Solution to a class of advanced-retarded differential equations

Operator solution

Héctor M. Moya-Cessa

Abstract We give an operator solution to an advanced-retarded differential equation. The application of the operators involved produces a solution in terms of Bessel functions.

Keywords Advanced-retarded differential equations · Operator techniques · Photonic circuits

1 Introduction

Recently it has been shown how photonic circuits may be constructed such that their dynamics obeys advanced-retarded (AR) differential equations [1]. Álvarez-Rodríguez *et al.* showed the similarities between the AR differential equations and discrete differential equations that arise in the propagation of classical or quantum light through waveguide arrays [2–5].

AR differential equations, also known as mixed functional differential equations, are equations for which the derivative of the function explicitly depends on the same function evaluated at different values of the variable [6–10]. They are useful to describe phenomena that contains feedback/feedforward interactions in their evolution [11–13].

2 Operator solution

We consider an AR differential equation of the following form

$$i\frac{dx(t)}{dt} = \alpha t x(t) + \lambda[x(t+\tau) + x(t-\tau)] \tag{1}$$

Héctor M. Moya-Cessa
Instituto Nacional de Astrofísica Óptica y Electrónica, Luis Enrique Erro 1, Santa María Tonantzintla, Puebla, 72840 Mexico
Tel.: +52-222-2663100
E-mail: hmmc@inaoep.mx

and, by defining the operator $D_t = \frac{d}{dt}$, with commutator $[D_t, t] = 1$, we can rewrite (1) as

$$i\frac{dx(t)}{dt} = [\alpha t + \lambda(e^{\tau D_t} + e^{-\tau D_t})]x(t) \tag{2}$$

that has as the simple solution

$$x(t) = \frac{1}{N} e^{\beta \sinh(\tau D_t)} e^{-i\alpha \frac{t^2}{2}} x(0), \tag{3}$$

with $x(0)$ the initial condition, N a normalization constant and we have defined $\beta = \frac{2\lambda}{\alpha\tau}$. It is not difficult to show that this is a solution because

$$\frac{dx(t)}{dt} = -\frac{i}{N}\alpha e^{\beta \sinh(\tau D_t)} t e^{-i\alpha \frac{t^2}{2}} x(0), \tag{4}$$

that, by inserting a unit operator, $e^{-\beta \sinh(\tau D_t)} e^{\beta \sinh(\tau D_t)}$ after the linear term gives

$$\frac{dx(t)}{dt} = -\frac{i}{N}\alpha e^{\beta \sinh(\tau D_t)} t e^{-\beta \sinh(\tau D_t)} x(t), \tag{5}$$

and finally applying the relation $e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \ldots$, with A and B two arbitrary operators, equation (2) is recovered

$$\frac{dx(t)}{dt} = -\frac{i}{N}\alpha[t + \beta\tau \cosh(\tau D_t)]x(t). \tag{6}$$

By writing $e^{i\alpha \sinh(\tau D_t)}$ in terms of Bessel functions

$$x(t) = \frac{x(0)}{N} \sum_{n=-\infty}^{\infty} J_n(\beta)\, e^{n\tau D_t} e^{-i\alpha \frac{t^2}{2}} \tag{7}$$

we end up with the final form

$$x(t) = \frac{x(0)}{N} \sum_{n=-\infty}^{\infty} J_n(\beta)\, e^{-i\alpha \frac{(t+n\tau)^2}{2}}, \tag{8}$$

so that the normalization constant takes the value

$$N = \sum_{n=-\infty}^{\infty} J_n(\beta)\, e^{-i\alpha \frac{n^2\tau^2}{2}}. \tag{9}$$

We plot in Figures 1-3 the absolute value squared of the amplitude $x(t)$, $|x(t)|^2$ for different values of α and β. It may be seen the periodic behaviour of the solutions, but also strong variations in $|x(t)|^2$ given relatively small differences between the parameters used. This is clear in Fig. 1. On the other hand, Fig. 3 (a) shows the solution given in Subsection 3.1.

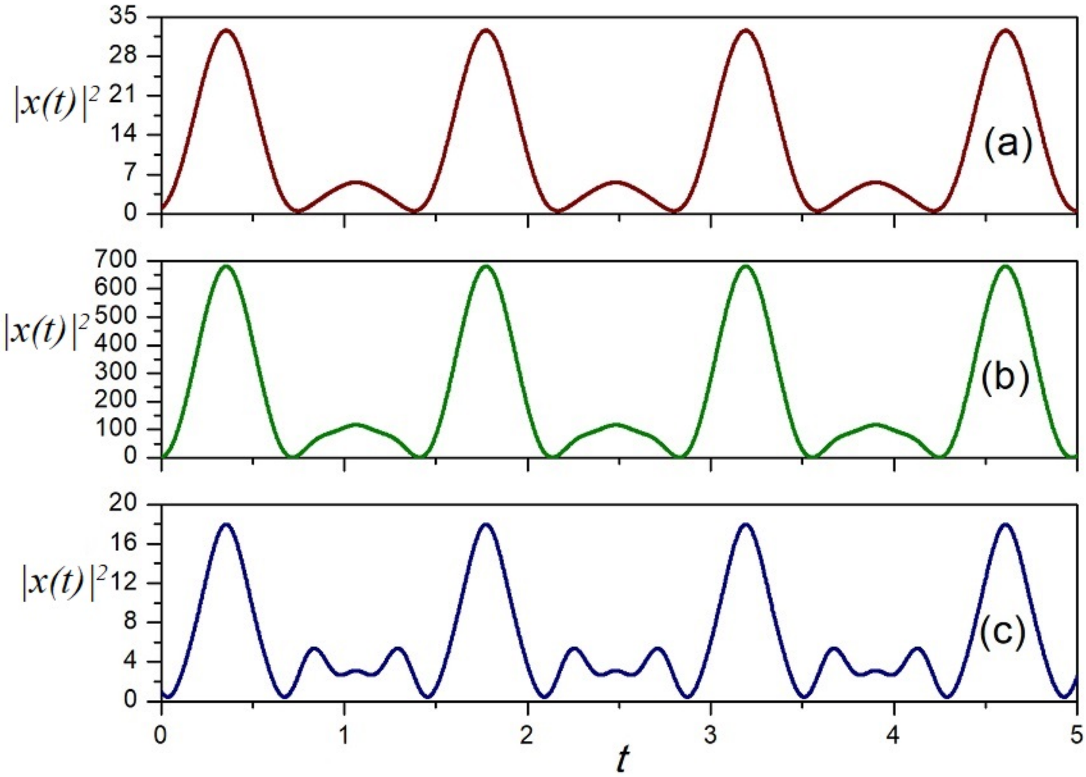

Fig. 1 $\alpha = 1.25$, $\tau = \sqrt{\pi}$ and (a) $\beta = 1.25$, (b) $\beta = 1.5$ and (c) $\beta = 2$

3 Some special cases

For some sets of parameters the solution (8) may take some closed forms. In this Section we look at those cases.

3.1 $\alpha\tau^2 = 2\pi$

In this case we may write the solution as

$$x(t) = \frac{x(0)}{N} \sum_{n=-\infty}^{\infty} J_n(\beta)\, e^{-i\alpha\frac{(t^2+2nt\tau)}{2}} e^{-in^2\pi}, \tag{10}$$

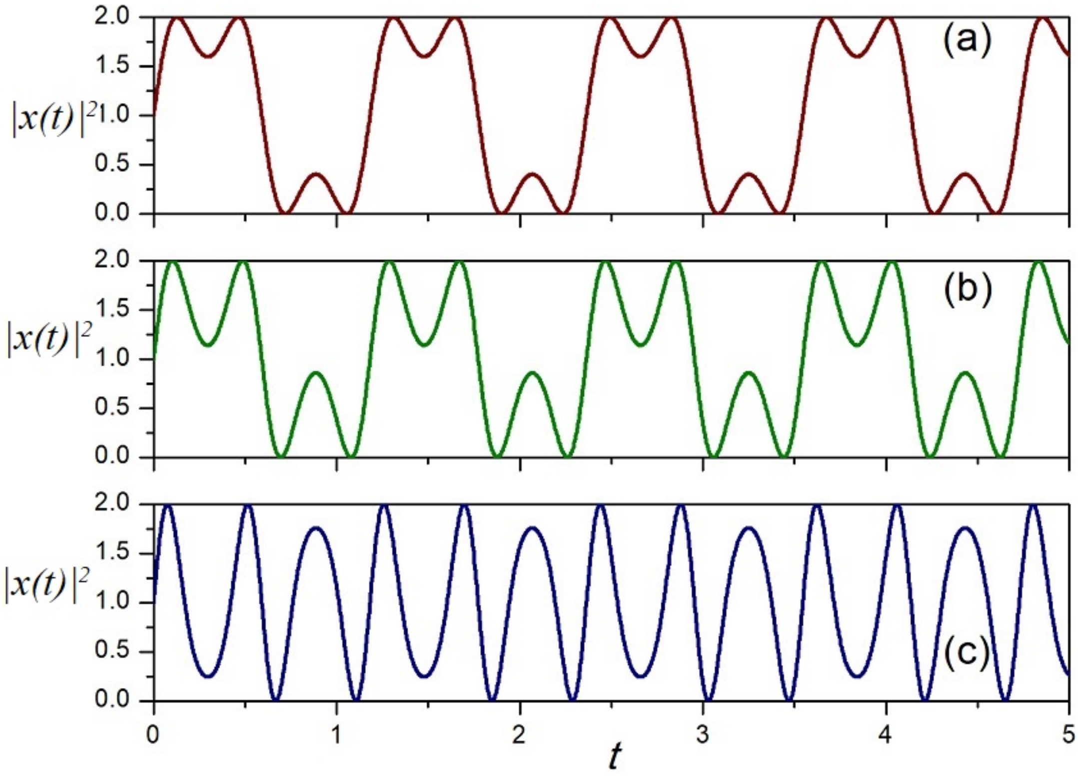

Fig. 2 $\alpha = 1.5$, $\tau = \sqrt{\pi}$ and (a) $\beta = 1.25$, (b) $\beta = 1.5$ and (c) $\beta = 2$

and, because $e^{-in^2\pi} = e^{-in\pi}$ we rewrite the above equation as

$$x(t) = \frac{x(0)}{N} \sum_{n=-\infty}^{\infty} J_n(\beta)\, e^{-i\alpha\frac{t^2}{2}} e^{-in(t\alpha\tau-\pi)}, \tag{11}$$

that may be added by using the generating function of Bessel functions [14–17]

$$x(t) = \frac{x(0)}{N} e^{-i\alpha\frac{t^2}{2}} e^{i\beta\sin(t\alpha\tau)}. \tag{12}$$

3.2 $\alpha\tau^2 = \pi$

The solution in this case takes the form

$$x(t) = \frac{x(0)}{N} \sum_{n=-\infty}^{\infty} J_n(\beta)\, e^{-i\alpha\frac{(t^2+2nt\tau)}{2}} e^{-in^2\frac{\pi}{2}}. \tag{13}$$

We see that the relevant term is $e^{-i\frac{\pi}{2}n^2} = (-i)^{n^2}$ that for n odd gives $-i$ while for even gives 1, such that we can split (13) in even and odd series

$$x(t) = \frac{x(0)e^{-i\alpha\frac{t^2}{2}}}{N} \sum_{n=-\infty}^{\infty} J_{2n}(\beta)\, e^{-i\alpha 2nt\tau} - i\frac{x(0)e^{-i\alpha\frac{t^2}{2}}}{N} \sum_{n=-\infty}^{\infty} J_{2n+1}(\beta)\, e^{-i\alpha(2n+1)t\tau}. \tag{14}$$

We rewrite the above equation in the form

$$x(t) = \frac{x(0)e^{-i\alpha\frac{t^2}{2}}}{N} \sum_{n=-\infty}^{\infty} J_n(\beta)\, e^{-i\alpha nt\tau}\, [1+(-1)^n] - i\frac{x(0)e^{-i\alpha\frac{t^2}{2}}}{N} \sum_{n=-\infty}^{\infty} J_n(\beta)\, e^{-i\alpha nt\tau}\, [1-(-1)^n], \tag{15}$$

that, by using the generating functions of Bessel functions [14–17] gives finally

$$x(t) = \frac{x(0)e^{-i\alpha\frac{t^2}{2}}}{N} \left([1-i]e^{-i\beta\sin(t\alpha\tau)} + [1+i]e^{i\beta\sin(t\alpha\tau)}\right). \tag{16}$$

4 Conclusion

We have shown that, for the specific form of equation (1), we can use operator techniques to solve it. The solution produces infinite series of Bessel functions that for particular values of parameters may be given a closed form. The series solution we have provided may be a hint to find solutions to some other types of AR differential equations (different time dependent parameters).

Acknowledgement. I would like to thank Dr. Armando Perez-Leija for useful discussions.

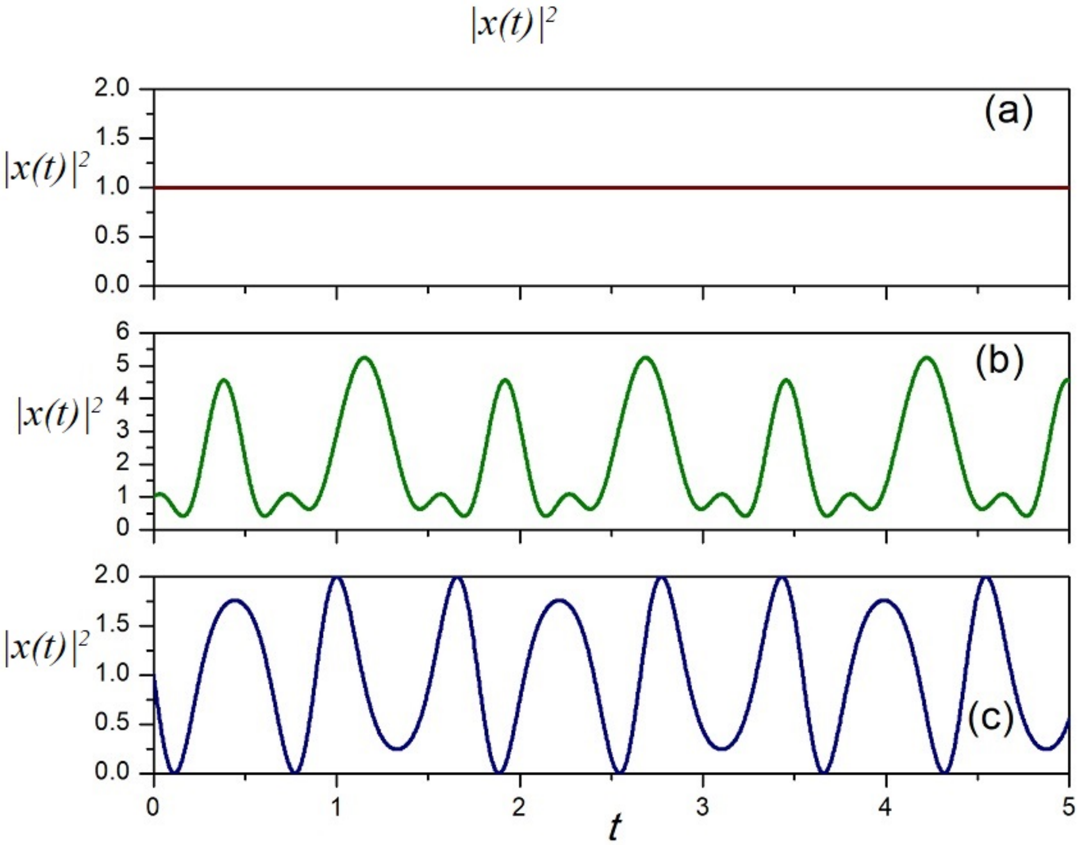

Fig. 3 $\alpha = 2$, $\beta = 2$ and (a) $\tau = \sqrt{\pi}$, (b) $\tau = \sqrt{\pi/3}$ and (c) $\tau = \sqrt{\pi/4}$

A

In this appendix we show, by using properties of the Bessel functions, that indeed, equation (8) is a solution to equation (1). Given the equation

$$x(t) = \frac{x(0)}{N} \sum_{n=-\infty}^{\infty} J_n(\beta)\, e^{-i\alpha \frac{(t+n\tau)^2}{2}}, \tag{17}$$

its derivative gives

$$\frac{dx(t)}{dt} = \frac{-i\alpha x(0)}{N} \sum_{n=-\infty}^{\infty} (t + n\tau) J_n(\beta)\, e^{-i\alpha \frac{(t+n\tau)^2}{2}}, \tag{18}$$

that, by applying the identity $\frac{2n}{x} J_n(x) = J_{n+1}(x) + J_{n-1}(x)$ may be rewritten as

$$\frac{dx(t)}{dt} = -i\alpha t x(t) \tag{19}$$

$$-\frac{i\alpha\tau\beta x(0)}{2N}\sum_{n=-\infty}^{\infty}[J_{n+1}(\beta)+J_{n-1}(\beta)]e^{-i\alpha\frac{(t+n\tau)^2}{2}}.$$

Changing the indices of the sums gives

$$\frac{dx(t)}{dt}=-i\alpha tx(t)-i\lambda\frac{x(0)}{N}\left[\sum_{n=-\infty}^{\infty}J_n(\beta)\,e^{-i\alpha\frac{[t+(n-1)\tau]^2}{2}}\right. \tag{20}$$
$$\left.-\sum_{n=-\infty}^{\infty}J_n(\beta)\,e^{-i\alpha\frac{[t+(n+1)\tau]^2}{2}}\right],$$

that gives equation (1).

References

1. Alvarez-Rodriguez, U., Perez-Leija, A., Egusquiza, I.L., Gräfe, M., Sanz, M., Lamata, L., Szameit, A. and Solano, E. Advanced-Retarded Differential Equations in Quantum Photonic Systems, Scientific Reports, 7, 42933 (2016)
2. Perez-Leija, A., Keil, R., Moya-Cessa, H., Szameit, A., and Christodoulides, D.N. Perfect transfer of path-entangled photons in J_x photonic lattices. Physical Review A, 87, 022303 (2013).
3. Perez-Leija, A., Hernandez-Herrejon, J.C., Moya-Cessa, H., Szameit, A., and Christodoulides, D.N. Generating photon encoded W states in multiport waveguide array systems. Physical Review A, 87, 013842 (2013).
4. Keil, R., Perez-Leija, A., Aleahmad, P., Moya-Cessa, H., Christodoulides, D.N., and Szameit, A. Observation of Bloch-like revivals in semi-infinite Glauber-Fock lattices. Optics Letters, 37, 3801–3803 (2012).
5. Rodriguez-Lara, B.M., Soto-Eguibar, F. Zarate-Cardenas, A. and Moya-Cessa, H.M. A Classical simulation of nonlinear Jaynes-Cummings and Rabi models in photonic lattices, Optics Express, 21, 12888-12898 (2013).
6. Myshkis, A. D. Mixed Functional Differential Equations. J. Math. Sci. 129, 5 (2005).
7. Rustichini, A. Functional Differential Equations of Mixed Type: The Linear Autonomous Case. J. Dyn. Diff. Eq. 1, 2 (1989).
8. Mallet-Paret, J. The Fredholm alternative for functional differential equations of mixed type. J. Dyn. Diff. Eq. 11, 1 (1999).
9. Berezansky, L., Braverman, E. and Pinelas, S. On nonoscillation of mixed advanced-delay differential equations with positive and negative coefficients. Comput. Math. Appl. 58, 766 (2009).
10. Ford, N. J., Lumb, P. M., Lima, P. M. and Teodoro, M. F. The numerical solution of forward-backward differential equations: Decomposition and related issues. J. Comput. Appl. Math. 234, 2745 (2010).
11. Lucero, J. C. Advanced-delay differential equation for aeroelastic oscillations in physiology. Biophys. Rev. Lett. 3, 125 (2008).
12. Collard, F., Licandro, O. and Puc, L. A. The short-run dynamics of optimal growth models with delays. Ann. Econ. Stat. 90, 127 (2008).
13. Chi, H., Bell, J. and Hassard, B. Numerical solution of a nonlinear advance-delay-differential equation from nerve conduction theory. J. Math. Biol. 24, 583 (1986).
14. Arfken G B and Weber H J 2005 Mathematical Methods for Physicists 6th edn (Burlington, MA: Elsevier Academic Press)
15. Moya-Cessa, H.M. and Soto-Eguibar, F. Differential equations: an operational approach, (Rinton Press, New Jersey, 2011).
16. G. Dattoli, L. Giannessi, M. Richetta, and A. Torre, Il Nuovo Cimento 103B (1989) 149.
17. G. Dattoli, A. Torre, S. Lorenzutta, G. Maino, C. Chiccoli, Il Nuovo Cimento 106B (1991) 21.