Discrepancy between power radiated and the power loss due to radiation reaction for an accelerated charge

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Abstract: We examine here the discrepancy between the radiated power, calculated from the Poynting flux at infinity, and the power loss due to radiation reaction for an accelerated charge. It is emphasized that one needs to maintain a clear distinction between the electromagnetic power received by a set of far-off observers and the instantaneous mechanical power loss of the charge. In literature both quantities are treated as not only equal but almost synonymous, however, the two, in general, need not be so. It is shown that in the case of a periodic motion, the two formulations do yield the same result for the power loss in a time averaged sense, though, the instantaneous rates could be quite different. It is demonstrated that the difference in the two power formulas is nothing but the difference in the rate of change of energy in self-fields of the charge between the retarded and present times. In particular, in the case of a uniformly accelerated charge, power going into the self-fields at the present time is equal to the power that was going into the self-fields at the retarded time plus the power going in acceleration fields, usually called radiation. From a comparison of far fields with the instantaneous position of the uniformly accelerated charge, it is shown that all its fields, including the acceleration fields, remain around the charge and are not *radiated away* from it.

Keywords: Classical electromagnetism; Applied classical electromagnetism. Radiation by moving charges; radiation or classical fields

1. Introduction

In electromagnetic radiation by a point charge, the radiated power is proportional to the square of the acceleration, known as Larmor's formula. On the other hand, the consequent radiation reaction on the charge turns out to be proportional to the rate of change of the acceleration of the charge. The two formulations do not seem to be conversant with each other. This apparent discrepancy in the two formulations has remained without a proper, universally acceptable, solution for more than a century. The conventional wisdom is that Larmor's formulation is more rigorous than the radiation-reaction formulation. We critically examine the relation between the two formulations and demonstrate that a mathematical subtlety in the application of Poynting's theorem is being missed when we try to use the energy-momentum conservation laws to compare the two formulas.

We shall, unless otherwise specified, confine ourselves only to non-relativistic motion, as the same set of disparities get carried over to the relativistic case [1]. Further, we shall assume a one-dimensional motion with acceleration parallel to the velocity and also throughout use the cgs system of units.

2. Two discrepant formulations for radiation losses from an accelerated charge

The electromagnetic field ($\mathbf{E}$, $\mathbf{B}$) of an arbitrarily moving charge e at a time t is given by [2–5],

$$
\begin{aligned}
\mathbf{E} &= \left[\frac{e(\mathbf{n} - \mathbf{v}/c)}{\gamma^2 r^2 (1 - \mathbf{n} \cdot \mathbf{v}/c)^3} + \frac{e\mathbf{n} \times \{(\mathbf{n} - \mathbf{v}/c) \times \dot{\mathbf{v}}\}}{rc^2 (1 - \mathbf{n} \cdot \mathbf{v}/c)^3} \right]_{t'}, \\
\mathbf{B} &= \mathbf{n} \times \mathbf{E},
\end{aligned}
\tag{1}
$$

where all quantities in square brackets are to be evaluated at the retarded time $t' = t - r/c$.

As the acceleration contributes only to the transverse fields, we shall, unless otherwise specified, leave the radial fields aside and consider, henceforth, only the transverse fields. It is to be emphasized that not only the acceleration fields, even the velocity fields have a transverse field component, normal to the radial direction along $\mathbf{n}$.

With the help of the vector identity $\mathbf{v} = \mathbf{n}(\mathbf{v.n}) - \mathbf{n} \times \{\mathbf{n} \times \mathbf{v}\}$, transverse components of the electromagnetic field of a charge, having a non-relativistic motion and therefore comprising only linear terms in velocity ($\mathbf{v}$) and acceleration ($\dot{\mathbf{v}}$), can be written from Eq. (1) as

$$
\begin{aligned}
\mathbf{E} &= \left[\frac{e\mathbf{n} \times (\mathbf{n} \times \mathbf{v})}{cr^2} + \frac{e\mathbf{n} \times (\mathbf{n} \times \dot{\mathbf{v}})}{c^2 r} \right]_{t'} = \left[\frac{e\mathbf{n} \times (\mathbf{n} \times (\mathbf{v} + \dot{\mathbf{v}}r/c))}{cr^2} \right]_{t'}, \\
\mathbf{B} &= \left[-\frac{e\mathbf{n} \times \mathbf{v}}{cr^2} - \frac{e\mathbf{n} \times \dot{\mathbf{v}}}{c^2 r} \right]_{t'} = -\left[\frac{e\mathbf{n} \times (\mathbf{v} + \dot{\mathbf{v}}r/c)}{cr^2} \right]_{t'}.
\end{aligned}
\tag{2}
$$

To calculate the radiated electromagnetic power, we make use of the radial component of the Poynting vector [2–4],

$$
\mathbf{n} \cdot \mathcal{S} = \frac{c}{4\pi} \mathbf{n} \cdot (\mathbf{E} \times \mathbf{B}) = \frac{c}{4\pi} (\mathbf{n} \times \mathbf{E}) \cdot \mathbf{B} = \frac{c}{4\pi} (B)^2.
\tag{3}
$$

Accordingly, one gets for the the radial component of the Poynting vector

$$
\mathbf{n} \cdot \mathcal{S} = \frac{e^2 \left[(\mathbf{v} + \dot{\mathbf{v}}r/c)^2 \right]_{t'}}{4\pi r^4 c} \sin^2 \theta.
\tag{4}
$$

The $\sin^2 \theta$ pattern implies that the rate of momentum being carried in the electromagnetic radiation is zero.

$$
\dot{\mathbf{p}}_r = 0.
\tag{5}
$$

However, the net Poynting flow through a spherical surface, Σ of radius r, around the charge, for a large r, is

$$
P_r = \int_{r \to \infty} d\Sigma \, (\mathbf{n} \cdot \mathcal{S}) = \frac{e^2}{2c} \int_0^\pi d\theta \, \sin^3 \theta \, \left. \frac{\left[(\mathbf{v} + \dot{\mathbf{v}}r/c)^2 \right]_{t'}}{r^2} \right|_{r \to \infty} = \frac{2e^2}{3c^3} \left[\dot{\mathbf{v}}^2 \right]_{t'}.
\tag{6}
$$

This is Larmor's famous result for power radiated from an accelerated charged particle [2–4]. Since the contribution of the velocity fields ($\propto 1/r^2$) appears to be negligible for a large enough value of r, the common perception is that in all cases, the acceleration fields ($\propto 1/r$) alone represent the radiation from a charge, with the Poynting flow due to the acceleration fields being independent of r.

Presumably, using the energy-momentum conservation laws [6,7], we can compute the mechanical energy-momentum losses of the radiating charged particle. For instance, the momentum of the charge would not change due to radiation damping,

$$
\mathbf{F} = -\dot{\mathbf{p}}_r = 0,
\tag{7}
$$

while the kinetic energy, $\mathcal{T}$, of the charged particle should change due to radiation losses at a rate

$$
\frac{d\mathcal{T}}{dt} = -\mathcal{P}_r.
\tag{8}
$$

Now, the above two equations do not seem consistent as the charged particle cannot lose kinetic energy without losing momentum. In fact, some problem is inherently present in Eq. (8) itself, as in the rest frame of the charge, the energy loss rate is finite ($\propto \dot{\mathbf{v}}^2$) even when the charged particle has no kinetic energy ($\mathbf{v} = 0$) to lose. It may be pointed out here that such a power loss into radiation can happen, without any change in the kinetic energy of the emitting charge, only if there were a loss of internal

(rest mass!) energy, without an accompanying loss of momentum [8]. But we do not contemplate a radiating charged particle to be converting its rest mass energy into electromagnetic radiation; after all a radiating electron still remains an electron at the end of the emission of radiation.

Somewhere something is amiss!

2.1. An inappropriate usage of the Poynting theorem

Actually, in the above formulation, which is the standard text-book approach, one is equating the Poynting flux at time t to the rate of kinetic energy loss of the charge at a retarded time $t - r/c$, purportedly using Poynting's theorem of energy conservation. However, a fallacy lies in this particular step. Poynting's theorem does not relate Poynting flux through a surface at some time t to energy loss rate by the enclosed charge at a retarded time $t - r/c$. It is true that the electromagnetic fields at r at time t do get determined by the charge motion at the retarded time $t' = t - r/c$ and intuitively we may then equate the electromagnetic power represented by the Poynting flux at r at time t to the mechanical power *loss* of the charge at the retarded time $t' = t - r/c$. Sometimes our common-sense notion of causality may be in conflict with the strict mathematical definition of Poynting's theorem and the ensuing application of energy-momentum conservation laws to electromechanical systems could lead us astray. It is such an oversight in this case that has mostly been the cause of confusion in this long drawn out controversy. In Poynting's theorem all quantities need to be calculated, strictly for the *same instant of time* [2–4].

Applying the Poynting's theorem in an appropriate manner, using real time values of the charge motion [9], one gets the instantaneous rate of loss of the mechanical energy by the charge as

$$\mathcal{P}_{\mathrm{c}} = -\frac{2e^2}{3c^3}\ddot{\mathbf{v}} \cdot \mathbf{v}\,, \tag{9}$$

where all quantities are evaluated for the same instant, say, t.

Now, one should clearly distinguish between the electromagnetic power received by a set of far-off observers and the instantaneous loss of mechanical power by the charge. In literature both power rates are treated as not only equal but almost synonymous. However, the two need not be the same as seen from Eqs. (6) and (9).

The difference in the two power formulas is

$$\mathcal{P}_{\mathrm{c}} - \mathcal{P}_{\mathrm{r}} = -\frac{2e^2}{3c^3}\ddot{\mathbf{v}} \cdot \mathbf{v} - \frac{2e^2}{3c^3}\left[\dot{\mathbf{v}} \cdot \dot{\mathbf{v}}\right]_{t'} = -\frac{2e^2}{3c^3}\frac{\mathrm{d}(\dot{\mathbf{v}} \cdot \mathbf{v})}{\mathrm{d}t}\,. \tag{10}$$

The last most term in Eq. (10) is known as the Schott term, after Schott [10] who first pointed it out, and is thought in literature to arise from an acceleration-dependent energy, $-2e^2(\dot{\mathbf{v}} \cdot \mathbf{v})/3c^3$, in electromagnetic fields, lying somewhere in the vicinity of the charge [11–16]. This elusive, century-old term does not seem to make an appearance elsewhere in physics. We shall later demonstrate that the Schott term is not any real electromagnetic energy and makes an appearance in the above equation merely because of the changing self-field energy of an accelerated charge between "real" and "retarded" times.

In the same way, the momentum conservation theorem, using Maxwell's stress tensor, directly leads to a rate of change of momentum of the charge [17]

$$\dot{\mathbf{p}}_{\mathrm{c}} = \frac{2e^2}{3c^3}\ddot{\mathbf{v}}\,. \tag{11}$$

The result in Eq. (11), known as the Abraham-Lorentz radiation reaction formula, has been obtained earlier from the self-force of the charge, calculated albeit in a rather cumbersome manner [2,3,10,18–20].

2.2. Applicability of Larmor's formula to compute radiative power losses in case of a periodic motion

Does the discrepancy in two formulations imply that Larmor's formula cannot be applied for computing mechanical power losses for a radiating charge?

In the case of a periodic motion of period T, there is no difference in the radiated power integrated or averaged between t to $t + T$ and t' to $t' + T$, therefore Larmor's formula, does yield a correct average power loss by the charge for a periodic case.

Let us write the motion of a harmonically oscillating charge (like in a radio antenna) as

$$\mathbf{x} = \mathbf{x}_0 \sin(\omega t + \phi) . \tag{12}$$

Then

$$\mathbf{v} = \dot{\mathbf{x}} = \omega \mathbf{x}_0 \cos(\omega t + \phi) , \tag{13}$$

$$\dot{\mathbf{v}} = \ddot{\mathbf{x}} = -\omega^2 \mathbf{x}_0 \sin(\omega t + \phi) = -\omega^2 \mathbf{x} , \tag{14}$$

$$\ddot{\mathbf{v}} = -\omega^3 \mathbf{x}_0 \cos(\omega t + \phi) = -\omega^2 \mathbf{v} . \tag{15}$$

Then Larmor's formula yields radiative power $\propto \dot{\mathbf{v}}^2 = \omega^4 \mathbf{x}_0^2 \sin^2(\omega t + \phi)$ while the power loss from the radiation reaction turns out $\propto -\ddot{\mathbf{v}} \cdot \mathbf{v} = \omega^4 \mathbf{x}_0^2 \cos^2(\omega t + \phi)$. Though the two expressions yield equal radiated energy when integrated or averaged over a complete cycle, the instantaneous rates are quite different. As an actual motion of the charge could be Fourier analysed, then the power spectrum, which gives average power in the cycle for each frequency component, would be the same. Of course in a non-periodic case like that of a uniformly accelerated charge, a Fourier analysis is not possible, and in such cases the two formulas could yield discordant answers.

2.3. Discrepancy in two power formulas is due to the difference in power going in self-fields at 'real' and retarded times

In order to understand the genesis of the difference between Eqs. (6) and (9), which respectively are at the retarded and real times, we consider the effect of the self-force of an accelerated charge on itself. We assume the charge to be a spherical shell of a sufficiently small radius ϵ, though, as we shall see, the final results turn out to be independent of the value of ϵ. Each infinitesimal element of the spherical shell experiences a force due to the time-retarded fields from the remainder parts of the charged shell and the total force on the charge is obtained by a double integration over the shell [2]. The net self-force on the accelerated charged spherical shell at a time t turns out to be proportional to the acceleration the charge had at a retarded time $t' = t - \epsilon/c$ [21].

$$\mathbf{f}_t = -\frac{2e^2}{3\epsilon c^2} \dot{\mathbf{v}}_{t'} , \tag{16}$$

where $\dot{\mathbf{v}}_{t'}$ is the acceleration of the charge at the retarded time t'. Then, for the charge moving with velocity v_t at time t, the work being done against self-force of the charge is

$$\mathcal{P}_t = -\mathbf{f}_t \cdot \mathbf{v}_t = \frac{2e^2}{3\epsilon c^2} \dot{\mathbf{v}}_{t'} \cdot \mathbf{v}_t . \tag{17}$$

Because work is done against the self-force, this is the rate at which energy is being put into the self-fields of the charge.

By expressing the acceleration at time t' in terms of its real-time value at t, to a first order, we have

$$\dot{\mathbf{v}}_{t'} = \dot{\mathbf{v}}_t - \ddot{\mathbf{v}}\epsilon/c. \tag{18}$$

Then from Eq. (16) for the self-force, in terms of real-time values, we can write

$$\mathbf{f}_t = -\frac{2e^2}{3\epsilon c^2}\dot{\mathbf{v}}_t + \frac{2e^2}{3c^3}\ddot{\mathbf{v}}_t \,, \tag{19}$$

where the last term is the well-known Abraham-Lorentz radiation reaction (Eq. (11)).

From this we get the formula for power going into self-fields (Eq. (17)), but now expressed in terms of the real time values, as

$$\mathcal{P}_t = \frac{2e^2}{3\epsilon c^2}(\dot{\mathbf{v}} \cdot \mathbf{v})_t - \frac{2e^2}{3c^3}(\ddot{\mathbf{v}} \cdot \mathbf{v})_t \,. \tag{20}$$

On the other hand, if we express the velocity itself in terms of its value at the retarded time t', to a first order in ϵ/c, we have

$$\mathbf{v}_t = \mathbf{v}_{t'} + \dot{\mathbf{v}}_{t'}\,\epsilon/c. \tag{21}$$

Substituting in (Eq. (17)), we get

$$\mathcal{P}_t = \frac{2e^2}{3\epsilon c^2}[\dot{\mathbf{v}} \cdot \mathbf{v}]_{t'} + \frac{2e^2}{3c^3}[\dot{\mathbf{v}} \cdot \dot{\mathbf{v}}]_{t'} \,. \tag{22}$$

The first term on the right hand side shows the rate of change of self-field energy of the accelerated charge *at the retarded time t'*, and the second term is Larmor's formula, again evaluated at t'.

From Eqs. (20) and (22), we get

$$-\frac{2e^2}{3c^3}(\ddot{\mathbf{v}} \cdot \mathbf{v})_t - \frac{2e^2}{3c^3}[\dot{\mathbf{v}} \cdot \dot{\mathbf{v}}]_{t'} = \frac{2e^2}{3\epsilon c^2}[\dot{\mathbf{v}} \cdot \mathbf{v}]_{t'} - \frac{2e^2}{3\epsilon c^2}(\dot{\mathbf{v}} \cdot \mathbf{v})_t \tag{23}$$

It shows that the difference in the two power formulas, Eqs. (9) and (6), which respectively are at real and retarded times, is nothing but the difference in the rate of change of energy in self-fields of the charge between retarded and present times.

Now, we can write the right hand side of Eq. (23) as

$$\frac{2e^2}{3\epsilon c^2}[\dot{\mathbf{v}} \cdot \mathbf{v}]_{t'} - \frac{2e^2}{3\epsilon c^2}(\dot{\mathbf{v}} \cdot \mathbf{v})_t = -\frac{2e^2}{3\epsilon c^2}\frac{\mathrm{d}(\dot{\mathbf{v}} \cdot \mathbf{v})}{\mathrm{d}t}\frac{\epsilon}{c} = -\frac{2e^2}{3c^3}\frac{\mathrm{d}(\dot{\mathbf{v}} \cdot \mathbf{v})}{\mathrm{d}t} \,, \tag{24}$$

a result independent of ϵ. This demonstrates that the elusive Schott term is not some actually energy hidden in the near fields but shows up in Eq. (10) merely due to the different rates of energy change in the self-fields between retarded and present times of an accelerated charge. This is consistent with the findings from a critical examination of the electromagnetic fields of a uniformly accelerated charge [22], where, contrary to the suggestions in the literature [11–16], no Schott energy term was found anywhere in the near vicinity of the charge, or for that matter, even in the far-off regions.

3. A uniformly accelerated charge

In the derivation of Larmor's formula (Eq. (5)), which is a standard text-book material [2–4], it is assumed that any contribution of velocity fields could be neglected. This assumption holds true in almost all cases, a notable exception being where the velocity of the accelerated charge may be a monotonic function of time, e.g., in the case of a uniformly accelerated charge [23].

Actually, in the case of a uniform acceleration, in the expressions for the fields (Eq. (2)), the *retarded* velocity of the charge would be related to the *present value* of velocity, $\mathbf{v}_0 = [\mathbf{v} + \dot{\mathbf{v}}r/c]_{t'}$. Then the transverse components of the electromagnetic fields become

$$
\begin{aligned}
\mathbf{E} &= \frac{e\mathbf{n} \times (\mathbf{n} \times \mathbf{v}_0)}{cr^2} , \\
\mathbf{B} &= \frac{-e\mathbf{n} \times \mathbf{v}_0}{cr^2} .
\end{aligned}
\tag{25}
$$

Thus we see that what all the acceleration fields do in this case is to make the instantaneous transverse fields *everywhere* directly proportional to the instantaneous present velocity v_0 of the accelerated charge.

3.1. The contribution of acceleration fields to the energy-momentum of self-field

It is well known that the self-field energy of a charge moving with a uniform velocity is different for different values of the velocity (see e.g. [23]). After all when a charge is accelerated, depending upon the change in velocity, its self-field energy must change too.

But that change in self-field energy cannot come from the velocity fields alone (the first term on the right hand side of Eq. (1)) which contains no information about the change that might take place in the velocity of the charge. Therefore the acceleration fields, to some extent at least, must provide for the changes taking place in the energy in self-fields, which are attached to the charge. As the acceleration, $\dot{\mathbf{v}}$, changes the velocity of the charge to say, $\mathbf{v}_0 = \mathbf{v} + \dot{\mathbf{v}}r/c$, the acceleration fields ($\propto \dot{\mathbf{v}}/r$) ensure that the transverse fields accordingly remain 'updated' ($\propto \mathbf{v}_0/r^2$), to remain synchronized with the 'present' value of the velocity of the charge, and the energy in self-fields is always equal to that required because of the 'present' velocity of the accelerated charge. The conventional wisdom, on the other hand, is that the acceleration fields, exclusively and wholly, represent power irreversibly lost as radiation and that of course is the genesis of Larmor's formula of radiative losses. Thus we see that there may be something amiss in the standard picture which does not take into account whatsoever contribution of the Poynting flux, due to the acceleration fields, might be towards the changing self-field energy of the accelerating charge. After all, the Coulomb field energy in radial fields, has zero self-field energy in transverse fields, and the growth in the self-field energy as the charge picks up speed due to acceleration, could have come only from the acceleration fields. The radiation actually would be that part of the Poynting flux which is over and above the value determined by the change occurring in the instantaneous velocity of the charge.

Employing the formula for the electromagnetic field energy

$$
\mathcal{E} = \int_{\mathrm{V}} \frac{E^2 + B^2}{8\pi} \, \mathrm{d}V ,
\tag{26}
$$

it is possible to compute the electromagnetic field energy, not only for a charge moving with a uniform velocity, but even in the case of a charge moving with a uniform acceleration [23]. For instance, the transverse field energy of the uniformly accelerated charge, in a shell of volume $4\pi r^2 \mathrm{d}r$, enclosed between spheres Σ and Σ_1 of radii r and $r + \mathrm{d}r$, is

$$
\mathrm{d}\mathcal{E} = \frac{e^2}{2} \left(\frac{4v_0^2}{3c^2} \right) \frac{\mathrm{d}r}{r^2} .
\tag{27}
$$

We can integrate over r to get the total energy in the transverse fields outside a sphere of radius ϵ as,

$$
\mathcal{E} = \frac{2e^2}{3c^2\epsilon} v_0^2 .
\tag{28}
$$

Since the integral diverges for $r \to 0$, we restricted the lower limit of r to a small ϵ, which may represent the radius of the charged particle.

One can also calculate the energy in fields of a charge moving with a uniform velocity v_0 and exactly the same amount of field energy is found around the charge. Thus it is clear that the acceleration fields in the case of a uniformly accelerated charge add just sufficient energy in the self-fields so as to make the total field energy equal to that required because of the 'present' velocity of the accelerating charge. This is true even in the case of the charges moving with relativistic velocities [23].

That the Poynting flux in the acceleration fields feeds the self-field energy in the case of a uniformly accelerated charge, is further seen from a comparison of the self energy changes between the real and the retarded times. Since in the case of a uniformly accelerated charge $\ddot{\mathbf{v}} = 0$, then from Eq. (23), we get

$$\frac{2e^2}{3\epsilon c^2}(\dot{\mathbf{v}} \cdot \mathbf{v})_t = \frac{2e^2}{3\epsilon c^2}[\dot{\mathbf{v}} \cdot \mathbf{v}]_{t'} + \frac{2e^2}{3c^3}[\dot{\mathbf{v}} \cdot \dot{\mathbf{v}}]_{t'} . \tag{29}$$

From Eq. (29), it is obvious that in the case of a uniformly accelerated charge, power going into the self-fields at the present time t is equal to the power that was going into the self-fields at the retarded time t' plus the power going in acceleration fields, usually called Larmor's formula for radiative losses. Instead of any losses being suffered by the charge, the energy in its self-fields is being constantly augmented by the acceleration fields. There is no other power term in the formulation that could be called *radiation emitted* by the uniformly accelerated charge.

We can compute the net momentum as well, in the self-fields of a uniformly accelerated charge, from the volume integral

$$\mathbf{p} = \int_V \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} \, dV . \tag{30}$$

Due to the azimuth symmetry about the direction of motion, the transverse component of the electric field (Eq. (25)) makes a nil contribution to the momentum, when integrated over the solid angle. However, the radial component, $e\mathbf{n}/r^2$, does make a net finite contribution, which would be along the direction of motion. Accordingly, we get

$$\mathbf{p} = \frac{e^2 \mathbf{v}_0}{2\epsilon c^2} \int_0^\pi d\theta \, \sin^3 \theta = \frac{2e^2}{3\epsilon c^2} \mathbf{v}_0 = m_{el} \mathbf{v}_0 , \tag{31}$$

where $m_{el} = 2e^2/3\epsilon c^2$ is the electromagnetic mass of the charge [24]. Thus we see that as the charge velocity changes to $\mathbf{v}_0$ due to the acceleration, the acceleration fields contribute to the self-fields of the charge, so that the field momentum becomes $m_{el}\mathbf{v}_0$, in accordance with the instantaneous velocity $\mathbf{v}_0$.

Thus both the energy and momentum in the self-fields of the uniformly accelerated charge are getting constantly updated by its acceleration fields in accordance with its 'present' velocity at any instant.

3.2. Poynting flux in the case of a uniformly accelerated charge

In the derivation of Larmor's formula (Eq. (6)), one assumed that the velocity fields would *always* make a negligible contribution to the Poynting flow, for large r. However, in the case of a uniformly accelerated charge, the contribution of velocity fields could match that of the acceleration fields, for all r. From Eq. (25), we find the Poynting flux to be

$$\mathcal{P} = \frac{2e^2 \mathbf{v}_0^2}{3r^2 c}. \tag{32}$$

The power passing through the spherical surface in the case of a uniformly accelerated charge is $\propto v_0^2/r^2$.

The transverse component of the electromagnetic field here (Eq. (25)) is the same as would be that of a charge moving with a uniform velocity $\mathbf{v}_o$, equal to the "present" velocity of the accelerated charge. Therefore, a Poynting flux exactly similar to Eq. (32) is also present in the case of a uniformly moving

charge, where we know there are no radiation losses and the Poynting flow through a surface around time-retarded position of the charge is merely due to the "convective" flow of fields, along with the moving charge. However, with respect to the 'present' position of a charge, there is no radial Poynting flux in this case. Taking a cue from this, even for a uniformly accelerated charge, one should examine the Poynting flux vis-à-vis the 'present' position of the accelerated charge, to find out if there indeed is some radiation taking place. As the energy in the self-fields must be "co-moving" with the charge, (otherwise the self-fields would lag behind, and no longer remain about the charge to qualify as its self-fields), and there should accordingly be a Poynting flow. Therefore not all of the Poynting flow may constitute radiation. The radiated power would be the part of the Poynting flow that is detached from the charge [4], i.e., it should be over and above the energy changes in the self-fields of the charge, as determined from the changing velocity of the charge. As we saw from the energy-momentum in the fields in section 3.1, there is no such excess energy in fields to be termed as radiation in the case of a uniformly accelerated charge.

It is evident from Eq. (25) that the transverse component of electromagnetic field, at least in the instantaneous rest frame ($\mathbf{v}_o = 0$) of a uniformly accelerated charge, is nil. This happens due to a systematic cancellation of acceleration fields by the transverse component of velocity fields, in the instantaneous rest frame, both for the electric and magnetic fields, *at all distances*. That the magnetic field is zero everywhere in this case was first pointed out by Pauli [25], using Born's solutions [26], who inferred from it that no wave zone would be formed and hence there is no radiation from a uniformly accelerated charge.

3.2.1. A definition of radiation at infinity incompatible with Green's theorem

It has been claimed that Pauli's statement, that contradicts Larmor's formula, is invalid on the grounds that a limit to large r at a fixed time, say, $t = 0$, is implied therein [27,28]. It has been asserted that the radiation should instead be defined by the total rate of energy emitted by the charge at the retarded time t', and is to be calculated by integrating over the surface of the light sphere in the limit of infinite $r = c(t - t')$ for a *fixed emission time t'*, with both $t \to \infty$ and $r \to \infty$ [27,28]. The two limiting procedures, one with t fixed and the other with t' fixed, do not yield the same result and from that it has been concluded that Pauli's observation that $B = 0$ everywhere at some fixed time t is a mere curiosity that may be of some interest but does not imply an absence of radiation [27,28].

If we carefully examine the reason why a *fixed emission time t'* is being chosen for defining 'radiation' [27,28], we can see that this choice makes the contribution to the Poynting flow, from the velocity fields at t', for a large enough r, negligible. However, for a uniformly accelerated charge, one cannot ignore the contribution of the velocity fields to the Poynting flow, as $v(t') \propto -\dot{v}r/c$. Moreover, in this case, there is something unusual happening about the fields at large r vis-à-vis the charge location at large t, which we shall discuss in Section 3.3.

Actually Green's retarded solution, where, for instance, the scalar potential at a field point $\mathbf{x}$, at time t, is determined from the volume integral

$$\phi(\mathbf{x}, \mathbf{t}) \;=\; \int \left[\frac{\rho(\mathbf{x}')}{r} \right]_{t'} \mathrm{d}^3 x' \,, \tag{33}$$

with the charge density $\rho(\mathbf{x}')$ at the location $\mathbf{x}'$, as well as the distance $r = |\mathbf{x} - \mathbf{x}'|$, within the square brackets, are to be determined at the retarded time t' [2], with a similar expression for the vector potential.

Thus here $\mathbf{x}$ and t are first specified and the volume integral of ρ/r at the corresponding retarded time is then computed. Pauli's argument is consistent with this procedure. In fact, the radiation defined by first fixing the emission time, t' [27,28], strictly speaking, may not be in tune with Green's retarded time solution, and could sometime lead to wrong conclusions.

It may be pointed out that for a "point" charge e moving with velocity $\mathbf{v}$, first fixing the point charge position $\mathbf{x}'$ at the retarded-time t', to determine the potential this way, yields $\phi = e/r$, while
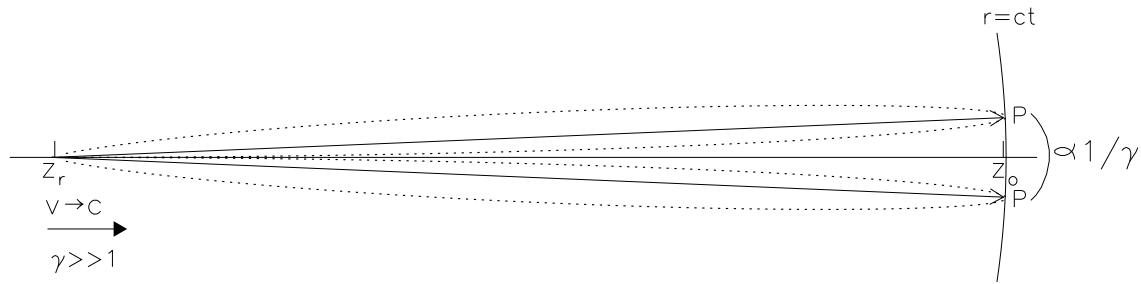
Figure 1. Angular distribution of the electric field strength with respect to the time-retarded position z_r of the uniformly accelerated charge, moving along the z-axis with velocity $v \to c$ and the corresponding Lorentz factor $\gamma \gg 1$. Due to the relativistic beaming, the field strength is mostly appreciable only within a cone of angle $\theta \sim 1/\gamma$ about the direction of motion. When at time t, the fields from the retarded position z_r are at the spherical front of radius $r = ct$, the charge meanwhile has moved to z_o, quite close to the spherical front. The circle represented by points P on the spherical front $r = ct$ where the field strength is maximum as a function of θ, lies almost vertically above z_o, the 'present' position of the charge, and thus are not very far from it, implying that the field at large r is still around the 'present' location of the charge.

the more correct approach of first fixing the field point **x** at time t, leads to $\phi = e/[r(1 - \mathbf{n} \cdot \mathbf{v}/c)]$, the correct expression for the potential [24].

3.3. Far fields and the relative location of the uniformly accelerated charge

Since we want to examine far fields at large r, this would also imply large values of $t = r/c$. Now a uniform acceleration for long durations could make the motion of the charge relativistic. Accordingly, in this section, we shall no longer assume the motion to be non-relativistic.

In a typical radiation scenario, the radiated power moves away ($r \to \infty$), with the charge responsible remaining behind, perhaps not very far from its location at the corresponding retarded time, e.g., in localized charge or current distributions in a radiating antenna. This of course necessarily implies that not only the motion of the charge is bound, its velocity and acceleration are having, some sort of oscillatory behaviour, even if not completely regular. However, in the case of a uniform acceleration, such is not the case. Due to a constant acceleration, the charge picks up speed, and after a long time its motion will become relativistic, with $v \to c$ and the corresponding Lorentz factor becoming very large ($\gamma \gg 1$). Then, due to the relativistic beaming, the distant fields of the charge as well as the associated Poynting flux is appreciable only within a narrow cone opening angle, $\theta_m \sim 1/\gamma$ [2–4], about the direction of motion. Moreover, the charge, moving with a velocity $v \to c$, is not very far behind the spherical wave-front of radius $r = ct$. Thus the charge, with $v \approx c(1 - 1/2\gamma^2)$, moves a distance $\sim ct(1 - 1/2\gamma^2)$ along the z-axis, while the maxima of the field at $r = ct$ has moved along the z-axis $r \cos\theta_m \approx ct(1 - \theta_m^2/2) \approx ct(1 - 1/2\gamma^2)$, thus the field maxima lies in a plane normal to the z-axis that passes nearly through the 'present' position of the charge on the z-axis (Fig. 1), and the fields are all around the charge. The electric field, in fact, very much resembles that of a charge moving with a uniform velocity equal to the 'present' velocity of the uniformly accelerated charge, with the field in a plane normal to the direction of motion. Thus as the fields move toward infinity, so does the charge. The fields actually are the self-fields of the charge that due to the acceleration fields, increase in strength, as the charge picks up speed, to a value expected from that of the charge moving with a uniform velocity equal to the 'present' velocity of uniformly accelerated charge. Accordingly there is no radiation being 'emitted' by the charge.

We can verify the above statements explicitly by a comparison of the fields of a uniformly accelerated charge, which may have a relativistic 'present' velocity $v_0 \to c$ and a corresponding Lorentz factor $\gamma_0 \gg 1$, with those of a charge moving with a uniform motion, with exactly the same velocity v_0 and thus the same Lorentz factor γ_0.
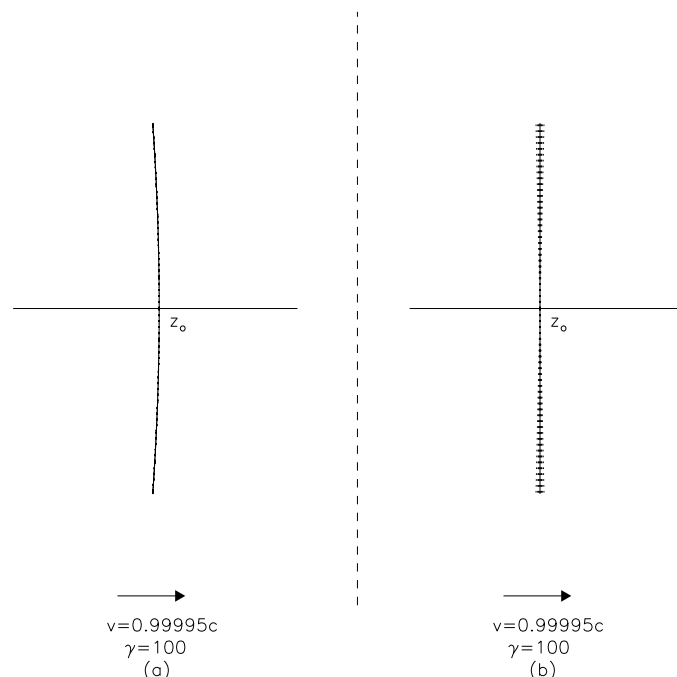
Figure 2. The electric field distribution (a) of a uniformly accelerated charge, with a 'present' velocity $v = 0.99995c$, corresponding to $\gamma = 100$ (b) of a charge moving with a uniform velocity $v = 0.99995c$, corresponding to $\gamma = 100$.

Let the charge moving with a uniform acceleration, $a \equiv \gamma^3 \dot{v}$ along $+z$ axis, was momentarily stationary at time $t = 0$ at a point $z = \alpha$, chosen, without any loss of generality, so that $\alpha = c^2/a$. The position and velocity of the charge, before or after, at any other time t are then given by $z_0 = (\alpha^2 + c^2t^2)^{1/2}$, $v_0 = c^2t/z_0$ and $\gamma_0 = z_0/\alpha$.

The electric field of the charge moving with a uniform acceleration, is given in cylindrical coordinates (z, ρ), as [27,29]

$$
\begin{aligned}
E_z &= -4e\alpha^2(z_0^2 + \rho^2 - z^2)/\xi^3 \\
E_\rho &= 8e\alpha^2\rho z/\xi^3 \, ,
\end{aligned}
\tag{34}
$$

where $\xi = [(z_0^2 - \rho^2 - z^2)^2 + 4\alpha^2\rho^2]^{1/2}$. This electric field expression is equivalent to the field expressions in terms of retarded-time quantities, and can be derived in the case of a uniformly accelerated charge starting from Eqs. (1), using algebraic transformations [30].

On the other hand, the electromagnetic field of the charge moving with a uniform velocity $\mathbf{v_0}$, can be written in a spherical coordinates (R, θ), centered at the "present" charge position [2–4], as

$$
E_R = \frac{e}{R^2\gamma_0^2(1 - (v_0/c)^2 \sin^2\theta)^{3/2}} \, .
\tag{35}
$$

The magnetic field in both cases is given by $\mathbf{B} = \mathbf{v_0} \times \mathbf{E}$.

Now if we plot the electric field for a large $r = ct$, which also implies for the uniformly accelerated charge, $v_0 \to c$ and $\gamma_0 \gg 1$, and compare it with a charge moving with a uniform velocity $\mathbf{v_0}$ and having the same γ_0, we find that the fields are very similar in both cases. Figure 2 shows a comparison of the electric fields in both cases for $\gamma_0 = 100$, corresponding to $v_0 = 0.99995c$. In both cases fields are almost indistinguishable and extend from the charge in direction normal to the direction of motion.

Figure 3 shows the corresponding Poynting flow in both cases, and is again almost indistinguishable, with the overall Poynting flow in each case being along the direction of motion of
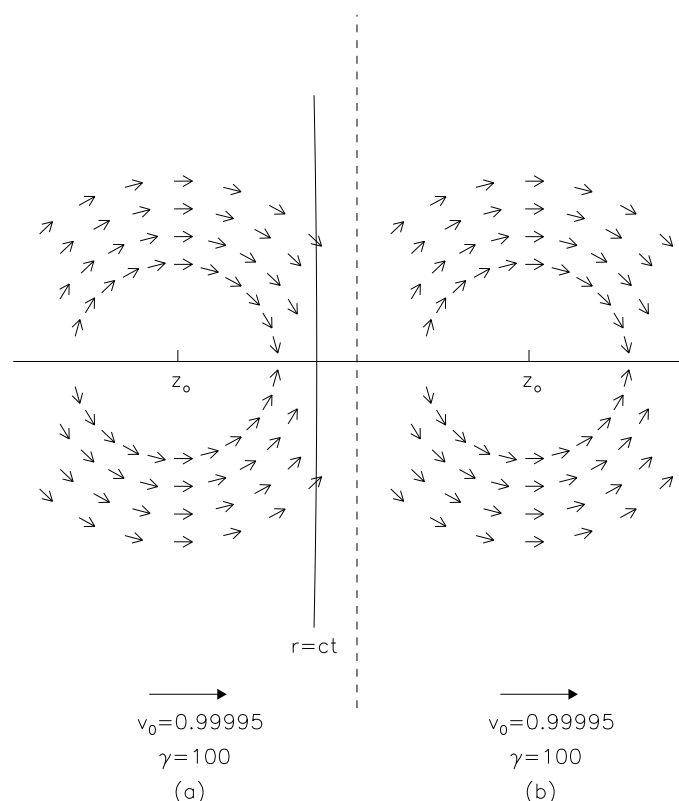
Figure 3. The Poynting vector for a charge (a) moving with a uniform proper acceleration, and is presently at z_o moving with a 'present' velocity $v_0 = 0.99995c$, corresponding to $\gamma_0 = 100$ (b) moving with a uniform velocity $v = 0.99995c$, corresponding to $\gamma = 100$. The spherical wave-front $r = ct$ is shown in the case of uniformly accelerated charge. The overall Poynting flow is along the direction of motion of the charge.

the charge, confirming that the Poynting flow for a uniformly accelerated charge merely represents the "convective" flow of self-fields, along with the moving charge, like in the case of a charge moving with a uniform velocity. Of course, in the case of a uniformly accelerated charge, the self-field strength continuously keeps getting 'updated' due to acceleration fields, in tune with the changing charge velocity due to its uniform acceleration. Naturally, there is no radiation reaction in the case of a uniformly accelerated charge since no field energy is being 'radiated away' from such a charge. This, of course, also makes the case of a uniformly accelerated charge fully conversant with the strong principle of equivalence.

It is a misconception that the radiation emitted from the uniformly accelerated charge goes beyond the horizon, the regions of space-time inaccessible to an observer co-accelerating with charge [29,31]. The only radiated power that goes beyond the horizon is that in the δ-fields, causally related to the charge during its uniform velocity before an acceleration was imposed at an infinite past. It has been explicitly demonstrated [30] that all the energy that goes into δ-fields is neatly explained by the radiation losses (Eqs. (9)), owing to the Lorentz-Dirac radiation reaction, because of *a rate of change of acceleration* the charge, that previously was moving with a uniform velocity, undergoes at that event. In fact as we demonstrated above, all the fields, including the acceleration fields, having a genesis from the uniform accelerated charge, remain around the moving charge and are not *radiated away* or dissociated from the charge as long as it continues moving with a uniform acceleration.

Funding: This research received no external funding.

Conflicts of Interest: The author declares no conflict of interest.

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