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Article

Note on the conformable boundary value problems: Sturm's theorems and Green's function

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Abstract: Recently, the conformable derivative and its properties have been introduced. In this paper, we propose and prove some new results on conformable Boundary Value Problems. First, we introduce a conformable version of classical Sturm´s separation, and comparison theorems. For a conformable Sturm-Liouville problem, Green's function is constructed, and its properties are also studied. In addition, we propose the applicability of the Green´s Function in solving conformable inhomogeneous linear differential equations with homogeneous boundary conditions, whose associated homogeneous boundary value problem has only trivial solution. Finally, we prove the generalized Hyers-Ulam stability of the conformable inhomogeneous boundary value problem.

Keywords: Conformable fractional derivative; Conformable fractional integral; Conformable fractional differential equations; Sturm´s Theorems; Green´s Function

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1. Introduction

The idea of fractional derivative was first raised by L'Hospital in 1695. Since then, several related new definitions have been proposed. The most common ones are Riemann-Liouville and Caputo definitions. For more information about the most known fractional definitions, we refer to [1,2]. A new definition of fractional derivative and fractional integral has been recently proposed by Khalil et al. in [3]. As a result, several important elements of the mathematical analysis of functions of a real variable have been formulated such as: chain rule, fractional power series expansion and fractional integration by parts formulas, Rolle's Theorem, Mean Value Theorem, [3-5]. The conformable partial derivative of the order $\alpha \in (0.1]$ of the real-valued functions of several variables and conformable gradient vector are also defined. In addition, a conformable version of Clairaut´s Theorem for partial derivative is investigated in [6]. In [7], conformable Jacobian matrix is defined, and chain rule for multivariable conformable derivative is proposed. In [8], the conformable version of Euler´s Theorem on homogeneous is introduced. Furthermore, in a short time, various research studies have been conducted on the theory and applications of fractional differential equations in the context of this newly introduced fractional derivative, [9-18].

This paper is organized as follows: In Section 2, the main concepts of conformable fractional calculus are presented. In Section 3, we proved a conformable version of the conformable second-order Sturm-Picone identity. From this result, we establish the conformable Sturm-Liouville comparison and separation theorems. In Section 4, for a conformable Sturm-Liouville problem, the Green function is constructed, and its properties are studied. At the end, we prove the generalized Hyers-Ulam stability of conformable inhomogeneous linear differential equations with homogeneous boundary conditions..

2. Basic definitions and tools

Definition 1. *Given a function* $f:[0,\infty) \to R$. *Then, the conformable fractional derivative of order* α, [3], *is defined by*

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$$(T_\alpha f)(t) = \lim_{\epsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon}, \quad (1)$$

for all $t>0$, $0<\alpha\le 1$. *If* f *is* α*-differentiable in some* $(0,a)$, $a>0$, *and* $\lim_{t\to 0^+}(T_\alpha f)(t)$ *exists, then it is defined as*

$$(T_\alpha f)(0) = \lim_{t\to 0^+}(T_\alpha f)(t), \quad (2)$$

Theorem 1. [3]. *If a function* $f:[0,\infty)\to R$ *is* α*-differentiable at* $t_0>0$, $0<\alpha\le 1$, *then* f *is continuous at* t_0.

Theorem 2. [3]. *Let* $0<\alpha\le 1$, *and let* f,g *be* α*-differentiable at a point* $t>0$. *Then*

(i) $T_\alpha(af+bg) = a\,(T_\alpha f)+b\,(T_\alpha g),\ \forall a,b\in R.$
(ii) $T_\alpha(t^p) = pt^{p-\alpha},\ \forall p\in R.$
(iii) $T_\alpha(\lambda)=0$, *for all constant functions* $f(t)=\lambda$.
(iv) $T_\alpha(fg) = f(T_\alpha g)+g(T_\alpha f).$
(v) $T_\alpha\left(\frac{f}{g}\right) = \frac{g(T_\alpha f)-f(T_\alpha g)}{g^2}.$
(vi) *If, in addition,* f *is differentiable, then* $(T_\alpha f)(t) = t^{1-\alpha}\frac{df}{dt}(t).$

The conformable fractional derivative of certain functions for the above definition is given as:

(i) $T_\alpha(1)=0,$
(ii) $T_\alpha(sin(at)) = at^{1-\alpha}cos(at),$
(iii) $T_\alpha(cos(at)) = -at^{1-\alpha}sin(at),$
(iv) $T_\alpha(e^{at}) = ae^{at},\ a\in R.$

Definition 2. *The (left) conformable derivative starting from* a *of a given function* $f:[a,\infty)\to R$ *of order* $0<\alpha\le 1$, [4], *is defined by*

$$(T_\alpha^a f)(t) = \lim_{\epsilon\to 0}\frac{f(t+\varepsilon(t-a)^{1-\alpha})-f(t)}{\varepsilon}, \quad (3)$$

When $a=0$, *it is written as* $(T_\alpha f)(t)$. *If* f *is* α*-differentiable in some* (a,b), *then the following can be defined as:*

$$(T_\alpha^a f)(a) = \lim_{t\to a^+}(T_\alpha^a f)(t), \quad (4)$$

Theorem 3 ***(Chain Rule).*** [4]. *Assume* $f,g:(a,\infty)\to R$ *be (left)* α*-differentiable functions, where* $0<\alpha\le 1$. *By letting* $h(t)=f(g(t))$, $h(t)$ *is* α*-differentiable for all* $t\ne a$ *and* $g(t)\ne 0$, *therefore, we have the following:*

$$(T_\alpha^a h)(t) = (T_\alpha^a f)(g(t))\cdot(T_\alpha^a g)(t).(g(t))^{\alpha-1}, \quad (5)$$

If $t=a$, *then*

$$(T_\alpha^a h)(a) = \lim_{t\to a^+}(T_\alpha^a f)(g(t))\cdot(T_\alpha^a g)(t).(g(t))^{\alpha-1}, \quad (6)$$

Theorem 4 ***(Rolle´s Theorem).*** [3]. *Let* $a>0$, $\alpha\in(0,1]$ *and* $f:[a,\infty)\to$ *be a given function that satisfies the following:*

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- f *is continuous on* $[a,b]$.
- f *is* α *–differentiable on* (a,b).
- $f(a) = f(b)$.

Then, there exists $c \in (a,b)$, *such that* $(T_\alpha f)(c)=0$.

Corollary 1. *Let* $I \subset [0.\infty)$, $\alpha \in (0,1]$ *and* $f: I \to R$ *be a given function that satisfies*

- f *is* α *–differentiable on* I.
- $f(a) = f(b) = 0$ *for certain* $c \in I$.

Then, there exists $c \in (a,b)$, *such that* $(T_\alpha f)(c)=0$.

Theorem 5. ***(Mean Value Theorem).*** [3]. *Let* $a > 0$, $\alpha \in (0,1]$ *and* $f: [a,\infty) \to R$ *be a given function that satisfies*

- f *is continuous in* $[a,b]$.
- f *is* α*-differentiable on* (a,b).

Then, exists $c \in (a,b)$ *such that*

$$(T_\alpha f)(c) = \frac{f(b)-f(a)}{\frac{b^\alpha}{\alpha}-\frac{a^\alpha}{\alpha}}, \tag{7}$$

Theorem 6. [5]. *Let* $a > 0$, $\alpha \in (0,1]$ *and* $f: [a,\infty) \to R$ *be a given function that satisfies*

- f *is continuous in* $[a,b]$.
- f *is* α*-differentiable on* (a,b).

If $(T_\alpha f)(c) = 0$ *for all* $t \in (a,b)$, *then* f *is a constant on* $[a,b]$.

Corollary 7. [5]. *Let* $a > 0$, $\alpha \in (0,1]$ *and* $F, G: [a,\infty) \to R$ *be functions such that* $(T_\alpha F)(t) = (T_\alpha G)(t)$ *for all* $t \in (a,b)$. *Then, there exists a constant* C *such that*

$$F(t) = G(t) + C, \tag{8}$$

The following definition is the α-fractional integral of a function f starting from $a \geq 0$:

Definition 3. $I_\alpha^a(f)(t) = \int_a^t \frac{f(x)}{x^{1-\alpha}} \cdot dx$, *where the integral is the usual Riemann improper integral, and* $\alpha \in (0,1]$, [2].

According to the above definition, the following can be shown:

Theorem 8. $T_\alpha^a I_\alpha^a(f)(t) = f(t)$, for $t \geq a$, where f is any continuous function in the domain of I_α.

Lemma 9. *Let* $f: (a,b) \to R$ *be differentiable and* $\alpha \in (0,1]$. *Then, for all* $a > 0$, *we have,* [3],

$$I_\alpha^a T_\alpha^a(f)(t) = f(t) - f(a), \tag{9}$$

Finally, we give the definition of non-conformable α-Wronskian, which is necessary in the next section.

Definition 4. *Let* x *and* y *be given conformable* α*-differentiable functions on* $[a,b]$ *with* $a \geq 0$ *and* $\alpha \in (0,1]$. *We set the following:*

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$$W^{\alpha}(x,y)(t) = \begin{vmatrix} x(t) & y(t) \\ (T_{\alpha}x)(t) & (T_{\alpha}y)(t) \end{vmatrix}, \tag{10}$$

3. Sturm's theorems

In this section, we consider the scalar fractional differential equation of second order of the following form:

$$T_{\alpha}T_{\alpha}x(t) + p(t)T_{\alpha}x(t) + q(t)x(t) = 0, \tag{11}$$

with continuous functions p and q, and $\alpha \in (0,1]$. Traditionally, from [19], two functions x and y that are continuous on $[a,b]$ for some $0 \le a < b$, will be called linearly dependent if there exist $c_1, c_2 \in R$ such that $|c_1| + |c_1| > 0$ and $c_1x(t) + c_2y(t) \equiv 0$ for all $t \in [a,b]$. In the other case, they are linearly independent.

Remark 1. We can write

$$W^{\alpha}(x,y)(t) = e^{-\int_{t_0}^{t}\frac{p(x)}{x^{1-\alpha}}dx}W^{\alpha}(x,y)(t_0), \tag{12}$$

for two solutions x and y of [5] and some $t_0 \in (a,b)$. In fact, we apply the operator T_{α} on $W^{\alpha}(x,y)(t)$ to obtain

$$T_{\alpha}\big(W^{\alpha}(x,y)(t)\big) = T_{\alpha}\big(x(t)T_{\alpha}y(t) - y(t)T_{\alpha}x(t)\big)$$
$$= T_{\alpha}x(t)T_{\alpha}y(t) + x(t)T_{\alpha}T_{\alpha}y(t) - T_{\alpha}y(t)T_{\alpha}x(t) - y(t)T_{\alpha}T_{\alpha}x(t)$$

However, x and y satisfies (11). Hence, we have:

$$T_{\alpha}T_{\alpha}x(t) = -p(t)T_{\alpha}x(t) - q(t)x(t)$$

and

$$T_{\alpha}T_{\alpha}y(t) = -p(t)T_{\alpha}y(t) - q(t)y(t)$$

Therefore, we get

$$T_{\alpha}\big(W^{\alpha}(x,y)(t)\big) = -\big(x(t)T_{\alpha}y(t) - y(t)T_{\alpha}x(t)\big)p(t) = -\big(W^{\alpha}(x,y)(t)\big)p(t)$$

Thus

$$\frac{T_{\alpha}\big(W^{\alpha}(x,y)(t)\big)}{W^{\alpha}(x,y)(t)} = -p(t)$$

Consequently, we have

$$W^{\alpha}(x,y)(t) = e^{-\int_{t_0}^{t}\frac{p(x)}{x^{1-\alpha}}dx}W^{\alpha}(x,y)(t_0)$$

This completes the proof. □

Similar to the classical case, by using the above formula, we can immediately obtain the following equivalent condition of linear independence:

Theorem 10. *Two solutions x and y of equation (11) defined on $[a,b]$ for some $0 \le a < b$ are linearly independent if and only if $W^{\alpha}(x,y)(t) \neq 0$ for all $t \in [a,b]$.*

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Now, we propose a conformable version of three classical results, the second order Sturm-Picone identity and Sturm's comparison, and separation theorems, [20].

Let us now introduce the non-conformable self-adjoint Sturm-Liouville equation as follows:

$$-T_\alpha(p_1(t)T_\alpha x(t)) + p_0(t)x(t) = 0, \tag{13}$$

$$-T_\alpha(q_1(t)T_\alpha y(t)) + q(t)y(t) = 0, \tag{14}$$

where $p_0, p_1, q_0, q_1, T_\alpha p_1, T_\alpha q_1$ are continuous on some closed interval $I \subset [0, +\infty)$, $p_1 > 0, q_1 > 0$ on I and $\alpha \in (0,1]$.

Theorem 11 (*Conformable Picone Identity*). *If $x(t), y(t)$ and $p_1(t)T_\alpha x(t), q_1(t)T_\alpha y(t)$ are α-differentiable for $t \in I$ and $y(t) \neq 0$ in I, then we obtain*

$$T_\alpha\left(\frac{x(t)}{y(t)}\left(p_1(t)y(t)T_\alpha x(t) - q_1(t)x(t)T_\alpha y(t)\right)\right)$$
$$= x(t)T_\alpha(p_1(t)T_\alpha x(t)) - \frac{(x(t))^2}{y(t)}T_\alpha(q_1(t)T_\alpha y(t)) + (p_1(t) - q_1(t))(T_\alpha x(t))^2$$
$$+ q_1(t)\left(T_\alpha x(t) - \frac{x(t)}{y(t)}T_\alpha y(t)\right)^2$$
$$(15)$$

Proof. This arises from the straightforward α-differentiation. □

Theorem 12 (*Conformable Sturm's Comparison Theorem*). *Let $0 \le a < b$ be two consecutive zeros of a nontrivial solution $x(t)$ of equation (3.3). Suppose that*

(i) $0 < q_1(t) \le p_1(t)$,

and

(ii) $q_0(t) \le p_0(t)$

for all $t \in [a, b]$. Then, every solution $y(t)$ of equation (14) has at least one zero in the closed interval $[a, b]$.

Proof. If $x(t)$ and $y(t)$ are solutions of (13) and (14), respectively, and $y(t) \neq 0$ for all $t \in [a, b]$, then the conformable Picone identity (15) yields on substitution of (131.4) as follows:

$$T_\alpha\left(\frac{x(t)}{y(t)}\left(p_1(t)y(t)T_\alpha x(t) - q_1(t)x(t)T_\alpha y(t)\right)\right)$$
$$= (p_0(t) - q_0(t))(x(t))^2 + (p_1(t) - q_1(t))(T_\alpha x(t))^2 + q_1(t)\left(T_\alpha x(t) - \frac{x(t)}{y(t)}T_\alpha y(t)\right)^2$$

Integrating over $[a, b]$; therefore, we have (see Lemma 9),

$$\int_a^b \left[(p_0(t) - q_0(t))(x(t))^2 + (p_1(t) - q_1(t))(T_\alpha x(t))^2 + q_1(t)\left(T_\alpha x(t) - \frac{x(t)}{y(t)}T_\alpha y(t)\right)^2\right]\frac{1}{t^{1-\alpha}}dt$$
$$= \left[\frac{x(t)}{y(t)}\left(p_1(t)y(t)T_\alpha x(t) - q_1(t)x(t)T_\alpha y(t)\right)\right]_{t=a}^{t=b}$$
$$(16)$$

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The right-hand side of equation (16) evaluates to zero by assuming $x(a) = x(b) = 0$, and $y(a) \neq 0$, $y(b) \neq 0$. Since $q_1(t) > 0$ in $[a,b]$, the third term of the integrand is nonnegative over $[a,b]$. Hence, we must have either

(i) $T_\alpha x(t) - \frac{x(t)}{y(t)} T_\alpha y(t) \equiv 0$ in $[a,b]$

or

(ii) $\int_a^b \left[(p_0(t) - q_0(t))(x(t))^2 + (p_1(t) - q_1(t))(T_\alpha x(t))^2 \right] \frac{1}{t^{1-\alpha}} dt < 0$

However, Case (ii) gives an immediate contradiction since $p_0(t) - q_0(t) \geq 0$ and $p_1(t) - q_1(t) \geq 0$ by assumption. In Case (i), we are also led to a contradiction since (i) implies

$$\frac{y(t)T_\alpha x(t) - x(t)T_\alpha y(t)}{(y(t))^2} = T_\alpha\left(\frac{x(t)}{y(t)}\right) \equiv 0, \text{ or}$$

$x(t) \equiv ky(t)$ for all $t \in [a,b]$, for some $k \neq 0$, but $y(a) = y(b) = 0$ which is a contrary to our assumption. □

Theorem 13 (*Conformable Sturm´s Separation Theorem*). *Let $0 \leq a < b$ be two consecutive zeros of a nontrivial solution $x(t)$ of equation (13). Let $y(t)$ be any other solution of equation (13) which is linearly independent of $x(t)$. Then, $y(t)$ has exactly one zero of the interval (a,b). In other words, the zeros of any two linearly independent solutions of (13) are interlaced.*

Proof. On the contrary, suppose that $y(t) \neq 0$ for all $t \in (a,b)$. Since $x(t)$ and $y(t)$ are linearly independent, it follows that $y(a) \neq 0$; otherwise, we would have

$$W^\alpha(x,y)(a) = \begin{vmatrix} x(a) & y(a) \\ T_\alpha x(a) & T_\alpha y(a) \end{vmatrix} = 0$$

which implies that the conformable Wronskian, $W^\alpha(x,y)(t)$, is zero for all t and that $x(t)$ and $y(t)$ are linearly dependent. For the same reason, we know that $y(b) \neq 0$, but when $q_1(t) \equiv p_1(t)$ and $q_0(t) \equiv p_0(t)$, equation (16) becomes

$$\int_a^b p_1(t)\left(T_\alpha x(t) - \frac{x(t)}{y(t)} T_\alpha y(t)\right)^2 \frac{1}{t^{1-\alpha}} dt = \left[\frac{x(t)}{y(t)} p_1(t)\big(y(t)T_\alpha x(t) - x(t)T_\alpha y(t)\big)\right]_{t=a}^{t=b}$$

Since $y(a) \neq 0$ and $y(b) \neq 0$, the right-hand side evaluates to zero. Since $p_1(t) > 0$ in $[a,b]$, it follows that $T_\alpha x(t) - \frac{x(t)}{y(t)} T_\alpha y(t) \equiv 0$, or

$$W^\alpha(x,y)(t) = y(t)T_\alpha x(t) - x(t)T_\alpha y(t) \equiv 0$$

for all $t \in (a,b)$. Hence, $x(t)$ and $y(t)$ are linearly dependent on (a,b) which is a contrary to our assumption.

Remark 2.

(i) Conformable Sturm's Comparison Theorem guarantees the existence of at least one zero.

(ii) The assumption $q_0(t) \leq p_0(t)$ cannot be dropped. Consider the equation on $t \geq 0$, $T_\alpha T_\alpha x(t) + x(t) = 0$ $(p_1(t) = 1, p_0(t) = -1)$ and $T_\alpha T_\alpha y(t) - y(t) = 0$ $(q_1(t) = 1, q_0(t) = 1)$ and let $x(t)$ and $y(t)$ be their non-trivial solutions, respectively. Between any two zeros of $x(t), y(t)$ does not admit a zero.

(iii) Consider the equation on $t \geq 0$, $T_\alpha T_\alpha x(t) + x(t) = 0$ $(p_1(t) = 1, p_0(t) = -1)$ and $T_\alpha T_\alpha y(t) + 4y(t) = 0$ $(q_1(t) = 1, q_0(t) = -4)$, and let $x(t) = sin\left(\frac{t^\alpha}{\alpha}\right)$ and $y(t) =$

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$sin\left(2\,\frac{t^{\alpha}}{\alpha}\right)$ be their non-trivial solutions, respectively. However, there is no zero of $x(t)$ between two consecutive zeros of $y\,(t)$.

Remark 3. An important application of Sturm´s Comparison Theorem is to provide a good understanding of the zero set on non-trivial solutions of Conformable Bessel's Equation. The Conformable Bessel's Equation is given by

$$t^{2\alpha}T_{\alpha}T_{\alpha}y(t) + \alpha t^{\alpha}T_{\alpha}y(t) + \alpha^{2}(t^{2\alpha} - p^{2})y(t) = 0, \tag{17}$$

where $\alpha \in (0,1]$ and $p \geq 0$. Clearly, if $\alpha = 0$, the above equation is just the classical Bessel Equation, [19]. For more information about the conformable Bessel's function in the solution of wave equation, we refer to [21].

For $t > 0$, making a change variable $y = \frac{v}{t^{\frac{\alpha}{2}}}$, the equation (17) transforms into

$$T_{\alpha}T_{\alpha}\,y(t) + \alpha^{2}\left(1 + \frac{1-4p^{2}}{4t^{2\alpha}}\right)v(t) = 0, \tag{18}$$

(To obtain the above equation, we start N-differentiating the equation $t^{\frac{\alpha}{2}}y = v$)

Case 1: $p > \frac{1}{2}$. In this case, compare (18) with

$$T_{\alpha}T_{\alpha}y(t) + \alpha^{2}y(t) = 0$$

which has a solution $sin(t^{\alpha})$ with zeros at $t = (n\pi)^{\frac{1}{\alpha}}$, $n \in N$. Therefore, a solution of (3.8) has at least one zero on each of the open interval $\left(((n-1)\pi)^{\frac{1}{\alpha}}, (n\pi)^{\frac{1}{\alpha}}\right)$, $n \in N$.

Case 2: $0 < p < \frac{1}{2}$. In this case, compare (18) with

$$T_{\alpha}T_{\alpha}y(t) + \alpha^{2}y(t) = 0$$

and conclude that between any two consecutives zeros, a and b of $v(t)$, there exists one zero of $sin(t^{\alpha})$. Thus, we have $a < (n\pi)^{\frac{1}{\alpha}} < b$ for some $n \in N$.

4. The study of conformable Green´s Functions

4.1. Conformable Green´s Functions

In this section, we consider the conformable Sturm- Liouville system

$$\left.\begin{array}{ll} T_{\alpha}(p(t)T_{\alpha}x(t)) + (\lambda\rho(t) - q(t))x(t) = 0 & (19a) \\ a_{1}x(a) + a_{2}T_{\alpha}x(a) = 0 & (19b) \\ b_{1}x(b) + b_{2}T_{\alpha}x(b) = 0 & (19c) \end{array}\right\}, \tag{19}$$

$$|a_{1}| + |a_{2}| \neq 0, |b_{1}| + |b_{2}| \neq 0$$

with continuous functions $p(t)$, $q(t)$ and $\rho(t)$ on $[a,b]$ for some $0 \leq a < b$, such that $\rho(t) \geq 0$ and $p(t) \geq 0$ for all $t \in [a,b]$ and $\alpha \in (0,1]$.

Definition 5. *Let Q denote the square $Q = [a,b] \times [a,b]$ for some $0 \leq a < b$, in the $t\varepsilon$-plane. A function $G^{\alpha}(t,\varepsilon)$ defined in Q is called conformable Green´s Function of Sturm-Liouville system (19), if it has the following properties:*

(i) *The function $G^{\alpha}(t,\varepsilon)$ is continuous in Q.*

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(ii) *Let $\varepsilon \in (a,b)$ be fixed. Then, $G^\alpha(t,\varepsilon)$ has conformable partial derivatives of left and right with respect to variable t, for $t=\varepsilon$, and it is verified as follows:*

$$\frac{\partial^\alpha}{\partial t^\alpha} G^\alpha(\varepsilon^+,\varepsilon) - \frac{\partial^\alpha}{\partial t^\alpha} G^\alpha(\varepsilon^-,\varepsilon) = -\frac{1}{p(\varepsilon)}$$

(iii) *Let $\varepsilon \in [a,b]$ be fixed. Then, $G^\alpha(t,\varepsilon)$ has continuous conformable partial derivatives of first and second order with respect to variable t, if $t \neq \varepsilon$, and it is verified as follows:*

$$\frac{\partial^\alpha}{\partial t^\alpha}\left(p(t)T_\alpha G^\alpha(t,\varepsilon)\right) + \left(\lambda\rho(t) - q(t)\right)G^\alpha(t,\varepsilon) = 0$$

(iv) *Let $\varepsilon \in (a,b)$ be fixed. Then, $G^\alpha(t,\varepsilon)$ satisfies the boundary conditions (19b) and (19c).*

Theorem 14. *Let $x_1(t)$ and $x_2(t)$ be two solutions of (19a) that verify condition (19b). Then, $x_1(t)$ and $x_2(t)$ are linearly dependent.*

Proof. Since $|a_1| + |a_2| \neq 0$, it follows from

$$a_1 x_1(a) + a_2 T_\alpha x_1(a) = 0$$
$$a_1 x_2(a) + a_2 T_\alpha x_2(a) = 0$$

that

$$W^\alpha(x,y)(a) = \begin{vmatrix} x_1(a) & x_2(a) \\ T_\alpha x_1(a) & T_\alpha x_2(a) \end{vmatrix} = 0$$

Therefore, $x_1(t)$ and $x_2(t)$ are linearly dependent. □

Theorem 15. *Let $x_1(t)$ and $x_2(t)$ be two solutions of (19a) that verify condition (19c). Then, $x_1(t)$ and $x_2(t)$ are linearly dependent.*

Proof. It is analogous to the proof of the above theorem. □

Theorem 16. *System (19) has no Green´s Function if λ is an eigenvalue.*

Proof. Let $x_1(t)$ be an eigenfunction of system (19). Let $x_2(t)$ be a solution of (19a) linearly independent of $x_1(t)$. From Theorems 14 and 15, it turns out that $x_2(t)$ does not verify the conditions (19b) and (19c).
According to the condition (iii) of $G^\alpha(t,\varepsilon)$, the said function is a solution of (19a) in the intervals $a \leq t < \varepsilon$ and $\varepsilon < t \leq b$, so it has the form

$$G^\alpha(t,\varepsilon) = \begin{cases} A_1(\varepsilon)x_1(t) + A_2(\varepsilon)x_2(t) & a \leq t < \varepsilon \\ B_1(\varepsilon)x_1(t) + B_2(\varepsilon)x_2(t) & \varepsilon < t \leq b \end{cases}$$

Let us now express that $G^\alpha(t,\varepsilon)$ meets the condition (iv)

$$a_1\left(A_1(\varepsilon)x_1(a) + A_2(\varepsilon)x_2(a)\right) + a_2\left(A_1(\varepsilon)T_\alpha x_1(a) + A_2(\varepsilon)T_\alpha x_2(a)\right) = 0$$
$$b_1\left(B_1(\varepsilon)x_1(b) + B_2(\varepsilon)x_2(b)\right) + b_2\left(B_1(\varepsilon)T_\alpha x_1(b) + B_2(\varepsilon)T_\alpha x_2(b)\right) = 0$$

Since $x_1(t)$ meets both conditions (19b) and (19c), the above equalities are reduced to

$$A_2(\varepsilon)\left(a_1 x_2(a) + a_2 T_\alpha x_2(a)\right) = 0$$
$$B_2(\varepsilon)\left(b_1 x_2(b) + b_2 T_\alpha x_2(b)\right) = 0$$

On the contrary, we have

$$a_1 x_2(a) + a_2 T_\alpha x_2(a) \neq 0$$

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$$b_1x_2(b)+b_2T_\alpha x_2(b)\neq 0$$

so that

$$A_2(\varepsilon)=0, a\leq t<\varepsilon$$
$$B_2(\varepsilon)=0, \varepsilon<t\leq b$$

From here, we have

$$G^\alpha(t,\varepsilon)=\begin{cases}A_1(\varepsilon)x_1(t) & a\leq t<\varepsilon\\ B_1(\varepsilon)x_1(t) & \varepsilon<t\leq b\end{cases}$$

Since $G^\alpha(t,\varepsilon)$ is a continuous function, we obtain

$$\lim_{t\to\varepsilon-}G^\alpha(t,\varepsilon)=A_1(\varepsilon)x_1(\varepsilon)=\lim_{t\to\varepsilon+}G^\alpha(t,\varepsilon)=B_1(\varepsilon)x_1(\varepsilon)$$

so that

$$A_1(\varepsilon)=B_1(\varepsilon),\qquad a<\varepsilon<b$$

From here, it follows that

$$\frac{\partial^\alpha}{\partial t^\alpha}G^\alpha(\varepsilon^+,\varepsilon)-\frac{\partial^\alpha}{\partial t^\alpha}G^\alpha(\varepsilon^-,\varepsilon)=0$$

which contradicts condition (ii). □

Theorem 17. *System (19) has one, and only one, Green's Function if λ is not an eigenvalue.*

Proof. Let $x_1(t)$ and $x_2(t)$ two solutions of (19) such that

$$x_1(a)=a_2, T_\alpha x_1(a)=-a_1, x_2(b)=b_2, T_\alpha x_2(b)=-b_1$$

Since $|a_1|+|a_2|\neq 0, |b_1|+|b_2|\neq 0$, $x_1(t)$ and $x_2(t)$ are not null, they are also satisfying conditions (19b) and (19c), respectively.
These solutions are linearly independent, since otherwise it would be

$$x_1(t)=\mu x_2(t), \mu\neq 0$$

Therefore, we have

$$b_1x_1(b)+b_2T_\alpha x_1(b)=\mu[b_1x_2(b)+b_2T_\alpha x_2(b)]=0$$

As a result, $x_1(t)$ would comply with (19b) and (19c). This is not possible since $x_1(t)$ is not an eigenfunction.
The reasoning as in the proof of Theorem 16, we have to

$$G^\alpha(t,\varepsilon)=\begin{cases}A_1(\varepsilon)x_1(t)+A_2(\varepsilon)x_2(t) & a\leq t<\varepsilon\\ B_1(\varepsilon)x_1(t)+B_2(\varepsilon)x_2(t) & \varepsilon<t\leq b\end{cases}$$

Expressing that $G^\alpha(t,\varepsilon)$ meets the condition (iv), and it turns out that

$$a_1\left(A_1(\varepsilon)x_1(a)+A_2(\varepsilon)x_2(a)\right)+a_2\left(A_1(\varepsilon)T_\alpha x_1(a)+A_2(\varepsilon)T_\alpha x_2(a)\right)=0$$
$$b_1\left(B_1(\varepsilon)x_1(b)+B_2(\varepsilon)x_2(b)\right)+b_2\left(B_1(\varepsilon)T_\alpha x_1(b)+B_2(\varepsilon)T_\alpha x_2(b)\right)=0$$

that is reduced to

$$A_2(\varepsilon)\left(a_1x_2(a)+a_2T_\alpha x_2(a)\right)=0$$

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$$B_1(\varepsilon)\big(b_1x_1(b)+b_2T_\alpha x_1(b)\big)=0$$

from where it follows, remembering that $x_1(t)$ and $x_2(t)$ are not eigenfunctions and, therefore, $a_1x_2(a)+a_2T_\alpha x_2(a)\neq 0,\ b_1x_1(b)+b_2T_\alpha x_1(b)\neq 0,$

$$\left.\begin{array}{l}A_2(\varepsilon)=0\\B_1(\varepsilon)=0\end{array}\right\}\ a<\varepsilon<b$$

Now, by applying conditions (i) and (ii), it turns out that

$$A_1(\varepsilon)x_1(\varepsilon)+B_2(\varepsilon)x_2(\varepsilon)=0$$
$$A_1(\varepsilon)T_\alpha x_1(\varepsilon)+B_2(\varepsilon)T_\alpha x_2(\varepsilon)=\frac{1}{p(\varepsilon)}$$

which allows us to calculate the following:

$$A_1(\varepsilon)=\frac{-x_2(\varepsilon)}{p(\varepsilon)[x_1(\varepsilon)T_\alpha x_2(\varepsilon)-x_2(\varepsilon)T_\alpha x_1(\varepsilon)]}$$
$$B_2(\varepsilon)=\frac{-x_1(\varepsilon)}{p(\varepsilon)[x_1(\varepsilon)T_\alpha x_2(\varepsilon)-x_2(\varepsilon)T_\alpha x_1(\varepsilon)]}$$

Note that $x_1(\varepsilon)T_\alpha x_2(\varepsilon)-x_2(\varepsilon)T_\alpha x_1(\varepsilon)$ is nonzero since it is conformable Wronskian of two linearly independent solutions of equation (19).
Given the following:

$$T_\alpha\big(p(t)T_\alpha x_1(t)\big)+\big(\lambda\rho(t)-q(t)\big)x_1(t)=0$$
$$T_\alpha\big(p(t)T_\alpha x_2(t)\big)+\big(\lambda\rho(t)-q(t)\big)x_2(t)=0$$

By multiplying the first equation by $x_2(t)$, the second by $x_1(t)$, and subtracting, we have

$$x_2(t)T_\alpha\big(p(t)T_\alpha x_1(t)\big)-x_1(t)T_\alpha\big(p(t)T_\alpha x_2(t)\big)=0$$

that can be written in the form

$$p(t)\big(x_2(t)T_\alpha x_1(t)-x_1(t)T_\alpha x_2(t)\big)=0$$

So, $p(\varepsilon)\big(x_2(\varepsilon)T_\alpha x_1(\varepsilon)-x_1(\varepsilon)T_\alpha x_2(\varepsilon)\big)$ is a constant K that does not depend on ε.
Hence, we have

$$G^\alpha(t,\varepsilon)=\begin{cases}\dfrac{1}{K}x_1(t)x_2(\varepsilon) & a\le t<\varepsilon\\[2mm] \dfrac{1}{K}x_1(\varepsilon)x_2(t) & \varepsilon<t\le b\end{cases}$$

The conformable Green´s Function $G^\alpha(t,\varepsilon)$ has the properties (i) - (iv). The uniqueness of this function is easily deduced from the method that we have followed to determine $G^\alpha(t,\varepsilon)$. □

Example 1. Consider the system

$$\left.\begin{array}{l}T_\alpha T_\alpha x(t)+x(t)=0,\ t\in\left[0,(\alpha\pi)^{\frac{1}{\alpha}}\right]\\ x(0)+T_\alpha x(0)=0\\ x\left((\alpha\pi)^{\frac{1}{\alpha}}\right)=0\end{array}\right\}$$

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for some $\alpha \in (0,1]$, we will find the corresponding conformable Green's Function. In this case $p(t) = 1,\ q(t) = -1,\ \lambda = 0,\ \rho(t)$ is any positive continuous function in $\left[0, (\alpha\pi)^{\frac{1}{\alpha}}\right]$, $a_1 = 1,\ a_2 = 1,$ $b_1 = 1,\ b_2 = 0.$

The general solution of $T_\alpha T_\alpha x(t) + x(t) = 0$ is

$$x(t) = A\cos\frac{t^\alpha}{\alpha} + B\sin\frac{t^\alpha}{\alpha}$$

Then, we have

$$x(0) + T_\alpha x(0) = A + B = 0$$
$$x\left((\alpha\pi)^{\frac{1}{\alpha}}\right) = -A = 0$$

From here $A = 0, B = 0$, so there was the conformable Green's Function of the given system. The solutions of $T_\alpha T_\alpha x(t) + x(t) = 0;\ x_1(t) = \cos\frac{t^\alpha}{\alpha} + \sin\frac{t^\alpha}{\alpha}, x_2(t) = \sin\frac{t^\alpha}{\alpha}$ satisfy the conditions $x(0) + T_\alpha x(0) = 0, x\left((\alpha\pi)^{\frac{1}{\alpha}}\right) = 0$. The conformable Green's Function has the form

$$G^\alpha(t,\varepsilon) = \begin{cases} \frac{1}{K}x_2(\varepsilon)x_1(t) & 0 \le t < \varepsilon \\ \frac{1}{K}x_1(\varepsilon)x_2(t) & \varepsilon < t \le (\alpha\pi)^{\frac{1}{\alpha}} \end{cases}$$

so that

$$K = p(\varepsilon)\big(x_2(\varepsilon)T_\alpha x_1(\varepsilon) - x_1(\varepsilon)T_\alpha x_2(\varepsilon)\big) = \left(-\sin\frac{\varepsilon^\alpha}{\alpha} + \cos\frac{\varepsilon^\alpha}{\alpha}\right)\sin\frac{\varepsilon^\alpha}{\alpha} - \left(\cos\frac{\varepsilon^\alpha}{\alpha} + \sin\frac{\varepsilon^\alpha}{\alpha}\right)\cos\frac{\varepsilon^\alpha}{\alpha} = -1$$

Therefore, we obtain

$$G^\alpha(t,\varepsilon) = \begin{cases} -\sin\frac{\varepsilon^\alpha}{\alpha}\left(\cos\frac{t^\alpha}{\alpha} + \sin\frac{t^\alpha}{\alpha}\right) & 0 \le t < \varepsilon \\ -\left(\cos\frac{\varepsilon^\alpha}{\alpha} + \sin\frac{\varepsilon^\alpha}{\alpha}\right)\sin\frac{t^\alpha}{\alpha} & \varepsilon < t \le (\alpha\pi)^{\frac{1}{\alpha}} \end{cases}$$

4.2. The applicability of Conformable Green's Function

In this section, we consider the system

$$\left.\begin{aligned} &T_\alpha\big(p(t)T_\alpha x(t)\big) - q(t)x(t) = 0 && (20a) \\ &a_1x(a) + a_2T_\alpha x(a) = 0 && (20b) \\ &b_1x(b) + b_2T_\alpha x(b) = 0 && (20c) \end{aligned}\right\}, \qquad (20)$$

obtained from (19) for $\lambda = 0$. We now propose to solve the inhomogeneous system

$$\left.\begin{aligned} &T_\alpha\big(p(t)T_\alpha x(t)\big) - q(t)x(t) = -f(t) \\ &a_1x(a) + a_2T_\alpha x(a) = 0 \\ &b_1x(b) + b_2T_\alpha x(b) = 0 \end{aligned}\right\}, \qquad (21)$$

where $f(t)$ is a real continuous function in the interval $[a, b]$ for some $0 \le a < b$.

Theorem 18. *If the homogeneous system (20) has its only solution as the identically null function, then (21) has only one solution, which is given by*

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$$x(t) = \int_a^b G^\alpha(t,\varepsilon) f(\varepsilon) \frac{1}{\varepsilon^{1-\alpha}} d\varepsilon$$

where $G^\alpha(t,\varepsilon)$ is the conformable Green´s Function of (20).

Proof. That homogeneous system (20) has its unique solution as the identically null function which is equivalent to saying that $\lambda = 0$ is not an eigenvalue of (19); therefore, there is the conformable Green´s Function of (20).
Let $x_1(t)$ and $x_2(t)$ be two linearly independent solutions of (20a) that verify (20b) and (20c), respectively. Let us apply the conformable version of the method of variation of the parameters to solve (20a). Then, we have

$$x(t) = A(t)x_1(t) + B(t)x_2(t)$$

$$T_\alpha\Big(p(t)\big(x_1(t)T_\alpha A(t) + x_2(t)T_\alpha B(t) + A(t)T_\alpha x_1(t) + B(t)T_\alpha x_2(t)\big)\Big) - q(t)\big(A(t)x_1(t) + B(t)x_2(t)\big) = -f(t)$$

that is to say

$$A(t)T_\alpha\big(p(t)T_\alpha x_1(t) - A(t)q(t)x_1(t)\big) + B(t)T_\alpha\big(p(t)T_\alpha x_2(t) - B(t)q(t)x_2(t)\big) + p(t)\big(T_\alpha A(t)T_\alpha x_1(t) + T_\alpha B(t)T_\alpha x_2(t)\big) + T_\alpha\Big(p(t)\big(x_1(t)T_\alpha A(t) + x_2(t)T_\alpha B(t)\big)\Big) = -f(t)$$

that is

$$p(t)\big(T_\alpha A(t)T_\alpha x_1(t) + T_\alpha B(t)T_\alpha x_2(t)\big) + T_\alpha\Big(p(t)\big(x_1(t)T_\alpha A(t) + x_2(t)T_\alpha B(t)\big)\Big) = -f(t)$$

We make

$$x_1(t)T_\alpha A(t) + x_2(t)T_\alpha B(t) = 0,$$

and we have

$$p(t)\big(T_\alpha A(t)T_\alpha x_1(t) + T_\alpha B(t)T_\alpha x_2(t)\big) = -f(t)$$

so that

$$T_\alpha A(t) = \frac{-x_2(t)f(t)}{p(t)\big(x_2(t)T_\alpha x_1(t) + x_1(t)T_\alpha x_2(t)\big)}$$

$$T_\alpha B(t) = \frac{-x_1(t)f(t)}{p(t)\big(x_2(t)T_\alpha x_1(t) + x_1(t)T_\alpha x_2(t)\big)}$$

We know, from the proof of Theorem 17, that $p(t)\big(x_2(t)T_\alpha x_1(t) + x_1(t)T_\alpha x_2(t)\big)$ is a constant, and it is equal to K. On the contrary, we have

$$\begin{aligned} &a_1x(a) + a_2T_\alpha x(a) \\ &= a_1\big(A(a)x_1(a) + B(a)x_2(a)\big) \\ &+ a_2\big(x_1(a)T_\alpha A(a) + x_2(a)T_\alpha B(a) + A(a)T_\alpha x_1(a) + B(a)T_\alpha x_2(a)\big) \\ &= A(a)\big(a_1x_1(a) + a_2T_\alpha x_1(a)\big) + B(a)\big(a_1x_2(a) + a_2T_\alpha x_2(a)\big) \\ &= B(a)\big(a_1x_2(a) + a_2T_\alpha x_2(a)\big) = 0 \end{aligned}$$

and since $x_2(t)$ is not an eigenfunction of (20) it turns out that

$$a_1x_2(a) + a_2T_\alpha x_2(a) \neq 0$$

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so that $B(a) = 0$.
By writing now the following:

$$b_1 x(b) + b T_\alpha x(b) = 0$$

Similarly, we obtain $A(b) = 0$.
So, we have

$$A(t) = \int_a^t \frac{x_2(\varepsilon)}{K} f(\varepsilon) \frac{1}{\varepsilon^{1-\alpha}} d\varepsilon + C_1$$

and since $A(b) = 0$, we have to

$$A(t) = -\int_a^t \frac{x_2(\varepsilon)}{K} f(\varepsilon) \frac{1}{\varepsilon^{1-\alpha}} d\varepsilon + \int_a^b \frac{x_2(\varepsilon)}{K} f(\varepsilon) \frac{1}{\varepsilon^{1-\alpha}} d\varepsilon = \int_t^b \frac{x_2(\varepsilon)}{K} f(\varepsilon) \frac{1}{\varepsilon^{1-\alpha}} d\varepsilon$$

Analogously

$$B(t) = \int_a^t \frac{x_1(\varepsilon)}{K} f(\varepsilon) \frac{1}{\varepsilon^{1-\alpha}} d\varepsilon$$

Thus, we obtain

$$x(t) = A(t) x_1(t) + B(t) x_2(t) = \int_t^b \frac{x_1(t) x_2(\varepsilon)}{K} f(\varepsilon) \frac{1}{\varepsilon^{1-\alpha}} d\varepsilon + \int_a^t \frac{x_1(\varepsilon) x_2(t)}{K} f(\varepsilon) \frac{1}{\varepsilon^{1-\alpha}} d\varepsilon$$
$$= \int_a^b G_3^\alpha(t,\varepsilon) f(\varepsilon) \frac{1}{\varepsilon^{1-\alpha}} d\varepsilon$$

where we have the following

$$G^\alpha(t,\varepsilon) = \begin{cases} \frac{1}{K} x_1(t) x_2(\varepsilon) & a \le t < \varepsilon \\ \frac{1}{K} x_1(\varepsilon) x_2(t) & \varepsilon < t \le b \end{cases}$$

which is the Green´s Function. □

Example 2. By using the Green´s Function, we want to solve the following system

$$\left. \begin{array}{c} T_\alpha T_\alpha x(t) + x(t) = e^{\frac{t^\alpha}{\alpha}} \quad t \in \left[0, (\alpha\pi)^{\frac{1}{\alpha}}\right] \\ x(0) = 0 \\ T_\alpha x\left((\alpha\pi)^{\frac{1}{\alpha}}\right) = 0 \end{array} \right\}$$

First, we find the conformable Green´s Function of the homogeneous system.
We have following:

$$x(t) = A cos\left(\frac{t^\alpha}{\alpha}\right) + B sin\left(\frac{t^\alpha}{\alpha}\right)$$
$$x(0) = 0 = A$$
$$T_\alpha x\left((\alpha\pi)^{\frac{1}{\alpha}}\right) = 0 = B$$

Therefore, the conformable Green's Function exists. This function can be written as

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$$G^{\alpha}(t,\varepsilon)=\begin{cases} cos\left(\frac{\varepsilon^{\alpha}}{\alpha}\right) sin\left(\frac{t^{\alpha}}{\alpha}\right) & 0\le t<\varepsilon \\ sin\left(\frac{\varepsilon^{\alpha}}{\alpha}\right) cos\left(\frac{t^{\alpha}}{\alpha}\right) & \varepsilon<t\le(\alpha\pi)^{\frac{1}{\alpha}} \end{cases}$$

Therefore, our intended solution can be written as follows:

$$x(t)=-\int_0^{(\alpha\pi)^{\frac{1}{\alpha}}} G^{\alpha}(t,\varepsilon)e^{\frac{\varepsilon^{\alpha}}{\alpha}}\frac{1}{\varepsilon^{1-\alpha}}d\varepsilon$$

$$=-\int_0^t sin\left(\frac{\varepsilon^{\alpha}}{\alpha}\right)cos\left(\frac{t^{\alpha}}{\alpha}\right)e^{\frac{\varepsilon^{\alpha}}{\alpha}}\frac{1}{\varepsilon^{1-\alpha}}d\varepsilon-\int_t^{(\alpha\pi)^{\frac{1}{\alpha}}} cos\left(\frac{\varepsilon^{\alpha}}{\alpha}\right)sin\left(\frac{t^{\alpha}}{\alpha}\right)e^{\frac{\varepsilon^{\alpha}}{\alpha}}\frac{1}{\varepsilon^{1-\alpha}}d\varepsilon$$

$$=-\frac{1}{2}cos\left(\frac{t^{\alpha}}{\alpha}\right)\left[e^{\frac{\varepsilon^{\alpha}}{\alpha}}\left(sin\left(\frac{\varepsilon^{\alpha}}{\alpha}\right)-cos\left(\frac{\varepsilon^{\alpha}}{\alpha}\right)\right)\right]_{\varepsilon=0}^{\varepsilon=t}$$

$$-\frac{1}{2}sin\left(\frac{t^{\alpha}}{\alpha}\right)\left[e^{\frac{\varepsilon^{\alpha}}{\alpha}}\left(sin\left(\frac{\varepsilon^{\alpha}}{\alpha}\right)-cos\left(\frac{\varepsilon^{\alpha}}{\alpha}\right)\right)\right]_{\varepsilon=t}^{\varepsilon=(\alpha\pi)^{\frac{1}{\alpha}}}$$

$$=-\frac{1}{2}cos\left(\frac{t^{\alpha}}{\alpha}\right)\left[e^{\frac{\varepsilon^{\alpha}}{\alpha}}\left(sin\left(\frac{t^{\alpha}}{\alpha}\right)-cos\left(\frac{t^{\alpha}}{\alpha}\right)\right)+1\right]$$

$$-\frac{1}{2}sin\left(\frac{t^{\alpha}}{\alpha}\right)\left[e^{\pi}-e^{\frac{t^{\alpha}}{\alpha}}\left(sin\left(\frac{t^{\alpha}}{\alpha}\right)-cos\left(\frac{t^{\alpha}}{\alpha}\right)\right)\right]$$

$$=e^{\pi}\left[-sin\left(\frac{t^{\alpha}}{\alpha}\right)cos\left(\frac{t^{\alpha}}{\alpha}\right)+\frac{1}{2}\right]-\frac{1}{2}cos\left(\frac{t^{\alpha}}{\alpha}\right)-\frac{1}{2}e^{\pi}sin\left(\frac{t^{\alpha}}{\alpha}\right)$$

Finally, we investigate the generalized Hyers-Ulam stability of the conformable linear inhomogeneous differential equation of second order (21) in the class of continuously twice α-differentiable functions.

Theorem 19. *Let $p,q,f:[a,b]\to R$ be continuous functions and let p be α-differentiable function on $[a,b]$. Assume that the conformable homogeneous differential equation (20) has its only solution as the identically null function. If a twice continuously α-differentiable function $x:[a,b]\to R$ satisfies the inequality*

$$|T_{\alpha}(p(t)T_{\alpha}x(t))-q(t)x(t)+f(t)|\le\varphi(t), \tag{22}$$

for all $t\in[a,b]$, where $\varphi:[a,b]\to[0,\infty)$ is given that such of the following integrals exists, then there exists a solution $x_0:[a,b]\to R$ of (21) such that

$$|x(t)-x_0(t)|\le\frac{1}{|K|}\left(|x_1(t)|\int_b^t|x_2(\varepsilon)|\varphi(\varepsilon)\frac{1}{\varepsilon^{1-\alpha}}d\varepsilon+|x_2(t)|\int_a^t|x_1(\varepsilon)|\varphi(\varepsilon)\frac{1}{\varepsilon^{1-\alpha}}d\varepsilon\right), \tag{23}$$

where K is a nonzero constant and $x_1(t)$ and $x_2(t)$ are two linearly independent solutions of (20a) that verify (20b) and (20c), respectively (see Theorem 18).

Proof. If we define a continuous function $s:[a,b]\to R$ by

$$s(t)=T_{\alpha}(p(t)T_{\alpha}x(t)-q(t)x(t)), \tag{24}$$

for all $t\in[a,b]$, then it follows (22) that

$$|s(t)+f(t)|\le\varphi(t), \tag{25}$$

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for all $t \in [a,b]$. In view of Theorem 18 and (24), we have

$$x(t) = -\int_a^b G^\alpha(t,\varepsilon)s(\varepsilon)\frac{1}{\varepsilon^{1-\alpha}}d\varepsilon = -\int_t^b \frac{x_1(t)x_2(\varepsilon)}{K}s(\varepsilon)\frac{1}{\varepsilon^{1-\alpha}}d\varepsilon - \int_a^t \frac{x_1(\varepsilon)x_2(t)}{K}s(\varepsilon)\frac{1}{\varepsilon^{1-\alpha}}d\varepsilon \tag{26}$$

where K is a nonzero constant because $x_1(t)$ and $x_2(t)$ are two linearly independent solutions of (20a) that verify (20b) and (20c), respectively (see Theorem 18).

We now define a function $x_0:[a,b] \to R$ by

$$x_0(t) = \int_t^b \frac{x_1(t)x_2(\varepsilon)}{K}f(\varepsilon)\frac{1}{\varepsilon^{1-\alpha}}d\varepsilon + \int_a^t \frac{x_1(\varepsilon)x_2(t)}{K}f(\varepsilon)\frac{1}{\varepsilon^{1-\alpha}}d\varepsilon, \tag{27}$$

for all $t \in [a,b]$. According to Theorem18, it is obvious that x_0 is a solution of (21). Moreover, it follows from (25), (26) and (27) that

$$|x(t) - x_0(t)| \le \left|-\int_t^b \frac{x_1(t)x_2(\varepsilon)}{K}(s(\varepsilon)+f(\varepsilon))\frac{1}{\varepsilon^{1-\alpha}}d\varepsilon - \int_a^t \frac{x_1(\varepsilon)x_2(t)}{K}(s(\varepsilon)+f(\varepsilon))\frac{1}{\varepsilon^{1-\alpha}}d\varepsilon\right|$$
$$\le \frac{1}{|K|}\left(|x_1(t)|\int_b^t |x_2(\varepsilon)|\varphi(\varepsilon)\frac{1}{\varepsilon^{1-\alpha}}d\varepsilon + |x_2(t)|\int_a^t |x_1(\varepsilon)|\varphi(\varepsilon)\frac{1}{\varepsilon^{1-\alpha}}d\varepsilon\right)$$

for all $t \in [a,b]$. □

Remark 4. Theorem 19 reduces to [22] (Theorem 3.2) in the case $\alpha = 0$ and using the Green´s Function.

5. Conclusions

In this research paper, we have proposed some results referring to the conformable boundary value problems. The conformable second order Sturm-Picone identity has been proven, and its Sturm's theorems of comparison and separation have been successfully established. As in the classical case, an important application of the Sturm's comparison theorem is to provide a clear understanding of the zero set of non-trivial solutions of the conformable Bessel's equation. For a conformable Sturm-Liouville system, we have defined the Green´s function and established its properties. The conformable Green's function is applied to construct the solution of the inhomogeneous problem of Sturm-Liouville, whose associated homogeneous problem has its only solution as the identically null function. Finally, we have proved the generalized Hyers-Ulam stability of the conformable linear inhomogeneous differential equation of second order (21) in the class of continuously twice α-differentiable functions.

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