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Stabilization and synchronization of hyper-chaotic financial system involved in positive interest rate

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Abstract: In real financial market, the delayed market feedback and the delayed effect of government macro-control are inevitable. And both the delay of market feedback and the delay of macro-control effect bring about the mathematical difficulties in studying stabilization and synchronization of the hyper-chaotic financial system. However, employing Lyapunov function method, differential mean value theorem, suitable bounded hypotheses and pulse control technology results in the globally asymptotical stabilization and synchronization criteria. It is the first paper to drive the stabilization and synchronization criteria under the assumptions of the double delays. Finally, numerical examples illuminate the effectiveness of the proposed methods.

Keywords: delayed feedback financial system; asymptotical stability; Lyapunov function; synchronization; impulse

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1. Introduction

The complexity of economic systems often leads to unpredictable dynamic behavior. The periodic economic crisis reminds me that it is necessary to study and control the instability and chaos of financial system. Recently, many scholars have investigated the stability and synchronization of a class of chaotic finical system that is composed of the production sub-block, currency sub-block, securities sub-block, and labor sub-block (see, e.g. [1-9]). In many related literature([1-9]), the following finical system was investigated, which is composed of the production sub-block, currency sub-block, securities sub-block, and labor sub-block,

$$\begin{cases} \dot{x} =z + (y - a)x \\ \dot{y} =1 - by - x^2 \\ \dot{z} = - x - cz, \end{cases} \tag{1.1}$$

where x represents the interest rate, y represents the investment demand, z represents the price index, a represents savings, b represents the unit investment cost, and c represents the elasticity of commodity demand.

Based on the improved financial chaos system model, the authors of [10] took the global economic crisis caused by American subprime mortgage crisis in 2007 as the background of the

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market model, adding a state variable (average profit rate) to the system (1.1), and proposed a new kind of hyper-chaotic financial system :

$$\begin{cases} \dot{x} =z+(y-a)x+u \\ \dot{y} =1-by-x^2 \\ \dot{z} =-x-cz \\ \dot{u} =-dxy-ku, \end{cases} \tag{1.2}$$

which is hyper-chaotic. Indeed, set $a=0.8989, b=0.1989, c=1.499, d=0.2001, k=0.1699$, I can compute that there are three equilibrium points $P_0(0,5.0277,0,0)$, P_1(1.6590,-8.8101, -1.1067,17.2141) and P_2(-1.6590,-8.8101, 1.1067,17.2141) for the system (1.2). And three equilibrium points all are unstable saddle points.

On the other hand, both the market feedback delay and the delayed effect of the government's macroeconomic control are inevitable, but seldom papers involved in the above-mentioned multiple delays, which inspires me to write this paper. For the first time, the stabilization and synchronization are simultaneously considered for the hyper-chaotic financial system with the above double delays. And the newly-obtained results will provide some theoretical guidance for the actual financial market.

The rest of this paper is arranged as follows, in the second section, some system descriptions and preparations are presented. And the main results on stabilization and synchronization are given in the third section. Besides, in the fourth section, two numerical examples are proposed to illuminate the effectiveness of the new criteria. In final section, some conclusions are derived to elaborate further the main purpose and significance of this article.

2. System descriptions and preparations

From [10], there exist three equilibrium points for the system (1.2):

$$P_0(0,\frac{1}{b},0,0),\, P_{1,2}(\pm\theta,\frac{k+ack}{c(k-d)},\mp\frac{\theta}{c},\frac{d\theta(1+ac)}{cd-ck}) \text{ with } \theta=\sqrt{\frac{kb+abck}{c(d-k)}+1}. \tag{2.1}$$

Since the delayed feedback is a common phenomenon in the real market,I get the following delayed feedback model for the system (1.2),

$$\begin{cases} \dot{x} =z+(y-a)x+u+k_1(x-x(t-\tau_1(t))) \\ \dot{y} =1-by-x^2+k_2(y-y(t-\tau_2(t))) \\ \dot{z} =-x-cz+k_3(z-z(t-\tau_3(t))) \\ \dot{u} =-dxy-ku+k_4(u-u(t-\tau_4(t))). \end{cases} \tag{2.2}$$

Let $X=(x_1,x_2,x_3,x_4)^T$, and

$$\begin{cases} x_1 =x-\theta \\ x_2 =y-\frac{k+ack}{c(k-d)} \\ x_3 =z+\frac{\theta}{c} \\ x_4 =u-\frac{d\theta(1+ac)}{cd-ck}, \end{cases} \tag{2.3}$$

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then the equilibrium point $P_1(\theta, \frac{k+ack}{c(k-d)}, -\frac{\theta}{c}, \frac{d\theta(1+ac)}{cd-ck})$ with the positive interest rate $\theta > 0$ of the system (1.2) corresponds to the zero solution of the following system

$$\begin{cases} \dot{x_1} =x_3 - \frac{\theta}{c} + (x_2 + \frac{k+ack}{c(k-d)} - a)(x_1+\theta) + x_4 + \frac{d\theta(1+ac)}{cd-ck} + k_1(x - x(t-\tau_1(t))) \\ \dot{x_2} =1 - b(x_2 + \frac{k+ack}{c(k-d)}) - (x_1+\theta)^2 + k_2(y - y(t-\tau_2(t))) \\ \dot{x_3} =-(x_1+\theta) - c(x_3 - \frac{\theta}{c}) + k_3(z - z(t-\tau_3(t))) \\ \dot{x_4} =-d(x_1+\theta)(x_2 + \frac{k+ack}{c(k-d)}) - k(x_4 + \frac{d\theta(1+ac)}{cd-ck}) + k_4(u - u(t-\tau_4(t))), \end{cases}$$

or

$$\begin{cases} \dot{x_1} =(\frac{k+ack}{c(k-d)} - a)x_1 + \theta x_2 + x_3 + x_4 + x_1x_2 + k_1(x - x(t-\tau_1(t))) \\ \dot{x_2} =-2\theta x_1 - bx_2 - x_1^2 + k_2(y - y(t-\tau_2(t))) \\ \dot{x_3} =-x_1 - cx_3 + k_3(z - z(t-\tau_3(t))) \\ \dot{x_4} =-\frac{d(k+ack)}{c(k-d)}x_1 - d\theta x_2 - kx_4 - dx_1x_2 + k_4(u - u(t-\tau_4(t))), \\ X(s) =\phi(s), \quad s \in [-\tau, 0], \end{cases} \tag{2.4}$$

where $X = (x_1, x_2, x_3, x_4)^T$. Furthermore, the system (2.4) can be rewritten as the following system in form of vector-matrix,

$$\begin{cases} \dot{X}(t) =AX(t) + f(X(t)) + K(X(t) - X(t-\tau(t))), \quad t \geqslant 0, \\ X(s) =\phi(s), \quad s \in [-\tau, 0], \end{cases} \tag{2.5}$$

where

$$A = \begin{pmatrix} (\frac{k+ack}{c(k-d)} - a) & \theta & 1 & 1 \\ -2\theta & -b & 0 & 0 \\ -1 & 0 & -c & 0 \\ -\frac{d(k+ack)}{c(k-d)} & -d\theta & 0 & -k \end{pmatrix}, \quad f(X) = \begin{pmatrix} x_1x_2 \\ -x_1^2 \\ 0 \\ -dx_1x_2 \end{pmatrix}, \quad K = \begin{pmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{pmatrix}. \tag{2.6}$$

Remark 1. Obviously, $f(X)$ is not Lipschitz continuous.

Under the delayed impulse control on the system (2.5), I can get

$$\begin{cases} \dot{X}(t) =AX(t) + f(X(t)) + K(X(t) - X(t-\tau(t))), \quad t \geqslant 0, t \neq t_k, \\ X(t_k^+) - X(t_k^-) = D_kX(t_k - \rho_k), \quad k = 1, 2, \cdots \\ X(s) =\phi(s), \quad s \in [-\tau, 0], \end{cases} \tag{2.7}$$

or

$$\begin{cases} \dot{x} =z + (y-a)x + u + k_1(x - x(t-\tau_1(t))), \quad t \geqslant 0, t \neq t_k, \\ \dot{y} =1 - by - x^2 + k_2(y - y(t-\tau_2(t))), \quad t \geqslant 0, t \neq t_k, \\ \dot{z} =-x - cz + k_3(z - z(t-\tau_3(t))), \quad t \geqslant 0, t \neq t_k, \\ \dot{u} =-dxy - ku + k_4(u - u(t-\tau_4(t))), \quad t \geqslant 0, t \neq t_k, \\ X(t_k^+) - X(t_k^-) = D_kX(t_k - \rho_k), \quad k = 1, 2, \cdots \\ X(s) =\phi(s), \quad s \in [-\tau, 0], \end{cases} \tag{2.8}$$

where X is defined in (2.3). That is, the zero solution of the system of (2.7) corresponds to the equilibrium point $P_1(\theta, \frac{k+ack}{c(k-d)}, -\frac{\theta}{c}, \frac{d\theta(1+ac)}{cd-ck})$ of the hyper-chaotic financial system (2.8).

Construct the following response system for the drive system (2.7),

$$\begin{cases} \dot{Y}(t) = AY(t) + f(Y(t)) + K(Y(t) - Y(t-\tau(t))), \quad t \geqslant 0, t \neq t_k, \\ Y(t_k^+) - Y(t_k^-) = D_k Y(t_k - \rho_k), \quad k = 1, 2, \cdots \\ Y(s) = \psi(s), \quad s \in [-\tau, 0], \end{cases} \tag{2.9}$$

and the error system of pulse synchronization is given as follows,

$$\begin{cases} \dot{e}(t) = Ae(t) + \check{f}(e(t)) + K(e(t) - e(t-\tau(t))), \quad t \geqslant 0, t \neq t_k, \\ e(t_k^+) - e(t_k^-) = D_k e(t_k - \rho_k), \quad k = 1, 2, \cdots \\ e(s) = \psi(s) - \phi(s), \quad s \in [-\tau, 0], \end{cases} \tag{2.10}$$

where $e = (e_1, e_2, e_3)^T = (Y_1 - X_1, Y_2 - X_2, Y_3 - X_3)^T$ is the error of synchronization, and

$$\check{f}(e) = f(Y) - f(X) = \begin{pmatrix} y_1 y_2 - x_1 x_2 \\ x_1^2 - y_1^2 \\ 0 \\ d x_1 x_2 - d y_1 y_2 \end{pmatrix} \tag{2.11}$$

Lemma 2.1([11, Theorem 3.2]). Under the assumptions of Theorem 3.1, the following fuzzy system (2.17) is bounded under the meaning of L^∞:

$$\begin{cases} \dot{X}(t) = \sum_{r=1}^{n} \varrho_r(\hat{\omega}(t)) H_{\sigma r} X(t) + f(X(t)) - D_\sigma X(t) + \varphi_\sigma(X(t), X(t-\tau(t))) dw(t), \quad t \geqslant 0 \\ X(s) = \xi(s), \quad s \in [-\tau, 0]. \end{cases} \tag{2.12}$$

For convenience, I employ the following notations:

- For a symmetric matrix A, I denote by $\lambda_{\max} A$ the maximum eigenvalue of A;
- For a vector $v = (v_1, v_2, v_3)^T \in R^3$, I denote by $\|v\| = \sqrt{\sum_{i=1}^{3} v_i^2}$ the norm of v;
- For a matrix A, I denote by $\|A\| = \sqrt{\lambda_{\max}(A^T A)}$ the norm of A.

3. Stabilization by impulse control

At first, I may assume in this section that $X(t_k^-) = X(t_k)$ for all $k \in \mathbb{Z}^+$.

Secondly, from Lemma 2.1, I can similarly derive the boundedness of the system (2.7) and (2.8) under mild conditions. So in this paper, I may give the following boundedness assumptions:

(H1) Suppose that time delays $\tau_i(t) \in [0, \tau]$, $i = 1, 2, 3$. Assume that there are two positive scalars M_1, M_2 such that

$$0 < M_1 \leqslant \|X(s)\|^2 \leqslant M_2, \quad \forall s \in [-\tau, +\infty). \tag{3.1}$$

If the upper limit of delays τ is appropriately small, the following continuity hypothesis is natural due to the boundedness.

(H2) For any given $\tau > 0$, there exists the corresponding positive number $c_\tau > 0$ such that

$$\left| \|X(t)\|^2 - \|X(t-\tau(t))\|^2 \right| < c_\tau. \tag{3.2}$$

In this section, I assume that time delays $\tau(t) \in [-\tau, 0]$, and $\rho_k \in [-\rho, 0]$ for all $k \in \mathbb{Z}^+$. In order to obtain the stability of the system, a certain pulse frequency is required, so I assume a smaller pulse interval as follows,

$$\sup_{k\in\mathbb{Z}^+} (t_k - t_{k-1}) < c_0, \tag{3.3}$$

where c_0 is a positive number.

Mainly inspired by some methods and ideas of my another [20], I present the following Theorem.

Theorem 3.1. Suppose that the conditions $(H1)$,$(H2)$ and (3.3) hold. And if

$$\left[\|D_k + I\| + \rho_k\|D_k\| \cdot \left(\|A\| + \sqrt{M_2(1+d^2)} + 2\|K\|\frac{\sqrt{M_2}}{\sqrt{M_1}}\right)\right] e^{\frac{c_\lambda c_0}{2}} \leqslant d_0 < 1, \quad \forall k \in \mathbb{Z}^+, \tag{3.4}$$

where d_0 is a positive scalar,

$$c_\lambda = \lambda_{\max}\left[A + A^T + 2K + r_1 I + \frac{1}{r_1}M_2(1+d^2)I + r_2K^2 + \frac{1}{r_2}\left(1 + \frac{c_\tau}{M_1}\right)I\right], \tag{3.5}$$

and I represents the identity matrix, $\rho_k < t_k - t_{k-1}$, for all $k \in \mathbb{Z}^+$, then the zero solution of the system (2.7) is globally asymptotically stable, and the equilibrium solution P_1 with the positive interest rate θ of the system (2.8) is globally asymptotically stable.

Proof. Consider the following Lyapunov function

$$V(t) = \|X\|^2. \tag{3.6}$$

It follows from (H1) that

$$\|f(X)\|^2 \leqslant M_2(1+d^2)\|X\|^2. \tag{3.7}$$

Next, (H1) and (H2) yield

$$\begin{aligned}
&D^+V(t,X) = 2X^T[AX(t) + f(X(t)) + K(X(t) - X(t-\tau(t)))]\\
\leqslant &X^T[A + A^T + 2K + r_1 I + \frac{1}{r_1}M_2(1+d^2)I]X + [r_2|X|^T K^2|X| + \frac{1}{r_2}|X(t-\tau(t)))|^T|X(t-\tau(t)))|]\\
\leqslant &\lambda_{\max}\left[A + A^T + 2K + r_1 I + \frac{1}{r_1}M_2(1+d^2)I + r_2K^2 + \frac{1}{r_2}\left(1+\frac{c_\tau}{M_1}\right)I\right]V(t,X),\ t \in (t_{k-1}, t_k],\ k \in \mathbb{Z}^+.
\end{aligned} \tag{3.8}$$

Then I get

$$\|X(t)\|^2 \leqslant \|X(t_{k-1}^+)\|^2 e^{c_\lambda(t - t_{k-1})}, \quad t \in (t_{k-1}, t_k]. \tag{3.9}$$

According to the impulsive condition, I have

$$X(t_k^+) = X(t_k^-) + D_kX(t_k - \rho_k) = X(t_k) + D_kX(t_k - \rho_k)$$

So I can conclude that there exists $\eta_{ki}(i = 1,2,3)$ with $\eta_{ki} \in (t_k - \rho_k, t_k)$ such that

$$\begin{aligned}
&\|X(t_k^+)\| = \|(D_k + I)X(t_k) - \rho_k D_k\dot{X}(\eta_k)]\|\\
\leqslant &\|D_k + I\| \cdot \|X(t_k)\| + \rho_k\|D_k\| \cdot \|AX(\eta_k) + f(X(\eta_k)) + K(X(\eta_k) - X(\eta_k - \tau(\eta_k))\|\\
\leqslant &\|D_k + I\| \cdot \|X(t_k)\| + \rho_k\|D_k\|\left[\left(\|A\| + \sqrt{M_2(1+d^2)}\right)\|X(\eta_k)\| + \|K\| \cdot \|X(\eta_k) - X(\eta_k - \tau(\eta_k))\|\right]\\
\leqslant &\left[\|D_k + I\| + \rho_k\|D_k\| \cdot \left(\|A\| + \sqrt{M_2(1+d^2)} + \|K\|\frac{2\sqrt{M_2}}{\sqrt{M_1}}\right)\right] e^{\frac{c_\lambda c_0}{2}}\|X(t_{k-1}^+)\|,
\end{aligned}$$

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which together with (3.4) implies that $\{\|X(t_k^+)\|\}_{k=1}^{\infty}$ is a convergent sequence with its limit being zero, where $X(\eta_k) = (x_1(\eta_{k1}), x_2(\eta_{k2}), x_3(\eta_{k3}))^T$. For any given $t \in (t_k, t_{k+1}]$, I get by (3.9)

$$0 \leqslant \|X(t)\| \leqslant \|X(t_{k-1}^+)\| e^{\frac{c_\lambda}{2}(t-t_{k-1})} \leqslant e^{\frac{c_\lambda}{2}c_0}\|X(t_{k-1}^+)\| \to 0, \ \ k \to \infty,$$

which completes the proof. □

Remark 2. The condition $\rho_k < t_k - t_{k-1}$ implies that every macro-control measure (pulse) of the government should be effective enough to see the pulse effect within each pulse interval. Besides, the condition $\sup\limits_{k\in\mathbb{Z}^+}(t_k - t_{k-1}) < c_0$ guarantees pulse (Macro-control) of a certain frequency if $c_0 > 0$ is appropriate small. No matter how complex and chaos the financial system is, high-frequency active macro-control is conducive to the global asymptotical stability of the economic system.

To derive the synchronization criterion, I may consider the following boundedness assumptions:
(H3) There are two positive scalars N_1, N_2 such that

$$0 < N_1 \leqslant \|e(s)\|^2 \leqslant N_2, \quad \forall s \in [-\tau, +\infty).$$

(H4) For any given $\tau > 0$, there exists the corresponding positive number $d_\tau > 0$ such that

$$\Big| \|e(t)\|^2 - \|e(t-\tau(t))\|^2 \Big| < d_\tau.$$

Theorem 3.2. Assume that X, Y satisfy the boundedness conditions (H1) and (H2), and the error variable e satisfies the boundedness assumptions (H3)-(H4). If, in addition, the following condition holds,

$$\left[\|D_k + I\| + \rho_k\|D_k\| \cdot \left(\|A\| + \sqrt{4M_2(2+d^2)} + \|K\|\frac{2\sqrt{N_2}}{\sqrt{N_1}}\right)\right] e^{\frac{d_\lambda c_0}{2}} \leqslant d_0 < 1, \tag{3.10}$$

then the system (2.9) can be globally exponentially synchronized onto the system (2.7), where

$$d_\lambda = \lambda_{\max}\left[A + A^T + 2K + r_1 I + \frac{4}{r_1}N_2(2+d^2)I + r_2K^2 + \frac{1}{r_2}\left(1 + \frac{d_\tau}{N_1}\right)I\right]. \tag{3.11}$$

Proof. Consider the following Lyapunov function

$$V(t) = \|e\|^2. \tag{3.12}$$

Similarly, I can conclude from the assumptions of Theorem 3.2 that

$$\begin{aligned} \|\check{f}(e)\|^2 \leqslant& 2(1+d^2)[\|X\|^2 + \|Y\|^2]\|e\|^2 + 2(\|X\|^2 + \|Y\|^2)\|e\|^2 \\ \leqslant& 4(1+d^2)M_2\|e\|^2 + 4M_2\|e\|^2 = 4(2+d^2)M_2\|e\|^2, \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} &D^+V(t,e) = 2e^T(t)[Ae(t) + \check{f}(e(t)) + K(e(t) - e(t-\tau(t)))] \\ \leqslant& e^T\left[A + A^T + 2K + r_1 I + \frac{4}{r_1}N_2(2+d^2)I + r_2K^2 + \frac{1}{r_2}\left(1+\frac{d_\tau}{N_1}\right)I\right]e \\ \leqslant& \lambda_{\max}\left[A + A^T + 2K + r_1 I + \frac{4}{r_1}N_2(2+d^2)I + r_2K^2 + \frac{1}{r_2}\left(1+\frac{d_\tau}{N_1}\right)I\right]V(t,e), \, t \in (t_{k-1}, t_k], \, k \in \mathbb{Z}^+. \end{aligned} \tag{3.14}$$

Then I get

$$\|e(t)\|^2 \leqslant \|e(t_{k-1}^+)\|^2 e^{d_\lambda(t-t_{k-1})} \Rightarrow \|X(t)\| \leqslant \|X(t_{k-1}^+)\| e^{\frac{d_\lambda}{2}(t-t_{k-1})}, \quad t \in (t_{k-1}, t_k]. \tag{3.15}$$

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According to the impulsive condition, I have

$$e(t_k^+) = e(t_k^-) + D_k e(t_k - \rho_k) = e(t_k) + D_k e(t_k - \rho_k)$$

So I deduce that there exists $\eta_{ki}(i = 1, 2, 3)$ with $\eta_{ki} \in (t_k - \rho_k, t_k)$ such that

$$\begin{aligned}
&\|e(t_k^+)\| \leqslant \|D_k + I\| \cdot \|e(t_k)\| + \rho_k \|D_k\| \cdot \|Ae(\eta_k) + \check{f}(e(\eta_k)) + K(e(\eta_k) - e(\eta_k - \tau(\eta_k))\| \\
&\leqslant \|D_k + I\| \cdot \|e(t_k)\| + \rho_k \|D_k\| \Big[\Big(\|A\| + \sqrt{4M_2(2+d^2)} \Big) \|e(\eta_k)\| + \|K\| \cdot \|e(\eta_k) - e(\eta_k - \tau(\eta_k))\| \Big] \\
&\leqslant \|D_k + I\| \cdot \|e(t_{k-1}^+)\| e^{\frac{d_\lambda}{2}(t - t_{k-1})} + \rho_k \|D_k\| \Big(\|A\| + \sqrt{4M_2(2+d^2)} + \|K\| \frac{2\sqrt{N_2}}{\sqrt{N_1}} \Big) \|e(t_{k-1}^+)\| e^{\frac{d_\lambda}{2}(t - t_{k-1})} \\
&\leqslant \Big[\|D_k + I\| + \rho_k \|D_k\| \cdot \Big(\|A\| + \sqrt{4M_2(2+d^2)} + \|K\| \frac{2\sqrt{N_2}}{\sqrt{N_1}} \Big) \Big] e^{\frac{d_\lambda c_0}{2}} \|e(t_{k-1}^+)\|,
\end{aligned}$$

which together with (3.10) implies that $\{\|e(t_k^+)\|\}_{k=1}^{\infty}$ is a convergent sequence with its limit being zero, where $e(\eta_k) = (x_1(\eta_{k1}), x_2(\eta_{k2}), x_3(\eta_{k3}))^T$. For any given $t \in (t_k, t_{k+1}]$, I get by (3.15)

$$0 \leqslant \|e(t)\| \leqslant \|e(t_{k-1}^+)\| e^{\frac{d_\lambda}{2}(t - t_{k-1})} \leqslant e^{\frac{d_\lambda}{2} c_0} \|e(t_{k-1}^+)\| \to 0, \quad k \to \infty,$$

which completes the proof. □

Remark 3. It is the first paper to derive the synchronization criterion for the systems (2.9) and (2.7) under the assumptions of the double delays.

4. Numerical examples

Example 4.1. Consider the system (2.7) or (2.8) with the following data:

$$M_1 = 1, M_2 = 8, \tau = 0.5, c_\tau = 0.3, \rho_k \equiv 0.01, D_k \equiv -\frac{4}{5} I, r_1 = r_2 = r_3 = r_4 = 1, \tag{4.1}$$

$$K = \begin{pmatrix} 0.05 & 0 & 0 & 0 \\ 0 & 0.07 & 0 & 0 \\ 0 & 0 & 0.03 & 0 \\ 0 & 0 & 0 & 0.033 \end{pmatrix}. \tag{4.2}$$

Below I may consider two different data for comparisons:
Case 1.

$$a = 0.3, b = 0.2, c = 0.5, d = 0.11, k = 0.81, c_0 = 0.15, \tag{4.3a}$$

then direct computation yields that $\theta = \sqrt{\frac{kb+abck}{c(d-k)} + 1} = 0.6839$,

$$A = \begin{pmatrix} 2.3614 & 0.6839 & 1.0000 & 1.0000 \\ -1.3678 & -0.2000 & 0 & 0 \\ -1.0000 & 0 & -0.5000 & 0 \\ -0.2928 & -0.0752 & 0 & -0.8100 \end{pmatrix}.$$

Further computation derives that $\|A\| = 3.2952$, $\|K\| = 0.07$, $c_\lambda = 18.0898$. Let $d_0 = 0.99$ and direct computation results in

$$\Big[\|D_k + I\| + \rho_k \|D_k\| \cdot \Big(\|A\| + \sqrt{M_2(1+d^2)} + 2\|K\| \frac{\sqrt{M_2}}{\sqrt{M_1}} \Big) \Big] e^{\frac{c_\lambda c_0}{2}} = 0.8021 \leqslant 0.99 = d_0 < 1, \quad \forall k \in \mathbb{Z}^+,$$

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which implies that the condition (3.4) is satisfied, and the equilibrium solution P_1 with the positive interest rate $\theta = 68.39\%$ of the system (2.8) is globally asymptotically stable due to Theorem 3.1.

Case 2.

$$a = 0.9, b = 0.2, c = 0.5, d = 0.11, k = 0.81, c_0 = 0.17, \tag{4.3b}$$

then direct computation yields that $\theta = \sqrt{\frac{kb+abck}{c(d-k)} + 1} = 0.5735$,

$$A = \begin{pmatrix} 2.4557 & 0.5735 & 1.0000 & 1.0000 \\ -1.1469 & -0.2000 & 0 & 0 \\ -1.0000 & 0 & -0.5000 & 0 \\ -0.3691 & -0.0631 & 0 & -0.8100 \end{pmatrix}.$$

Further computation derives that $\|A\| = 3.2922$, $\|K\| = 0.07$, $c_\lambda = 18.2325$. Let $d_0 = 0.99$ and direct computation results in

$$\left[\|D_k + I\| + \rho_k\|D_k\| \cdot \left(\|A\| + \sqrt{M_2(1+d^2)} + 2\|K\|\frac{\sqrt{M_2}}{\sqrt{M_1}}\right)\right] e^{\frac{c_\lambda c_0}{2}} = 0.9729 \leqslant 0.9 = d_0 < 1, \quad \forall k \in \mathbb{Z}^+,$$

which implies that the condition (3.4) is satisfied, and the equilibrium solution P_1 with the positive interest rate $\theta = 57.35\%$ of the system (2.8) is globally asymptotically stable due to Theorem 3.1.

Table 1. Comparisons of positive interest rate and pulse interval

	interest rate θ	pulse interval c_0
Case 1	68.39%	0.15
Case 2	57.35%	0.17

Remark 4. Table 1 illuminates that when the system reach stable, the higher the interest rate, the smaller the pulse interval. This shows, in order to reach a balance of higher interest rates in the financial market, the government should speed up the pace of macro-control of the economy.

Example 4.2. Consider the following data for the systems (2.7),(2.9) and (2.10):

$$N_1 = 0.1,\ N_2 = 0.5, M_1 = 1, M_2 = 8, \tau = 0.5, c_\tau = 0.3, \rho_k \equiv 0.001, D_k \equiv -\frac{4}{5}I, r_1 = r_2 = r_3 = r_4 = 1,$$
$$a = 0.9, b = 0.2, c = 0.5, d = 0.11, k = 0.81, c_0 = 0.01,$$

$$K = \begin{pmatrix} 0.05 & 0 & 0 & 0 \\ 0 & 0.07 & 0 & 0 \\ 0 & 0 & 0.03 & 0 \\ 0 & 0 & 0 & 0.033 \end{pmatrix}.$$

and then

$$A = \begin{pmatrix} 2.3614 & 0.6839 & 1.0000 & 1.0000 \\ -1.3678 & -0.2000 & 0 & 0 \\ -1.0000 & 0 & -0.5000 & 0 \\ -0.2928 & -0.0752 & 0 & -0.8100 \end{pmatrix}.$$

Further computations derive

$$\left[\|D_k + I\| + \rho_k\|D_k\| \cdot \left(\|A\| + \sqrt{4M_2(2+d^2)} + \|K\|\frac{2\sqrt{N_2}}{\sqrt{N_1}}\right)\right] e^{\frac{d_\lambda c_0}{2}} = 0.9055 \leqslant 0.99 = d_0 < 1,$$

which implies that the condition (3.10) holds. And hence, Theorem 3.2 tells me that the system (2.9) can be globally exponentially synchronized onto the system (2.7).

Remark 5. Example 4.2 shows the feasibility of Theorem 3.2.

5. Conclusions

Interest rates are always positive at most of countries in the world when the economies are in balance. So my Theorem 3.1 and Theorem 3.2 illustrate theoretical guidance significance for the actual financial market. In particular, Theorem 3.1 shows that positive and correct macroeconomic control measures with a certain frequency are conducive to market balance and high positive interest rates. Finally, two numerical examples shows the effectiveness and feasibility of stabilization and synchronization criteria.

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