Appendix:

Supplementary Information

(Mathematica v12.1.1.0 code of TraditionalForm)

Part 1. The Square of the Norm of the Average Velocity is Proportional to the Number of Vectors

Definition: Particles with a higher mass level composed of k particles are called kth-order particles. Then, the velocity of a kth-order particle is the velocity of the overall center of mass of the k particles, which is the average of the velocity vectors of all these particles.
Assumption: Each particle is moving at the same speed and in a random direction in space.
Thus, the projection of the velocity vector of a kth-order particle onto one of the three equivalent coordinate axes of the 3-dimensional Cartesian coordinate system is the mean value of the projection (onto the same axis) of the velocity vectors of the 1st-order particles forming the kth-order particle, which follow the same distribution; therefore, it approximately follows a normal distribution (central limit theorem).
There are three equivalent (approximate) normal distributions, one on each of the three axes, which are not completely independent. However, James Clerk Maxwell and Ludwig Boltzmann proved that these distribution can, in fact, be equivalently treated as completely independent. This is because randomly selecting a vector is equivalent to randomly determining a three-axis coordinate; moreover, the problem of the momentum transfer of gas molecules participating in random collisions is also equivalent to the problem discussed in this article.

First, the probability density of the norm of the 3-dimensional vectors formed by three normal distribution $N(0, \sigma_2)$ components that are independent on three coordinate axes is calculated.

In[∘]:=
```
Clear["Global`*"];
𝒟 = Simplify[PDF[TransformedDistribution[x^2 + y^2 + z^2,
        {x, y, z} ≈ ProductDistribution[{NormalDistribution[0, σ2], 3}]], x], Assumptions → σ2 > 0];
𝒟1 = PDF[TransformedDistribution[√x, x ≈ ProbabilityDistribution[𝒟, {x, 0, +∞}]], x]
```

Out[∘]=
$$\begin{cases} \frac{\sqrt{\frac{2}{\pi}}\, x^2\, e^{-\frac{x^2}{2\sigma_2^2}}}{\sigma_2^3} & x > 0 \\ 0 & \text{True} \end{cases}$$

Then, we find the probability density of the Maxwell distribution with scale parameter σ_2:

In[∘]:=
```
𝒟2 = PDF[MaxwellDistribution[σ2], x]
```

Out[∘]=
$$\begin{cases} \frac{\sqrt{\frac{2}{\pi}}\, x^2\, e^{-\frac{x^2}{2\sigma_2^2}}}{\sigma_2^3} & x > 0 \\ 0 & \text{True} \end{cases}$$

Therefore, these two probability densities are equal:

In[∘]:=
```
𝒟1 − 𝒟2
```

Out[∘]= 0

We verify the above conclusion (c is the speed of 1st-order particle; n is the number of vectors) (This code takes approximately 13 hours):

```
In[∘]:= c = 1;
n = 1000;
m = 3 000 000;
dd = {};
ProgressIndicator[Dynamic[i], {1, m}]
For[i = 1, i < m, i++,
    𝓗 = RandomPoint[Sphere[{0, 0, 0}, c], n];
    𝓗𝓗 = Norm[Total /@ Transpose[𝓗]];
    dd = AppendTo[dd, 𝓗𝓗]];
𝓓 = SmoothKernelDistribution[dd, {"Adaptive", Automatic, Automatic}];
s1 = Plot[{PDF[𝓓, x], PDF[MaxwellDistribution[c/√3 √n], x]},
    {x, 0, 100 c}, PlotStyle → {{Red, Thickness → 0.0032}, {Blue, Thickness → 0.0032}},
    Frame → {{True, False}, {True, False}}, FrameLabel → {"Momentum", "Probability Density"},
    FrameStyle → Directive[Black, Thickness → 0.0017],
    LabelStyle → Directive[Black, FontFamily → "Arial", FontSize → 14],
    Epilog → Inset[LineLegend[{Directive[Blue, Thickness[0.0032]], Directive[Red, Thickness[0.0032]]},
          {Style["Theoretical", FontFamily → "Arial", FontSize → 14],
            Style["Simulated", FontFamily → "Arial", FontSize → 14]}, LegendFunction →
            (Framed[#, RoundingRadius → 4, FrameStyle → GrayLevel[0.6]] &)], Scaled[{0.732, 0.644}]]]
Export["/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure S1.png",
    s1, Background → None, ImageResolution → 600];
```

Out[∘]=

Out[∘]=

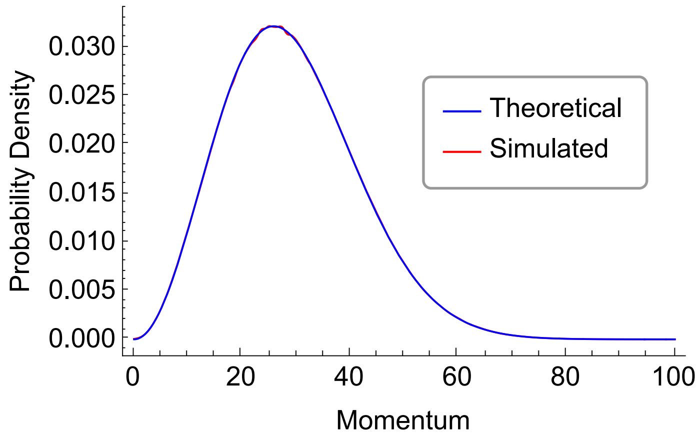

Figure S1. Probability density of momentum norm formed by 1000 randomly moving particles with c = 1 (theoretical and simulated results).

Accordingly, the norm of the 3-dimensional vectors formed by three normal distribution $N(0, \sigma_2)$ components which are independent on three coordinate axes follows the Maxwell distribution with the scale parameter σ_2.

Suppose that the standard deviation of the projection of the velocity of any one of the k equivalent particles forming a kth-order particle onto each equivalent coordinate axis is σ. Then, the standard deviation of the projection of the velocity of a kth-order particle onto each equivalent coordinate axis (i.e., the mean value of the projection of the velocity of 1st-order particle) is $\frac{\sigma}{\sqrt{k}}$, namely, the projection onto each coordinate axis (approximate) follows a normal distribution with a mean value of 0 and a standard deviation of $\frac{\sigma}{\sqrt{k}}$. As a result, the speed of kth-order particles follows the Maxwell distribution with scale parameter $\frac{\sigma}{\sqrt{k}}$.

Then, the average velocity of the kth-order particles is

In[∘]:= $\bar{v}$ = Mean[MaxwellDistribution[$\frac{\sigma}{\sqrt{k}}$]]

Out[∘]= $$\frac{2\sqrt{\frac{2}{\pi}}\,\sigma}{\sqrt{k}}$$

For the kth-order particles in different reference frames ($\mathcal{R}_u$ and $\mathcal{R}_0$) and with different standard deviations (σ_u and σ_0), the ratio of their average velocity $\bar{v}_u / \bar{v}_0$ =

In[∘]:= $$\frac{2\sqrt{\frac{2}{\pi}}\,\sigma_u}{\sqrt{k}} \Big/ \frac{2\sqrt{\frac{2}{\pi}}\,\sigma_0}{\sqrt{k}}$$

Out[∘]= $$\frac{\sigma_u}{\sigma_0}$$

Therefore, the ratio of σ_u to σ_0 is the ratio between the average speeds of particles of higher mass levels in $\mathcal{R}_u$ and $\mathcal{R}_0$.

For k_1th- and k_2th-order particles, the ratio of their average velocity $\bar{v}_1 / \bar{v}_2$ =

In[∘]:= $$\frac{2\sqrt{\frac{2}{\pi}}\,\sigma}{\sqrt{k_1}} \Big/ \frac{2\sqrt{\frac{2}{\pi}}\,\sigma}{\sqrt{k_2}}$$

Out[∘]= $$\frac{\sqrt{k_2}}{\sqrt{k_1}}$$

And because: $m_1 = \mu\, k_1$ and $m_2 = \mu\, k_2$, where μ is the scale factor or the mass of 1st-order particle. $\bar{v}_1 / \bar{v}_2$ is also equal to

In[∘]:= Simplify[$\frac{\sqrt{\frac{m_2}{\mu}}}{\sqrt{\frac{m_1}{\mu}}}$, Assumptions → $\mu > 0$]

Out[∘]= $$\frac{\sqrt{m_2}}{\sqrt{m_1}}$$

Therefore, the square of the average velocity of particles is directly proportional to the mass of particles or the number of 1st-order particles forming it.

Part 2. Special Relativistic Effects on Infinitesimal Particles

Correspondence:

The mixed distribution of $\mathcal{D}_1$and $\mathcal{D}_2$ is represented by $\mathcal{D}$12;

The mixed distribution of $\mathcal{D}_3$and $\mathcal{D}_4$ is represented by $\mathcal{D}$34;

The rest of the symbols are consistent with those in the main text.

In[∘]:=
```
Clear["Global`*"];
𝒟 = TransformedDistribution[c Cos[θ] Sin[ArcCos[η]],
    {θ ≈ UniformDistribution[{-π, π}], η ≈ UniformDistribution[{-1, 1}]}];
𝒟1 = TransformedDistribution[c Cos[θ] Sin[ArcCos[η]],
    {θ ≈ UniformDistribution[{-π, π}], η ≈ UniformDistribution[{u/c, 1}]}];
𝒟2 = TransformedDistribution[c Cos[θ] Sin[ArcCos[η]],
    {θ ≈ UniformDistribution[{-π, π}], η ≈ UniformDistribution[{-1, u/c}]}];
𝒟3 = TruncatedDistribution[{u, c}, UniformDistribution[{-c, c}]];
𝒟4 = TruncatedDistribution[{-c, u}, UniformDistribution[{-c, c}]];
𝒟34 = MixtureDistribution[{w, 1 - w}, {𝒟3, 𝒟4}];
Simplify[Mean[𝒟34], Assumptions → 0 < u < c]
```

Out[∘]= $\frac{1}{2}(c(2w-1)+u)$

Let the mean value expression be $\frac{1}{2}(c(2w-1)+u) = u$, and then we find the weight w

In[∘]:=
```
Reduce[1/2 (c (2 w - 1) + u) == u, w]
```

Out[∘]= $(u = 0 \wedge c = 0) \vee \left(c \neq 0 \wedge w = \frac{c+u}{2c}\right)$

Then, the mixed distribution $\mathcal{D}12$ consisting of $\mathcal{D}_1$ and $\mathcal{D}_2$ can be calculated in accordance with this weight w. The analytical form of $\mathcal{D}12$ cannot be given by Mathematica. Therefore, the standard deviation of $\mathcal{D}12$ is calculated directly (This code takes approximately 72 seconds).

In[∘]:=
```
w = (c + u)/(2 c);
𝒟12 = MixtureDistribution[{w, 1 - w}, {𝒟1, 𝒟2}];
σu = Simplify[StandardDeviation[𝒟12], Assumptions → 0 < u < c]
```

Out[∘]= $\frac{\sqrt{c^2-u^2}}{\sqrt{3}}$

The standard deviation of $\mathcal{D}34$ is the same.

In[∘]:=
```
Simplify[StandardDeviation[𝒟34], Assumptions → 0 < u < c]
```

Out[∘]= $\frac{\sqrt{c^2-u^2}}{\sqrt{3}}$

Then, the ratio between σ_u and the velocity components on the x-axis of the particles in $\mathcal{R}_0$ can be obtained.

In[∘]:=
```
Simplify[σu / StandardDeviation[𝒟], Assumptions → 0 < u < c]
```

Out[∘]= $\frac{\sqrt{c^2-u^2}}{c}$

The same factor can also be obtained by evaluating the ratio of the standard deviation of $\mathcal{D}34$ to the standard deviation of the velocity components on the z-axis in $\mathcal{R}_0$.

In[∘]:=
```
Simplify[StandardDeviation[𝒟34] / StandardDeviation[UniformDistribution[{−c, c}]],
  Assumptions → 0 < u < c]
```

Out[∘]=
$$\frac{\sqrt{c^2 - u^2}}{c}$$

When $c = 10$ and $u = 6$, the distribution of $\mathcal{D}12$ on x- or y-axes is like this (This code takes approximately 350 seconds):

In[∘]:=
```
c = 10;
u = 6;
data = RandomVariate[𝒟12, 300 000 000];
𝒟0 = SmoothKernelDistribution[data, {"Adaptive", Automatic, Automatic}];
s2 = Plot[PDF[𝒟0, x], {x, −10, 10}, PlotRange → Full, PlotStyle → {Blue, Thickness → 0.0032},
   AxesLabel → {HoldForm[Speed], HoldForm[Probability Density]},
   AxesStyle → Directive[Black, Thickness → 0.001],
   LabelStyle → Directive[Black, FontFamily → "Arial", FontSize → 14]]
Export["/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure S2.png",
   s2, Background → None, ImageResolution → 600];
```

Out[∘]=

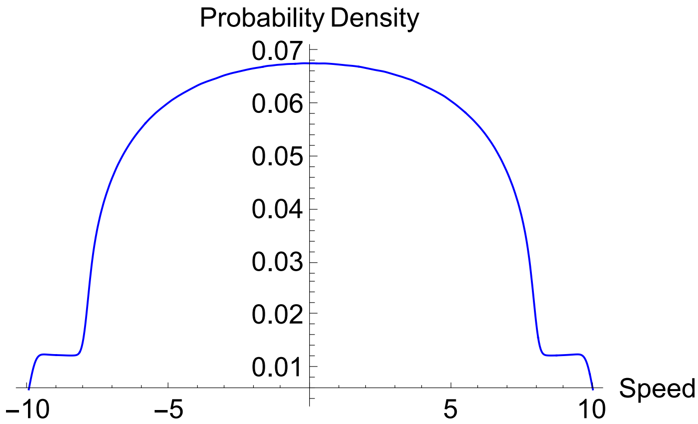

Figure S2. Simulated probability density of the mixed distribution $\mathcal{D}12$ when $c = 10$ and $u = 6$.

Part 3. Position aggregation being dominant is equivalent to velocity direction aggregation being dominant

Suppose that the standard deviation of the projection of the velocity of the 1st-order particle swarm onto each equivalent coordinate axis is σ. Then, the Maxwell speed density function related to mass is as follows:

In[∘]:=
```
𝒟 = MaxwellDistribution[σ/√k];
PDF[𝒟, x]
```

Out[∘]=
$$\begin{cases} \dfrac{\sqrt{\frac{2}{\pi}}\, k^{3/2}\, x^2\, e^{-\frac{k x^2}{2 \sigma^2}}}{\sigma^3} & x > 0 \\ 0 & \text{True} \end{cases}$$

In the main text, $\mathcal{Y}$ is generally used to represent the magnitude of the momentum in a microdomain $\mathcal{V}$. However, due to the characteristics of Mathematica, we replace $\mathcal{Y}$ with Y and substitute it with x in the above formula, namely,

$$x = \frac{Y}{k};$$

Then, the probability density of the magnitude of the momentum (Y) is

In[∘]:= $\mathcal{D} = \text{TransformedDistribution}\left[k\,x,\ x \approx \text{ProbabilityDistribution}\left[\frac{\sqrt{\frac{2}{\pi}}\,k^{3/2}\,x^2\,e^{-\frac{k x^2}{2\sigma^2}}}{\sigma^3}, \{x, -\infty, \infty\}\right]\right];$

Simplify[PDF[𝒟, Y], Assumptions → k > 0]

Out[∘]= $$\frac{\sqrt{\frac{2}{\pi}}\,Y^2\,e^{-\frac{Y^2}{2 k \sigma^2}}}{k^{3/2}\,\sigma^3}$$

Find the value of k at the maximum value of the above formula:

In[∘]:= $\text{Reduce}\left[D\left[\frac{\sqrt{\frac{2}{\pi}}\,Y^2\,e^{-\frac{Y^2}{2 k \sigma^2}}}{k^{3/2}\,\sigma^3}, \{k, 1\}\right] == 0, k\right]$

Out[∘]= $$(k \neq 0 \wedge \sigma \neq 0 \wedge Y = 0) \vee \left(\sigma \neq 0 \wedge Y \neq 0 \wedge k = \frac{Y^2}{3\,\sigma^2}\right)$$

And this formula is expanded by a Taylor series about the point $k = \frac{Y^2}{3\,\sigma^2}$ according to k.

In[∘]:= $\text{Series}\left[\frac{\sqrt{\frac{2}{\pi}}\,Y^2\,e^{-\frac{Y^2}{2 k \sigma^2}}}{k^{3/2}\,\sigma^3}, \left\{k, \frac{Y^2}{3\,\sigma^2}, 3\right\}\right]$

Out[∘]= $$\frac{3\,\sqrt{\frac{6}{\pi}}\,Y^2}{e^{3/2}\,\sigma^3\left(\frac{Y^2}{\sigma^2}\right)^{3/2}} - \frac{81\left(\sqrt{\frac{3}{2\pi}}\,\sigma\right)\left(k - \frac{Y^2}{3\sigma^2}\right)^2}{2\left(e^{3/2}\,Y^2\left(\frac{Y^2}{\sigma^2}\right)^{3/2}\right)} + \frac{81\,\sqrt{\frac{6}{\pi}}\,\sigma^3\left(k - \frac{Y^2}{3\sigma^2}\right)^3}{e^{3/2}\,Y^4\left(\frac{Y^2}{\sigma^2}\right)^{3/2}} + O\left(\left(k - \frac{Y^2}{3\,\sigma^2}\right)^4\right)$$

Therefore, this formula is a parabola with its opening facing downward about the point $k = \frac{Y^2}{3\,\sigma^2}$, which is symmetric! This result shows that the two aggregation effects are equivalent about the point $k = \frac{Y^2}{3\,\sigma^2}$.

Each mass level particle can be seen as being formed by particles of lower mass level. Regardless of how much mass aggregation or velocity direction aggregation the particles exhibit, it can be regarded as a slight one with a lower mass level. This is carried out step by step. Finally, the minimal deviation of the aggregation behavior of the position or velocity direction for infinitesimal particles can be achieved. The above results show that when the slightest aggregation behavior occurs, the difficulty of the two aggregation behaviors is equivalent. Therefore, the two aggregation behaviors can be replaced with each other for the statistical influence of the diffusion behavior on the infinitesimal particles. When the particles of higher mass level are investigated, their dynamic behaviors are affected by the dynamic behaviors of the lower-mass-level particles forming them (this is not contradictory to the viewpoint that each mass level particle can be treated equally, that is, when the behavior of particles of higher quality level is investigated, these particles can be regarded as free particles with a statistical effect, while the lower-mass-level particles forming them can be completely ignored. See Section **3.3.5.4** of the main text for more details. When the number of particles of higher mass level is small and does not meet the statistical conditions, their dynamic behavior should be investigated according to the dynamic behaviors of the particles of lower mass level. When the number of particles of higher mass level is large enough, their dynamic behaviors can be investigated separately, that is, they are not affected by the dynamic behaviors of lower-mass-level particles, and they show the minimal deviation of the aggregation behavior. However, when there are artificial regulations, even if the number of particles of higher mass level is large enough, they will still be affected by the dynamic behavior of particles of lower mass level), thus following the dynamic behaviors of the lower-mass-level particles. Therefore, the two aggregation behaviors are equivalent to that of the higher mass-level-particles. In summary, the two aggregation effects are interchangeable for any mass level particles and thus affect the diffusion behavior as a single statistical effect of the aggregation behavior.

Part 4. The Norm of the Component Vector is Proportional to the Number of Vectors Forming It

When the total vector value of a specified vector swarm is determined, the mean norms between different component vectors should be proportional to the number forming them. The following proves this viewpoint in detail.

It has been proven that the degree of slowdown on all three axes is the same in Part 2 of the Supplementary Information. Then, let $\mathcal{M}\mathrm{k}$ being the norm of momentum of k particles observed from $\mathcal{R}_u$, it follows Maxwell distribution with scale parameter $\frac{\sqrt{k}\ \sqrt{c^2-u^2}}{\sqrt{3}}$ when observing from $\mathcal{R}_u$. And when observing all of the moving particles in $\mathcal{R}_u$ from $\mathcal{R}_0$, all the randomly moving particles in $\mathcal{R}_u$ can be considered to have an additional velocity component u along the z-axis. Therefore, according to cosine theorem, the probability density of momentum norm formed by k particles in $\mathcal{R}_u$ observed in $\mathcal{R}_0$ can be expressed as (This code takes approximately 54 seconds):

In[ ]:=
```
Clear["Global`*"];
𝒟 = TransformedDistribution[√((k u)² + ℳk² − 2 k u ℳk Cos[ArcCos[η]]),
    {ℳk ≈ MaxwellDistribution[(√k √(c² − u²))/√3], η ≈ UniformDistribution[{−1, 1}]}];
FullSimplify[PDF[𝒟, x], Assumptions → c > 0 ∧ 0 < u < c]
```

Out[ ]=

$$\begin{cases} \dfrac{\sqrt{3}\ x\left(e^{\frac{6 u x}{c^2-u^2}}-1\right)e^{-\frac{3(k u+x)^2}{2k\left(c^2-u^2\right)}}}{k\,u\ \sqrt{2\pi c^2 k-2\pi k u^2}} & k>0\land((x>0\land k\,u>x)\lor k\,u<x) \\ -\dfrac{\sqrt{6\pi}\ \sqrt{c^2 k-u x}\left(5 u x-2 c^2 k\right)\mathrm{erf}\left(\frac{\sqrt{6}\ x}{\sqrt{c^2 k-u x}}\right)+4 x\, e^{\frac{6x^2}{u x-c^2 k}}\left(c^2(6k+2)-u(2u+3x)\right)-8x(c-u)(c+u)}{4\ \sqrt{6\pi}\ k^{5/2}\,u\,((c-u)(c+u))^{3/2}} & k\,u=x\land k>0 \end{cases}$$

The meaningful part (first branch) is selected to be verified. Note that the sampling with the replacement method in the particle swarm with a mean speed of u can simulate all of the cases of the particle swarm with a mean speed of u. (The following code takes averagely 4.2 + 0.5 hours)

```
In[∘]:= c = 1;
n = 1 000 000;
𝓗𝓗 = 0;
While[𝓗𝓗 < 2700,
   𝓗 = RandomPoint[Sphere[{0, 0, 0}, c], n];
   𝓗𝓗 = Norm[Total /@ Transpose[𝓗]]];
m = 1 000 000;
dd = {};
ProgressIndicator[Dynamic[j], {1, m}]
For[j = 1, j < m, j++,
   𝓗0 = RandomChoice[𝓗, 0.3 n];
   𝓗𝓗0 = Norm[Total /@ Transpose[𝓗0]];
   dd = AppendTo[dd, 𝓗𝓗0]];
𝒟 = SmoothKernelDistribution[dd, {"Adaptive", Automatic, Automatic}];
k = 0.3 n;
u = 𝓗𝓗/n;
s3 = Plot[{PDF[𝒟, x], (Sqrt[3] x (E^((6 u x)/(c^2 - u^2)) - 1) E^(-(3 (k u + x)^2)/(2 k (c^2 - u^2))))/(k u Sqrt[2 π c^2 k - 2 π k u^2])}, {x, 0, 2500},
   PlotStyle → {{Red, Thickness → 0.0032}, {Blue, Thickness → 0.0032}},
   Frame → {{True, False}, {True, False}}, FrameLabel → {"Momentum", "Probability Density"},
   FrameStyle → Directive[Black, Thickness → 0.0017],
   LabelStyle → Directive[Black, FontFamily → "Arial", FontSize → 14],
   Epilog → Inset[LineLegend[{Directive[Blue, Thickness[0.0032]], Directive[Red, Thickness[0.0032]]},
         {Style["Theoretical", FontFamily → "Arial", FontSize → 14],
          Style["Simulated", FontFamily → "Arial", FontSize → 14]}, LegendFunction →
          (Framed[#, RoundingRadius → 4, FrameStyle → GrayLevel[0.6]] &)], Scaled[{0.756, 0.644}]]]
Export["/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure S3.png",
   s3, Background → None, ImageResolution → 600];

Out[∘]=
Out[∘]=
```

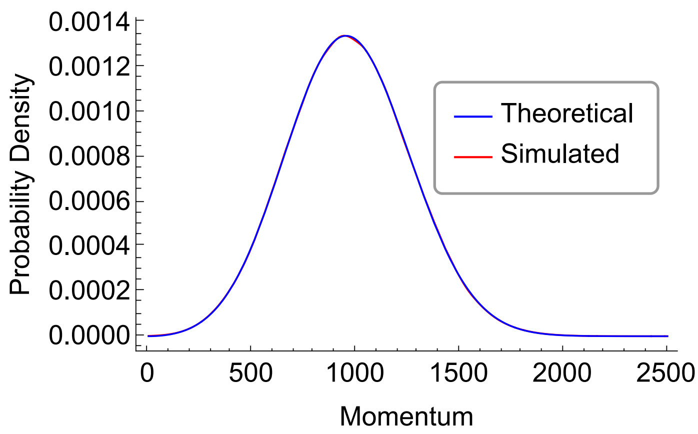

Figure S3. Probability density of momentum norm formed by 3×10^5 particles in $\mathcal{R}_u$ observed in $\mathcal{R}_0$ when $c = 1$ and u is a fixed value.

In view of the above conclusions, we find the mean value of this distribution (This code takes approximately 150 seconds).

In[∘]:= $\overline{\mathcal{Y}}_k$ = **FullSimplify**[

$$\textbf{Mean}\left[\textbf{ProbabilityDistribution}\left[\frac{\sqrt{3}\,x\left(e^{\frac{6ux}{c^2-u^2}}-1\right)e^{-\frac{3(ku+x)^2}{2k(c^2-u^2)}}}{k\,u\,\sqrt{2\pi c^2 k-2\pi k u^2}},\{x,0,+\infty\}\right]\right],\ \textbf{Assumptions}\to c>u>0\land k>0\right]$$

Out[∘]=
$$\frac{\left(c^2+(3k-1)u^2\right)\operatorname{erf}\left(\frac{\sqrt{\frac{3}{2}}\,k\,u}{\sqrt{k(c-u)(c+u)}}\right)+\sqrt{\frac{6}{\pi}}\,u\,e^{\frac{3ku^2}{2(u^2-c^2)}}\sqrt{k(c-u)(c+u)}}{3u}$$

We find the limit of the ratio of this mean value $\overline{\mathcal{Y}}_k$ and k when k approaches $+\infty$.

$$\textbf{Simplify}\left[\textbf{Limit}\left[\frac{\overline{\mathcal{Y}}_k}{k},\ k\to+\infty\right],\ \textbf{Assumptions}\to u>0\right]$$

Out[∘]=
$$\begin{cases} -u & \arg\left(c^2-u^2\right)\geq\pi \\ u & \text{True}\end{cases}$$

The second brunch is meaningful. Therefore, when k is a large number, the norm of the mean value $\overline{\boldsymbol{\mathcal{Y}}}_k$ is directly proportional to the number k forming $\overline{\boldsymbol{\mathcal{Y}}}_k$, namely $\overline{\mathcal{Y}}_k = k\cdot u$.

Eq. 21 in the main text determines the proportion of particle number distributed in various boxes partitioned by k, and these particles are distributed in each box of $\mathcal{V}$ with equal probability. That is, the particles are randomly extracted from the micro domain $\mathcal{V}$ to be distributed in each box. When the number of extractions is large enough, the norm of each component vector partitioned by k should be directly proportional to the number of particles according to the probability and the scale factor is u.

In view of the fact that the unique expansion of scalar $\mathcal{M}$ in the form of including power series is

$$\mathcal{M}=\sum_{k=1}^{\infty}\frac{e^{-\mathcal{M}}\mathcal{M}^k}{(k-1)!}$$

If the corresponding terms marked by k is directly proportional between the expansion of the norm $|\boldsymbol{\mathcal{M}}|$ of vector $\boldsymbol{\mathcal{M}}$ and the expansion of the scalar $\mathcal{M}$ representing the number of particles or let the numbers of particles be proportional to the norms of vectors they form, the number $\mathcal{M}$ of particles must be equal to the norm $|\boldsymbol{\mathcal{M}}|$ of the vector $\boldsymbol{\mathcal{M}}$ they form besides they are required to obey Poisson distribution. According to the above conclusion $\overline{\mathcal{Y}}_k = k\cdot u$, so the average speed $u = 1$ is needed in the system.

Next, we verify the standard deviations of this distribution in the three axes (This code takes averagely 291 seconds).

In[∘]:=
```
c = 1;
n = 10 000 000;
ℋℋ = 0;
While[ℋℋ < 6600,
    ℋ = RandomPoint[Sphere[{0, 0, 0}, c], n];
    ℋℋ = Norm[Total /@ Transpose[ℋ]]];
ℋ0 = RandomChoice[ℋ, 0.3 n];
ℋℋx = StandardDeviation[Transpose[ℋ0][[1]]]
ℋℋy = StandardDeviation[Transpose[ℋ0][[2]]]
ℋℋz = StandardDeviation[Transpose[ℋ0][[3]]]
```

Out[∘]= 0.57735

Out[∘]= 0.577374

Out[∘]= 0.577327

The standard deviation in theory is:

$$u = \frac{\mathcal{HH}}{n};$$

$$\frac{\sqrt{c^2 - u^2}}{\sqrt{3}}$$

Out[∘]= 0.57735

This result also verifies that the conclusions in Part 2 and Part 3 are both correct.

Part 5. The 2-Dimensional Situation Under the Same Conditions

This code takes approximately 22 seconds.

In[∘]:=
```
Clear["Global`*"];
Needs["NDSolve`FEM`"];
Ω = ImplicitRegion[16/10000 ≤ x^2 + y^2 ≤ 16, {x, y}];
mesh = ToElementMesh[Ω, MeshRefinementFunction →
    Function[{vertices, area}, area > 3/100000 (1/10 + 80 Norm[Mean[vertices]])]];

uif = NDSolveValue[{∂²u(x, y)/∂x² + ∂²u(x, y)/∂y² − (∂u(x, y)/∂x)^2 − (∂u(x, y)/∂y)^2 == 0,
    DirichletCondition[u[x, y] == 1 + 2 i, x^2 + y^2 == 16/10000],
    DirichletCondition[u[x, y] == 0, x^2 + y^2 == 16]}, u, {x, y} ∈ mesh];
s4 = Plot3D[(Abs[uif[x, y]])^2, {x, y} ∈ mesh, PlotRange → {0, 5}, ColorFunction → (Hue[0.65, #3] &),
   MeshStyle → GrayLevel[0.4], AxesLabel → {Style["x", 15, FontFamily → "Arial", Black, Italic, Bold],
      Style["y", 15, FontFamily → "Arial", Black, Italic, Bold], Rotate["Density", π/2]},
   AxesStyle → Directive[Black, FontFamily → "Arial", FontSize → 15],
   BoxStyle → Directive[Black, Thickness → 0.002], BoxRatios → Automatic, ViewPoint → {15, −26, 16}];

ImageResize[s4, 700]
Export["/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure S4.png",
  s4, Background → None, ImageResolution → 600];
```

Out[∘]=

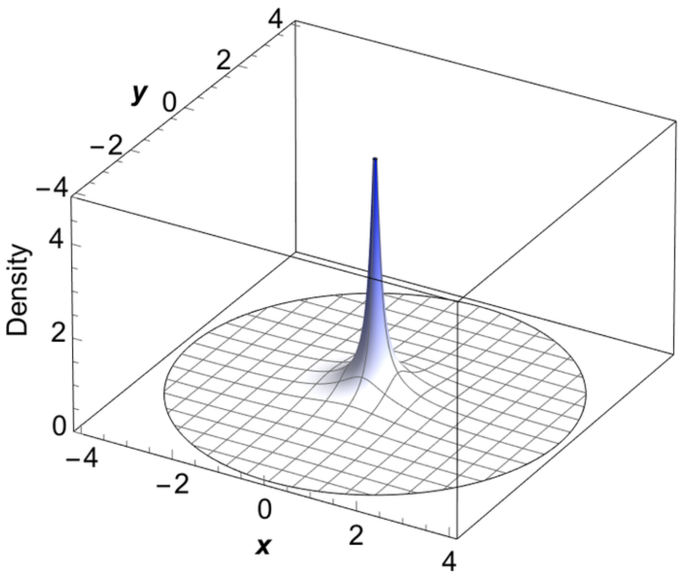

Figure S4. Distribution of mass density of a particle swarm meeting the conditions $\mathcal{M}(0, 0) = 1 + 2i \wedge (\mathcal{M}(x, y) = 0 \wedge x^2 + y^2 = 4^2)$.

It can be seen from Figure S4 that it is a circular symmetrical structure.

Part 6. Differences Between the Two Solving Methods (Schrödinger Equation and Eq. 40)

This code takes approximately 45 hours.

In[∘]:= **Clear["Global`*"];**

usol = DSolveValue$\left[\left\{i\,\dfrac{\partial\psi(x,t)}{\partial t} == -\dfrac{1}{2}\,\dfrac{\partial^2\psi(x,t)}{\partial x^2},\ \psi(x,0) == e^{-2x^2}\right\},\ \psi,\ \{x,t\}\right]$;

F**[x_] :=** e^{-x}**;** $L = 20$;

vsol = NDSolveValue$\left[\left\{i\,\dfrac{\partial\mathcal{M}(x,t)}{\partial t} == -\dfrac{1}{2}\,F[\mathcal{M}(x,t)]\left(\dfrac{\partial^2\mathcal{M}(x,t)}{\partial x^2} - \left(\dfrac{\partial\mathcal{M}(x,t)}{\partial x}\right)^2\right),\right.\right.$

$\left.\left.\mathcal{M}(x,0) == 10^{-2}\,e^{-2x^2},\ \mathcal{M}(-L,t) == \mathcal{M}(L,t)\right\},\ \mathcal{M},\ \{x,-L,L\},\ \{t,0,3\},\ \text{WorkingPrecision}\to 40\right]$;

s5 = Plot3D[Abs[usol[*x*, *t*]] − 10^2 Abs[vsol[*x*, *t*]], {*t*, 0, 1.6}, {*x*, −8, 8}, PlotPoints → 60,
MaxRecursion → 3, PlotRange → {{0, 1.6}, {−8, 8}, {−0.002, 0.003}},
MeshStyle → GrayLevel[0.4], BoundaryStyle → GrayLevel[0.4],
AxesLabel → {Style["*t* ", 15, FontFamily → "Arial", Black, Italic, Bold],
Style["x", 15, FontFamily → "Arial", Black, Italic, Bold], Rotate["Deviation ", $\frac{\pi}{2}$**]},**
AxesStyle → Directive[Black, Thickness → 0.002], BoxStyle → Directive[Black, Thickness → 0.002],
Ticks → {{{0, "0.0"}, {0.5, 0.5, {0.01, 0}, Thickness → 0.003}, {1.0, "1.0", {0.01, 0}, Thickness → 0.003},
{1.5, 1.5, {0.01, 0}, Thickness → 0.003}}, {{−6, −6, {0.011, 0}, Thickness → 0.003},
{−3, −3, {0.011, 0}, Thickness → 0.003}, {0, 0, {0.011, 0}, Thickness → 0.003},
{3, 3, {0.011, 0}, Thickness → 0.003}, {6, 6, {0.011, 0}, Thickness → 0.003}},
{{−0.002, −0.002, {0.012, 0}, Thickness → 0.003}, {−0.001, −0.001, {0.012, 0}, Thickness → 0.003},
{0, "0.000", {0.012, 0}, Thickness → 0.003}, {0.001, 0.001, {0.012, 0}, Thickness → 0.003},
{0.002, "0.002", {0.012, 0}, Thickness → 0.003}, {0.003, "0.003", {0.012, 0}, Thickness → 0.003}}},
LabelStyle → Directive[Black, FontFamily → "Arial", FontSize → 15], ViewPoint → {1, −2, 2.1}];

FindMaxValue[{(Abs[usol[*x*, *t*]] − 10^2 Abs[vsol[*x*, *t*]]), *x* > 0, *t* > 0}, {*x*, *t*}, WorkingPrecision → 34] /
(Abs[usol[*x*, *t*]] /. Last[
FindMaximum[{Abs[usol[*x*, *t*]] − 10^2 Abs[vsol[*x*, *t*]], *x* > 0, *t* > 0}, {*x*, *t*}, WorkingPrecision → 34]])

ImageResize[s5, 700]
Export["/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure S5.png",
s5, Background → None, ImageResolution → 600];

Out[∘]= 0.01137609304650582034220637885507277

Out[∘]=

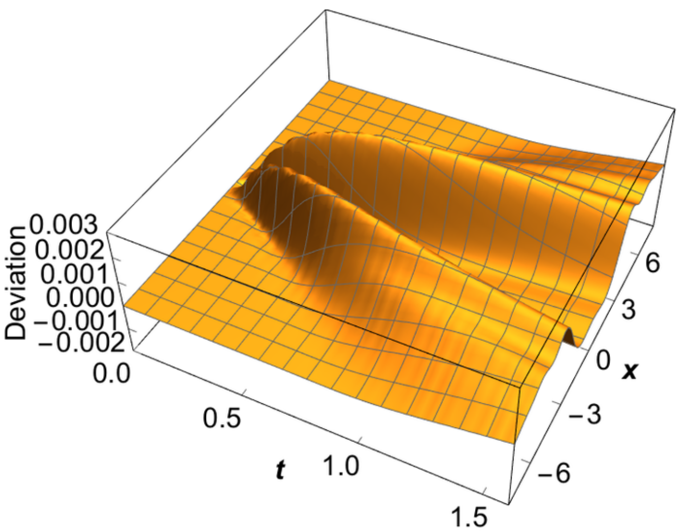

Figure S5. Deviation of the contours computed by the Schrödinger equation and Eq. 40 with the initial wave function being $10^{-2}\,e^{-2x^2}$.

Part 7. Another Comparison When the Initial Wave Function Is $1.4\, e^{-2 x^2}$

This code takes approximately 3 hours.

In[∘]:=

```
Clear["Global`*"];
Off[NDSolveValue::eerr];
L = 20;
F[x_] := ⅇ^(-x);
usol = NDSolveValue[{ⅈ ∂M(x, t)/∂t == -1/2 F[M(x, t)] (∂²M(x, t)/∂x² - (∂M(x, t)/∂x)^2),
    M(x, 0) == 6/5 ⅇ^(-2 x^2), M(-L, t) == M(L, t)}, M, {x, -L, L}, {t, 0, 3}, WorkingPrecision → 22];
vsol = NDSolveValue[{ⅈ ∂M(x, t)/∂t == -1/2 F[M(x, t)] (∂²M(x, t)/∂x² - (∂M(x, t)/∂x)^2), M(x, 0) == 7/5 ⅇ^(-2 x^2),
    M(-L, t) == M(L, t)}, M, {x, -L, L}, {t, 0, 3}, WorkingPrecision → 26];
χ[x, t] = {u[x, t], v[x, t]};
σ3 = (1  0
      0 -1); σ1 = (0 1
                   1 0);
xsol = NDSolve[{ⅈ D[χ[x, t], t] == -σ1.χ(x, t) - ⅈ σ3.D[χ[x, t], x], u[x, 0] == √2/2 ⅇ^(-2 x^2), v[x, 0] == √2/2 ⅇ^(-2 x^2),
    u[L, t] == u[-L, t], v[L, t] == v[-L, t]}, {u, v}, {x, -L, L}, {t, 0, 3}, WorkingPrecision → 14];
G1 = Plot3D[5/6 Abs[usol[x, t]], {t, 0, 1.6}, {x, -8, 8}, PlotPoints → 60, MaxRecursion → 3,
    PlotRange → {{0, 1.6}, {-8, 8}, {0, 1.23}}, MeshStyle → GrayLevel[0.4],
    BoundaryStyle → GrayLevel[0.4], AxesLabel → {Style["t ", 22, FontFamily → "Arial",
        Black, Italic, Bold], Style["x", 22, FontFamily → "Arial", Black, Italic, Bold], ""},
    AxesStyle → Directive[Black, Thickness → 0.002], BoxStyle → Directive[Black, Thickness → 0.002],
    Ticks → {{{0, "0.0"}, {0.5, 0.5, {0.01, 0}, Thickness → 0.003}, {1.0, "1.0", {0.01, 0}, Thickness → 0.003},
        {1.5, 1.5, {0.01, 0}, Thickness → 0.003}}, {{-6, -6, {0.011, 0}, Thickness → 0.003},
        {-3, -3, {0.011, 0}, Thickness → 0.003}, {0, 0, {0.011, 0}, Thickness → 0.003},
        {3, 3, {0.011, 0}, Thickness → 0.003}, {6, 6, {0.011, 0}, Thickness → 0.003}},
      {{0, "0.0"}, {0.5, 0.5, {0.012, 0}, Thickness → 0.003}, {1, "1.0", {0.012, 0}, Thickness → 0.003}}},
    LabelStyle → Directive[Black, FontFamily → "Arial", FontSize → 21], ViewPoint → {1, -2, 2.1},
    Epilog → Text[Style["a", 22, FontFamily → "Arial", Bold, Black], {-0.09, 0.88}, {-1, 1}]];
G2 = Plot3D[5/7 Abs[vsol[x, t]], {t, 0, 1.6}, {x, -8, 8}, PlotPoints → 60, MaxRecursion → 3,
    PlotRange → {{0, 1.6}, {-8, 8}, {0, 1.23}}, MeshStyle → GrayLevel[0.4],
    BoundaryStyle → GrayLevel[0.4], AxesLabel → {Style["t ", 22, FontFamily → "Arial",
        Black, Italic, Bold], Style["x", 22, FontFamily → "Arial", Black, Italic, Bold], ""},
    AxesStyle → Directive[Black, Thickness → 0.002], BoxStyle → Directive[Black, Thickness → 0.002],
    Ticks → {{{0, "0.0"}, {0.5, 0.5, {0.01, 0}, Thickness → 0.003}, {1.0, "1.0", {0.01, 0}, Thickness → 0.003},
        {1.5, 1.5, {0.01, 0}, Thickness → 0.003}}, {{-6, -6, {0.011, 0}, Thickness → 0.003},
        {-3, -3, {0.011, 0}, Thickness → 0.003}, {0, 0, {0.011, 0}, Thickness → 0.003},
        {3, 3, {0.011, 0}, Thickness → 0.003}, {6, 6, {0.011, 0}, Thickness → 0.003}},
      {{0, "0.0"}, {0.5, 0.5, {0.012, 0}, Thickness → 0.003}, {1, "1.0", {0.012, 0}, Thickness → 0.003}}},
    LabelStyle → Directive[Black, FontFamily → "Arial", FontSize → 21], ViewPoint → {1, -2, 2.1},
    Epilog → Text[Style["b", 22, FontFamily → "Arial", Bold, Black], {-0.09, 0.88}, {-1, 1}]];
G3 = Plot3D[Norm[Evaluate[{u[x, t], v[x, t]} /. xsol]], {t, 0, 1.6}, {x, -8, 8},
    PlotPoints → 60, MaxRecursion → 3, PlotRange → {{0, 1.6}, {-8, 8}, {0, 1.23}},
    MeshStyle → GrayLevel[0.4], BoundaryStyle → GrayLevel[0.4],
```

```
    AxesLabel → {Style["t ", 22, FontFamily → "Arial", Black, Italic, Bold],
       Style["x", 22, FontFamily → "Arial", Black, Italic, Bold], ""},
    AxesStyle → Directive[Black, Thickness → 0.002], BoxStyle → Directive[Black, Thickness → 0.002],
    Ticks → {{{0, "0.0"}, {0.5, 0.5, {0.01, 0}, Thickness → 0.003}, {1.0, "1.0", {0.01, 0}, Thickness → 0.003},
         {1.5, 1.5, {0.01, 0}, Thickness → 0.003}}, {{−6, −6, {0.011, 0}, Thickness → 0.003},
         {−3, −3, {0.011, 0}, Thickness → 0.003}, {0, 0, {0.011, 0}, Thickness → 0.003},
         {3, 3, {0.011, 0}, Thickness → 0.003}, {6, 6, {0.011, 0}, Thickness → 0.003}},
       {{0, "0.0"}, {0.5, 0.5, {0.012, 0}, Thickness → 0.003}, {1, "1.0", {0.012, 0}, Thickness → 0.003}}},
    LabelStyle → Directive[Black, FontFamily → "Arial", FontSize → 21], ViewPoint → {1, −2, 2.1},
    Epilog → Text[Style["c", 22, FontFamily → "Arial", Bold, Black], {−0.09, 0.88}, {−1, 1}]];
G4 = Plot[{5/6 Norm[usol[x, 1]], 5/7 Norm[vsol[x, 1]], Norm[Evaluate[{u[x, 1], v[x, 1]} /. xsol]]}, {x, −3.8, 3.8},
    PlotStyle → {{Red, Thickness → 0.005}, {Blue, Thickness → 0.005}, {Black, Thickness → 0.005}},
    PlotRange → {{−3, 3}, {−0.02, 0.9}}, Frame → {{False, False}, {True, False}},
    FrameStyle → Directive[Black, Thickness → 0.002],
    AxesStyle → Directive[GrayLevel[0.3], Thickness → 0.0012], FrameTicks →
      {{{−3, −3, {0.01, 0}, Thickness → 0.003}, {−2, −2, {0.01, 0}, Thickness → 0.003}, {−1, −1, {0.01, 0},
           Thickness → 0.003}, {0, 0, {0.01, 0}, Thickness → 0.003}, {1, 1, {0.01, 0}, Thickness → 0.003},
         {2, 2, {0.01, 0}, Thickness → 0.003}, {3, 3, {0.01, 0}, Thickness → 0.003}}, {{0, 0}}},
    FrameLabel → {Style["x", 22, FontFamily → "Arial", Black, Italic, Bold]},
    LabelStyle → Directive[Black, FontFamily → "Arial", FontSize → 22], PlotLegends →
      Placed[LineLegend[{Directive[Red, Thickness[0.0036]], Directive[Blue, Thickness[0.0036]], ,
           Directive[Black, Thickness[0.0036]]}, {"1.2", "1.4", "Dirac"}], {0.873, 0.72}]];
s6 = GraphicsGrid[{{G1, G2}, {G3, G4}}, Alignment → Bottom, ImageSize → 700,
    Epilog → Text[Style["d", 22, FontFamily → "Arial", Black, Bold], Scaled[{0.6053, 0.7776}]]];

ImageResize[s6, 1000]
Export["/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure S6.png",
   s6, Background → None, ImageResolution → 600];
```

Out[∘]=

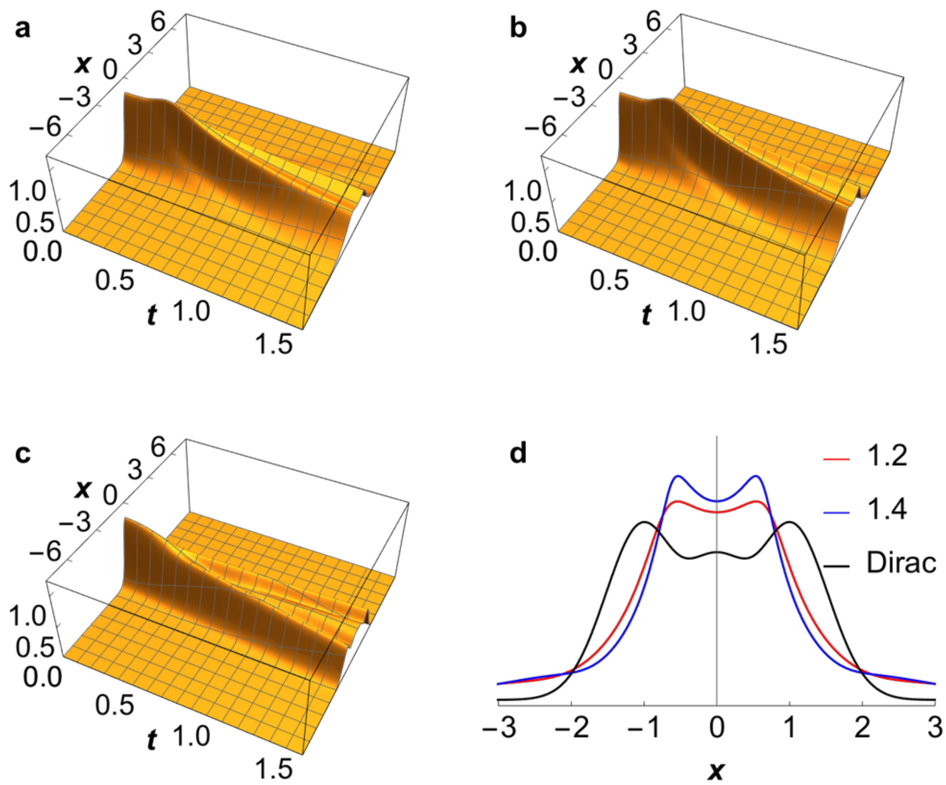

Figure S6. Illustrations of the 1-dimensional time-dependent diffusion of the Gaussian wave packet e^{-2x^2} as obtained using various methods in natural units. **a**, Computation result of Eq. 40 when the initial condition is $\mathcal{M}_0(x, t) = 1.2\, e^{-2x^2}$. The norm has been reduced ($\times \frac{5}{6}$) to facilitate the shape comparison. **b**, Computation result of Eq. 40 when the initial condition is $\mathcal{M}_0(x, t) = 1.4\, e^{-2x^2}$. The norm has been reduced ($\times \frac{5}{7}$) to facilitate the shape comparison. **c**, Computation result of the Dirac equation. **d**, Comparison between them at $t = 1$.

Part 8. Figures Used in This Article

Figure1

```
In[•]:= Clear["Global`*"];
aa = Graphics[{{RGBColor[178/255, 252/255, 61/255], Rectangle[{0, 0}, {1, 1}]},
    {RGBColor[178/255, 252/255, 61/255, 0.5], Rectangle[{1, 0}, {2, 1}]},
    {RGBColor[250/255, 200/255, 0], Arrowheads[0.06], {Thickness[0.006], Arrow[{{0.7, 0.54}, {1.3, 0.54}}]}},
    {RGBColor[250/255, 200/255, 0], Arrowheads[0.06], {Thickness[0.006], Arrow[{{1.3, 0.46}, {0.7, 0.46}}]}},
    {RGBColor[178/255, 252/255, 61/255, 0.5], Arrowheads[0.06],
      {Thickness[0.006], Arrow[{{0, -1.3}, {1.2, -1.3}}]}},
    {RGBColor[178/255, 252/255, 61/255], Arrowheads[0.06], {Thickness[0.006], Arrow[{{0, -1.3}, {0.8, -0.3}}]}},
    {Orange, {Thickness[0.002], DotDashed, Line[{{1, -0.05}, {1, 1.05}}]}},
    {Orange, {Thickness[0.004], Dashed, Line[{{0.8, -0.3}, {2, -0.3}}]}},
    {Orange, {Thickness[0.004], Dashed, Line[{{1.2, -1.3}, {2, -0.3}}]}},
    {Blue, Arrowheads[0.06], {Thickness[0.006], Arrow[{{1.2, -1.3}, {0.8, -0.3}}]}},
    {Blue, Arrowheads[0.06], {Thickness[0.006], Arrow[{{0, -1.3}, {2, -0.3}}]}},
    {Blue, Arrowheads[0.06], {Thickness[0.006], Arrow[{{0, -1.3}, {1, -0.8}}]}},
    Text[Style["V", 24, FontFamily → "Euclid Math One", White], {0.45, 0.5}],
    Text[Style["A", 17, FontFamily → "Arial", White], {0.513, 0.456}],
    Text[Style["V", 24, FontFamily → "Euclid Math One", White], {1.55, 0.5}],
    Text[Style["B", 17, FontFamily → "Arial", White], {1.616, 0.455}],
    Text[Style["D", 24, FontFamily → "Arial", Orange, Italic], {0.982, 0.63}],
    Text[Style["A", 17, FontFamily → "Arial", Orange], {1.063, 0.59}],
    Text[Style["D", 24, FontFamily → "Arial", Orange, Italic], {0.982, 0.38}],
    Text[Style["B", 17, FontFamily → "Arial", Orange], {1.065, 0.34}],
    Text[Style["Φ", 24, FontFamily → "Arial", Orange], {1.06, 1.08}],
    Text[Style["O", 24, FontFamily → "Arial", Orange], {0, -1.39}],
    Text[Style["B", 24, FontFamily → "Arial", RGBColor[178/255, 252/255, 61/255, 0.5]], {1.2, -1.39}],
    Text[Style["A", 24, FontFamily → "Arial", RGBColor[178/255, 252/255, 61/255]], {0.7, -0.28}],
    Text[Style["C", 24, FontFamily → "Arial", Orange], {2.02, -0.4}],
    Text[Style["M", 24, FontFamily → "Arial", Orange], {0.972, -0.93}],
    Inset[Style["a", Black, Bold, FontFamily → "Arial", FontSize → 24], {0.034, 1.12}],
    Inset[Style["b", Black, Bold, FontFamily → "Arial", FontSize → 24], {0.034, -0.2}]}];
Export["/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure 1.png",
  aa, Background → None, ImageResolution → 600];
```

######## Figure1 ##

######## Figure2 ##

```
In[∘]:= Clear["Global`*"];
text = Graphics[{Gray, Line[{{1, 0}, {1, 10}}], Line[{{2, 0}, {2, 10}}],
    Line[{{3, 0}, {3, 10}}], Line[{{4, 0}, {4, 10}}], Line[{{5, 0}, {5, 10}}],
    Line[{{6, 0}, {6, 10}}], Line[{{7, 0}, {7, 10}}], Line[{{8, 0}, {8, 10}}], Line[{{9, 0}, {9, 10}}],
    Line[{{0, 1}, {10, 1}}], Line[{{0, 2}, {10, 2}}], Line[{{0, 3}, {10, 3}}], Line[{{0, 4}, {10, 4}}],
    Line[{{0, 5}, {10, 5}}], Line[{{0, 6}, {10, 6}}], Line[{{0, 7}, {10, 7}}], Line[{{0, 8}, {10, 8}}],
    Line[{{0, 9}, {10, 9}}], Orange, Rectangle[{6, 4}, {7, 5}]}, PlotRangePadding → 1/1000];
bb = Show[{Plot3D[Sin[x + Cos[y]], {x, −3, 3}, {y, −3, 3}, PlotPoints → 60, MaxRecursion → 3,
     PlotStyle → Texture[text], Mesh → None, Lighting → "Neutral",
     PlotLabels → Placed[Style["dS", 14, FontFamily → "Arial", Orange],
       {0.9, −0.9}, Background → None], BoundaryStyle → None, Boxed → False,
     Axes → None, ViewPoint → {1, −1.9, 1.4}], Graphics3D[{{Thickness[0.007], Black,
       Arrow[{{0, 0, 0}, {−Evaluate[D[Sin[x + Cos[y]], x] /. {x → 0.88, y → −0.3}],
             −Evaluate[D[Sin[x + Cos[y]], y] /. {x → 0.88, y → −0.3}], 1}} +
          {{0.88, −0.3, Sin[0.88 + Cos[−0.3]]}, {0.88, −0.3, Sin[0.88 + Cos[−0.3]]}}]},
      {Text[Style["N", 14, FontFamily → "Arial", Bold, Italic, Black],
        {−Evaluate[D[Sin[x + Cos[y]], x] /. {x → 0.88, y → −0.3}],
           −Evaluate[D[Sin[x + Cos[y]], y] /. {x → 0.88, y → −0.3}], 1} +
          {0.88, −0.3, Sin[0.88 + Cos[−0.3]]} + {0.02, 0.03, 0.23}]},
      {Thickness[0.007], Blue, Arrow[{{0.88, −0.3, Sin[0.88 + Cos[−0.3]]}, {1.88, −0.5, 2}}]},
      {Text[Style["𝒳", 14, FontFamily → "Arial", Bold, Italic, Blue], {2.01, −0.5, 2.01}]},
      {Text[Style["Σ", 14, FontFamily → "Arial", Italic, Gray], {−2.14, −1.5, 0.7}]}}]}];
Export["/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure 2.png",
   bb, Background → None, ImageResolution → 600];
bfg =
   Import[
     "/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure 2.png"];
bfg = ImageTake[bfg, {290, 1870}, {110, 2920}];
Export[
   "/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure 2.png", bfg];
```

######## Figure2 ##

######## Figure3 ##

This code takes approximately 227 seconds.

```
In[∘]:= Clear["Global`*"];
Needs["NDSolve`FEM`"];
Ω = ImplicitRegion[16/10000 ≤ x^2 + y^2 + z^2 ≤ 16, {x, y, z}];
nr = 50; nφ = 50; nθ = 50;
coordinates = Flatten[ Table[{r Sin[φ] Cos[θ], r Sin[φ] Sin[θ], r Cos[φ]}, {r, 4/100, 4, (4 - 4/100)/(nr - 1)},
      {φ, 1/10000, π - 1/10000, (π - 2/10000)/(nφ - 1)}, {θ, 0, 2 π, 2 π/(nθ - 1)}], 2];
incidents = Flatten[Table[Block[{p1 = (j - 1)*nr + i, p2 = j*nr + i, p3 = p2 + 1, p4 = p1 + 1, p5, p6, p7, p8},
        {p5, p6, p7, p8} = {p1, p2, p3, p4} + k*nr*nφ;
        {p1, p2, p3, p4} += (k - 1)*nr*nφ;
        {p1, p2, p3, p4, p5, p6, p7, p8}], {i, 1, nr - 1}, {j, 1, nφ - 1}, {k, 1, nθ - 1}], 2];
mesh =
   ToElementMesh["Coordinates" → coordinates, "MeshElements" → {HexahedronElement[incidents]}];
uif = NDSolveValue[{∂²𝓜(x, y, z)/∂x² + ∂²𝓜(x, y, z)/∂y² + ∂²𝓜(x, y, z)/∂z² - (∂𝓜(x, y, z)/∂x)^2 - (∂𝓜(x, y, z)/∂y)^2 -
       (∂𝓜(x, y, z)/∂z)^2 == 0, DirichletCondition[𝓜[x, y, z] == 1 + 2 i, x^2 + y^2 + z^2 == 16/10000],
     DirichletCondition[𝓜[x, y, z] == 0, x^2 + y^2 + z^2 == 16]}, 𝓜, {x, y, z} ∈ mesh];
G1 = SliceDensityPlot3D[(Norm[uif[x, y, z]])^2, "CenterPlanes", {x, -4, 4}, {y, -4, 4},
     {z, -4, 4}, Boxed → False, Axes → None, ColorFunction → (Hue[0.65, #1] &),
     BoundaryStyle → Directive[Thickness[0.001], Gray], PlotRange → {0, 5}, PlotPoints → 100,
     Epilog → Text[Style["a", 23, FontFamily → "Arial", Bold, Black], {0.2478, 0.93}, {0.5, 4}]];
G2 = Plot3D[(Norm[uif[x, y, 0]])^2, {x, -4, 4}, {y, -4, 4},
     PlotRange → {{-4.52, 4.52}, {-4.52, 4.52}, {-1.7, 5}}, MeshStyle →
       {Directive[GrayLevel[0.4], Thickness[0.001]], Directive[GrayLevel[0.4], Thickness[0.001]]},
     BoundaryStyle → Directive[Thickness[0.001], Gray], Boxed → False, Axes → None,
     ColorFunction → (Hue[0.65, #3] &), BoxRatios → Automatic, MeshStyle → Gray,
     ImageSize → {360, 360}, PlotPoints → 30, ViewPoint → {1.2, -2, 0.7},
     Epilog → Text[Style["b", 23, FontFamily → "Arial", Bold, Black], {0.2478, 0.93}, {0.5, 4}]];
G3 = DensityPlot[(Norm[uif[x, y, 0]])^2, {x, -4, 4}, {y, -4, 4}, PlotRange → {{-7.2, 7.2}, {-7.2, 7.2}, {0, 5}},
     ColorFunction → (Hue[0.65, #1] &), Frame → False, PlotPoints → 180,
     Epilog → {Text[Style["c", 23, FontFamily → "Arial", Bold, Black], {-3.75, 4.24}],
         {Directive[Thickness[0.001], Gray], Circle[{0, 0}, 4]}}];
G4 = Plot[(Norm[uif[x, 0, 0]])^2, {x, -4, 4}, PlotRange → {{-7.3, 7.3}, {-0.95, 6}},
     ColorFunction → (Hue[0.65, #2] &), PlotPoints → 180, PlotStyle → {Thickness → 0.0036},
     Frame → False, Axes → None, AspectRatio → Automatic];
G4 = Show[{G4, Plot[(Norm[uif[x, 0, 0]])^2, {x, -4, 4}, PlotRange → {{-7.3, 7.3}, {-0.95, 0.9}},
         PlotStyle → Directive[GrayLevel[0.66], Thickness → 0.0036, Dashed],
         Frame → False, Axes → None, AspectRatio → Automatic]}];
cc = GraphicsGrid[{{G1, G2}, {G3, G4}}, Spacings → {-70, -70}, ImageSize → 700,
     Epilog → Text[Style["d", 22, FontFamily → "Arial", Black, Bold], Scaled[{0.6684, 0.3371}]]];
Export["/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure 3.png",
   cc, Background → None, ImageResolution → 500];
cfg =
   Import[
     "/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure 3.png"];
cfg = ImageTake[cfg, {570, 4670}, {560, 4300}];
Export[
   "/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure 3.png", cfg];
```

Figure3

Figure4

This code takes approximately 12 minutes.

```
In[∘]:= Clear["Global`*"];
L = 20;
F[x_] := ⅇ^(-x);
usol = NDSolveValue[{ⅈ D[ℳ[x, t], t] == -1/2 F[ℳ[x, t]] (D[ℳ[x, t], {x, 2}] - (D[ℳ[x, t], x])^2),
      ℳ[x, 0] == 10^-13 ⅇ^(-2 x^2), ℳ[-L, t] == ℳ[L, t]}, ℳ, {x, -L, L}, {t, 0, 3}, WorkingPrecision → 66];
F[x_] := ⅇ^(-x);
vsol = NDSolveValue[{ⅈ D[ℳ[x, t], t] == -1/2 F[ℳ[x, t]] (D[ℳ[x, t], {x, 2}] - (D[ℳ[x, t], x])^2),
      ℳ[x, 0] == ⅇ^(-2 x^2), ℳ[-L, t] == ℳ[L, t]}, ℳ, {x, -L, L}, {t, 0, 3}, WorkingPrecision → 20];
wsol = NDSolveValue[{ⅈ D[ψ[x, t], t] == -1/2 D[ψ[x, t], {x, 2}], ψ[x, 0] == ⅇ^(-2 x^2), ψ[-L, t] == ψ[L, t]},
    ψ, {x, -L, L}, {t, 0, 3}, WorkingPrecision → 18];
χ[x, t] = {u[x, t], v[x, t]};
σ3 = {{1, 0}, {0, -1}}; σ1 = {{0, 1}, {1, 0}};
xsol = NDSolve[{ⅈ D[χ[x, t], t] == -σ1.χ[x, t] - ⅈ σ3.D[χ[x, t], x], u[x, 0] == Sqrt[2]/2 ⅇ^(-2 x^2), v[x, 0] == Sqrt[2]/2 ⅇ^(-2 x^2),
      u[L, t] == u[-L, t], v[L, t] == v[-L, t]}, {u, v}, {x, -L, L}, {t, 0, 3}, WorkingPrecision → 14];
G1 = Plot3D[10^13 Abs[usol[x, t]], {t, 0, 1.6}, {x, -8, 8}, PlotPoints → 60,
    MaxRecursion → 3, PlotRange → {{0, 1.6}, {-8, 8}, {0, 1.05}}, MeshStyle →
      {Directive[GrayLevel[0.4], Thickness[0.0014]], Directive[GrayLevel[0.4], Thickness[0.0014]]},
    BoundaryStyle → Directive[GrayLevel[0.4], Thickness[0.0014]],
    AxesLabel → {Style["t ", 20, FontFamily → "Arial", Black, Italic, Bold],
        Style["x", 20, FontFamily → "Arial", Black, Italic, Bold], ""},
    AxesStyle → Directive[Black, Thickness → 0.002], BoxStyle → Directive[Black, Thickness → 0.002],
    Ticks → {{{0, "0.0"}, {0.5, 0.5, {0.01, 0}, Thickness → 0.003}, {1.0, "1.0", {0.01, 0}, Thickness → 0.003},
          {1.5, 1.5, {0.01, 0}, Thickness → 0.003}}, {{-6, -6, {0.011, 0}, Thickness → 0.003},
          {-3, -3, {0.011, 0}, Thickness → 0.003}, {0, 0, {0.011, 0}, Thickness → 0.003},
          {3, 3, {0.011, 0}, Thickness → 0.003}, {6, 6, {0.011, 0}, Thickness → 0.003}},
        {{0, "0.0"}, {0.5, 0.5, {0.012, 0}, Thickness → 0.003}, {1, "1.0", {0.012, 0}, Thickness → 0.003}}},
    LabelStyle → Directive[Black, FontFamily → "Arial", FontSize → 20], ViewPoint → {1,  -2,  2.1},
    Epilog → Text[Style["a", 20, FontFamily → "Arial", Bold, Black], {-0.09, 0.88}, {-1, 1}]];
G2 = Plot3D[Abs[vsol[x, t]], {t, 0, 1.6}, {x, -8, 8}, PlotPoints → 60, MaxRecursion → 3,
    PlotRange → {{0, 1.6}, {-8, 8}, {0, 1.05}}, MeshStyle →
      {Directive[GrayLevel[0.4], Thickness[0.0014]], Directive[GrayLevel[0.4], Thickness[0.0014]]},
    BoundaryStyle → Directive[GrayLevel[0.4], Thickness[0.0014]],
    AxesLabel → {Style["t ", 20, FontFamily → "Arial", Black, Italic, Bold],
        Style["x", 20, FontFamily → "Arial", Black, Italic, Bold], ""},
    AxesStyle → Directive[Black, Thickness → 0.002], BoxStyle → Directive[Black, Thickness → 0.002],
    Ticks → {{{0, "0.0"}, {0.5, 0.5, {0.01, 0}, Thickness → 0.003}, {1.0, "1.0", {0.01, 0}, Thickness → 0.003},
          {1.5, 1.5, {0.01, 0}, Thickness → 0.003}}, {{-6, -6, {0.011, 0}, Thickness → 0.003},
          {-3, -3, {0.011, 0}, Thickness → 0.003}, {0, 0, {0.011, 0}, Thickness → 0.003},
          {3, 3, {0.011, 0}, Thickness → 0.003}, {6, 6, {0.011, 0}, Thickness → 0.003}},
        {{0, "0.0"}, {0.5, 0.5, {0.012, 0}, Thickness → 0.003}, {1, "1.0", {0.012, 0}, Thickness → 0.003}}},
    LabelStyle → Directive[Black, FontFamily → "Arial", FontSize → 20], ViewPoint → {1,  -2,  2.1},
```

```
    Epilog → Text[Style["b", 20, FontFamily → "Arial", Bold, Black], {−0.09, 0.88}, {−1, 1}]];
G3 = Plot3D[Abs[wsol[x, t]], {t, 0, 1.6}, {x, −8, 8}, PlotPoints → 60, MaxRecursion → 3,
    PlotRange → {{0, 1.6}, {−8, 8}, {0, 1.05}}, MeshStyle →
      {Directive[GrayLevel[0.4], Thickness[0.0014]], Directive[GrayLevel[0.4], Thickness[0.0014]]},
    BoundaryStyle → Directive[GrayLevel[0.4], Thickness[0.0014]],
    AxesLabel → {Style["t ", 20, FontFamily → "Arial", Black, Italic, Bold],
        Style["x", 20, FontFamily → "Arial", Black, Italic, Bold], ""},
    AxesStyle → Directive[Black, Thickness → 0.002], BoxStyle → Directive[Black, Thickness → 0.002],
    Ticks → {{{0, "0.0"}, {0.5, 0.5, {0.01, 0}, Thickness → 0.003}, {1.0, "1.0", {0.01, 0}, Thickness → 0.003},
          {1.5, 1.5, {0.01, 0}, Thickness → 0.003}}, {{−6, −6, {0.011, 0}, Thickness → 0.003},
          {−3, −3, {0.011, 0}, Thickness → 0.003}, {0, 0, {0.011, 0}, Thickness → 0.003},
          {3, 3, {0.011, 0}, Thickness → 0.003}, {6, 6, {0.011, 0}, Thickness → 0.003}},
        {{0, "0.0"}, {0.5, 0.5, {0.012, 0}, Thickness → 0.003}, {1, "1.0", {0.012, 0}, Thickness → 0.003}}},
    LabelStyle → Directive[Black, FontFamily → "Arial", FontSize → 20], ViewPoint → {1, −2, 2.1},
    Epilog → Text[Style["c", 20, FontFamily → "Arial", Bold, Black], {−0.09, 0.88}, {−1, 1}]];
G4 = Plot3D[Norm[Evaluate[{u[x, t], v[x, t]} /. xsol]], {t, 0, 1.6}, {x, −8, 8}, PlotPoints → 60,
    MaxRecursion → 3, PlotRange → {{0, 1.6}, {−8, 8}, {0, 1.05}}, MeshStyle →
      {Directive[GrayLevel[0.4], Thickness[0.0014]], Directive[GrayLevel[0.4], Thickness[0.0014]]},
    BoundaryStyle → Directive[GrayLevel[0.4], Thickness[0.0014]],
    AxesLabel → {Style["t ", 20, FontFamily → "Arial", Black, Italic, Bold],
        Style["x", 20, FontFamily → "Arial", Black, Italic, Bold], ""},
    AxesStyle → Directive[Black, Thickness → 0.002], BoxStyle → Directive[Black, Thickness → 0.002],
    Ticks → {{{0, "0.0"}, {0.5, 0.5, {0.01, 0}, Thickness → 0.003}, {1.0, "1.0", {0.01, 0}, Thickness → 0.003},
          {1.5, 1.5, {0.01, 0}, Thickness → 0.003}}, {{−6, −6, {0.011, 0}, Thickness → 0.003},
          {−3, −3, {0.011, 0}, Thickness → 0.003}, {0, 0, {0.011, 0}, Thickness → 0.003},
          {3, 3, {0.011, 0}, Thickness → 0.003}, {6, 6, {0.011, 0}, Thickness → 0.003}},
        {{0, "0.0"}, {0.5, 0.5, {0.012, 0}, Thickness → 0.003}, {1, "1.0", {0.012, 0}, Thickness → 0.003}}},
    LabelStyle → Directive[Black, FontFamily → "Arial", FontSize → 20], ViewPoint → {1, −2, 2.1},
    Epilog → Text[Style["d", 20, FontFamily → "Arial", Bold, Black], {−0.09, 0.88}, {−1, 1}]];
dd = GraphicsGrid[{{G1, G2}, {G3, G4}}, Spacings → {Scaled[−0.01], Scaled[−0.01]}, ImageSize → 700];
Export["/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure 4.png",
   dd, Background → None, ImageResolution → 600];
dfg =
   Import[
     "/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure 4.png"];
dfg = ImageTake[dfg, {220, 5009}, {120, 5620}];
Export[
   "/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure 4.png", dfg];
```

```
## ## ## Figure4 ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ##
```

```
## ## ## Figure5 ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ##
```

This code takes approximately 11 minutes.

```
Clear["Global`*"];
L = 20;
```

$F[\mathrm{x\_}] := e^{-x};$

$$\mathrm{usol} = \mathrm{NDSolveValue}\Big[\Big\{i\,\frac{\partial \mathcal{M}(x,t)}{\partial t} == -\frac{1}{2}F[\mathcal{M}(x,t)]\left(\frac{\partial^2 \mathcal{M}(x,t)}{\partial x^2} - \left(\frac{\partial \mathcal{M}(x,t)}{\partial x}\right)^2\right),$$
$$\mathcal{M}(x,0) == 10^{-13}\,e^{-2x^2},\ \mathcal{M}(-L,t) == \mathcal{M}(L,t)\Big\},\ \mathcal{M},\ \{x,-L,L\},\ \{t,0,3\},\ \mathrm{WorkingPrecision} \to 66\Big];$$

$F[\mathrm{x\_}] := e^{-x};$

$$\mathrm{vsol} = \mathrm{NDSolveValue}\Big[\Big\{i\,\frac{\partial \mathcal{M}(x,t)}{\partial t} == -\frac{1}{2}F[\mathcal{M}(x,t)]\left(\frac{\partial^2 \mathcal{M}(x,t)}{\partial x^2} - \left(\frac{\partial \mathcal{M}(x,t)}{\partial x}\right)^2\right),$$
$$\mathcal{M}(x,0) == e^{-2x^2},\ \mathcal{M}(-L,t) == \mathcal{M}(L,t)\Big\},\ \mathcal{M},\ \{x,-L,L\},\ \{t,0,3\},\ \mathrm{WorkingPrecision} \to 20\Big];$$

```
wsol = NDSolveValue[{i ∂ψ(x, t)/∂t == -1/2 ∂²ψ(x, t)/∂x², ψ(x, 0) == ⅇ^(-2 x²), ψ(-L, t) == ψ(L, t)},
    ψ, {x, -L, L}, {t, 0, 3}, WorkingPrecision → 18];
χ[x, t] = {u[x, t], v[x, t]};
σ3 = {{1, 0}, {0, -1}}; σ1 = {{0, 1}, {1, 0}};
xsol = NDSolve[{i D[χ[x, t], t] == -σ1.χ(x, t) - i σ3.D[χ[x, t], x], u[x, 0] == √2/2 ⅇ^(-2 x²), v[x, 0] == √2/2 ⅇ^(-2 x²),
    u[L, t] == u[-L, t], v[L, t] == v[-L, t]}, {u, v}, {x, -L, L}, {t, 0, 3}, WorkingPrecision → 14];
G1 = Plot[ⅇ^(-2 x²), {x, -3.8, 3.8}, PlotStyle → {Gray, Thickness → 0.005, Dashed},
    PlotRange → {{-3, 3}, {-0.02, 1.1}}, Frame → {{False, False}, {True, False}},
    FrameStyle → Directive[Black, Thickness → 0.002],
    AxesStyle → Directive[GrayLevel[0.3], Thickness → 0.0016], FrameTicks →
     {{{-3, -3, {0.01, 0}, Thickness → 0.003}, {-2, -2, {0.01, 0}, Thickness → 0.003}, {-1, -1, {0.01, 0},
          Thickness → 0.003}, {0, 0, {0.01, 0}, Thickness → 0.003}, {1, 1, {0.01, 0}, Thickness → 0.003},
        {2, 2, {0.01, 0}, Thickness → 0.003}, {3, 3, {0.01, 0}, Thickness → 0.003}}, {{0, 0}}},
    FrameLabel → {Style["x", 22, FontFamily → "Arial", Black, Italic, Bold]},
    LabelStyle → Directive[Black, FontFamily → "Arial", FontSize → 22],
    Epilog → Text[Style["t = 0.0", 22, FontFamily → "Arial", Black, Bold], Scaled[{0.1, 0.96}]]];
datau = Table[{x, 10^13 Norm[usol[x, 0.2]]}, {x, -4, 4}];
fu = Normal[NonlinearModelFit[datau, a Exp[b x²], {a, b}, x]];
σu = NumberForm[StandardDeviation[ProbabilityDistribution[(∫_-∞^∞ fu ⅆx)^-1 fu, {x, -∞, ∞}]], {3, 2}];
σv = NumberForm[√(N[∫_-10^10 N[(∫_-10^10 Norm[vsol[x, 0.2]] ⅆx)^-1] x² Norm[vsol[x, 0.2]] ⅆx]), {3, 2}];
dataw = Table[{x, Norm[wsol[x, 0.2]]}, {x, -4, 4}];
fw = Normal[NonlinearModelFit[datau, a Exp[b x²], {a, b}, x]];
σw = NumberForm[StandardDeviation[ProbabilityDistribution[(∫_-∞^∞ fw ⅆx)^-1 fw, {x, -∞, ∞}]], {3, 2}];
σx = NumberForm[√(N[∫_-10^10 N[(∫_-10^10 Norm[Evaluate[{u[x, 0.2], v[x, 0.2]} /. xsol]] ⅆx)^-1]
          x² Norm[Evaluate[{u[x, 0.2], v[x, 0.2]} /. xsol]] ⅆx]), {3, 2}];
G2 = Plot[{ⅇ^(-2 x²), Callout[10^13 Norm[usol[x, 0.2]], StringForm["σ = ``", σu], {0.7, 0.64}, {-0.21, 0.83},
      CalloutStyle → {Red, None}, LabelStyle → {FontFamily → "Arial", FontSize → 22, Red},
      Background → None], Callout[Norm[vsol[x, 0.2]], StringForm["σ = ``", σv],
      {0.7, 1.03}, {0.11, 1.01}, CalloutStyle → {Blue, None},
      LabelStyle → {FontFamily → "Arial", FontSize → 22, Blue}, Background → None],
    Callout[Norm[wsol[x, 0.2]], StringForm["σ = ``", σw], {0.7, 0.9}, {0.23, 0.84},
      CalloutStyle → {Orange, None}, LabelStyle → {FontFamily → "Arial", FontSize → 22, Orange},
      Background → None], Callout[Norm[Evaluate[{u[x, 0.2], v[x, 0.2]} /. xsol]],
      StringForm["σ = ``", σx], {0.7, 0.77}, {0.42, 0.74}, CalloutStyle → {Green, None},
      LabelStyle → {FontFamily → "Arial", FontSize → 22, Green}, Background → None]}, {x, -3.8, 3.8},
    PlotStyle → {{Gray, Thickness → 0.005, Dashed}, {Red, Thickness → 0.005}, {Blue, Thickness → 0.005},
      {Orange, Thickness → 0.005}, {Green, Thickness → 0.005}}, PlotRange → {{-3, 3}, {-0.02, 1.1}},
    FrameLabel → {Style["x", 22, FontFamily → "Arial", Black, Italic, Bold]},
    Frame → {{False, False}, {True, False}}, FrameStyle → Directive[Black, Thickness → 0.002],
    AxesStyle → Directive[GrayLevel[0.3], Thickness → 0.0016], FrameTicks →
     {{{-3, -3, {0.01, 0}, Thickness → 0.003}, {-2, -2, {0.01, 0}, Thickness → 0.003}, {-1, -1, {0.01, 0},
```

```
        Thickness → 0.003}, {0, 0, {0.01, 0}, Thickness → 0.003}, {1, 1, {0.01, 0}, Thickness → 0.003},
      {2, 2, {0.01, 0}, Thickness → 0.003}, {3, 3, {0.01, 0}, Thickness → 0.003}}, {{0, 0}}},
   Epilog → Text[Style["t = 0.2", 22, FontFamily → "Arial", Black, Bold], Scaled[{0.1, 0.96}]]];
G2 = Show[G2, LabelStyle → {FontFamily → "Arial", 22, GrayLevel[0]}];
datau = Table[{x, 10^13 Norm[usol[x, 0.4]]}, {x, -4, 4}];
fu = Normal[NonlinearModelFit[datau, a Exp[b x^2], {a, b}, x]];
σu = NumberForm[StandardDeviation[ProbabilityDistribution[(∫_{-∞}^{∞} fu dx)^-1 fu, {x, -∞, ∞}]], {3, 2}];
σv = NumberForm[Sqrt[N[∫_{-10}^{10} N[(∫_{-10}^{10} Norm[vsol[x, 0.4]] dx)^-1] x^2 Norm[vsol[x, 0.4]] dx]], {3, 2}];
dataw = Table[{x, Norm[wsol[x, 0.4]]}, {x, -4, 4}];
fw = Normal[NonlinearModelFit[datau, a Exp[b x^2], {a, b}, x]];
σw = NumberForm[StandardDeviation[ProbabilityDistribution[(∫_{-∞}^{∞} fw dx)^-1 fw, {x, -∞, ∞}]], {3, 2}];
dataw = Table[{x, Norm[wsol[x, 0.4]]}, {x, -4, 4}];
fw = Normal[NonlinearModelFit[datau, a Exp[b x^2], {a, b}, x]];
σw = NumberForm[StandardDeviation[ProbabilityDistribution[(∫_{-∞}^{∞} fw dx)^-1 fw, {x, -∞, ∞}]], {3, 2}];
σx = NumberForm[Sqrt[N[∫_{-10}^{10} N[(∫_{-10}^{10} Norm[Evaluate[{u[x, 0.4], v[x, 0.4]} /. xsol]] dx)^-1]
        x^2 Norm[Evaluate[{u[x, 0.4], v[x, 0.4]} /. xsol]] dx]], {3, 2}];
G3 = Plot[{e^(-2 x^2), Callout[10^13 Norm[usol[x, 0.4]], StringForm["σ = ``", σu], {0.8, 0.54}, {-0.22, 0.7},
      CalloutStyle → {Red, None}, LabelStyle → {FontFamily → "Arial", FontSize → 22, Red},
      Background → None], Callout[Norm[vsol[x, 0.4]], StringForm["σ = ``", σv],
      {0.8, 0.93}, {0.157, 0.93}, CalloutStyle → {Blue, None},
      LabelStyle → {FontFamily → "Arial", FontSize → 22, Blue}, Background → None],
    Callout[Norm[wsol[x, 0.4]], StringForm["σ = ``", σw], {0.8, 0.67}, {0.23, 0.713},
      CalloutStyle → {Orange, None}, LabelStyle → {FontFamily → "Arial", FontSize → 22, Orange},
      Background → None], Callout[Norm[Evaluate[{u[x, 0.4], v[x, 0.4]} /. xsol]],
      StringForm["σ = ``", σx], {0.8, 0.8}, {0.14, 0.756}, CalloutStyle → {Green, None},
      LabelStyle → {FontFamily → "Arial", FontSize → 22, Green}, Background → None]},
   {x, -3.8, 3.8}, PlotStyle → {{Gray, Thickness → 0.005, Dashed}, {Red, Thickness → 0.005},
      {Blue, Thickness → 0.005}, {Orange, Thickness → 0.005}, {Green, Thickness → 0.005}},
   PlotRange → {{-3, 3}, {-0.02, 1.1}}, Frame → {{False, False}, {True, False}},
   FrameStyle → Directive[Black, Thickness → 0.002],
   AxesStyle → Directive[GrayLevel[0.3], Thickness → 0.0016], FrameTicks →
    {{{-3, -3, {0.01, 0}, Thickness → 0.003}, {-2, -2, {0.01, 0}, Thickness → 0.003}, {-1, -1, {0.01, 0},
        Thickness → 0.003}, {0, 0, {0.01, 0}, Thickness → 0.003}, {1, 1, {0.01, 0}, Thickness → 0.003},
      {2, 2, {0.01, 0}, Thickness → 0.003}, {3, 3, {0.01, 0}, Thickness → 0.003}}, {{0, 0}}},
   FrameLabel → {Style["x", 22, FontFamily → "Arial", Black, Italic, Bold]},
   Epilog → Text[Style["t = 0.4", 22, FontFamily → "Arial", Black, Bold], Scaled[{0.1, 0.96}]]];
G3 = Show[G3, LabelStyle → {FontFamily → "Arial", 22, GrayLevel[0]}];
datau = Table[{x, 10^13 Norm[usol[x, 0.6]]}, {x, -4, 4}];
fu = Normal[NonlinearModelFit[datau, a Exp[b x^2], {a, b}, x]];
σu = NumberForm[StandardDeviation[ProbabilityDistribution[(∫_{-∞}^{∞} fu dx)^-1 fu, {x, -∞, ∞}]], {3, 2}];
σv = NumberForm[Sqrt[N[∫_{-10}^{10} N[(∫_{-10}^{10} Norm[vsol[x, 0.6]] dx)^-1] x^2 Norm[vsol[x, 0.6]] dx]], {3, 2}];
```

```
dataw = Table[{x, Norm[wsol[x, 0.6]]}, {x, -4, 4}];
fw = Normal[NonlinearModelFit[datau, a Exp[b x^2], {a, b}, x]];
σw = NumberForm[StandardDeviation[ProbabilityDistribution[(∫_{-∞}^{∞} fw dx)^-1 fw, {x, -∞, ∞}]], {3, 2}];
dataw = Table[{x, Norm[wsol[x, 0.6]]}, {x, -4, 4}];
fw = Normal[NonlinearModelFit[datau, a Exp[b x^2], {a, b}, x]];
σw = NumberForm[StandardDeviation[ProbabilityDistribution[(∫_{-∞}^{∞} fw dx)^-1 fw, {x, -∞, ∞}]], {3, 2}];
σx = NumberForm[√(N[∫_{-10}^{10} N[(∫_{-10}^{10} Norm[Evaluate[{u[x, 0.6], v[x, 0.6]} /. xsol]] dx)^-1]
        x^2 Norm[Evaluate[{u[x, 0.6], v[x, 0.6]} /. xsol]] dx]), {3, 2}];
G4 = Plot[{e^(-2 x^2), Callout[10^13 Norm[usol[x, 0.6]], StringForm["σ = ``", σu], {1.1, 0.49}, {0.79, 0.52},
      CalloutStyle → {Red, None}, LabelStyle → {FontFamily → "Arial", FontSize → 22, Red},
      Background → None], Callout[Norm[vsol[x, 0.6]], StringForm["σ = ``", σv],
      {1.1, 0.88}, {0.157, 0.815}, CalloutStyle → {Blue, None},
      LabelStyle → {FontFamily → "Arial", FontSize → 22, Blue}, Background → None],
    Callout[Norm[wsol[x, 0.6]], StringForm["σ = ``", σw], {1.1, 0.62}, {0.79, 0.52},
      CalloutStyle → {Orange, None}, LabelStyle → {FontFamily → "Arial", FontSize → 22, Orange},
      Background → None], Callout[Norm[Evaluate[{u[x, 0.6], v[x, 0.6]} /. xsol]],
      StringForm["σ = ``", σx], {1.1, 0.75}, {0.64, 0.69}, CalloutStyle → {Green, None},
      LabelStyle → {FontFamily → "Arial", FontSize → 22, Green}, Background → None]},
   {x, -3.8, 3.8}, PlotStyle → {{Gray, Thickness → 0.005, Dashed}, {Red, Thickness → 0.005},
      {Blue, Thickness → 0.005}, {Orange, Thickness → 0.005}, {Green, Thickness → 0.005}},
   PlotRange → {{-3, 3}, {-0.02, 1.1}}, Frame → {{False, False}, {True, False}},
   FrameStyle → Directive[Black, Thickness → 0.002],
   AxesStyle → Directive[GrayLevel[0.3], Thickness → 0.0016], FrameTicks →
     {{{-3, -3, {0.01, 0}, Thickness → 0.003}, {-2, -2, {0.01, 0}, Thickness → 0.003}, {-1, -1, {0.01, 0},
         Thickness → 0.003}, {0, 0, {0.01, 0}, Thickness → 0.003}, {1, 1, {0.01, 0}, Thickness → 0.003},
       {2, 2, {0.01, 0}, Thickness → 0.003}, {3, 3, {0.01, 0}, Thickness → 0.003}}, {{0, 0}}},
   FrameLabel → {Style["x", 22, FontFamily → "Arial", Black, Italic, Bold]},
   Epilog → Text[Style["t = 0.6", 22, FontFamily → "Arial", Black, Bold], Scaled[{0.1, 0.96}]]];
G4 = Show[G4, LabelStyle → {FontFamily → "Arial", 22, GrayLevel[0]}];
ee = GraphicsGrid[{{G1, G2}, {G3, G4}}, ImageSize → 800, Spacings → {Scaled[-0.2], Scaled[0.16]},
   Epilog → Inset[LineLegend[{Directive[Red, Thickness[0.004]], Directive[Blue, Thickness[0.004]],
         Directive[Orange, Thickness[0.004]], Directive[Green, Thickness[0.004]]},
        {Style["Eq. 40^1", FontFamily → "Arial", FontSize → 22], Style["Eq. 40^2", FontFamily → "Arial",
           FontSize → 22], Style["Schrödinger", FontFamily → "Arial", FontSize → 22],
         Style["Dirac", FontFamily → "Arial", FontSize → 22]}, LegendFunction →
         (Framed[#, RoundingRadius → 5, FrameStyle → GrayLevel[0.3]] &)], Scaled[{0.5, 1.081}]]];
Export["/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure 5.png",
  ee, Background → None, ImageResolution → 600];
efg =
  Import[
   "/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure 5.png"];
efg = ImageTake[efg, {0, 4650}, {260, 6400}];
Export[
  "/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure 5.png", efg];
## ## ## Figure5 ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ##
## ## ## Figure6 ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ##
```

This code takes approximately 3 hours.

```
Clear["Global`*"];
Off[NDSolveValue::eerr];
L = 20;
F[x_] := ⅇ^(-x);
usol = NDSolveValue[{ⅈ ∂M(x, t)/∂t == -1/2 F[M(x, t)] (∂²M(x, t)/∂x² - (∂M(x, t)/∂x)²),
    M(x, 0) == 10^-13 ⅇ^(-2 x²), M(-L, t) == M(L, t)}, M, {x, -L, L}, {t, 0, 3}, WorkingPrecision → 66];
vsol = NDSolveValue[{ⅈ ∂M(x, t)/∂t == -1/2 F[M(x, t)] (∂²M(x, t)/∂x² - (∂M(x, t)/∂x)²), M(x, 0) == ⅇ^(-2 x²),
    M(-L, t) == M(L, t)}, M, {x, -L, L}, {t, 0, 3}, WorkingPrecision → 20];
wsol = NDSolveValue[{ⅈ ∂M(x, t)/∂t == -1/2 F[M(x, t)] (∂²M(x, t)/∂x² - (∂M(x, t)/∂x)²),
    M(x, 0) == 6/5 ⅇ^(-2 x²), M(-L, t) == M(L, t)}, M, {x, -L, L}, {t, 0, 3}, WorkingPrecision → 22];
xsol = NDSolveValue[{ⅈ ∂M(x, t)/∂t == -1/2 F[M(x, t)] (∂²M(x, t)/∂x² - (∂M(x, t)/∂x)²), M(x, 0) == 7/5 ⅇ^(-2 x²),
    M(-L, t) == M(L, t)}, M, {x, -L, L}, {t, 0, 3}, WorkingPrecision → 26];
G1 = Plot[{10^13 Norm[usol[0, t]], Norm[vsol[0, t]], FindMaxValue[Norm[vsol[x, t]], {x, 0, 3}],
      5/6 Norm[wsol[0, t]], FindMaxValue[5/6 Norm[wsol[x, t]], {x, 0, 3}],
      5/7 Norm[xsol[0, t]], FindMaxValue[5/7 Norm[xsol[x, t]], {x, 0, 3}]},
    {t, 0, 3}, PlotStyle → {{Orange, Thickness → 0.005}, {Green, Thickness → 0.005},
      {Green, Thickness → 0.005, Dashed}, {Blue, Thickness → 0.005},
      {Blue, Thickness → 0.005, Dashed}, {Red, Thickness → 0.005}, {Red, Thickness → 0.005, Dashed}},
    PlotRange → {{0, 3}, {0, 1.23}}, Frame → {{True, False}, {True, False}},
    FrameStyle → Directive[Black, Thickness → 0.002], FrameTicks →
     {{{0, "0.0"}, {0.5, 0.5, {0.01, 0}, Thickness → 0.003}, {1.0, "1.0", {0.01, 0}, Thickness → 0.003},
        {1.5, 1.5, {0.01, 0}, Thickness → 0.003}, {2.0, "2.0", {0.01, 0}, Thickness → 0.003},
        {2.5, "2.5", {0.01, 0}, Thickness → 0.003}, {3.0, "3.0", {0.01, 0}, Thickness → 0.003}},
      {{0, "0.0"}, {0.2, 0.2, {0.012, 0}, Thickness → 0.003}, {0.4, "0.4", {0.012, 0}, Thickness → 0.003},
        {0.6, "0.6", {0.012, 0}, Thickness → 0.003}, {0.8, "0.8", {0.012, 0}, Thickness → 0.003},
        {1.0, "1.0", {0.012, 0}, Thickness → 0.003}, {1.2, "1.2", {0.012, 0}, Thickness → 0.003}}},
    FrameLabel → {Style["t", 22, FontFamily → "Arial", Black, Italic, Bold]},
    LabelStyle → Directive[Black, FontFamily → "Arial", FontSize → 22]];
xv = NArgMax[Norm[vsol[0, t]], {t, 0.1, 0.5}];
xw = NArgMax[Norm[wsol[0, t]], {t, 0.1, 0.5}];
xx = NArgMax[Norm[xsol[0, t]], {t, 0.1, 0.5}];
G2 = Plot[{10^13 Norm[usol[x, 0]], Norm[vsol[x, xv]], 5/6 Norm[wsol[x, xw]], 5/7 Norm[xsol[x, xx]]},
    {x, -3, 3}, PlotStyle → {{Orange, Thickness → 0.005}, {Green, Thickness → 0.005},
      {Blue, Thickness → 0.005}, {Red, Thickness → 0.005}}, PlotRange → {{-3, 3}, {-0.02, 1.23}},
    Frame → {{False, False}, {True, False}}, FrameStyle → Directive[Black, Thickness → 0.002],
    AxesStyle → Directive[GrayLevel[0.3], Thickness → 0.0012], FrameTicks →
     {{{-3, -3, {0.01, 0}, Thickness → 0.003}, {-2, -2, {0.01, 0}, Thickness → 0.003}, {-1, -1, {0.01, 0},
          Thickness → 0.003}, {0, 0, {0.01, 0}, Thickness → 0.003}, {1, 1, {0.01, 0}, Thickness → 0.003},
        {2, 2, {0.01, 0}, Thickness → 0.003}, {3, 3, {0.01, 0}, Thickness → 0.003}}, {{0, 0}}},
    FrameLabel → {Style["x", 22, FontFamily → "Arial", Black, Italic, Bold]},
```

```
    LabelStyle → Directive[Black, FontFamily → "Arial", FontSize → 22]];
ff = Labeled[GraphicsRow[{G1, G2}, ImageSize → 800, Spacings → Scaled[0.15],
      Epilog → Inset[LineLegend[{Directive[Orange, Thickness[0.004]], Directive[Green, Thickness[0.004]],
            Directive[Blue, Thickness[0.004]], Directive[Red, Thickness[0.004]]},
          {Style["10^-13 ", FontFamily → "Arial", FontSize → 22],
            Style["1.0 ", FontFamily → "Arial", FontSize → 22], Style["1.2 ", FontFamily → "Arial",
              FontSize → 22], Style["1.4", FontFamily → "Arial", FontSize → 22]},
          LegendFunction → (Framed[#, RoundingRadius → 5, FrameStyle → GrayLevel[0.3]] &),
          LegendLayout → "Row"], Scaled[{0.5, 0.8}]]],
    Text[Style["    a                                                    b", 22, FontFamily → "Arial", Bold, Black]],
    {{Top, Left}},
    Spacings → {0, -0.3}];
Export["/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure 6.png",
   ff, Background → None, ImageResolution → 600];
ffg =
   Import[
     "/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure 6.png"];
ffg = ImageTake[ffg, {0, 2500}, {180, 6500}];
Export[
   "/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure 6.png", ffg];
```

Figure6