A note on type 2 degenerate poly-Fubini polynomials and numbers

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Abstract. In this paper, we construct the degenerate poly-Fubini polynomials, called the type 2 degenerate poly-Fubini polynomials, by using the modified degenerate polyexponential function and derive several properties on the degenerate poly-Fubini polynomials and numbers. In the last section, we introduce type 2 degenerate unipoly-Fubini polynomials attached to an arithmetic function, by using the modified degenerate polyexponential function and investigate some identities for those polynomials. Furthermore, we give some new explicit expressions and identities of degenerate unipoly polynomials related to special numbers and polynomials.

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1. Introduction

Special function posses a lot of importance in numerous field of physics, mathematics, applied sciences and other related areas including functional analysis, differential equations, quantum mechanics, mathematical analysis, mathematical physics, and so on on (see [30, 31, 34, 35, 37]) and see the references cited therein. For example; Riemann zeta function is closely related with the Bernoulli numbers and zeros of special polynomials is one of the most useful and applicable family of special functions. Some of the most considerable polynomials in the theory of special polynomials are the Fubini polynomials (see [3, 4]), the type 2 poly-Bernoulli polynomials (see [14, 24]), the type 2 poly-Genocchi polynomials (see [7, 25]) and the degenerate poly-Bernoulli polynomials (see [11, 14, 18]), the degenerate poly-Euler polynomials (see [33]), the degenerate poly-Genocchi polynomials (see [16]). Recently, the aforementioned polynomials and their several extensions have been densely studied and investigated by diverse mathematicians and physicists (see [1-37]) and see also each of the references cited therein.

The classical Bernoulli $B_n(x)$, Euler $E_n(x)$ and Genocchi $G_n(x)$ polynomial are defined by means of the following generating function as follows

$$\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}, \mid t \mid< 2\pi, \quad \frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}, \mid t \mid< \pi,$$

and

$$\frac{2t}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!}, \mid t \mid< \pi, \text{(see [12, 13, 14])} \tag{1.1}$$

respectively.

Kargin [29] defined the two-variable Fubini polynomials by the following generating function

$$\frac{e^{xt}}{1-y(e^t-1)}=\sum_{n=0}^{\infty}F_n(x,y)\frac{t^n}{n!}, \text{(see [1, 4])}. \tag{1.2}$$

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When $x = 0$ in (1.2), the two variable Fubini polynomials $F_n(x, y)$ reduce to the usual Fubini polynomials given by

$$\frac{1}{1 - y(e^t - 1)} = \sum_{n=0}^{\infty} F_n(y)\frac{t^n}{n!}, \text{(see [5, 11])}. \tag{1.3}$$

It is easy to see that

$$F_n\left(x, -\frac{1}{2}\right) = E_n(x), F_n\left(-\frac{1}{2}\right) = E_n \tag{1.4}$$

and

$$F_n(y) = \sum_{k=0}^{n} S_2(n, k)k!y^k.$$

For $y = 1$ in (1.3), we get the familiar Fubini numbers $F_n(1) = F_n$ as follows

$$\frac{1}{2 - e^t)} = \sum_{n=0}^{\infty} F_n\frac{t^n}{n!}, \text{(see [1, 2])}. \tag{1.5}$$

For more information about the applications of the usual Fubini polynomials and numbers (see [1, 4, 7, 12, 29, 30, 37]).

The degenerate exponential function is defined by (see [16, 18, 19])

$$e_\lambda^x(t) = (1 + \lambda t)^{\frac{x}{\lambda}}, e_\lambda(t)e_\lambda^1(t) = (1 + \lambda t)^{\frac{1}{\lambda}}, \lambda \in \mathbb{R}. \tag{1.6}$$

Here we note that

$$e_\lambda^x(t) = \sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}, \tag{1.7}$$

where $(x)_{0,\lambda} = 1, (x)_{n,\lambda} = x(x - \lambda)(x - 2\lambda)\cdots(x - (n - 1)\lambda), (n \geq 1)$.

In [2, 3], Carlitz considered the degenerate Bernoulli polynomials which are given by

$$\frac{t}{e_\lambda(t) - 1}e_\lambda^x(t) = \frac{t}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1}(1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty}\beta_{n,\lambda}(x)\frac{t^n}{n!}. \tag{1.8}$$

On setting $x = 0$, $\beta_{n,\lambda} = \beta_{n,\lambda}(0)$ are called degenerate Bernoulli numbers.

For $k \in \mathbb{Z}$, the modified degenerate polyexponential function [27] is defined by Kim-Kim to be

$$\mathrm{Ei}_{k,\lambda}(x) = \sum_{n=1}^{\infty}\frac{(1)_{n,\lambda}x^n}{(n - 1)!n^k}, (\mid x \mid < 1). \tag{1.9}$$

Note that

$$\mathrm{Ei}_{1,\lambda}(x) = \sum_{n=1}^{\infty}\frac{(1)_{n,\lambda}x^n}{n!} = e_\lambda(x) - 1. \tag{1.10}$$

In [27], Kim et al. considered the degenerate poly-Genocchi polynomials are defined by means of the following generating function

$$\frac{\mathrm{Ei}_{k,\lambda}(\log_\lambda(1 + t))}{e_\lambda(t) + 1}e_\lambda^x(t) = \sum_{n=0}^{\infty}G_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}, (k \in \mathbb{Z}). \tag{1.11}$$

In the case when $x = 0$, $G_{n,\lambda}^{(k)} = G_{n,\lambda}^{(k)}(0)$ are called the degenerate poly-Genocchi numbers.

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The two variable degenerate Fubini polynomials $F_{n,\lambda}(x;y)$ are defined by

$$\frac{1}{1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} F_{n,\lambda}(x;y)\frac{t^n}{n!}, \text{(see [22, 23, 24])}. \tag{1.12}$$

When $x=0$ and $y=1$, $F_{n,\lambda}=F_{n,\lambda}(0;1)$ are called the degenerate Fubini numbers.

In [36], the degenerate Daehee polynomials $D_{n,\lambda}(x)$ are defined by

$$\frac{\log_\lambda(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_{n,\lambda}(x)\frac{t^n}{n!}, \text{(see [10, 21])} \tag{1.13}$$

For $x=0, D_{n,\lambda}=D_{n,\lambda}(0)$ are called the degenerate Daehee numbers.

The degenerate Stirling numbers of the first kind are defined by

$$\frac{1}{k!}(\log_\lambda(1+t))^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{t^n}{n!}, \ (k\geq 0), \text{(see [13, 15, 20])}. \tag{1.14}$$

Note here that $\lim_{\lambda\to 0} S_{1,\lambda}(n,k) = S_1(n,k)$, where $S_1(n,k)$ are the Stirling numbers of the first kind given by

$$\frac{1}{k!}(\log(1+t))^k = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!}, \ (k\geq 0), \text{(see [6, 7])}. \tag{1.15}$$

The degenerate Stirling numbers of the second kind are given by

$$\frac{1}{k!}\left(e_\lambda(t)-1\right)^k = \sum_{n=l}^{\infty} S_{2,\lambda}(n,l)\frac{t^n}{n!}, \text{(see [16])}. \tag{1.16}$$

Observe here that $\lim_{\lambda\to 0} S_{2,\lambda}(n,k) = S_2(n,k)$, where $S_2(n,k)$ are the Stirling numbers of the second kind given by

$$\frac{1}{k!}(e^t-1)^k = \sum_{n=l}^{\infty} S_2(n,l)\frac{t^n}{n!}, \text{(see [1-37])}. \tag{1.17}$$

By the motivation of the works of Kim et al. [], we first define the type 2 degenerate poly-,Fubini polynomials by using the modified degenerate polyexponential functions. We investigate some new properties of these numbers and polynomials and derive some new identities and relations between the new type of degenerate poly-Fubini polynomials. Furthermore, we consider the type 2 degenerate unipoly-Fubini polynomials and discuss some identities of them.

2. Type 2 degenerate poly-Fubini polynomials and numbers

In this section, we define the degenerate Fubini polynomials by using the modified degenerate polyexponential function which are called the type 2 degenerate poly-Fubini numbers and polynomials. We then investigate many relations and formulas for these polynomials and numbers, which covers several summation formulas, addition identities, recurrence relationships. We also give some formulas associated with degenerate Stirling numbers of the first and second kind.

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Let $\lambda \in \mathbb{C}$ and $k \in \mathbb{Z}$, by using the modified degenerate polyexponential function, we consider the type 2 degenerate poly-Fubini polynomials are defined by means of the following generating function

$$\frac{\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t))}{t(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1))}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} F_{n,\lambda}^{(k)}(x;y)\frac{t^n}{n!}. \tag{2.1}$$

In the special case, $x = 0$ and $y = 1$, $F_{n,\lambda}^{(k)} = F_{n,\lambda}^{(k)}(0;1)$ are called the type 2 degenerate poly-Fubini numbers, where $\log_\lambda(t) = \frac{1}{\lambda}(t^\lambda - 1)$ is the compositional inverse of $e_\lambda(t)$ satisfying

$$\log_\lambda e_\lambda(t) = e_\lambda(\log_\lambda(1+t)) = t.$$

For $k = 1$ in (2.1), we get

$$\frac{1}{1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} F_{n,\lambda}(x)\frac{t^n}{n!}, \tag{2.2}$$

where $F_{n,\lambda}(x;y)$ are called the degenerate Fubini polynomials (see Eq. (1.12)).

Obviously

$$\lim_{\lambda\longrightarrow 0}\left(\frac{\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t))}{t(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1))}(1+\lambda t)^{\frac{x}{\lambda}}\right) = \sum_{n=0}^{\infty}\lim_{\lambda\longrightarrow 0} F_{n,\lambda}^{(k)}(x;y)\frac{t^n}{n!}$$

$$= \frac{\mathrm{Ei}_k(\log(1+t))}{t(1-y(e^t-1))}e^{xt} = \sum_{n=0}^{\infty} F_n^{(k)}(x;y)\frac{t^n}{n!}. \tag{2.3}$$

Thus, by (2.1) and (2.3), we have

$$\lim_{\lambda\longrightarrow 0} F_{n,\lambda}^{(k)}(x;y) = F_n^{(k)}(x;y), (n \geq 0) \tag{2.4}$$

where $F_n^{(k)}(x;y)$ are called the type 2 poly-Fubini polynomials.

By using equations (1.9), (1.14) and (2.1), we observe that

$$\sum_{n=0}^{\infty} F_{n,\lambda}^{(k)}(y)\frac{t^n}{n!} = \frac{\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t))}{t(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1))}$$

$$= \frac{1}{1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)}\frac{1}{t}\sum_{m=1}^{\infty}\frac{(\log_\lambda(1+t))^m}{(m-1)!m^k}$$

$$= \frac{1}{1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)}\frac{1}{t}\sum_{m=0}^{\infty}\frac{(\log_\lambda(1+t))^{m+1}}{(m+1)!(m+1)^{k-1}}$$

$$= \frac{1}{1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)}\frac{1}{t}\sum_{m=0}^{\infty}\frac{1}{(m+1)^{k-1}}\sum_{l=m+1}^{\infty} S_{1,\lambda}(l,m+1)\frac{t^l}{l!}$$

$$= \frac{1}{1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)}\frac{1}{t}\sum_{m=0}^{\infty}\frac{1}{(m+1)^{k-1}}\sum_{l=m}^{\infty} S_{1,\lambda}(l+1,m+1)\frac{t^l}{(l+1)!}$$

$$= \left(\sum_{s=0}^{\infty} F_{n,\lambda}(y)\frac{t^n}{n!}\right)\left(\sum_{l=0}^{\infty}\sum_{m=0}^{l}\frac{1}{(m+1)^{k-1}}\frac{S_{1,\lambda}(l+1,m+1)}{l+1}\frac{t^l}{l!}\right)$$

$$L.H.S = \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}F_{n-l,\lambda}(y)\frac{S_{1,\lambda}(l+1,m+1)}{l+1(m+1)^{k-1}}\right)\frac{t^n}{n!}. \tag{2.5}$$

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Therefore, by (2.5), we obtain the following theorem.

Theorem 2.1. For $n \geq 0$ and $k \in \mathbb{Z}$, we have

$$F_{n,\lambda}^{(k)}(y) = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} F_{n-l,\lambda}(y) \frac{S_{1,\lambda}(l+1, m+1)}{l+1(m+1)^{k-1}}. \tag{2.6}$$

Corollary 2.1. For $n \geq 0$ and $k \in \mathbb{Z}$, we have

$$F_{n,\lambda}^{(1)}(y) = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} F_{n-l,\lambda}(y) \frac{S_{1,\lambda}(l+1, m+1)}{l+1}. \tag{2.7}$$

Moreover,

$$\sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} F_{n-l,\lambda}(y) \frac{S_{1,\lambda}(l+1, m+1)}{l+1} = 0, n \geq 1.$$

From (2.1), we observe that

$$\begin{aligned}
\sum_{n=0}^{\infty} F_{n,\lambda}^{(k)}(x;y) \frac{t^n}{n!} &= \frac{\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t))}{t(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1))}(1+\lambda t)^{\frac{x}{\lambda}} \\
&= \left(\sum_{n=0}^{\infty} F_{n,\lambda}^{(k)}(y) \frac{t^n}{n!}\right) \left(\sum_{m=0}^{\infty} (x)_{m,\lambda} \frac{t^m}{m!}\right) \\
&= \left(\sum_{n=0}^{\infty} F_{n,\lambda}^{(k)}(y) \frac{t^n}{n!}\right) \left(\sum_{m=0}^{\infty} (x)_{m,\lambda} \frac{t^m}{m!}\right) \\
L.H.S &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} F_{n-m,\lambda}^{(k)}(y)(x)_{m,\lambda}\right) \frac{t^n}{n!}. \qquad (2.8)
\end{aligned}$$

By comparing the coefficients on both sides of (2.8), we obtain the following theorem.

Theorem 2.2. Let $n \geq 0$ and $k \in \mathbb{Z}$. Then, we have

$$F_{n,\lambda}^{(k)}(x;y) = \sum_{m=0}^{n} \binom{n}{m} F_{n-m,\lambda}^{(k)}(y)(x)_{m,\lambda}. \tag{2.9}$$

In [26], it is well known that the degenerate Bernoulli polynomials of the second kind are defined by

$$\frac{t}{\log_\lambda(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_{n,\lambda}(x) \frac{t^n}{n!}. \tag{2.10}$$

For $x = 0$, $b_{n,\lambda} = b_{n,\lambda}(0)$ are called degenerate Bernoulli numbers of the second kind.

From (1.9), we note that

$$\frac{d}{dx}\mathrm{Ei}_{\mathrm{k},\lambda}(\log_\lambda(1+\mathrm{x})) = \frac{\mathrm{d}}{\mathrm{dx}} \sum_{\mathrm{n}=1}^{\infty} \frac{(1)_{\mathrm{n},\lambda}(\log_\lambda(1+\mathrm{x}))^{\mathrm{n}}}{(\mathrm{n}-1)!\mathrm{n}^{\mathrm{k}}}$$

$$= \frac{(1+x)^{\lambda-1}}{\log_\lambda(1+x)} \sum_{n=1}^{\infty} \frac{(1)_{n,\lambda}(\log_\lambda(1+x))^n}{(n-1)!n^{k-1}} = \frac{(1+x)^{\lambda-1}}{\log_\lambda(1+x)}\mathrm{Ei}_{\mathrm{k}-1,\lambda}(\log_\lambda(1+x)). \tag{2.11}$$

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Thus, from (2.1) and (2.11), we have

$$\sum_{n=0}^{\infty} F_{n,\lambda}^{(k)}(y)\frac{x^n}{n!} = \frac{1}{x(1-y((1+\lambda x)^{\frac{1}{\lambda}}-1))}\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+x)\right)$$

$$= \frac{1}{x(1-y((1+\lambda x)^{\frac{1}{\lambda}}-1))}\int_0^x \underbrace{\frac{(1+t)^{\lambda-1}}{\log_\lambda(1+t)}\int_0^t \cdots \frac{(1+t)^{\lambda-1}}{\log_\lambda(1+t)}\int_0^t}_{(k-2)-\text{times}} \frac{(1+t)^{\lambda-1}}{\log_\lambda(1+t)} t dt \cdots dt$$

$$= \frac{1}{x(1-y((1+\lambda x)^{\frac{1}{\lambda}}-1))}\sum_{m=0}^{\infty}\sum_{m_1+\cdots+m_{k-1}=m}\binom{m}{m_1+\cdots+m_{k-1}}$$

$$\times \frac{b_{m_1,\lambda}(\lambda-1)}{m_1+1}\frac{b_{m_2,\lambda}(\lambda-1)}{m_1+m_2+1}\cdots\frac{b_{m_{k-1},\lambda}(\lambda-1)}{m_1+\cdots+m_{k-1}+1}\frac{x^m}{m!}$$

$$\sum_{n=0}^{\infty} F_{n,\lambda}^{(k)}(y)\frac{x^n}{n!} = \frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\sum_{m_1+\cdots+m_{k-1}=m}\binom{m}{m_1+\cdots+m_{k-1}}$$

$$\times \frac{b_{m_1,\lambda}(\lambda-1)}{m_1+1}\frac{b_{m_2,\lambda}(\lambda-1)}{m_1+m_2+1}\cdots\frac{b_{m_{k-1},\lambda}(\lambda-1)}{m_1+\cdots+m_{k-1}+1}F_{n-m,\lambda}(y)\frac{x^n}{n!}. \tag{2.12}$$

Therefore, by (2.12), we obtain the following theorem.

Theorem 2.3. For $n \geq 0$, we have

$$F_{n,\lambda}^{(k)}(y) = \frac{1}{2}\sum_{m=0}^{n}\binom{n}{m}\sum_{m_1+\cdots+m_{k-1}=m}\binom{m}{m_1+\cdots+m_{k-1}}$$

$$\times \frac{b_{m_1,\lambda}(\lambda-1)}{m_1+1}\frac{b_{m_2,\lambda}(\lambda-1)}{m_1+m_2+1}\cdots\frac{b_{m_{k-1},\lambda}(\lambda-1)}{m_1+\cdots+m_{k-1}+1}F_{n-m,\lambda}(y). \tag{2.13}$$

Corollary 2.2. For $k \geq 2$, we have

$$F_{n,\lambda}^{(2)}(y) = \frac{1}{2}\sum_{m=0}^{n}\binom{n}{m}\frac{b_{m,\lambda}(\lambda-1)}{m+1}F_{n-m,\lambda}(y).$$

Let $k \geq 1$, be an integer. For $s \in \mathbb{C}$, we define the function $\eta_{k,\lambda}(s)$ as

$$\eta_{k,\lambda}(s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}}{t(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1))}\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+t)\right)dt$$

$$= \frac{1}{\Gamma(s)}\int_0^1 \frac{t^{s-1}}{t(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1))}\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+t)\right)dt$$

$$+\frac{1}{\Gamma(s)}\int_1^\infty \frac{t^{s-1}}{t(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1))}\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+t)\right)dt. \tag{2.14}$$

The second integral converges absolutely for any $s \in \mathbb{C}$ and hence, the second term on the right hand side vanishes at non-positive integers. That is,

$$\lim_{s\to -m}\left|\frac{1}{\Gamma(s)}\int_1^\infty \frac{t^{s-1}}{t(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1))}\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+t)\right)dt\right| \leq \frac{1}{\Gamma(-m)}M = 0. \tag{2.15}$$

On the other hand, for $\Re(s) > 0$, the first integral in (2.15) can be written as

$$\frac{1}{\Gamma(s)}\sum_{l=0}^{\infty}\frac{F_{l,\lambda}^{(k)}(y)}{l!}\frac{1}{s+l},$$

which defines an entire function of s. Thus, we may include that $\eta_{k,\lambda}(s)$ can be continued to an entire function of s.

Further, from (2.14) and (2.15), we obtain

$$\begin{aligned}
\eta_{k,\lambda}(-m) &= \lim_{s\to -m}\frac{1}{\Gamma(s)}\int_0^1 \frac{t^{s-1}}{t(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1))}\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+t)\right)dt \\
&= \lim_{s\to -m}\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\sum_{l=0}^{\infty}\frac{F_{l,\lambda}^{(k)}(y)t^l}{l!}dt = \lim_{s\to -m}\frac{1}{\Gamma(s)}\sum_{l=0}^{\infty}\frac{F_{l,\lambda}^{(k)}(y)}{s+l}\frac{1}{l!} \\
&= \cdots+0+\cdots+0+\lim_{s\to -m}\frac{1}{\Gamma(s)}\frac{1}{s+m}\frac{F_{m,\lambda}^{(k)}(y)}{m!}+0+0+\cdots \qquad (2.16)\\
&= \lim_{s\to -m}\frac{\left(\frac{\Gamma(1-s)\sin\pi s}{\pi}\right)}{s+m}\frac{F_{m,\lambda}^{(k)}(y)}{m!} = \Gamma(1+m)\cos(\pi m)\frac{F_{m,\lambda}^{(k)}(y)}{m!} \\
&= (-1)^m F_{m,\lambda}^{(k)}(y).
\end{aligned}$$

Therefore, by (2.16), we obtain the following theorem.

Theorem 2.4. Let $k\geq 1$ and $m\in\mathbb{N}\bigcup\{0\}$, $s\in\mathbb{C}$, we have

$$\eta_{k,\lambda}(-m) = (-1)^m F_{m,\lambda}^{(k)}(y).$$

From (2.1), we note that

$$\begin{aligned}
\frac{\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t))}{t} &= (1-y((1+\lambda t)^{\frac{1}{\lambda}}-1))\sum_{n=0}^{\infty}F_{n,\lambda}^{(k)}(y)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}F_{n,\lambda}^{(k)}(y)\frac{t^n}{n!} - y\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}(1)_{m,\lambda}F_{n-m,\lambda}^{(k)}(y)\frac{t^n}{n!} + y\sum_{n=0}^{\infty}F_{n,\lambda}^{(k)}(y)\frac{t^n}{n!} \\
&= (1+y)\sum_{n=0}^{\infty}F_{n,\lambda}^{(k)}(y)\frac{t^n}{n!} - y\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}(1)_{m,\lambda}F_{n-m,\lambda}^{(k)}(y)\frac{t^n}{n!}. \qquad (2.17)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\frac{\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t))}{t} &= \frac{1}{t}\sum_{m=1}^{\infty}\frac{(1)_{m,\lambda}(\log_\lambda(1+t))^m}{(m-1)!m^k} \\
&= \frac{1}{t}\sum_{m=0}^{\infty}\frac{(1)_{m+1,\lambda}(\log_\lambda(1+t))^{m+1}}{m!(m+1)^k}\frac{(m+1)!}{(m+1)!} \\
&= \frac{1}{t}\sum_{m=0}^{\infty}\frac{(1)_{m+1,\lambda}}{(m+1)^{k-1}}\sum_{n=m+1}^{\infty}S_{1,\lambda}(n,m+1)\frac{t^n}{n!}
\end{aligned}$$

$$L.H.S = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\frac{1}{(m+1)^{k-1}}\frac{(1)_{m+1,\lambda}S_{1,\lambda}(n+1,m+1)}{n+1}\right)\frac{t^n}{n!}. \qquad (2.18)$$

Therefore, by (2.17) and (2.18), we obtain the following theorem.

Theorem 2.5. Let $k\geq 1$ and $m\in\mathbb{N}\bigcup\{0\}$, $s\in\mathbb{C}$, we have

$$F_{n,\lambda}^{(k)}(y) = \frac{1}{1+y}\left[y\sum_{m=0}^{n}\binom{n}{m}(1)_{m,\lambda}F_{n-m,\lambda}^{(k)}(y) + \sum_{m=0}^{n}\frac{1}{(m+1)^{k-1}}\frac{(1)_{m+1,\lambda}S_{1,\lambda}(n+1,m+1)}{n+1}\right].$$

For $k=1$ in Theorem 2.5., we get the following corollary

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Corollary 2.3. For $m \in \mathbb{N}\bigcup\{0\}$, we have

$$F_{n,\lambda}(y) = \frac{1}{1+y}\left[y\sum_{m=0}^{n}\binom{n}{m}(1)_{m,\lambda}F_{n-m,\lambda}(y) + \sum_{m=0}^{n}\frac{(1)_{m+1,\lambda}S_{1,\lambda}(n+1,m+1)}{n+1}\right].$$

From (2.1), we note that

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(F_{n,\lambda}^{(k)}(x+1;y) - F_{n,\lambda}^{(k)}(x;y)\right)\frac{z^n}{n!} \\
&= \frac{\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t))}{t(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1))}(1+\lambda t)^{\frac{x}{\lambda}}\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right) \\
&= \frac{\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t))}{t}\left(\frac{1}{y}\left(\left(\frac{(1+\lambda t)^{\frac{x}{\lambda}}}{1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)}\right) - (1+\lambda t)^{\frac{x}{\lambda}}\right)\right) \\
&= \frac{1}{y}\left(\frac{\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t))(1+\lambda t)^{\frac{x}{\lambda}}}{t(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1))} - \frac{\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t))(1+\lambda t)^{\frac{x}{\lambda}}}{t}\right) \\
&= \frac{1}{y}\left(\sum_{n=0}^{\infty}F_{n,\lambda}^{(k)}(x;y)\frac{t^n}{n!} - \sum_{l=0}^{\infty}\sum_{m=0}^{l}\frac{1}{(m+1)^{k-1}}\frac{S_{1,\lambda}(l+1,m+1)}{l+1}\frac{t^l}{l!}\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}\right) \\
&= \frac{1}{y}\sum_{n=0}^{\infty}\left(F_{n,\lambda}^{(k)}(x;y) - \sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}\frac{1}{(m+1)^{k-1}}\frac{S_{1,\lambda}(l+1,m+1)}{l+1}(x)_{n-l,\lambda}\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.19}$$

By comparing the coefficients on both sides of (2.19), we get the following theorem.

Theorem 2.6. Let $k \in \mathbb{Z}$ and $n \geq 0$, we have

$$yF_{n,\lambda}^{(k)}(x+1;y) = (y+1)F_{n,\lambda}^{(k)}(x;y) - \sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}\frac{1}{(m+1)^{k-1}}\frac{S_{1,\lambda}(l+1,m+1)}{l+1}(x)_{n-l,\lambda}.$$

When $x = 0$ and $x = -1$ in Theorem (2.6), we get

$$yF_{n,\lambda}^{(k)}(1;y) = (y+1)F_{n,\lambda}^{(k)}(y) - \sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}\frac{1}{(m+1)^{k-1}}\frac{S_{1,\lambda}(l+1,m+1)}{l+1}, (n \geq 0).$$

and

$$yF_{n,\lambda}^{(k)}(y) = (y+1)F_{n,\lambda}^{(k)}(-1;y) - \sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}\frac{1}{(m+1)^{k-1}}\frac{S_{1,\lambda}(l+1,m+1)}{l+1}(-1)_{n-l,\lambda}, (n \geq 0).$$

Now, we observe that

$$\begin{aligned}
&\left(\frac{\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t))}{t(1-y_1((1+\lambda t)^{\frac{1}{\lambda}}-1))}(1+\lambda t)^{\frac{x_1}{\lambda}}\right)\left(\frac{\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t))}{t(1-y_2((1+\lambda t)^{\frac{1}{\lambda}}-1))}(1+\lambda t)^{\frac{x_2}{\lambda}}\right) \\
&= \frac{\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t))}{t}\left(\frac{y_2}{y_2-y_1}\frac{(1+\lambda t)^{\frac{x_1+x_2}{\lambda}}}{1-y_2((1+\lambda t)^{\frac{1}{\lambda}}-1)} - \frac{y_1}{y_2-y_1}\frac{(1+\lambda t)^{\frac{x_1+x_2}{\lambda}}}{1-y_2((1+\lambda t)^{\frac{1}{\lambda}}-1)}\right) \\
&= \sum_{n=0}^{\infty}\left(\frac{y_2F_{n,\lambda}^{(k)}(x_1+x_2;y_2) - y_1F_{n,\lambda}^{(k)}(x_1+x_2;y_1)}{y_2-y_1}\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.20}$$

On the other hand,

$$\left(\frac{\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t))}{t(1-y_1((1+\lambda t)^{\frac{1}{\lambda}}-1))}(1+\lambda t)^{\frac{x_1}{\lambda}}\right)\left(\frac{\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t))}{t(1-y_2((1+\lambda t)^{\frac{1}{\lambda}}-1))}(1+\lambda t)^{\frac{x_2}{\lambda}}\right)$$

$$= \left(\sum_{n=0}^{\infty} F_{n,\lambda}^{(k)}(x_1;y_1)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty} F_{m,\lambda}^{(k)}(x_2;y_2)\frac{t^m}{m!}\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}F_{n-m,\lambda}^{(k)}(x_1;y_1)F_{m,\lambda}^{(k)}(x_2;y_2)\right)\frac{t^n}{n!}. \tag{2.21}$$

Therefore, by (2.20) and (2.21), we get the following theorem.

Theorem 2.7. Let $k \in \mathbb{Z}$ and $n \geq 0$, we have

$$\sum_{m=0}^{n}\binom{n}{m}F_{n-m,\lambda}^{(k)}(x_1;y_1)F_{m,\lambda}^{(k)}(x_2;y_2)$$

$$= \frac{y_2F_{n,\lambda}^{(k)}(x_1+x_2;y_2) - y_1F_{n,\lambda}^{(k)}(x_1+x_2;y_1)}{y_2-y_1}.$$

For $x_1 = x_2 = 0$ in Theorem 2.7, we have

Corollary 2.4. Let $k \in \mathbb{Z}$ and $n \geq 0$, we have

$$\sum_{m=0}^{n}\binom{n}{m}F_{n-m,\lambda}^{(k)}(y_1)F_{m,\lambda}^{(k)}(y_2) = \frac{y_2F_{n,\lambda}^{(k)}(y_2) - y_1F_{n,\lambda}^{(k)}(y_1)}{y_2-y_1}.$$

From (2.1), we note that

$$\sum_{n=0}^{\infty} F_{n,\lambda}^{(k)}(x;y-1)\frac{t^n}{n!} = \frac{\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t))}{t(1-(y-1)((1+\lambda t)^{\frac{1}{\lambda}}-1))}(1+\lambda t)^{\frac{x}{\lambda}}$$

$$= \frac{\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t))(1+\lambda t)^{\frac{x}{\lambda}}}{t(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)+(1+\lambda t)^{\frac{1}{\lambda}}-1)}$$

$$= \frac{\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t))(1+\lambda t)^{\frac{x-1}{\lambda}}}{t(1-y(1-\lambda t)^{\frac{1}{\lambda}})}$$

$$= \frac{\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t))(1-\lambda(-t))^{-\frac{1-x}{\lambda}}}{t(1+y((1-\lambda(-t))^{-\frac{1}{\lambda}}-1))}$$

$$= \sum_{n=0}^{\infty} F_{n,-\lambda}^{(k)}(1-x;-y)\frac{(-1)^n t^n}{n!}. \tag{2.22}$$

Comparing the coefficients on both sides of (2.22), we obtain the following theorem.

Theorem 2.8. Let $k \in \mathbb{Z}$ and $n \geq 0$, we have

$$F_{n,\lambda}^{(k)}(x;y-1) = (-1)^n F_{n,-\lambda}^{(k)}(1-x;-y).$$

On setting $x = 0$ in Theorem 2.8, we get

Corollary 2.5. Let $k \in \mathbb{Z}$ and $n \geq 0$, we have

$$F_{n,\lambda}^{(k)}(y-1) = (-1)^n F_{n,-\lambda}^{(k)}(1;-y).$$

From (2.1), we see that

$$\sum_{n=0}^{\infty} F_{n,\lambda}^{(k)}(y)\frac{t^n}{n!} = \frac{\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t))}{t(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1))}$$

$$= \left(\frac{\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t))}{t}\right)\left(\sum_{r=0}^{\infty} y^r((1+\lambda t)^{\frac{1}{\lambda}}-1)^r\right)$$

$$= \frac{1}{t}\left(\sum_{m=1}^{\infty}\frac{(1)_{m,\lambda}(\log_\lambda(1+t))^m}{(m-1)!m^k}\right)\left(\sum_{n=0}^{\infty}\sum_{r=0}^{n}y^r S_{2,\lambda}(n,r)\frac{t^n}{n!}\right)$$
$$\left(\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{(m+1)^{k-1}}\frac{(1)_{m+1,\lambda}S_{1,\lambda}(l+1,m+1)}{l+1}\frac{t^l}{l!}\right)\left(\sum_{n=0}^{\infty}\sum_{r=0}^{n}y^r S_{2,\lambda}(n,r)\frac{t^n}{n!}\right)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{r=0}^{n-l}\sum_{m=0}^{l}\binom{n}{l}\frac{1}{(m+1)^{k-1}}\frac{(1)_{m+1,\lambda}S_{1,\lambda}(l+1,m+1)}{l+1}y^r S_{2,\lambda}(n-l,r)\right)\frac{t^n}{n!}. \tag{2.23}$$

Therefore, by (2.23), we get the following theorem.

Theorem 2.9. Let $k \in \mathbb{Z}$ and $n \geq 0$, we have

$$F_{n,\lambda}^{(k)}(y) = \sum_{l=0}^{n}\sum_{r=0}^{n-l}\sum_{m=0}^{l}\binom{n}{l}\frac{1}{(m+1)^{k-1}}\frac{(1)_{m+1,\lambda}S_{1,\lambda}(l+1,m+1)}{l+1}y^r S_{2,\lambda}(n-l,r).$$

Corollary 2.6. Let $k \in \mathbb{Z}$ and $n \geq 0$, we have

$$F_{n,\lambda}(y) = \sum_{l=0}^{n}\sum_{r=0}^{n-l}\sum_{m=0}^{l}\binom{n}{l}\frac{(1)_{m+1,\lambda}S_{1,\lambda}(l+1,m+1)}{l+1}y^r S_{2,\lambda}(n-l,r).$$

3. Type 2 degenerate unipoly-Fubini polynomials and numbers

In this section, we define type 2 degenerate unipoly-Fubini polynomials by using modified degenerate polyexponential function. We derive some explicit expressions and multifarious properties.

Let p be any arithmetic function which is a real or complex valued function defined on the set of positive integers $\mathbb{N}$. Kim-Kim [15] defined the unipoly function attached to polynomials $p(x)$ by

$$u_k(x|p) = \sum_{n=1}^{\infty}\frac{p(n)}{n^k}x^n, (k \in \mathbb{Z}). \tag{3.1}$$

Moreover,

$$u_k(x|1) = \sum_{n=1}^{\infty}\frac{x^n}{n^k} = \mathrm{Li}_k(x), (\text{see [8]}), \tag{3.2}$$

is the ordinary polylogaritm function.

In this paper, we define the degenerate unipoly function attached to polynomials $p(x)$ as follows:

$$u_{k,\lambda}(x|p) = \sum_{i=1}^{\infty}p(i)\frac{(1)_{i,\lambda}x^i}{i^k}. \tag{3.3}$$

It is worthy to note that

$$u_{k,\lambda}\left(x|\frac{1}{\Gamma}\right) = \mathrm{Ei}_{k,\lambda}(x) \tag{3.4}$$

is the modified degenerate polyexponential function.

By using (3.3), we define the degenerate unipoly-Fubini polynomials as follows:

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$$\frac{u_{k,\lambda}(\log_\lambda(1+t)|p)}{t(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}F_{n,\lambda,p}^{(k)}(x;y)\frac{t^n}{n!}. \tag{3.5}$$

In the case when $x=0$ and $y=1$, $F_{n,\lambda,p}^{(k)}=F_{n,\lambda,p}^{(k)}(0;1)$ are called the type 2 degenerate unipoly-Fubini numbers. Let us take $p(n)=\frac{1}{\Gamma\lambda}$. Then we have

$$\begin{aligned}\sum_{n=0}^{\infty}F_{n,\lambda,\frac{1}{\Gamma}}^{(k)}(x;y)\frac{t^n}{n!}&=\frac{u_{k,\lambda}(\log_\lambda(1+t)|\frac{1}{\Gamma}p)}{t(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)}(1+\lambda t)^{\frac{x}{\lambda}}\\&=\frac{1}{t(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)}\sum_{m=1}^{\infty}\frac{(\log_\lambda(1+t))^m}{m^k(m+1)!}(1+\lambda t)^{\frac{x}{\lambda}}\\&=\frac{\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t))}{t(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)}(1+\lambda t)^{\frac{x}{\lambda}}\\&=\sum_{n=0}^{\infty}F_{n,\lambda}^{(k)}(x;y)\frac{t^n}{n!}.\end{aligned} \tag{3.6}$$

Thus, by (3.6), we have the following theorem.

Theorem 3.1. Let $n\geq 0$ and $k\in\mathbb{Z}$, and Γn be a Gamma function. Then, we have

$$F_{n,\lambda,\frac{1}{\Gamma}}^{(k)}(x;y)=F_{n,\lambda}^{(k)}(x;y).$$

From (3.5), we get

$$\begin{aligned}\sum_{n=0}^{\infty}F_{n,\lambda,p}^{(k)}(y)\frac{t^n}{n!}&=\frac{u_{k,\lambda}(\log_\lambda(1+t)|p)}{t(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)}\\&=\frac{1}{t(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)}\sum_{m=1}^{\infty}\frac{p(m)(1)_{m,\lambda}}{m^k}(\log_\lambda(1+t))^m\\&=\frac{1}{t(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)}\sum_{m=0}^{\infty}\frac{p(m+1)(1)_{m+1,\lambda}(m+1)!}{(m+1)^k}\sum_{l=m+1}^{\infty}S_{1,\lambda}(m+1,l)\frac{t^l}{l!}\\&=\left(\sum_{j=0}^{\infty}F_{j,\lambda}(y)\frac{t^j}{j!}\right)\left(\sum_{m=0}^{\infty}\sum_{l=0}^{m}\frac{p(m+1)(1)_{m+1,\lambda}(m+1)!}{(m+1)^k}\frac{S_{1,\lambda}(m+1,l+1)}{l+1}\frac{t^l}{l!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty}\sum_{m=0}^{l}\binom{n}{l}\frac{p(m+1)(1)_{m+1,\lambda}(m+1)!S_{1,\lambda}(m+1,l+1)F_{n-l,\lambda}(y)}{(m+1)^k(l+1)}\right)\frac{t^n}{n!}.\end{aligned} \tag{3.8}$$

Therefore, by comparing the coefficients on both sides of (3.8), we obtain the following theorem.

Theorem 3.2. Let $n\in\mathbb{N}$ and $k\in\mathbb{Z}$. Then we have

$$F_{n,\lambda,p}^{(k)}(y)=\sum_{l=0}^{\infty}\sum_{m=0}^{l}\binom{n}{l}\frac{p(m+1)(1)_{m+1,\lambda}(m+1)!S_{1,\lambda}(m+1,l+1)F_{n-l,\lambda}(y)}{(m+1)^k(l+1)}. \tag{3.9}$$

In particular,

$$F_{n,\lambda,\frac{1}{\Gamma}}^{(k)}(y)=F_{n,\lambda}^{(k)}(y)=\sum_{l=0}^{\infty}\sum_{m=0}^{l}\binom{n}{l}\frac{S_{1,\lambda}(m+1,l+1)F_{n-l,\lambda}(y)}{(m+1)^{k-1}(l+1)}. \tag{3.10}$$

From (3.5), we observe that

$$\sum_{n=0}^{\infty} F_{n,\lambda}^{(k,p)}(x;y)\frac{t^n}{n!} = \frac{u_{k,\lambda}(\log_\lambda(1+t)|p)}{t(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1))}\left(e_\lambda^{-1}(t)-1+1\right)^x$$

$$= \frac{u_{k,\lambda}(\log_\lambda(1+t)|p)}{t(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1))}\sum_{m=0}^{\infty}\binom{x+m-1}{m}\left(1-e_\lambda^{-1}(t)\right)^m$$

$$= \left(\sum_{n=0}^{\infty} F_{n,\lambda,p}^{(k)}(y)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}(x)^{(m)}\sum_{l=m}^{\infty} S_{2,\lambda}(l,m;-m)\frac{t^l}{l!}\right)$$

$$= \left(\sum_{n=0}^{\infty} F_{n,\lambda,p}^{(k)}(y)\frac{t^n}{n!}\right)\left(\sum_{l=0}^{\infty}\sum_{m=0}^{l}(x)^{(m)} S_{2,\lambda}(l,m;-m)\frac{t^l}{l!}\right)$$

$$L.H.S = \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}F_{n-l,\lambda,p}^{(k)}(y)(x)^{(m)}S_{2,\lambda}(l,m;-m)\right)\frac{t^n}{n!}. \tag{3.11}$$

From (3.11) , we obtain the following theorem.

Theorem 3.3. Let $n \geq 0$ and $k \in \mathbb{Z}$. Then we have

$$F_{n,\lambda,p}^{(k)}(x;y) = \sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}F_{n-l,\lambda,p}^{(k)}(y)(x)^{(m)}S_{2,\lambda}(l,m;-m). \tag{3.12}$$

From (3.5), we observe that

$$\sum_{n=0}^{\infty} F_{n,\lambda,p}^{(k)}(y)\frac{t^n}{n!} = \frac{u_{k,\lambda}(\log_\lambda(1+t)|p)}{t(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1))}$$

$$= \frac{1}{t(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1))}\sum_{m=0}^{\infty}\frac{p(m+1)(1)_{m+1,\lambda}m!}{(m+1)^k m!}(\log_\lambda(1+t))^{m+1}$$

$$= \frac{\log_\lambda(1+t)}{t(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1))}\sum_{m=0}^{\infty}\frac{p(m+1)(1)_{m+1,\lambda}m!}{(m+1)^k m!}(\log_\lambda(1+t))^{m}$$

$$= \frac{\log_\lambda(1+t)}{t}\frac{1}{1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)}\sum_{m=0}^{\infty}\frac{p(m+1)(1)_{m+1,\lambda}m!}{(m+1)^k}\sum_{l=m}^{\infty}S_{1,\lambda}(l,m)\frac{t^l}{l!}$$

$$= \left(\sum_{s=0}^{\infty}D_{s,\lambda}\frac{t^s}{s!}\right)\left(\sum_{a=0}^{\infty}F_{a,\lambda}(y)\frac{t^a}{a!}\right)\left(\sum_{l=0}^{\infty}\sum_{m=0}^{n}\frac{p(m+1)(1)_{m+1,\lambda}m!}{(m+1)^k}S_{1,\lambda}(l,m)\frac{t^l}{l!}\right)$$

$$= \left(\sum_{b=0}^{\infty}\sum_{a=0}^{b}\binom{b}{a}D_{b-a,\lambda}F_{a,\lambda}(y)\frac{t^b}{b!}\right)\left(\sum_{l=0}^{\infty}\sum_{m=0}^{n}\frac{p(m+1)(1)_{m+1,\lambda}m!}{(m+1)^k}S_{1,\lambda}(l,m)\frac{t^l}{l!}\right)$$

$$L.H.S = \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{a=0}^{n-l}\sum_{m=0}^{l}\binom{n}{l}D_{n-l-a,\lambda}F_{a,\lambda}(y)\frac{p(m+1)(1)_{m+1,\lambda}m!}{(m+1)^k}S_{1,\lambda}(l,m)\right)\frac{t^n}{n!}. \tag{3.13}$$

By comparing coefficients on both sides of (3.13), we obtain the following theorem.

Theorem 3.4. Let $n \geq 0$ and $k \in \mathbb{Z}$. Then we have

$$F_{n,\lambda,p}^{(k)}(y) = \sum_{l=0}^{n}\sum_{a=0}^{n-l}\sum_{m=0}^{l}\binom{n}{l}D_{n-l-a,\lambda}F_{a,\lambda}(y)\frac{p(m+1)(1)_{m+1,\lambda}m!}{(m+1)^k}S_{1,\lambda}(l,m). \tag{3.14}$$

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By applying the difference operator $\triangle_\lambda$ to both sides of equation (3.5), we get

$$\triangle_\lambda\left(\sum_{n=0}^{\infty}F_{n,\lambda,p}^{(k)}(x;y)\frac{t^n}{n!}\right)=\triangle_\lambda\left(\frac{u_{k,\lambda}(\log_\lambda(1+t)|p)}{t(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)}(1+\lambda t)^{\frac{x}{\lambda}}\right)$$

and then we have

$$\begin{aligned}\sum_{n=0}^{\infty}\triangle_\lambda F_{n,\lambda,p}^{(k)}(y)\frac{t^n}{n!}&=\frac{u_{k,\lambda}(\log_\lambda(1+t)|p)}{t(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)}\triangle_\lambda e_\lambda^{x}(t)\\&=\frac{u_{k,\lambda}(\log_\lambda(1+t)|p)}{t(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)}e_\lambda^{x}(t)t\\&=\sum_{n=0}^{\infty}F_{n,\lambda,p}^{(k)}(x;y)\frac{t^{n+1}}{n!}.\end{aligned}\tag{3.15}$$

Therefore, by (3.15), we obtain the following theorem.

Theorem 3.5. Let $n\geq 0$ and $k\in\mathbb{Z}$. Then we have

$$F_{n,\lambda,p}^{(k)}(x;y)=nF_{n-1,\lambda,p}^{(k)}(x;y).$$

By applying the derivative operator $\frac{\delta}{\delta x}$ with respect to x to both sides of equation (3.5), we have

$$\frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty}F_{n,\lambda,p}^{(k)}(x;y)\frac{t^n}{n!}\right)=\frac{\partial}{\partial x}\left(\frac{u_{k,\lambda}(\log_\lambda(1+t)|p)}{t(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)}(1+\lambda t)^{\frac{x}{\lambda}}\right)$$

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{\partial}{\partial x}F_{n,\lambda,p}^{(k)}(x;y)\frac{t^n}{n!}\\&=\frac{u_{k,\lambda}(\log_\lambda(1+t)|p)}{t(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)}\frac{\partial}{\partial x}(1+\lambda t)^{\frac{x}{\lambda}}\\&=\frac{u_{k,\lambda}(\log_\lambda(1+t)|p)}{t(1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t)^{\frac{1}{\lambda}}\\&=\left(\sum_{n=0}^{\infty}F_{n,\lambda,p}^{(k)}(x;y)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}(1)_{m,\lambda}\frac{t^m}{m!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}F_{n-m,\lambda,p}^{(k)}(x;y)(1)_{m,\lambda}\right)\frac{t^n}{n!}.\end{aligned}\tag{3.16}$$

By, comparing the coefficients of t^n on both sides, we get the following theorem.

Theorem 3.6. Let $n\geq 0$ and $k\in\mathbb{Z}$. Then we have

$$\frac{\partial}{\partial x}F_{n,\lambda,p}^{(k)}(x;y)=\sum_{m=0}^{n}\binom{n}{m}F_{n-m,\lambda,p}^{(k)}(x;y)(1)_{m,\lambda}.$$

4. Conclusions

Recently Kim-Kim [] considered the degenerate poly-Genocchi polynomials by making use of the modified degenerate polyexponential functions and Kim et al. [] introduced the two variable degenerate Fubini polynomials. By using these functions and polynomials, we defined the type 2 degenerate poly-Fubini polynomials and obtained some identities of the degenerate poly-Fubini numbers. In the final section, we defined the degenerate unipoly-Fubini polynomials by the modified degenerate poly-exponential functions and obtained some properties of the degenerate unipoly-Fubini numbers and polynomials and given multifarious properties including derivative and integral properties. Furthermore, we have provided a correlation between the unipoly-Fubini polynomials and the degenerate special polynomials.

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