(1-y(e^t-1))} \text{Ei}_k(\log(1+t)) dt, \quad (2.9)$$

We observe that  $\eta_k(s)$  is holomorphic function for  $\Re(s) > 0$  because of the comparison test as  $\text{Ei}_k(\log(1+2t)) \leq \text{Ei}_1(\log(1+2t))$  with the assumption  $2t \geq 0$ . By (2.9), we have

$$\eta_k(s) = \frac{1}{\Gamma(s)} \int_0^1 \frac{t^{s-1}}{t(1-y(e^t-1))} \text{Ei}_k(\log(1+t)) dt + \frac{1}{\Gamma(s)} \int_1^{\infty} \frac{t^{s-1}}{t(1-y(e^t-1))} \text{Ei}_k(\log(1+t)) dt. \quad (2.10)$$

The second integral in (2.10) converges absolutely for any  $s \in \mathbb{C}$  and thus, the second term on the right hand side vanishes at non positive integers. Hence, we get

$$\lim_{s \rightarrow -m} \left| \frac{1}{\Gamma(s)} \int_0^1 \frac{t^{s-1}}{t(1-y(e^t-1))} \text{Ei}_k(\log(1+t)) dt \right| \leq \frac{1}{\Gamma(-m)} M = 0 \quad (2.11)$$

since

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}.$$

Moreover, for  $\Re(s) > 0$ , the first integral in (2.10) can be written as

$$\begin{aligned} \frac{1}{\Gamma(s)} \int_0^1 \frac{t^{s-1}}{t(1-y(e^t-1))} \text{Ei}_k(\log(1+t)) dt &= \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{F_n^{(k)}(y)}{n!} \int_0^1 t^{n+s-1} dt \\ &= \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{F_n^{(k)}(y)}{n!} \frac{1}{n+s}, \end{aligned} \quad (2.12)$$

which defines an entire function of  $s$ . Hence, we arrive that  $\eta_k(s)$  can be continued to an entire function of  $s$ . From (2.10) and (2.11), we obtain

$$\begin{aligned} \eta_k(-m) &= \lim_{s \rightarrow -m} \frac{1}{\Gamma(s)} \int_0^1 \frac{t^{s-1}}{t(1-y(e^t-1))} \text{Ei}_k(\log(1+t)) dt \\ &= \lim_{s \rightarrow -m} \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{F_n^{(k)}(y)}{n!(n+s)} \\ &= \cdots + \cdots + 0 + \lim_{s \rightarrow -m} \frac{1}{\Gamma(s)} \frac{F_m^{(k)}}{m!(m+s)} + 0 + 0 + \cdots \\ &= \lim_{s \rightarrow -m} \frac{1}{m+s} \frac{\Gamma(1-s) \sin(\pi s)}{\pi} \frac{F_m^{(k)}(y)}{m!} \\ &= \Gamma(1+m) \cos(\pi m) \frac{F_m^{(k)}(y)}{m!} \end{aligned}$$

$$= (-1)^m F_m^{(k)}(y).$$

Thus, we get the following theorem.

**Theorem 2.3.** Let  $k \in \mathbb{N}$ . The function  $\eta_k(s)$  has an analytic continuation to an function of  $s \in \mathbb{C}$ , and the special values at non positive integers are given by

$$\eta_k(-m) = (-1)^m F_m^{(k)}(y), (m \in \mathbb{N}_0).$$

From (2.1), we have

$$\begin{aligned} \sum_{n=0}^{\infty} F_n^{(k)}(x+z; y) \frac{t^n}{n!} &= \left( \frac{\text{Ei}_k(\log(1+t))}{t(1-y(e^t-1))} \right) e^{(x+z)t} \\ &= \left( \sum_{n=0}^{\infty} F_n^{(k)}(x; y) \frac{t^n}{n!} \right) \left( \sum_{m=0}^{\infty} z^m \frac{t^m}{m!} \right) \\ L.H.S &= \sum_{n=0}^{\infty} \left( \sum_{m=0}^n \binom{n}{m} F_{n-m}^{(k)}(x; y) z^m \right) \frac{t^n}{n!}. \end{aligned} \quad (2.13)$$

By comparing the coefficients of  $t^n$ , we obtain the following theorem.

**Theorem 2.4.** For  $n \geq 0$ , we have

$$F_n^{(k)}(x+z; y) = \sum_{m=0}^n \binom{n}{m} F_{n-m}^{(k)}(x; y) z^m.$$

From (2.1), we note that

$$\begin{aligned} \frac{\text{Ei}_k(\log(1+t))}{t} &= (1-y(e^t-1)) \sum_{n=0}^{\infty} F_n^{(k)}(y) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} F_n^{(k)}(y) \frac{t^n}{n!} - y \sum_{n=0}^{\infty} \sum_{m=0}^n \binom{n}{m} F_{n-m}^{(k)}(y) \frac{t^n}{n!} + y \sum_{n=0}^{\infty} F_n^{(k)}(y) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left( F_n^{(k)}(y) - y \sum_{m=0}^n \binom{n}{m} F_{n-m}^{(k)}(y) + y F_n^{(k)}(y) \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left( (1+y) F_n^{(k)}(y) - y \sum_{m=0}^n \binom{n}{m} F_{n-m}^{(k)}(y) \right) \frac{t^n}{n!}. \end{aligned} \quad (2.14)$$

On the other hand,

$$\begin{aligned} \text{Ei}_k(\log(1+t)) &= \frac{1}{t} \sum_{m=1}^{\infty} \frac{(\log(1+t))^m}{(m-1)! m^k} \\ &= \frac{1}{t} \sum_{m=0}^{\infty} \frac{(\log(1+t))^{m+1}}{m!(m+1)^k} = \frac{1}{t} \sum_{m=0}^{\infty} \frac{(\log(1+t))^{m+1}}{(m+1)^{k-1} (m+1)!} \\ &= \frac{1}{t} \sum_{m=0}^{\infty} \frac{1}{(m+1)^{k-1}} \sum_{n=m+1}^{\infty} S_1(n, m+1) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^n \frac{1}{(m+1)^{k-1}} \frac{S_1(n+1, m+1)}{n+1} \frac{t^n}{n!}. \end{aligned} \quad (2.15)$$

Therefore, by (2.14) and (2.15), we obtain the following theorem.

**Theorem 2.5.** For  $n \geq 0$ , we have

$$F_n^{(k)}(y) = \frac{1}{1+y} \left[ y \sum_{m=0}^n \binom{n}{m} F_{n-m}^{(k)}(y) + \sum_{m=0}^n \frac{1}{(m+1)^{k-1}} \frac{S_1(n+1, m+1)}{n+1} \right].$$

By (2.1), we observe that

$$\begin{aligned} \sum_{n=0}^{\infty} \left[ F_n^{(k)}(x+1; y) - F_n^{(k)}(x; y) \right] \frac{t^n}{n!} &= \frac{\text{Ei}_k(\log(1+t))}{t(1-y(e^t-1))} e^{xt} (e^t - 1) \\ &= \frac{\text{Ei}_k(\log(1+t))}{t} \frac{e^{xt}}{1-y(e^t-1)} (e^t - 1) \\ &= \frac{\text{Ei}_k(\log(1+t))}{t} \frac{1}{y} \left[ \frac{e^{xt}}{1-y(e^t-1)} - e^{xt} \right] \\ &= \frac{1}{y} \left[ \frac{\text{Ei}_k(\log(1+t))}{t(1-y(e^t-1))} e^{xt} - \frac{\text{Ei}_k(\log(1+t))}{t} e^{xt} \right] \\ &= \frac{1}{y} \left[ \sum_{n=0}^{\infty} F_n^{(k)}(x; y) - \sum_{l=0}^n \sum_{m=0}^l \binom{n}{l} \frac{1}{(m+1)^{k-1}} \frac{S_1(l+1, m+1)}{l+1} x^{n-l} \right] \frac{t^n}{n!}. \quad (2.16) \end{aligned}$$

On comparing the coefficients of  $t^n$  on both sides, we get the following theorem.

**Theorem 2.6.** For  $n \geq 0$ , we have

$$yF_n^{(k)}(x+1; y) = (1+y)F_n^{(k)}(x; y) - \sum_{l=0}^n \sum_{m=0}^l \binom{n}{l} \frac{1}{(m+1)^{k-1}} \frac{S_1(l+1, m+1)}{l+1} x^{n-l}.$$

From (2.1), we note that

$$\begin{aligned} &\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} F_n^{(k)}(x_1; y_1) F_m^{(k)}(x_2; y_2) \frac{t^n}{n!} \frac{t^m}{m!} \\ &= \frac{\text{Ei}_k(\log(1+t))}{t(1-y_1(e^t-1))} e^{x_1 t} \frac{\text{Ei}_k(\log(1+t))}{t(1-y_2(e^t-1))} e^{x_2 t} \\ &= \frac{[\text{Ei}_k(\log(1+t))]^2}{t^2} \left( \frac{e^{x_1 t}}{1-y_1(e^t-1)} \frac{e^{x_2 t}}{1-y_2(e^t-1)} \right) \\ &= \frac{[\text{Ei}_k(\log(1+t))]^2}{t^2} \left( \frac{y_2}{y_2-y_1} \frac{e^{(x_1+x_2)t}}{1-y_2(e^t-1)} - \frac{y_1}{y_2-y_1} \frac{e^{(x_1+x_2)t}}{1-y_1(e^t-1)} \right) \\ &= \left( \frac{\sum_{n=0}^{\infty} \left[ y_2 F_n^{(k)}(x_1+x_2; y_2) - y_1 F_n^{(k)}(x_1+x_2; y_1) \right]}{y_2-y_1} \right) \frac{t^n}{n!}. \quad (2.17) \end{aligned}$$

On the other hand,

$$\begin{aligned} &\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} F_n^{(k)}(x_1; y_1) F_m^{(k)}(x_2; y_2) \frac{t^n}{n!} \frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty} \left( \sum_{m=0}^n \binom{n}{m} F_{n-m}^{(k)}(x_1; y_1) F_m^{(k)}(x_2; y_2) \right) \frac{t^n}{n!}. \quad (2.18) \end{aligned}$$

Therefore, by (2.17) and (2.18), we obtain the following theorem.

**Theorem 2.7.** For  $n \geq 0$ , we have

$$\sum_{m=0}^n \binom{n}{m} F_{n-m}^{(k)}(x_1; y_1) F_m^{(k)}(x_2; y_2)$$



$$= \frac{y_2 F_n^{(k)}(x_1 + x_2; y_2) - y_1 F_n^{(k)}(x_1 + x_2; y_1)}{y_2 - y_1}.$$

Using (2.1), we have

$$\begin{aligned} \sum_{n=0}^{\infty} F_n^{(k)}(y) \frac{t^n}{n!} &= \frac{\text{Ei}_k(\log(1+t))}{t(1-y(e^t-1))} \\ &= \frac{\text{Ei}_k(\log(1+t))}{t} \sum_{m=0}^{\infty} y^m \sum_{l=m}^{\infty} m! S_2(l, m) \frac{t^l}{l!} \\ &= \left( \sum_{n=0}^{\infty} \sum_{m=0}^n \frac{1}{(m+1)^{k-1}} \frac{S_1(n+1, m+1)}{n+1} \frac{t^n}{n!} \right) \left( \sum_{l=0}^{\infty} \sum_{r=0}^l y^r r! S_2(l, r) \frac{t^l}{l!} \right) \\ &= \sum_{n=0}^{\infty} \left( \sum_{l=0}^n \sum_{m=0}^{n-l} \sum_{r=0}^l \binom{n}{l} \frac{1}{(m+1)^{k-1}} \frac{S_1(n-l+1, m+1)}{n-l+1} y^r r! S_2(l, r) \right). \end{aligned} \quad (2.19)$$

Therefore, by (2.19), we obtain the following theorem.

**Theorem 2.8.** For  $n \geq 0$ , we have

$$F_n^{(k)}(y) = \sum_{l=0}^n \sum_{m=0}^{n-l} \sum_{r=0}^l \binom{n}{l} \frac{1}{(m+1)^{k-1}} \frac{S_1(n-l+1, m+1)}{n-l+1} y^r r! S_2(l, r).$$

From (2.1), we observe that

$$\frac{\text{Ei}_k(\log(1+t))}{t(1-y(e^t-1))} = \sum_{n=0}^{\infty} F_n^{(k)}(y) \frac{t^n}{n!}.$$

Replacing  $t$  by  $e^t - 1$ , we get

$$\begin{aligned} \sum_{m=0}^{\infty} F_m^{(k)}(y) \frac{(e^t - 1)^m}{m!} &= \frac{\text{Ei}_k(t)}{(e^t - 1)(1 - y(e^{e^t - 1} - 1))} \\ &= \frac{1}{1 - y(e^{e^t - 1} - 1)} \frac{t}{e^t - 1} \frac{\text{Ei}_k(t)}{t} \\ &= \sum_{p=0}^{\infty} F_p(y) \frac{(e^t - 1)^p}{p!} \sum_{q=0}^{\infty} B_q \frac{t^q}{q!} \sum_{n=0}^{\infty} \frac{t^n}{n!(n+1)^k} \\ &= \sum_{p=0}^{\infty} F_p(y) \sum_{p=i}^{\infty} S_1(i, p) \frac{t^i}{i!} \sum_{q=0}^{\infty} B_q \frac{t^q}{q!} \sum_{n=0}^{\infty} \frac{t^n}{n!(n+1)^k} \\ &= \sum_{i=0}^{\infty} \sum_{p=0}^i F_p(y) S_1(i, p) \frac{t^i}{i!} \sum_{q=0}^{\infty} B_q \frac{t^q}{q!} \sum_{n=0}^{\infty} \frac{t^n}{n!(n+1)^k} \\ &= \sum_{q=0}^{\infty} \left( \sum_{i=0}^q \sum_{p=0}^i \binom{q}{i} F_p(y) S_1(i, p) B_{q-i} \right) \sum_{n=0}^{\infty} \frac{t^n}{n!(n+1)^k} \\ &= \sum_{n=0}^{\infty} \left( \sum_{q=0}^n \sum_{i=0}^q \sum_{p=0}^i \binom{q}{i} \binom{n}{q} F_p(y) S_1(i, p) B_{q-i} \frac{1}{(n-q+1)^k} \right) \frac{t^n}{n!}. \end{aligned} \quad (2.20)$$

On the other hand,

$$\sum_{m=0}^{\infty} F_m^{(k)}(y) \frac{(e^t - 1)^m}{m!}$$

$$\begin{aligned}
&= \sum_{m=0}^{\infty} F_m^{(k)}(y) \sum_{n=m}^{\infty} S_1(n, m) \frac{t^n}{n!} \\
&\sum_{n=0}^{\infty} \left( \sum_{m=0}^n F_m^{(k)}(y) S_1(n, m) \right) \frac{t^n}{n!}.
\end{aligned} \quad (2.21)$$

Therefore, by (2.20) and (2.21), we get the following theorem.

**Theorem 2.9.** For  $n \geq 0$ , we have

$$\sum_{m=0}^n F_m^{(k)}(y) S_1(n, m) = \sum_{q=0}^n \sum_{i=0}^q \sum_{p=0}^i \binom{q}{i} \binom{n}{q} F_p(y) S_1(i, p) B_{q-i} \frac{1}{(n-q+1)^k}.$$

### 3. The type 2 unipoly-Fubini polynomials and numbers

In this section, we consider the type 2 unipoly-Fubini polynomials and numbers and investigate some properties of those polynomials.

Let  $p$  be any arithmetic function which is a real or complex valued function defined on the set of positive integers  $\mathbb{N}$ . Kim-Kim [9] defined the unipoly function attached to polynomials  $p(x)$  by

$$u_k(x|p) = \sum_{n=1}^{\infty} \frac{p(n)}{n^k} x^n, \quad (k \in \mathbb{Z}). \quad (3.1)$$

Moreover,

$$u_k(x|1) = \sum_{n=1}^{\infty} \frac{x^n}{n^k} = \text{Li}_k(x), \quad (\text{see [26]}), \quad (3.2)$$

is the ordinary polylogarithm function.

Now, we define the type 2 unipoly-Fubini polynomials attached to polynomials  $p(x)$  by

$$\frac{1}{t(1-y(e^t-1))} u_k(\log(1+t)|p) e^{xt} = \sum_{n=0}^{\infty} F_{n,p}^{(k)}(x; y) \frac{t^n}{n!}. \quad (3.3)$$

When  $x = 0$  and  $y = 1$ ,  $F_{n,p}^{(k)} = F_{n,p}^{(k)}(0; 1)$  are called the unipoly-Fubini numbers attached to  $p$ .

If we take  $p(n) = \frac{1}{\Gamma(n)}$ . Then, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} F_{n, \frac{1}{\Gamma}}^{(k)}(x; y) \frac{t^n}{n!} &= \frac{1}{t(1-y(e^t-1))} e^{xt} u_k\left(\log(1+t) \middle| \frac{1}{\Gamma}\right) \\
&= \frac{1}{t(1-y(e^t-1))} e^{xt} \sum_{m=1}^{\infty} \frac{(\log(1+t))^m}{m^k(m-1)!}.
\end{aligned} \quad (3.4)$$

In particular, for  $k = 1$ , we obtain

$$\sum_{n=0}^{\infty} F_{n, \frac{1}{\Gamma}}^{(1)}(x; y) \frac{t^n}{n!} = \frac{1}{t(1-y(e^t-1))} e^{xt} \sum_{m=1}^{\infty} \frac{(\log(1+t))^m}{m!} = \frac{1}{1-y(e^t-1)} e^{xt}. \quad (3.5)$$

Therefore by (3.3) and (3.5), we have

$$F_{n, \frac{1}{\Gamma}}^{(k)}(x; y) = F_n(x; y). \quad (3.6)$$

From (3.3), we have

$$\begin{aligned}
 \sum_{n=0}^{\infty} F_{n,p}^{(k)}(x; y) \frac{t^n}{n!} &= \frac{1}{t(1-y(e^t-1))} e^{xt} u_k(\log(1+t)|p) \\
 &= \frac{1}{t(1-y(e^t-1))} e^{xt} \sum_{m=1}^{\infty} \frac{p(m)}{m^k} (\log(1+t))^m \\
 &= \frac{1}{t(1-y(e^t-1))} e^{xt} \sum_{m=0}^{\infty} \frac{p(m+1)}{(m+1)^k} (\log(1+t))^{m+1} \\
 &= \frac{1}{t(1-y(e^t-1))} e^{xt} \sum_{m=0}^{\infty} \frac{p(m+1)(m+1)!}{(m+1)^k} \sum_{l=m+1}^{\infty} S_1(l, m+1) \frac{t^l}{l!} \\
 &= \frac{1}{1-y(e^t-1)} e^{xt} \sum_{m=0}^{\infty} \frac{p(m+1)(m+1)!}{(m+1)^k} \sum_{l=m}^{\infty} \frac{S_1(l+1, m+1)}{l+1} \frac{t^l}{l!} \\
 &= \sum_{n=0}^{\infty} F_n(x; y) \frac{t^n}{n!} \sum_{l=0}^{\infty} \left( \sum_{m=0}^l \frac{p(m+1)(m+1)!}{(m+1)^k} \frac{S_1(l+1, m+1)}{l+1} \right) \frac{t^l}{l!} \\
 L.H.S &= \sum_{n=0}^{\infty} \left( \sum_{l=0}^n \sum_{m=0}^l \binom{n}{l} \frac{p(m+1)(m+1)!}{(m+1)^k} \frac{S_1(l+1, m+1)}{l+1} F_{n-l}(x; y) \right) \frac{t^n}{n!}. \quad (3.7)
 \end{aligned}$$

Therefore, by comparing the coefficients on both sides of (3.7), we obtain the following theorem.

**Theorem 3.1.** For  $n \geq 0$  and  $k \in \mathbb{Z}$ , we have

$$F_{n,p}^{(k)}(x; y) = \sum_{l=0}^n \sum_{m=0}^l \binom{n}{l} \frac{p(m+1)(m+1)!}{(m+1)^k} \frac{S_{1,\lambda}(l+1, m+1)}{l+1} F_{n-l}(x; y).$$

Moreover,

$$F_{n,\frac{1}{p}}^{(k)}(x; y) = \sum_{l=0}^n \sum_{m=0}^l \binom{n}{l} \frac{1}{(m+1)^{k-1}} \frac{S_1(l+1, m+1)}{l+1} F_{n-l}(x; y).$$

From (3.3), we have

$$\begin{aligned}
 \sum_{n=0}^{\infty} F_{n,p}^{(k)}(x; y) \frac{t^n}{n!} &= \frac{1}{t(1-y(e^t-1))} u_k(\log(1+t)|p) e^{xt} \\
 &= \frac{1}{t(1-y(e^t-1))} u_k(\log(1+t)|p) (e^t-1+1)^x \\
 &= \frac{1}{t(1-y(e^t-1))} u_k(\log(1+t)|p) \sum_{m=0}^{\infty} (x)_m \frac{(e^t-1)^m}{m!} \\
 &= \sum_{n=0}^{\infty} F_{n,p}^{(k)}(y) \frac{t^n}{n!} \sum_{l=0}^{\infty} \sum_{m=0}^l (x)_m S_2(l, m) \frac{t^l}{l!} \\
 L.H.S &= \sum_{n=0}^{\infty} \left( \sum_{l=0}^n \sum_{m=0}^l \binom{n}{l} (x)_m S_2(l, m) F_{n-l,p}^{(k)}(y) \right) \frac{t^n}{n!}. \quad (3.8)
 \end{aligned}$$

By comparing the coefficients on both sides of (3.8), we get the following theorem.

**Theorem 3.2.** For  $n \geq 0$  and  $k \in \mathbb{Z}$ , we have

$$F_{n,p}^{(k)}(x; y) = \sum_{l=0}^n \sum_{m=0}^l \binom{n}{l} (x)_m S_2(l, m) F_{n-l,p}^{(k)}(y).$$

From (3.3), we have

$$\begin{aligned} \sum_{n=0}^{\infty} F_{n,p}^{(k)}(y) \frac{t^n}{n!} &= \frac{1}{t(1-y(e^t-1))} u_k(\log(1+t)|p) \\ &= \frac{1}{t(1-y(e^t-1))} \sum_{m=1}^{\infty} \frac{p(m)}{m^k} (\log(1+t))^m \\ &= \frac{1}{t(1-y(e^t-1))} \sum_{m=0}^{\infty} \frac{p(m+1)}{(m+1)^k} (\log(1+t))^{m+1} \\ &= \frac{\log(1+t)}{t} \frac{1}{1-y(e^t-1)} \sum_{m=0}^{\infty} \frac{p(m+1)m!}{(m+1)^k} \sum_{l=m}^{\infty} S_1(l, m) \frac{t^l}{l!} \\ &= \sum_{r=0}^{\infty} D_r \frac{t^r}{r!} \sum_{n=0}^{\infty} F_n(y) \frac{t^n}{n!} \sum_{l=0}^{\infty} \left( \sum_{m=0}^l \frac{p(m+1)m!}{(m+1)^k} S_1(l, m) 2^l \right) \frac{t^l}{l!} \\ &= \sum_{n=0}^{\infty} \sum_{r=0}^n \binom{n}{r} D_r F_{n-r}(y) \frac{t^n}{n!} \sum_{l=0}^{\infty} \left( \sum_{m=0}^l \frac{p(m+1)m!}{(m+1)^k} S_1(l, m) \right) \frac{t^l}{l!} \\ &= \sum_{n=0}^{\infty} \sum_{r=0}^n \sum_{l=0}^n \binom{n}{r} \binom{n}{l} D_r F_{n-r-l}(y) \sum_{m=0}^l \frac{p(m+1)m!}{(m+1)^k} S_1(l, m) \frac{t^n}{n!}. \end{aligned} \quad (3.9)$$

Therefore, by comparing the coefficients on both sides of (3.9), we obtain the following theorem.

**Theorem 3.3.** For  $n \geq 0$  and  $k \in \mathbb{Z}$ , we have

$$F_{n,p}^{(k)} = \sum_{r=0}^n \sum_{l=0}^n \binom{n}{r} \binom{n}{l} D_r F_{n-r-l}(y) \sum_{m=0}^l \frac{p(m+1)m!}{(m+1)^k} S_1(l, m).$$

We observe that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{d}{dx} F_{n,p}^{(k)}(x; y) \frac{t^n}{n!} &= \frac{1}{t(1-y(e^t-1))} u_k(\log(1+t)|p) \frac{d}{dx} e^{xt} \\ &= \sum_{n=0}^{\infty} F_{n,p}^{(k)}(x; y) \frac{t^{n+1}}{n!}. \end{aligned} \quad (3.10)$$

Therefore, we give the following theorem.

**Theorem 3.4.** For  $n \geq 0$  and  $k \in \mathbb{Z}$ , we have

$$\frac{d}{dx} F_{n,p}^{(k)}(x; y) = n F_{n-1,p}^{(k)}(x; y).$$

By theorem 3.4, we consider that

$$\begin{aligned} \int_{\alpha}^{\beta} F_{n,p}^{(k)}(x; y) dx &= \frac{1}{n+1} \int_{\alpha}^{\beta} \frac{d}{dx} F_{n,p}^{(k)}(x; y) dx \\ &= \frac{F_{n+1,p}^{(k)}(\beta; y) - F_{n+1,p}^{(k)}(\alpha; y)}{n+1}. \end{aligned}$$

Thus, we provide the following theorem.

**Theorem 3.5.** For  $n \geq 0$  and  $k \in \mathbb{Z}$ , we have

$$\int_{\alpha}^{\beta} F_{n,p}^{(k)}(x; y) dx = \frac{F_{n+1,p}^{(k)}(\beta; y) - F_{n+1,p}^{(k)}(\alpha; y)}{n+1}.$$

and

$$\int_{\alpha}^{\beta} F_{n,p}^{(k)}(x; y) dy = \frac{F_{n+1,p}^{(k)}(x; \beta) - F_{n+1,p}^{(k)}(x; \alpha)}{n+1}.$$

## 5. Conclusion

Motivated by the definition of the type 2 poly-Bernoulli introduced by Kim-Kim [9], in the present paper, we have considered a class of new generating function for the poly-Fubini polynomials, called the type 2 poly-Fubini polynomials, by means of the polyexponential function. Then, we have derived some useful relations and properties. We have showed that the type 2 poly-Fubini polynomials equal a linear combination of the classical Fubini polynomials and Stirlings numbers of the first kind. In a special case, we have given a relation between the type 2 poly-Fubini polynomials and Bernoulli polynomials of order  $r$ . Moreover, by inspired by definition of the unipoly-Bernoulli polynomials introduced by Kim-Kim [9], we have introduced the unipoly-Fubini polynomials by means of unipoly function and have given multifarious properties including derivative and integral properties. Moreover, we have provided a correlation between the unipoly-Fubini polynomials and the classical Fubini polynomials.

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