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*Article*

# Existence, uniqueness and input-to-state stability of ground-state stationary strong solution of a single-species model via Mountain Pass Lemma

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**Abstract:** In this paper, the authors employ Mountain Pass Lemma, the method of weak solution regularization and Lyapunov function method to derive the unique existence of globally exponential stable positive stationary solution of a single-species model with diffusion and delayed feedback. The obtained stability criterion illuminates that under some suitable conditions, a certain internal competition is conducive to the overall stability of the population, and a certain amount of family planning is conducive to the overall stability of the population. A numerical example and three tables show the effectiveness of the proposed methods.

**Keywords:** strong solution, regularity of weak Solutions, a single-species ecosystem; Lyapunov function; Mountain Pass Lemma; unique existence

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## 1. Introduction

Logistic system is one of the most classical models in ecology and mathematics, which is very important to the development of ecology ([1]). It is usually expressed as

$$\frac{dx}{dt} = rx(t)(1 - \frac{x(t)}{K}), \tag{1.1}$$

where Where $x(t)$ represents the density or quantity of population $x$ at time $t$, $r > 0$ and $K$ represent the intrinsic growth rate of population and environmental capacity, respectively. In 2011, Xiaoling Zou and Ke Wang investigated the long time behavior of the following stochastic ecosystem for a single-species ( [2, Theorem 2] ):

$$dx = x[a - bx]dt + \alpha x dB(t). \tag{1.2}$$

where $a > 0$ and $b > 0$ describe the growth rate and the intra-specific competition; $\alpha > 0$ measures the intensity of the environmental disturbances. In recent years, model (1.2) has been widely adopted in many applications (see, e.g. [3-5]). A large number of facts have shown that the spatial scale and structure of the environment can affect population interaction [6, 7] and community composition [8]. In the landmark document [9], Kellam gave a large number of observations, which had a profound impact on the study of spatial ecology. First, he linked random walk with diffusion equation, The former is a description of the individual movement of some theoretical biological species, and the latter is a description of the density distribution of biological populations. He uses the data of muskrat transmission in Central Europe to prove that this connection is reasonable for small animals. Secondly, he combines diffusion with population dynamics, and effectively introduces the reaction-diffusion equation into theoretical ecology.

In recent years, many dynamical systems, including reaction systems, have been considered as the theoretical branches of dynamical systems([10-12]). In addition, the competition within the population is the participation of the adult population, and there is a period from infancy to adulthood. At the same time, this time-delay problem is affected by many stochastic factors such as weather, temperature, humidity and so on. Besides, in real life, the factors that affect population growth do not change only at a fixed time, but also occur randomly. When these factors occur, the system will also change randomly. As is well known, the phenomenon of population clustering is widespread in nature, which is likely to be completely affected by environmental factors and human factors. In this case, the growth curve of mosquitoes or small fish will be different from the previous form. This phenomenon can be expressed as a switch between two environmental states, because the switching between different environments is not memory free, Therefore, one can use continuous time Markov chain to model the situation of environment switching ([13-16]).

Inspired by some ideas and method of related literature [17-23], I am to investigate the stability of stationary density of a single-species model with diffusion and delayed feedback under natural state. This paper has the following highlights:

★ As far as I know, it is the first paper to investigate the stability of stationary density of a single-species model with diffusion and delayed feedback under Dirichlet zero boundary value. And the Dirichlet boundary value can well simulate the fact that the species lives in its biosphere, while the population density tends to zero at the boundary of biosphere due to the harsh condition.

★ It is the first comprehensive application of Mountain Pass Lemma, variational technique and Lyapunov function method to derive the unique existence of globally exponentially stable positive stationary solution of a single-species model with diffusion and delayed feedback under Dirichlet zero boundary value.

★ The obtained stability criterion illuminates that under some suitable conditions, a certain internal competition is conducive to the overall stability of the population, and a certain amount of family planning is conducive to the overall stability of the population.

For convenience, throughout of this paper, I denote by $\lambda_1$ the first positive eigenvalue of Laplace operator $-\Delta$ in $H_0^1(\Omega)$. Denote $u^+ = \max\{u, 0\}$, $u^- = \min\{u, 0\}$. Denote by $\|u\| = \sqrt{\int_\Omega |\nabla u(x)|^2 dx}$ the norm of $H_0^1(\Omega)$, and by $\lambda_1$ the first positive eigenvalue of Laplace operator $-\Delta$ in $H_0^1(\Omega)$. Besides, I denote $|v| = (|v_1|, |v_2|)^T$ for $v = (v_1, v_2)^T \in \mathbb{R}^2$, and $|C| = (|c_{ij}|)_{2\times 2}$ for matrix $C = (c_{ij})_{2\times 2}$.

## 2. System descriptions

Denote by $(\Upsilon,\ \mathcal{F},\ \mathbb{P})$ the complete probability space with a natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Let $S = \{1,\ 2,\ \cdots,\ n_0\}$ and the random form process $\{r(t) : [0,\ +\infty) \to S\}$ be a homogeneous, finite-state Markovian process with right continuous trajectories with generator $\Pi = (\gamma_{ij})_{n_0\times n_0}$ and transition probability from mode $i$ at time $t$ to mode $j$ at time $t+\delta,\ i,\ j \in S$,

$$\mathbb{P}(r(t+\delta) = j \mid r(t) = i) = \begin{cases} \gamma_{ij}\delta + o(\delta), & j \neq i \\ 1 + \gamma_{ij}\delta + o(\delta), & j = i, \end{cases}$$

where $\gamma_{ij} \geqslant 0$ is transition probability rate from $i$ to $j(j \neq i)$ and $\gamma_{ii} = -\sum_{j=1, j\neq i}^{n_0} \gamma_{ij}$, $\delta > 0$ and $\lim\limits_{\delta\to 0} o(\delta)/\delta = 0$.

Consider the following ecosystem with diffusion and delayed feedback

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} =q\Delta u(t,x) + u(t,x)[a - bu(t,x)] + c(r(t))[u(t,x) - u(t-\tau(t),x)] + \Lambda(u), \quad t \geqslant 0,\ x \in \Omega, \\ u(t,x) =0, \quad x \in \Omega,\ t \geqslant 0. \end{cases} \tag{2.1}$$

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where $\Lambda(u)$ is the external input, $a > 0$ and $b > 0$ describe the growth rate and the intra-specific competition, and $\Omega \in R^3$ is bounded domain with its boundary $\partial\Omega$, and is also a $C^{2,\sigma}$ domain in $R^3$(see, e.g. [17]). It is also suitable to the case that the species lives in two dimensional plane (see [19, Remark 1]).

Throughout this paper, I assume

(H1) the positive function $\Lambda$ is only a **micro perturbation**. That is, there exists a positive number $\varepsilon > 0$ small enough such that

$$\lim_{u\to\infty} \frac{\Lambda(u)}{u^\theta} = \varepsilon = \lim_{u\to\infty} \frac{\Lambda(u)}{u}, \tag{2.2}$$

where $2^* - 1 > \theta = \frac{\theta_2}{\theta_1} > 2$ with $\theta_2$ and $\theta_1$ being a pair of coprime odd numbers. And $\Lambda(\cdot)$ is continuous and $\Lambda(u) \geqslant 0$ for all $u \geqslant 0$. Here, $2^*$ is the Sobolev critical exponent. In addition, $\Lambda(u) = 0$ for all $u \leqslant 0$.

Let $u_*(x)$ be a stationary solution of the system (2.1) implies that $u_*(x)$ is a solution of the following equation

$$-q\Delta u = au - bu^2 + \Lambda(u), \quad x \in \Omega; \quad u|_{\partial\Omega} = 0, \tag{2.3}$$

Of course, each solution of the equation (2.3) must be one of the solutions of the system (2.1).

**Definition 1.** The stationary solution $u_*(x)$ of the system (2.1) is called the ground-state stationary solution of the system (2.1) if $u_*(x)$ is the ground-state solution of the equation (2.3).

**Definition 2.** A solution $u_*(x)$ of the equation (2.3) is called the strong solution of the equation (2.3) if $u_*(x) \in C^2(\Omega)$.

To prove the main result of this paper, I need the following Lemmas (see, e.g. [17, 20]):

**Lemma 2.1.** Consider the following equation:

$$\begin{cases} -\Delta u = f(x,u), & x \in \Omega; \\ u|_{\partial\Omega} = 0, \end{cases} \tag{2.4}$$

where $\Omega$ is a $C^{k+2,\alpha}$ domain of $R^n$, and $f$ satisfies the following conditions:

(a) there exists $0 < r \leqslant 1$ such that for any given positive number $M$,

$$f(x,u) \in C^{k,r}(\overline{\Omega} \times [-M, M], R^1),$$

(b) if $n \geqslant 3$ and $s = 2^* - 1$, or $n \leqslant 2$ and $s > 1$, then $f(x,u) = O(|u|^s)$ (as $|u| \to \infty$) holds uniformly for $x \in \overline{\Omega}$,

(c) $\lim\limits_{u\to 0} \frac{f(x,u)}{u} = a(x) \in L^\infty(\Omega)$.

Then the solution of the equation (2.3) in $H_0^1(\Omega)$ is the strong solution. In addition, $u \in C^{k+2,\delta}$ for $\delta = \alpha r^{k+1}$.

**Lemma 2.2.** Let $u \in H_0^1(\Omega)$. Then there is a conclusion that $u^+, u^-, |u| \in H_0^1(\Omega)$. Besides, $\nabla u^+ = \nabla u$ if $u > 0$, and $\nabla u^+ = 0$ if $u \leqslant 0$; $\nabla u^- = 0$ if $u \geqslant 0$, and $\nabla u^- = \nabla u$ if $u < 0$; In addition, $\nabla|u| = \nabla u$ if $u > 0$, and $\nabla|u| = -\nabla u$ if $u < 0$. Besides, $\nabla|u| = 0$ if $u = 0$.

**Lemma 2.3** (Mountain Pass Lemma without the (PS) condition). Let $X$ is a Banach space, $\Psi \in C^1(X, \mathbb{R})$, satisfying $\Psi(0) = 0$ , and there exists $\rho > 0$ such that $\Psi|_{\partial B_\rho(0)} \geqslant \alpha > 0$. Besides, there is $e \in X \setminus \overline{B_\rho(0)}$ such that $\Psi(e) \leqslant 0$. Let $\Gamma$ be the set of all paths connecting 0 and $e$. That is,

$$\Gamma = \{\psi \in C([0,1], H_0^1(\Omega)) : \psi(0) = 0, \psi(1) = e\}. \tag{2.5}$$

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Set

$$c_* = \inf_{\psi \in \Gamma} \max_{s \in [0,1]} \Psi(\psi(s)). \tag{2.6}$$

Then $c_* \geqslant \alpha$, and $\Psi$ possesses a critical sequence on $c_*$.

**Remark 1.** Lemma 2.3 is the Mountain Pass Lemma without the (PS) condition (see,e.g. [18]). If, in addition, $\Psi$ satisfies the (PS) condition, then $c_*$ is a critical value of $\Psi$.

**Remark 2.** Let $\Psi$ be the functional corresponding to the equation (2.3), then $u^*(x)$ must be a ground-state solution of the equation (2.3) if $u_*(x)$ is a critical point of the functional $\Psi$ with $\Psi(u^*(x)) = c_*$ defined in (2.6) of Lemma 2.3.

## 3. Main result

Firstly, I may present the existence of a stationary strong solution $u^*(x)$ of the system 2.1. In addition, it is necessary to guarantee that $u_*(x) \geqslant 0$ and $u^*(x) \neq 0$, which may be proved as follows:

**Theorem 3.1.** Suppose the condition (H1) holds, and if

$$a < q\lambda_1, \tag{3.1}$$

Then there is a ground-state strong stationary solution for the system (2.1).

**Proof.** Let $u_*(x)$ be a positive stationary solution of the system (2.1), satisfying

$$-q\Delta u = au - bu^2 + \Lambda(u), \quad u(x) > 0, \quad x \in \Omega; \quad u|_{\partial\Omega} = 0, \tag{3.2}$$

whose functional is

$$\Psi(u) = \int_\Omega \frac{|\nabla u|^2 + \mu u^2 - \mu(u^+)^2}{2} dx - \frac{a}{2q}\int_\Omega (u^+)^2 dx + \frac{b}{3q}\int_\Omega (u^+)^3 dx - \int_\Omega \widetilde{\Lambda}(u^+)dx, \tag{3.3}$$

where $\mu > 0$ is a constant, and $\widetilde{\Lambda}(u) = \int_0^u \Lambda(s)ds$. It is obvious that $\Psi \in C^1(H_0^1(\Omega), R^1)$, and a critical point of the functional $\Psi$ is corresponding to the solution of the equation (3.2). Next, I claim that $\Psi$ satisfies the condition of the Mountain road geometry. In fact, obviously $\Psi(0) = 0$.

The micro perturbation condition (H1) yields that there are there positive constants $c_0, m_1, m_2$ with $m_1 < 1 < m_2$ such that

$$\varepsilon - \frac{1}{2}\varepsilon < |\frac{\Lambda}{u^\theta}| < \varepsilon + \frac{1}{2}\varepsilon, \, u \in (m_2, +\infty), \tag{3.4}$$

or

$$\frac{1}{2}\varepsilon u \leqslant \Lambda(u) \leqslant \frac{3}{2}\varepsilon u, \quad u \in [0, m_1); \quad \frac{1}{2}\varepsilon u^\theta \leqslant \Lambda(u) \leqslant \frac{3}{2}\varepsilon u^\theta, \quad u \in (m_2, +\infty); \quad 0 \leqslant \Lambda(u) \leqslant c_0, \quad u \in [m_1, m_2], \tag{3.5}$$

which implies

$$\Lambda(u) \leqslant \frac{3}{2}\varepsilon u, \quad u \in [0, m_1); \quad \Lambda(u) \leqslant \frac{3}{2}\varepsilon u^\theta, \quad u \in (m_2, +\infty); \quad \Lambda(u) \leqslant c_0, \quad u \in [m_1, m_2], \tag{3.6}$$

and then

$$\Lambda(u) \leqslant \frac{3}{2}\varepsilon u + \frac{3}{2}\varepsilon u^\theta + c_0 \leqslant \frac{3}{2}\varepsilon u + \frac{3}{2}\varepsilon u^\theta + c_0 \frac{u^\theta}{m_1^\theta}, \quad \forall u \geqslant 0. \tag{3.7}$$

Moreover,

$$\widetilde{\Lambda}(u) = \int_0^u \Lambda(s)ds \leqslant \frac{3}{4}\varepsilon u^2 + (\frac{3}{2}\varepsilon + \frac{c_0}{m_1^\theta})\frac{1}{1+\theta}u^{1+\theta}, \quad \forall u \geqslant 0. \tag{3.8}$$

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Next, (3.3) ,(3.8), Poincare inequality and Sobolev embedding theorem yield that there is a positive constant $c_1 > 0$ such that

$$\Psi(u) \geqslant \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{a}{2q}\int_\Omega u^2 dx - \int_\Omega \widetilde{\Lambda}(u^+)dx \geqslant \left(\frac{1}{2} - \frac{a}{2q\lambda_1} - \frac{3}{4}\varepsilon\right)\|u\|^2 - c_1\|u\|^{1+\theta} \tag{3.9}$$

Besides, (3.1) and small $\varepsilon > 0$ lead to

$$\frac{1}{2} - \frac{a}{2q\lambda_1} - \frac{3}{4}\varepsilon > 0, \tag{3.10}$$

which together with $\theta > 2$ means that there exists $\rho > 0$ small enough such that $J|_{\partial B_\rho(0)} \geqslant \alpha > 0$.

On the other hand, it follows by (3.5) and $m_1 < 1$ that

$$\frac{1}{2}\varepsilon u^\theta \leqslant \frac{1}{2}\varepsilon u \leqslant \Lambda(u), \quad u \in [0, m_1); \quad \frac{1}{2}\varepsilon u^\theta \leqslant \Lambda(u), \quad u \in (m_2, +\infty); \quad \Lambda(u) \geqslant 0, \quad u \in [m_1, m_2],$$

which implies

$$\Lambda(u) \geqslant \frac{1}{2}\varepsilon u^\theta, \quad \forall u \geqslant 0, \tag{3.11}$$

or

$$\widetilde{\Lambda}(u) \geqslant \frac{1}{2(1+\theta)}\varepsilon u^{1+\theta}, \quad \forall u \geqslant 0. \tag{3.12}$$

I may select $u \in H_0^1(\Omega)$ with $u \geqslant 0$, and then

$$\begin{aligned} \Psi(u) =& \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{a}{2q}\int_\Omega u^2 dx + \frac{b}{3q}\int_\Omega u^3 dx dx - \int_\Omega \widetilde{\Lambda}(u)dx \\ \leqslant& \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{a}{2q}\int_\Omega u^2 dx + \frac{b}{3q}\int_\Omega u^3 dx dx - \int_\Omega \frac{1}{2(1+\theta)}\varepsilon u^{1+\theta} dx. \end{aligned} \tag{3.13}$$

Let $\varphi_1(x) > 0$ with $\|\varphi\| = 1$ be the eigenfunction corresponding to the first positive eigenvalue $\lambda_1$ (see, e.g. [18]), and set $u = s\varphi$, then $\Psi(s\varphi) \to -\infty$ if $s \to +\infty$ so that there exists $s_0 > 0$ satisfying $\|s_0\varphi\| \geqslant \rho$ and $\Psi(s\varphi) < 0$. And then $\Psi$ satisfies the condition of the Mountain road geometry. According to Mountain Pass Lemma, Let $\Gamma$ be the set of all paths connecting 0 and $e = s_0\varphi$. That is,

$$\Gamma = \{\psi \in C([0,1], H_0^1(\Omega)) : \psi(0) = 0, \psi(1) = e\}. \tag{3.14}$$

Set

$$c_* = \inf_{\psi \in \Gamma} \max_{s \in [0,1]} \Psi(\psi(s)). \tag{3.15}$$

Then $c_* \geqslant \alpha$, and $\Psi$ possesses a critical sequence on $c_*$, say, $\{u_k\} \subset H_0^1(\Omega)$ with $\Psi(u_k) \to c_*$ and $\Psi'(u_k) \to 0$ in $(H_0^1(\Omega))^*$. That is, for any given $\epsilon > 0$, there exists $k$ big enough such that

$$\int_\Omega \frac{|\nabla u_k|^2 + \mu u_k^2 - \mu(u_k^+)^2}{2}dx - \frac{a}{2q}\int_\Omega (u_k^+)^2 dx + \frac{b}{3q}\int_\Omega (u_k^+)^3 dx - \int_\Omega \widetilde{\Lambda}(u_k^+)dx = \Psi(u_k) = c_* + o(1) \tag{3.17}$$

and

$$\int_\Omega \left(|\nabla u_k|^2 + \mu u_k^2 - \mu u_k^+ u_k - \frac{a}{q}u_k^+ u_k + \frac{b}{q}(u_k^+)^2 u_k - \Lambda(u_k^+)u_k\right)dx = \langle \Psi'(u_k), u_k\rangle. \tag{3.18}$$

$$\langle \Psi'(u_k), u_k\rangle \leqslant \epsilon\|u_k\|, \tag{3.19}$$

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(3.7) yields

$$\Lambda(u) \leqslant \frac{3}{2}\varepsilon u + \frac{3}{2}\varepsilon u^{\theta} + c_0 \tag{3.20}$$

Similarly as the methods of [17, (3.12)-(3.15)], employing (3.17)-(3.20) results in

$$(\frac{1}{2} - \frac{1}{1+\theta})\left(1 - \frac{a}{\lambda_1 q} - c_3\varepsilon\right)\|u_k\|^2 + c_4\int_{\Omega}(u^+)^3dx \leqslant c_* + o(1) + \frac{\epsilon}{1+\theta}\|u_k\|,$$

where $c_3, c_4$ both are positive constants, and $1 - \frac{a}{\lambda_1 q} - c_3\varepsilon > 0$ due to the small $\varepsilon$, which means the boundedness of $\{u_k\}$. Due to $\theta < 2^* - 1$, the equation (3.2) is the subcritical growth. It is a routine proof of the fact that $\{u_k\}$ sequently compact, say, $u_k \to u_*(x)$ in $H_0^1(\Omega)$ and $\Psi(u_*(x)) = c_* \geqslant \alpha > 0$, which implies $u_*(x) \neq 0$. Besides, $u_*(x)$ is the critical point of $\Psi$ so that

$$\int_{\Omega}\left(\nabla u_*(x)\nabla\vartheta + \mu u_*(x)\vartheta - \mu u_*(x)^+\vartheta - \frac{a}{q}u_*(x)^+\vartheta + \frac{b}{q}(u_*(x)^+)^2\vartheta - \Lambda(u_*(x)^+\vartheta)\right)dx = 0. \tag{3.21}$$

In (3.21), setting $\vartheta = u_*(x)^-$, Lemma 2.2 leads to

$$\mu\int_{\Omega}|u_*(x)^-|^2dx \leqslant \int_{\Omega}\left(|\nabla u_*(x)^-|^2 + \mu|u_*(x)^-|^2\right)dx = 0,$$

which implies that $u_*(x)^- = 0$ a.e. $x \in \Omega$. Now I have prove that $u_*(x) \geqslant 0$ and $u^*(x) \neq 0$.

Similarly as that of [17], now I claim that the above-mentioned $u_*(x)$ is the strong solution.

Indeed, $u_*(x) \neq 0$ is the non-negative solution of the following Dirichlet problem:

$$\begin{cases} -\Delta u = f(x,u), & x \in \Omega; \\ u|_{\partial\Omega} = 0 \end{cases} \tag{3.22}$$

where

$$f(x,u) = \frac{1}{q}[au - bu^2 + \Lambda(u)].$$

It is easy from the assumptions on $\Lambda$ to verify that $f$ satisfies the conditions (a)-(c), then Lemma 2.1 yields that $u^*(x)$ is the strong solution. □

Set $v(t,x) = u(t,x) - u_*(x)$. Since $u_*(x)$ is a stationary solution of the system (2.1), the system (2.1) is equivalent to the following system

$$\begin{cases} \frac{\partial v(t,x)}{\partial t} = q\Delta v(t,x) + (a + c_r - 2bu^*(x))v(t,x) + g(v(t,x)) - bv^2(t,x) - c_r v(t-\tau(t),x), & t \geqslant 0,\ x \in \Omega, \\ v(t,x) = 0, \quad x \in \Omega,\ t \geqslant 0, \end{cases} \tag{3.23}$$

where

$$g(v(t,x)) = \Lambda(u(t,x)) - \Lambda(u^*(x)). \tag{3.24}$$

Obviously, $u_*(x)$ of the system (2.1) is corresponding to the zero solution of the system (3.23).

Equipped the system (3.23) with the initial value:

$$v(s,x) = \phi(s,x),\ (s,x) \in [-\tau, 0] \times \Omega. \tag{3.25}$$

Moreover, I give some suitable assumptions as follows,

(H2) There are positive numbers $M_0, N_0$ such that

$$0 < N_0 \leqslant u \leqslant M_0. \tag{3.26}$$

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(H3) There is a positive number $M_1 > 0$ such that

$$|\Lambda(s_1) - \Lambda(s_2)| \leqslant M_1|s_1 - s_2|, \quad \forall s_1, s_2 \in R^1. \tag{3.27}$$

**Remark 3.** Everyone knows the fact that the population density of any species must have the bounded below, or the species will die out. For example, When the population density of whales is lower than a certain degree, it will be difficult for male and female whales to meet each other in the vast sea, leading to the extinction of the species. Besides, due to the limited resource, the population density of any species must have an supper boundedness. So the condition (H2) is a suitable assumption.

**Theorem 3.2.** If all the conditions of Theorem 3.1 hold, and if, in addition,

$$\lambda_1 q > a - 2bN_0 + M_1, \tag{3.28}$$

then $u_*(x)$ is the unique stationary solution of the system (3.1).

**Proof.** Let $u, w$ both are the stationary solutions of the system (2.1). Firstly, the conditions (H2) yields

$$(u-w)[-b(u^2-w^2)] = -b(u-w)^2(u+w) \leqslant -2bN_0(u-w)^2$$

and

$$(u-w)(\Lambda(u)-\Lambda(w)) \leqslant |u-w||\Lambda(u)-\Lambda(w)| \leqslant M_1(u-w)^2.$$

Since $u, w$ both are the stationary solutions of the system (2.1), then Pioncare inequality yields

$$\lambda_1 q\|u-w\|^2_{L^2(\Omega)} \leqslant \int_\Omega (u-w)[a(u-w) - b(u^2-w^2) + \Lambda(u) - \Lambda(w)]dx \leqslant \int_\Omega [a - 2bN_0 + M_1](u-w)^2 dx,$$

which proves $u = w$, and the proof is completed.
□

**Theorem 3.3.** Suppose the conditions (H1)-(H3) and (3.28) hold, and if there are positive numbers $p_r (r \in S), k_1 > 0$ such that

$$\min_{r\in S}\left(2\lambda_1 q + 2(2bN_0 - a - c_r) - 2M_1 - 2b(M_0 - N_0) - c_r k_1 - \frac{1}{p_r}\sum_{j\in S}\gamma_{rj}p_j\right) > \max_{r\in S}(c_r k_1^{-1}) \geqslant 0, \tag{3.29}$$

then the null solution of the system (3.23) with the initial value (3.25) is globally exponential input-to-state stability , at the same time, $u^*(x)$ is globally exponential input-to-state stability at the convergence rate $\frac{\lambda}{2}$, where $\alpha = \min_{r\in S}\Big(2\lambda_1 q + 2(2bN_0 - a - c_r) - 2M_1 - 2b(M_0 - N_0) - c_r k_1 - \frac{1}{p_r}\sum_{j\in S}\gamma_{rj}p_j\Big)$, $\beta = \max_{r\in S}(c_r k_1^{-1})$, $\lambda$ is the unique positive solution of $\lambda = \alpha - \beta e^{\lambda\tau}$.

**Proof.** Consider the following Lyapunov function:

$$V(t,x,v,r) = \int_\Omega p_r v^2(t,x)dx, \tag{3.30}$$

where $p_r$ is a positive number for each $r$.

Firstly, (H2) yields,

$$v = u - u_*(x) \geqslant N_0 - M_0 \Rightarrow v^3 \geqslant (N_0 - M_0)v^2$$

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(H3) yields

$$|vg(v)| \leqslant |v|M_1|v| = M_1 v^2.$$

Let $\mathcal{L}$ be the weak infinitesimal operator (see, e.g. [21]) such that

$$\begin{aligned}\mathcal{L}V &\leqslant \int_\Omega \Big( -2\lambda_1 q p_r v^2 + 2p_r(a + c_r - 2bu^*(x))v^2 + 2p_r v g(v(t,x)) - 2bp_r v^3 - 2p_r c_r v v(t-\tau(t),x) + \sum_{j\in S}\gamma_{rj}p_j \Big) dx \\ &\leqslant -\Big(2\lambda_1 q + 2(2bN_0 - a - c_r) - 2M_1 - 2b(M_0 - N_0) - c_r k_1 - \frac{1}{p_r}\sum_{j\in S}\gamma_{rj}p_j\Big) V + c_r k_1^{-1} V_\tau \\ &\leqslant -\Big[\min_{r\in S}\Big(2\lambda_1 q + 2(2bN_0 - a - c_r) - 2M_1 - 2b(M_0 - N_0) - c_r k_1 - \frac{1}{p_r}\sum_{j\in S}\gamma_{rj}p_j\Big)\Big] V + \max_{r\in S}(c_r k_1^{-1}) V_\tau \end{aligned} \tag{3.31}$$

where $V_\tau(t) = \sup_{t-\tau\leqslant s\leqslant t} V(s)$.

Now, [25, Lemma 2] yields that

$$\min_{r\in S} p_r \|v\|^2_{L^2(\Omega)} \leqslant V(t,r) \leqslant \max_{r\in S} p_r \|v\|^2_\tau e^{-\lambda\tau(t-t_0)}, \quad t \geqslant 0,$$

where $t_0 = 0$, $\|v(t)\|^2_\tau = \sup_{t-\tau\leqslant s\leqslant t} \|v(s)\|^2_{L^2(\Omega)}$. This completes the proof.

□

**Remark 4.** In this paper, I employ Mountain Pass Lemma and variational technique to derive the existence of positive stationary solution, which is different from the methods in my another paper [18]. Particularly, ground-state solution is more suitable to practical engineering (see,e.g. [26-29]).

## 4. Numerical example

**Example 4.1.** In the system (2.1), let $\Omega = (-\frac{1}{2},\frac{1}{2}) \times (-\frac{1}{2},\frac{1}{2}) \times (-\frac{1}{2},\frac{1}{2})$, then $\lambda_1 \geqslant 3$ (see [18, Remark 14]). $q = 0.1, a = 0.2$ then $a < q\lambda_1$, and the condition (3.1) holds. Set

$$\Lambda(u) = \begin{cases} 0, & u \in (-\infty, 0] \\ \varepsilon u, & u \in [0,1] \\ \varepsilon u^{\frac{7}{3}}, & u \in [1, +\infty), \end{cases} \tag{4.1}$$

where $\varepsilon = 0.00001$ , then (H1) holds. Let $b = 0.1, N_0 = 2, M_0 = 10$, then direct computation yields $M_1 = 0.003$, obviously (3.28) holds. Then Theorem 3.1-3.2 yields $u_*(x)$ is the unique stationary solution of the system (2.1), and $u_*(x)$ is the ground state stationary strong solution of the system (2.1).

Moreover, set $\tau = 0.5, S = \{1,2\}$, and $c_1 = 0.01, c_2 = 0.02$, $\gamma_{11} = -0.01, \gamma_{12} = 0.01; \gamma_{21} = -0.02, \gamma_{22} = 0.02, p_1 = 0.9999, p_2 = 1.0001, k_1 = 1$, direct computation yields (3.29) holds for $r = 1,2$, and $\lambda = 0.4769$. According to Theorem 3.3, the null solution of the system (3.23) with the initial value (3.25) is globally exponential input-to-state stability , at the same time, $u^*(x)$ is globally exponential input-to-state stability at the convergence rate 23.85%.

**Comparison 1.** In Example 4.1, replacing $\varepsilon = 0.00001$ with $\varepsilon = 0.0001$, and other data unchanged, direct computation yields the convergence rate $\frac{\lambda}{2} = 21.59\%$.

**Table 1.** Comparisons the influences on the convergence rate $\frac{\lambda}{2}$ under different perturbations with the same other data

| | $\varepsilon = 0.00001$ | $\varepsilon = 0.0001$ |
|---|---|---|
| Interference degree | smaller | bigger |
| Convergence rate $\frac{\lambda}{2}$ | 23.85% | 21.59% |

**Remark 5.** Table 1 illuminates that under some suitable conditions, the smaller the external input disturbance, the more stable the natural ecosystem of a single-species.

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**Comparison 2.** In Example 4.1, replacing $b = 0.1$ with $b = 0.2$, and other data unchanged, direct computation yields the convergence rate $\frac{\lambda}{2} = 28.98\%$.

**Table 2.** Comparisons the influences on the convergence rate $\frac{\lambda}{2}$ under different intra population competition intensities with the same other data

| | $b = 0.1$ | $b = 0.2$ |
|---|---|---|
| Interference degree | smaller | bigger |
| Convergence rate $\frac{\lambda}{2}$ | 23.85% | 28.98% |

**Remark 6.** Table 2 shows that a certain degree of inhibition and competition within the population is beneficial to the overall stability of the population for the natural ecosystem of a single-species.

**Comparison 3.** In Example 4.1, replacing $a = 0.2$ with $a = 0.23$, and other data unchanged, direct computation yields the convergence rate $\frac{\lambda}{2} = 19.08\%$.

**Table 3.** Comparisons the influences on the convergence rate $\frac{\lambda}{2}$ under different growth rates with the same other data

| | $a = 0.2$ | $a = 0.23$ |
|---|---|---|
| growth rates | smaller | bigger |
| Convergence rate $\frac{\lambda}{2}$ | 23.85% | 19.08% |

**Remark 7.** Table 3 verifies that due to the loss of natural enemies in a single species model , the higher the natural population growth rate is, the worse the population stability is.

## 5. Conclusions

In this paper, by using the mountain pass lemma, the method of weak solution regularization and the method of Lyapunov function, the global stability criterion of the ground state positive stationary strong solution of the single population model is given. As a single population loses the restriction of natural enemies, a certain internal competition is conducive to the overall stability of the population, and a certain amount of family planning is conducive to the overall stability of the population.

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