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Existence and local stability of stationary solutions for nonlinear Gilpin Ayala competition model with Dirichlet boundary value

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Abstract: In this paper, the existence of two nontrivial stationary solutions for the nonlinear Gilpin Ayala two species competition model is given by using the mountain pass lemma, and the local stability criterion of the trivial solution is given by using Lyapunov function method. Based on the local stability criterion, we give some suggestions on how to avoid the population extinction. This is, when the population is on the verge of extinction, we should try our best to avoid the diffusion behavior and reduce the diffusion coefficient, otherwise the species are easy to go extinct. Numerical example shows the effectiveness of the proposed method.

Keywords: Gilpin Ayala competition model; Lyapunov function; Mountain Pass Lemma; Palais Smale condition; Dirichlet boundary value

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1. Introduction

In 1920, Lotka and Volterra proposed the famous population competition model ([1,2]):

$$\begin{cases} \dot{x}_1(t) = x_1(t)[b_1 - a_{11}x_1(t) - a_{12}x_2(t)], \\ \dot{x}_2(t) = x_2(t)[b_2 - a_{21}x_1(t) - a_{22}x_2(t)], \end{cases} \tag{1.1}$$

where $x_i(t)$ represents the population density of the ith population at time $t(i = 1, 2)$, $b_i > 0$ represents the birth rate of the population of the ith population, $a_{ij} > 0$ represents the competition parameter of two populations, which is recognized and cited by many scholars. Diffusion is usually considered reasonably, for example in the reference [3] and related references:

$$\begin{cases} \frac{\partial u_1}{\partial t} = d_1\Delta u_1 + u_1(t)[b_1 - a_{11}u_1(t) - a_{12}u_2(t)], \quad x \in \Omega, t > 0, \\ \frac{\partial u_2}{\partial t} = d_2\Delta u_2 + u_2(t)[b_2 - a_{21}u_1(t) - a_{22}u_2(t)], \quad x \in \Omega, t > 0, \\ \frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = 0, \quad x \in \partial\Omega, t > 0, \\ u_1(x,0) = u_0(x), \ u_2(x,0) = v_0(x), \quad x \in \Omega. \end{cases} \tag{1.2}$$

and

$$\begin{cases} \frac{\partial u}{\partial t} =\Delta[(d_1+a_{11}u+a_{12}v)u]+\mu_1 u(1-u-a_1 v), & x\in\Omega, t>0,\\ \frac{\partial v}{\partial t} =\Delta[(d_2+a_{21}u+a_{22}v)v]+\mu_2 v(1-v-a_2 u), & x\in\Omega, t>0,\\ \frac{\partial u}{\partial \nu} =\frac{\partial v}{\partial \nu}=0, & x\in\partial\Omega, t>0,\\ u(x,0) =u_0(x),\ v(x,0)=v_0(x), & x\in\Omega. \end{cases} \tag{1.3}$$

In 2017, Yuanyuan Liu and Youshan Tao studied the linear competition model of cross diffusion of two populations under Neumann boundary conditions ([4]):

$$\begin{cases} \frac{\partial u}{\partial t} =\Delta[(d_1+a_{12}v)u]+\mu_1 u(1-u-a_1 v), & x\in\Omega, t>0,\\ 0 =\Delta v+\mu_2 v(1-v-a_2 u), & x\in\Omega, t>0,\\ \frac{\partial u}{\partial \nu} =\frac{\partial v}{\partial \nu}=0, & x\in\partial\Omega, t>0,\\ u(x,0) =u_0(x), & x\in\Omega. \end{cases} \tag{1.4}$$

In 1973, Gilpin and Ayala found that the linear competition model was not consistent with the experimental results ([5]). Through accurate data analysis, they proposed a nonlinear competition model of two populations:

$$\begin{cases} \dot{x}_1(t) =x_1(t)[b_1-a_{11}x_1^{\theta_1}(t)-a_{12}x_2(t)],\\ \dot{x}_2(t) =x_2(t)[b_2-a_{21}x_1(t)-a_{22}x_2^{\theta_2}(t)], \end{cases} \tag{1.5}$$

where θ_1, θ_2 represents the nonlinear density constraint parameter. As pointed out in [6-9], when the parameter θ_i is much less than 1, the nonlinear density constrained model can well simulate the population ecology of Drosophila melanogaster, and the diffusion type Gilpin Ayala competition model under Neumann boundary value condition has also been studied by scholars:

$$\begin{cases} \frac{\partial u_1}{\partial t} =d_1\Delta u_1+u_1(b_1-a_{11}u_1^{\theta_1}-a_{12}u_2),\\ \frac{\partial u_2}{\partial t} =d_2\Delta u_2+u_2(b_2-a_{22}u^{\theta_2}-a_{21}u_1),\\ \frac{\partial u_1}{\partial \nu} =\frac{\partial u_2}{\partial \nu}=0, \quad x\in\partial\Omega, t>0, \end{cases} \tag{1.7}$$

where μ_1, μ_2, a, b, c and d all are positive numbers.

It is noted that the diffusion ecosystem with Neumann boundary value has been widely studied ([3,4,8] and related references), but the diffusive ecosystem under Dirichlet boundary value is rarely studied. In fact, the Drichlet boundary value diffusion ecosystem can better reflect the actual population ecology. Therefore, this paper will study the dynamic behavior of nonlinear Gilpin Ayala competition model with Dirichlet zero boundary value, I will give the existence of two nonzero steady-state solutions for this model. Recently, the author has studied the double positive solutions of the following delay feedback Gilpin-Ayala competition model in [20], where the global stability of the positive solution was presented in [20].

$$\begin{cases} \frac{\partial u_1}{\partial t} =d_1\Delta u_1+u_1(b_1-a_{11}u_1^{\theta_1}-a_{12}u_2)+k_1(r(t))[u_1-u_1(t-\tau_1(t),x)]+\chi_1, & t\geqslant 0,\ x\in\Omega,\\ \frac{\partial u_2}{\partial t} =d_2\Delta u_2+u_2(b_2-a_{21}u_1-a_{22}u_2^{\theta_2})+k_2(r(t))[u_2-u_2(t-\tau_2(t),x)]+\chi_2, & t\geqslant 0,\ x\in\Omega,\\ u_1(t,x)=u_2(t,x)=0, & t\geqslant 0,\ x\in\partial\Omega, \end{cases} \tag{1.8}$$

So, in this paper, I only study the stability of zero solution, avoiding duplication with another article of mine

This paper, I denote by λ_1 the first positive eigenvalue of the Laplace operator $-\Delta$ in $H_0^1(\Omega)$. Denote by $\|u\| = \sqrt{\int_\Omega |\nabla u(x)|^2 dx}$ the norm of Sobolev space $H_0^1(\Omega)$.

2. Preparation

Consider the nonlinear Gilpin-Ayala competition model under Dirichlet boundary value:

$$\begin{cases} \dfrac{\partial u_1}{\partial t} =d_1\Delta u_1 + u_1(b_1 - a_{11}u_1^{\theta_1} - a_{12}u_2), & t \geqslant 0,\ x \in \Omega, \\ \dfrac{\partial u_2}{\partial t} =d_2\Delta u_2 + u_2(b_2 - a_{21}u_1 - a_{22}u_2^{\theta_2}), & t \geqslant 0,\ x \in \Omega, \\ u_1(t,x) = u_2(t,x) = 0, \quad t \geqslant 0,\ x \in \partial\Omega, \\ u_1(0,x) = \xi_1(x), \quad u_2(0,x) = \xi_2(x), \end{cases} \tag{2.1}$$

where Ω is a domain in $\mathbb{R}^3$ with the smooth boundary $\partial\Omega$.

Remark 1. Here, we assume $\Omega \subset \mathbb{R}^3$. And if two species live in two dimensional plane, we can assume $u_i(t,x) = u_i(t,x_1,x_2,x_3) = u_i(t,x_1,x_2,\cdot)$, independent of the third dimension, where $x = (x_1,x_2,x_3)^T \in \Omega$.

Besides, I need Mountain Pass Lemma as follows ([12]).

Lemma 2.1 (Mountain Pass Lemma without the (PS) condition). Let X is a Banach space, $\Psi \in C^1(X,\mathbb{R})$, satisfying $\Psi(0) = 0$, and there exists $\rho > 0$ such that $\Psi|_{\partial B_\rho(0)} \geqslant \alpha > 0$. Besides, there is $e \in X \setminus \overline{B_\rho(0)}$ such that $\Psi(e) \leqslant 0$. Let Γ be the set of all paths connecting 0 and e. That is,

$$\Gamma = \{\psi \in C([0,1], H_0^1(\Omega)) : \psi(0) = 0,\ \psi(1) = e\}.$$

Set

$$c_* = \inf_{\psi\in\Gamma} \max_{s\in[0,1]} \Psi(\psi(s)).$$

Then $c_* \geqslant \alpha$, and Ψ possesses a critical sequence on c_*.

Remark 2. Lemma 2.1 is the Mountain Pass Lemma without the (PS) condition (see,e.g. [11,12]). If, in addition, Ψ satisfies the (PS) condition, then c_* is a critical value of Ψ.

3. Main results

As pointed out in [6-9], As pointed out in [6-9], when the parameter θ_i is much less than 1, the nonlinear density constrained model can well simulate the population ecology of Drosophila melanogaster. So I assume $\theta_i \in (0,1)$, and

(H1) For each $i \in \{1,2\}$, there are positive numbers p_i, q_i such that $\frac{p_i}{q_i} - 2 = \theta_i \in (0,1)$, where p_i and q_i are a pair of Coprime odd numbers.

Theorem 3.1. Suppose (H1) holds, $b_i < d_i\lambda_1$ and $0 < \theta_i < 1, \forall\, i = 1,2$. Then the system (2.1) possesses at least three stationary solutions $(0,0)$, $(u_{1*}(x),0)$ and $(0,u_{2*}(x))$, where $u_{i*}(x) \neq 0, \forall\, i = 1,2$.

Proof. Firstly, $(0,0)$ is a trivial solution of the system (2.1).

Next, if $(u_1(x),0)$ is a stationary solution of the system (2.1),

$$\begin{cases} d_1\Delta u_1(x) + b_1u_1(x) - a_{11}u_1(x)^{1+\theta_1} = 0, & a.e.\ x \in \Omega, \\ u_1(x) = 0, \quad x \in \partial\Omega. \end{cases} \tag{3.1}$$

Similarly, if $(0,u_2(x))$ is a stationary solution of the system (2.1),

$$\begin{cases} d_2\Delta u_2(x) + b_2u_2(x) - a_{22}u_2(x)^{1+\theta_2} = 0, \\ u_2(x) = 0, \quad x \in \partial\Omega. \end{cases} \tag{3.2}$$

Obviously,

$$J(u_1) = \frac{1}{2}d_1\|u_1\|^2 - \frac{1}{2}b_1\int_\Omega u_1^2 dx + \frac{a_{11}}{2+\theta_1}\int_\Omega u_1^{2+\theta_1} dx \tag{3.3}$$

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is the functional corresponding to the equation (3.1), and $J \in C^1(H_0^1(\Omega), \mathbb{R}^1)$.

Besides, $J(0) = 0$. And Sobolev embedding theorem yields that there is $c > 0$ such that

$$\begin{aligned} J(u_1) =& \frac{1}{2} d_1 \|u_1\|^2 - \frac{1}{2} b_1 \int_\Omega u_1^2 dx + \frac{a_{11}}{2+\theta_1} \int_\Omega u_1^{2+\theta_1} dx \geqslant \frac{1}{2} d_1 \|u_1\|^2 - \frac{b_1}{2\lambda_1} \|u_1\|^2 - \frac{a_{11}}{2+\theta_1} \int_\Omega |u_1|^{2+\theta_1} dx \\ \geqslant& \frac{1}{2} d_1 (1 - \frac{b_1}{d_1 \lambda_1}) \|u_1\|^2 - \frac{c a_{11}}{2+\theta_1} \|u_1\|^{2+\theta_1}. \end{aligned} \tag{3.4}$$

Let $\rho > 0$ small enough such that

$$J|_{\partial B_\rho(0)} \geqslant \alpha, \tag{3.5}$$

where $\alpha = \frac{1}{2} d_1 (1 - \frac{b_1}{d_1 \lambda_1}) \rho^2 - \frac{c a_{11}}{2+\theta_1} \rho^{2+\theta_1} > 0$. Denote by $\varphi_1(x) > 0$ the eigenfunction of λ_1 , satisfying $\|\varphi_1\| = 1$ ([11, 17]). Then

$$J(-s\varphi_1) = \frac{1}{2} d_1 \| - s\varphi_1\|^2 - \frac{1}{2} b_1 \int_\Omega (-s\varphi_1)^2 dx + \frac{a_{11}}{2+\theta_1} \int_\Omega (-s\varphi_1)^{2+\theta_1} dx \to -\infty, \quad s \to +\infty, \tag{3.6}$$

Thereby, there is a s_0 such that $s_0 > \rho$ and $J(-s_0\varphi_1) < 0$, where $\| - s_0\varphi_1\| = s_0 > \rho$.

Let Γ be the set of all paths connecting 0 and $-s_0\varphi_1$, i.e.,

$$\Gamma = \{\psi \in C([0,1], H_0^1(\Omega)) : \psi(0) = 0, \psi(1) = -s_0\varphi_1\}. \tag{3.7}$$

Set

$$c_0 = \inf_{\psi\in\Gamma} \max_{s\in[0,1]} J(\psi(s)). \tag{3.8}$$

then

$$c_0 \geqslant \frac{1}{2} d_1 (1 - \frac{b_1}{d_1 \lambda_1}) \rho^2 - \frac{c a_{11}}{2+\theta_1} \rho^{2+\theta_1} > 0, \tag{3.9}$$

Lemma 2.1 yields that there is a sequence $\{u_{1n}\}_{n=1}^\infty \subset H_0^1(\Omega)$ such that

$$J(u_{1n}) \to c_0, \quad \text{and} \quad J'(u_{1n}) \to 0, \quad n \to \infty. \tag{3.10}$$

Below, similarly as those of [18], I will prove the sequence $\{u_{1n}\}_{n=1}^\infty \subset H_0^1(\Omega)$ satisfying (3.10) must be bounded.

In fact, (3.10) yields

$$\frac{1}{2} d_1 \|u_{1n}\|^2 - \frac{1}{2} b_1 \int_\Omega u_{1n}^2 dx + \frac{a_{11}}{2+\theta_1} \int_\Omega u_{1n}^{2+\theta_1} dx = c_0 + o(1) \tag{3.11}$$

and

$$d_1 \|u_{1n}\|^2 - b_1 \int_\Omega u_{1n}^2 dx + a_{11} \int_\Omega u_{1n}^{2+\theta_1} dx = \langle J'(u_{1n}), u_{1n} \rangle, \tag{3.12}$$

and for $\varepsilon > 0$ small enough suc that there exits a n big enough such that

$$|\langle J'(u_{1n}), u_{1n} \rangle| \leqslant \varepsilon \|u_{1n}\|. \tag{3.13}$$

So I have

$$d_1 (\frac{1}{2} - \frac{1}{2+\theta_1})(1 - \frac{b_1}{d_1 \lambda_1}) \|u_{1n}\|^2 \leqslant c_0 + o(1) - \frac{\varepsilon}{2+\theta_1} \|u_{1n}\|,$$

which means the boundedness of $\{u_{1n}\}_{n=1}^\infty$.

Now I shall prove that the bounded sequence $\{u_{1n}\}_{n=1}^\infty$ must be compact sequentially. This is only a conventional proof. However, in view of the completeness of the proof, I am willing to give the proof:

In fact, (H1) means $\frac{1}{d_1}(b_1 u_1(x) - a_{11} u_1(x)^{1+\theta_1})$ satisfies the Caratheodory condition:

$$\left| \frac{1}{d_1}(b_1 u_1(x) - a_{11} u_1(x)^{1+\theta_1}) \right| \leqslant c_1 + c_2 |u_1|^2, \quad \forall (x, u_1) \in \Omega \times \mathbb{R},$$

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where c_1, c_2 are positive numbers big enough. Due to $\Omega \subset \mathbb{R}^3$, the critical Sobolev exponent is 6, and hence the operator $\tilde{J}' : H_0^1(\Omega) \to (H_0^1(\Omega))^*$ is compact, where the functional

$$\tilde{J} = \int_\Omega \left(\frac{1}{2} b_1 u_1^2 - \frac{a_{11}}{2+\theta_1} u_1^{2+\theta_1} \right) dx.$$

Moreover,

$$\langle \tilde{J}'(u_1), \varphi \rangle = \int_\Omega \left(b_1 u_1(x)\varphi - a_{11} u_1(x)^{1+\theta_1} \varphi \right) dx, \quad \forall \varphi \in H_0^1(\Omega).$$

and then the bounded sequence $\{u_{1n}\}_{n=1}^\infty$ possesses a subsequence, say, $\{u_{1n}\}_{n=1}^\infty$, satisfying $J'(u_{1n}) \to J'(u_{1*})$ in $(H_0^1(\Omega))^*$, $n \to \infty$, where $u_{1*} \in H_0^1(\Omega)$. For any $\varphi \in H_0^1(\Omega)$,

$$\langle J'(u_{1n}) - J'(u_{1m}), \varphi \rangle = d_1 \int_\Omega (\nabla u_{1n} - \nabla u_{1m}) \cdot \nabla \varphi dx - \langle \tilde{J}'(u_{1n}) - \tilde{J}'(u_{1m}), \varphi \rangle,$$

which together with $\{u_{1n}\}_{n=1}^\infty \subset H_0^1(\Omega)$, (3.10) and the arbitrariness of φ implies

$$\begin{aligned} &\|u_{1n} - u_{1m}\|^2 \leqslant (\|J'(u_{1n})\| + \|J'(u_{1m}\|)\|u_{1n} - u_{1m}\| + \|\tilde{J}'(u_{1n}) - \tilde{J}'(u_{1m})\| \, \|u_{1n} - u_{1m}\| \\ \leqslant &(\|J'(u_{1n})\| + \|J'(u_{1m}\| + \|\tilde{J}'(u_{1n}) - \tilde{J}'(u_{1m})\|)(\|u_{1n}\| + \|u_{1m}\|) \to 0, \quad n \to \infty, m \to \infty, \end{aligned}$$

This shows that $\{u_{1n}\}_{n=1}^\infty$ is compact sequentially. And then there exists a subsequence of $\{u_{1n}\}_{n=1}^\infty$ convergent to a point in $H_0^1(\Omega)$, say, $u_{1*} \in H_0^1(\Omega)$, Due to $J(u_{1*}) = c_0 \geqslant \frac{1}{2} d_1 (1 - \frac{b_1}{d_1 \lambda_1}) \rho^2 - \frac{c a_{11}}{2+\theta_1} \rho^{2+\theta_1} > 0$, I see $u_{1*} \neq 0$, which shows that $(u_{1*}, 0) \neq (0,0)$. Similarly, I can similarly prove there is at least another stationary solution $(0, u_{2*}) \neq (0,0)$ for the system (2.1).

Theorem 3.2. Under the assumptions of Theorem 3.1, the zero solution $(0,0)$ is locally asymptotically stable .

Proof. Firstly, the condition $b_i < \lambda_1 d_i$ yields,

$$B < \lambda_1 D, \tag{3.14}$$

where

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \; B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}. \tag{3.15}$$

Next, consider the following linear system:

$$\begin{cases} \frac{\partial u_1}{\partial t} = d_1 \Delta u_1 + b_1 u_1, & t \geqslant 0, \, x \in \Omega, \\ \frac{\partial u_2}{\partial t} = d_2 \Delta u_2 + b_2 u_2, & t \geqslant 0, \, x \in \Omega, \\ u_1(t,x) = u_2(t,x) = 0, & t \geqslant 0, \, x \in \partial\Omega, \\ u_1(0,x) = \xi_1(x), \quad u_2(0,x) = \xi_2(x), \end{cases} \tag{3.16}$$

Consider the Lyapunov function:

$$V = \int_\Omega (u_1^2 + u_2^2) dx.$$

The condition (3.14) yields

$$\begin{aligned} \frac{dV}{dt}\Big|_{(3.16)} &= \int_\Omega \left(2d_1 u_1 \Delta u_1 + 2b_1 u_1^2 + 2d_2 u_2 \Delta u_2 + 2b_2 u_2^2 \right) dx \\ &\leqslant \int_\Omega u^T \Big(-2\lambda_1 D + 2B \Big) u dx \leqslant 0, \end{aligned} \tag{3.17}$$

where $u = (u_1, u_2)^T$. Then (3.17) yields that the zero solution $(0,0)$ of the linear system (3.16) is asymptotically stable ([19]). And hence, the zero solution $(0,0)$ of the nonlinear system (2.1) is locally asymptotically stable.

4. Numerical example

Example 4.1. In the system (2.1), I assume $\Omega = (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{2}, \frac{1}{2})$, then $\lambda_1 \geqslant 3$ ([11, Remark 14]). Set $\theta_1 = \frac{1}{3}, \theta_2 = \frac{1}{5}$, then the condition (H1) is satisfied. Assume, in addition,

$$D = \begin{pmatrix} 0.6 & 0 \\ 0 & 0.5 \end{pmatrix}, \; B = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.2 \end{pmatrix},$$

then $b_i < d_i\lambda_1$ and $0 < \theta_i < 1, \forall\, i = 1, 2$. Theorem 3.1 tells that the system (2.1) possesses at least three stationary solutions $(0, 0), (u_{1*}(x), 0)$ and $(0, u_{2*}(x))$, where $u_{i*}(x) \neq 0, \forall\, i = 1, 2$. Moreover , Theorem 3.2 yields that the zero solution $(0, 0)$ of the system (2.1) is locally asymptotically stable.

5. Conlusions

In this paper, the existence of two nontrivial stationary solutions of the nonlinear Gilpin-Ayala model of two species competition is derived by using the mountain pass lemma. The local stability criteria of the trivial solutions are given by using the Lyapunov function method. The local stability conclusion of the double zero solution fully indicates that when the population is on the verge of extinction, the diffusion behavior should be avoided and the diffusion coefficient should be reduced, otherwise the species will be prone to extinction

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