Article

LEAST SQUARES APPROXIMATION OF FLATNESS ON RIEMANNIAN MANIFOLDS

Iulia Hirica [1,‡] https://orcid.org/0000-0002-1339-995X,
Constantin Udriste [2,†,‡] https://orcid.org/0000-0002-7132-6900,
Gabriel Pripoae [1,‡] https://orcid.org/0000 - 0002 - 3315 - 3762
and Ionel Tevy [2,‡] https://orcid.org/0000 - 0001 - 9160 - 9185

1 University of Bucharest, Faculty of Mathematics and Computer Science 14 Academiei Str., RO-010014, Bucharest 1, Romania; ihirica@fmi.unibuc.ro ; gabrielpripoae@yahoo.com
2 University Politehnica of Bucharest, Faculty of Applied Sciences, Department of Mathematics and Informatics, Splaiul Independentei 313, RO-060042, Bucharest 6, Romania; udriste@mathem.pub.ro , vascatevy@yahoo.fr
* Correspondence: udriste@mathem.pub.ro; Tel.: 0040 745 371 684
† Second address: Academy of Romanian Scientists, 3 Ilfov RO-050044, Bucharest, Romania
‡ These authors contributed equally to this work.

Abstract: The purpose of this paper is threefold: (i) to introduce and study the Euler-Lagrange prolongations of flatness PDEs solutions (best approximation of flatness) via associated least squares Lagrangian densities and integral functionals on Riemannian manifolds; (ii) to analyze some decomposable multivariate dynamics represented by Euler-Lagrange PDEs of least squares Lagrangians generated by flatness PDEs and Riemannian metrics; (iii) to give examples of explicit flat extremals and non-flat approximations.

Keywords: geometric flatness; least squares Lagrangian densities; adapted metrics and connections

MSC: 58J99, 53C44, 53C21

---

1. Introduction and contributions

Least squares Lagrangians on Riemannian manifolds and the problem of best approximation of flatness have gained a lot of attention lately [2], especially when they come to optimization problems whose objectives are integral functionals. Combining this theory with decomposable multivariate dynamics [20], we get new results in differential geometry and global analysis.

Section 1 outlines the ground material regarding PDEs in differential geometry, least squares Lagrangian densities, dual variational principle, Riemannian volume form, and positive definite differential operators. Section 2 recalls the basic properties of ∇-flatness, introduces the crucial notion of least squares Lagrangian density attached to ∇-flatness and underlines that the Euclidean metrics extremals are stable with respect to conformal changing. In Section 3 comes the heart of the paper. Detailing the Riemann-flatness, we introduce the least squares Lagrangian density attached to Riemann-flatness and best approximations of Riemann-flatness solutions. Then the non-flatness extremals are analysed in detail. Section 4 shows how Ricci-flatness implies a least squares Lagrangian density and best approximations of Riemann-flatness solutions. Section 5 lists some analogues of least squares Lagrangian density attached to scalar curvature - flatness and confirms again that Einstein PDEs are extremals. Section 6 underlines that least squares technique is suitable for solving some problems in differential geometry.

 doi:10.20944/preprints202009.0234.v1

All Lagrangians we use are written in a local version which is of special interest for geometers and nonlinear analysts. Their explicit formulas reflect the properties usually needed for differential geometric constructions. In order to make the techniques in this paper available for a broad mathematical audience, we have tried to make the article as much self-contained as possible.

1.1. PDEs in Differential Geometry

The behavior of many different systems in nature and science are governed by a PDEs system. Usually such a system is thought of in terms of coordinates in order to prove existence of solutions or to find concrete ones. However, tensorial PDEs in differential geometry contain also information which is independent of the choice of coordinates. This is actually the most important information as it is independent of any external structure, artificially added to the PDE, and in this sense it is genuine. That is why, the differential geometry is often considered as an "art of manipulating PDEs" [9], [12], [14], [15], [17], [18], [24].

The most important geometric PDEs are those producing flatness (e.g. connection-flatness, curvature-flatness, Ricci-flatness, scalar curvature-flatness) and those producing constant curvature $(-1, 0, 1)$. The connection-flatness PDEs system is non-tensorial, while curvature-flatness, Ricci-flatness, scalar curvature-flatness PDEs systems are tensorial. Our ideas are coming also from the papers [3], [4], [5], [6], [7], [8], [21], [22], [23].

The connection-flatness and the curvature-flatness are interconnected.

In this paper, we present some specific features: (i) introducing those differential geometric structures needed to define and study geometric PDEs (some of them in a manifestly coordinate-independent way); (ii) defining PDEs and their signification within Differential Geometry and Global Analysis; (iii) developing techniques to find intrinsic properties of PDEs; (iv) discussing explicit examples to illustrate the importance of the choice of an appropriate context and language.

1.2. Riemannian volume form

Suppose $(M, g = (g_{ij}))$ is a smooth oriented Riemannian manifold. Then there is a consistent way to choose the sign of the square root $\sqrt{\det(g_{ij})}$ and define a volume form $d\mu = \sqrt{\det(g_{ij})}\ dx^1 \wedge \cdots \wedge dx^n$. We call it the Riemannian volume form of (M, g). Having a volume form allows us to integrate functions on M. In particular $vol(M) = \int_M d\mu$ is an important invariant of (M, g). It also allows us to define an inner product $\langle \phi, \psi \rangle = \int_M \langle \phi(x), \psi(x) \rangle_g\ d\mu$, on the space of differential forms and other tensors or objects on M, using the metric g and its inverse g^{-1}. This inner product induces the square of the norm $\|\phi\|^2 = \int_M |\phi(x)|_g^2\ d\mu$.

1.3. Least squares Lagrangian densities

Having in mind the so-called variational approach [1], [2], [16], in this Subsection we add typical functionals that appear in the theory of geometric and physical fields [20].

Let M be an oriented manifold of dimension n. Any differential operator (of vectorial form, tensorial or not) on the Riemannian manifold $(M, g = (g_{ij}))$ and the metric (geometric structure) g generate a least squares Lagrangian density L. The extremals of the Lagrangian $\mathcal{L} = L\sqrt{\det(g_{ij})}$, described by Euler-Lagrange PDEs, include the solutions of initial PDEs and other solutions which we call "Euler-Lagrange prolongations" of that solutions (best approximation of initial PDEs solutions).

Generally, the Euler-Lagrange equation provides the equation of motion for the dynamical field specified in the Lagrangian. If the Lagrangian attached to a PDE is that of the smallest squares, then the extremals give the best approximation of the PDE solutions.

The Euler-Lagrange PDEs are indexed related to the chosen fibered chart $(\mathbb{R}^n, \Psi)$, $\Psi = (f^I, x^i)$. However, since the Euler-Lagrange expressions are components of a global differential form (the Euler-Lagrange form), the solutions are independent of fibered charts [16].

Example 1. *(Compare with the paper [20]) Let $(M, g = (g_{ij}))$ be an n-dimensional Riemannian manifold, with local coordinates $x = (x^1, ..., x^n)$, and $T \subset M$ be a compact subset. Let I, J be multiindices, each subindex running in $\overline{1,n}$. Being given $I \times J$ Lagrangians $L^I_J(x, f(x), f_x(x))$, where $f(x)$ has multi-components, then the associated least squares Lagrangian with respect to the Riemannian metrics $G^J_I(x)G^L_K(x)$, induced by the Riemannian metric g_{ij}, is*

$$\mathcal{L} = \frac{1}{2}\, G^J_I(x)G^H_K(x)\; L^K_J(x, f(x), f_x(x))\; L^I_H(x, f(x), f_x(x))\sqrt{\det(g_{ij})}.$$

The extremals are solutions of the Euler-Lagrange PDE system

$$\left(\frac{1}{2}\frac{\partial(G^J_I G^H_K)}{\partial x^m}\; L^K_J L^I_H + G^J_I G^H_K L^K_J\; \frac{\partial L^I_H}{\partial x^m}\right)\sqrt{\det(g_{ij})}$$

$$+\frac{1}{2}G^J_I G^H_K L^K_J L^I_H \frac{\partial}{\partial x^m}\sqrt{\det(g_{ij})} - D_l\left(G^J_I G^H_K\; L^K_J \sqrt{\det(g_{ij})}\;\frac{\partial L^I_H}{\partial f^m_l}\right) = 0.$$

If the Lagrangian L^I_J is associated to the PDEs system $L^I_J(x, f(x), f_x(x)) = 0$, then the extremals contain the solutions of that system and the Euler-Lagrange dynamics is decomposable.

Remark 1. *If we need to subject the Euler-Lagrange PDEs to boundary conditions, then instead of M we use Ω as a compact, n-dimensional submanifold of M with boundary (a piece of M).*

1.4. Dual variational principle

Let (M, g) be a Riemannian manifold. Usually, the local components of the metric g are denoted by g_{ij} and the components of the inverse g^{-1} are denoted by g^{ij}. Due to the musical isomorphism between the tangent bundle TM and the cotangent bundle T^*M of a Riemannian manifold induced by its metric tensor g, the arbitrary variations of g_{ij} are equivalent to the arbitrary variations of g^{ij}, and any Lagrangian with respect to g_{ij} can be regarded as a Lagrangian in relation to g^{ij}, but the differential orders are different.

When calculating the variation with respect to g^{ij}, certain terms may appear whose integral over any domain Ω can be reduced via Divergence Theorem (integration by parts) to an integral over the boundary $\partial\Omega$, which vanish (variations vanish on boundary). Modulo this statement, the Euler-Lagrange PDEs are reduced to $\frac{\partial \mathcal{L}}{\partial g^{ij}} = 0$ (the formal partial derivatives equal to zero).

1.5. Positive definite differential operator

For an $n \times n$ matrix of numbers or functions, positive definiteness is equivalent to the fact that its leading principal minors are all positive (n inequalities).

For an $n \times n$ matrix of partial derivatives operators, positive definiteness is equivalent to the fact that its leading principal minors are all positive (n partial differential inequalities). For differential inequalities, see also [19].

2. Least squares Lagrangian density attached to ∇-flatness

Let (M, g) be a smooth oriented Riemannian manifold. The Riemannian metric g of components g_{ij} and its inverse g^{-1} of components g^{ij} determine (locally) the Christoffel symbols of the second kind

$$\Gamma^i_{jk} = \frac{1}{2}g^{il}\left(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\right) = \frac{1}{2}g^{il}\,(\delta^r_l\delta^s_j\delta^t_k + \delta^r_l\delta^s_k\delta^t_j - \delta^t_l\delta^r_j\delta^s_k)\frac{\partial g_{rs}}{\partial x^t},$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 10 September 2020 doi:10.20944/preprints202009.0234.v1

$i, j, k = \overline{1, n}$ (overdetermined elliptic partial differential operator). The ∇-flatness PDEs system $\Gamma^i_{jk} = 0$ is a (non-tensorial) PDEs system

$$(1) \qquad \frac{1}{2} g^{il} (\delta^r_l \delta^s_j \delta^t_k + \delta^r_l \delta^s_k \delta^t_j - \delta^t_l \delta^r_j \delta^s_k) \frac{\partial g_{rs}}{\partial x^t} = 0 \Leftrightarrow \frac{\partial g_{rs}}{\partial x^t} = 0$$

on the space of Riemannian metrics $S^2_+ T^*M$, i.e., $\frac{n^2(n+1)}{2}$ distinct first order non-linear nonhomogeneous PDEs whose unknowns are $\frac{n(n+1)}{2}$ functions g_{ij} (positive definite tensor); for $n > 1$, overdetermined system of PDEs; for $n = 1$, determined system. This PDEs system is symmetric in j, k. Imposing the initial condition $g_{ij}(0) = \delta_{ij}$, we find the solution $g_{ij}(x) = \delta_{ij}$ (Euclidean manifold).

The square of the norm $L = \|\nabla\|^2 = g_{ip} g^{jq} g^{kr} \Gamma^i_{jk} \Gamma^p_{qr}$ is a Lagrangian density of first order with respect to g_{ij} and of order zero with respect to g^{ij}. That is why, we have two kinds of writing the functional describing ∇-flatness deviation, either $I(g) = \int_M \|\nabla\|^2 d\mu$ or $I(g^{-1}) = \int_M \|\nabla\|^2 d\mu$. Though the second is more simple, from variational point of view, let us begin the study with $I(g)$ whose associated Lagrangian $\mathcal{L} = \|\nabla\|^2 \sqrt{\det(g_{ij})}$ is of first order in g_{ij}.

Theorem 1. *The extremals of $I(g)$, i.e., the solutions of PDEs*

$$\Gamma^i_{jk} \Gamma^p_{qr} \left[g^{jq} g^{kr} \delta^m_i \delta^n_p - g_{ip} (g^{mj} g^{nq} g^{kr} + g^{jq} g^{mk} g^{nr}) + \frac{1}{2} g_{ip} g^{jq} g^{kr} g^{mn} \right] \sqrt{\det(g_{ij})}$$

$$-D_{x^l} \left[g^{jq} g^{kr} (\delta^m_u \delta^n_j \delta^l_k + \delta^m_u \delta^n_k \delta^l_j - \delta^l_u \delta^m_j \delta^n_k) \Gamma^u_{qr} \sqrt{\det(g_{ij})} \right] = 0$$

split in two categories: $g_{ij}(x) = \delta_{ij}$ (global minimum points, i.e., solutions of ∇-flatness) and local minimum points of $I(g)$.

Proof. The extremals of the Lagrangian $\mathcal{L}$ are solutions of Euler-Lagrange PDEs

$$\frac{\partial \mathcal{L}}{\partial g_{mn}} - D_{x^l} \frac{\partial \mathcal{L}}{\partial(\partial_{x^l} g_{mn})} = 0.$$

This critical points are global (when $\mathcal{L}$=0) or local (when $\mathcal{L} \neq 0$).

Suppose $\mathcal{L} \neq 0$. Based on obvious formulas

$$\frac{\partial g_{jk}}{\partial g_{mn}} = \delta^m_j \delta^n_k, \quad \frac{\partial g^{jk}}{\partial g_{mn}} = -g^{mj} g^{nk},$$

$$\frac{\partial}{\partial g_{mn}} \det(g_{ij}) = \det(g_{ij}) g^{mn}, \quad \frac{\partial(\partial_{x^t} g_{rs})}{\partial(\partial_{x^l} g_{mn})} = \delta^l_t \delta^m_r \delta^n_s,$$

we obtain

$$\frac{\partial \mathcal{L}}{\partial g_{mn}} = \left[g^{jq} g^{kr} \Gamma^m_{jk} \Gamma^n_{qr} - g_{ip} \Gamma^i_{jk} \Gamma^p_{qr} (g^{mj} g^{nq} g^{kr} + g^{jq} g^{mk} g^{nr}) \right] \sqrt{\det(g_{ij})}$$

$$+ \frac{1}{2} g_{ip} g^{jq} g^{kr} \Gamma^i_{jk} \Gamma^p_{qr} \sqrt{\det(g_{ij})}\, g^{mn},$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_{x^l} g_{mn})} = 2 g_{ip} g^{jq} g^{kr} \Gamma^p_{qr} \sqrt{\det(g_{ij})} \frac{\partial \Gamma^i_{jk}}{\partial(\partial_{x^l} g_{mn})}$$

$$= g_{ip} g^{jq} g^{kr} \Gamma^p_{qr} g^{iu} (\delta^r_u \delta^s_j \delta^t_k + \delta^r_u \delta^s_k \delta^t_j - \delta^t_u \delta^r_j \delta^s_k) \sqrt{\det(g_{ij})} \frac{\partial(\partial_{x^t} g_{rs})}{\partial(\partial_{x^l} g_{mn})}$$

$$= g^{jq} g^{kr} \Gamma^u_{qr} (\delta^r_u \delta^s_j \delta^t_k + \delta^r_u \delta^s_k \delta^t_j - \delta^t_u \delta^r_j \delta^s_k) \delta^l_t \delta^m_r \delta^n_s \sqrt{\det(g_{ij})}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 10 September 2020 doi:10.20944/preprints202009.0234.v1

$$= g^{jq}g^{kr}(\delta^m_u\delta^n_j\delta^l_k + \delta^m_u\delta^n_k\delta^l_j - \delta^l_u\delta^m_j\delta^n_k)\Gamma^u_{qr}\sqrt{\det(g_{ij})}.$$

The explicit Euler-Lagrange PDEs are those in Theorem.

Now let us compute the Hessian matrix of components

$$H_{(lmn)(abc)} = \frac{\partial^2 L}{\partial(\partial_{x^l}g_{mn})\partial(\partial_{x^a}g_{bc})} = g^{jq}g^{kr}(\delta^m_u\delta^n_j\delta^l_k + \delta^m_u\delta^n_k\delta^l_j - \delta^l_u\delta^m_j\delta^n_k)\frac{\partial\Gamma^u_{qr}}{\partial(\partial_{x^a}g_{bc})}$$

$$= \frac{1}{2}g^{jq}g^{kr}(\delta^m_u\delta^n_j\delta^l_k + \delta^m_u\delta^n_k\delta^l_j - \delta^l_u\delta^m_j\delta^n_k)g^{uv}(\delta^r_v\delta^s_q\delta^t_r + \delta^r_v\delta^s_r\delta^t_q - \delta^t_v\delta^r_q\delta^s_r)\frac{\partial(\partial_{x^t}g_{rs})}{\partial(\partial_{x^a}g_{bc})}$$

$$= \frac{1}{2}g^{jq}g^{kr}g^{uv}(\delta^m_u\delta^n_j\delta^l_k + \delta^m_u\delta^n_k\delta^l_j - \delta^l_u\delta^m_j\delta^n_k)(\delta^b_v\delta^c_q\delta^a_r + \delta^b_v\delta^c_r\delta^a_q - \delta^a_v\delta^b_q\delta^c_r).$$

This matrix is invariant if one interchange l with a and the (un-ordered) pair m, n with the (un-ordered) pair b, c, what must happen with a mixed derivative. Since the matrix H is positive definite, all extremals are minimum points (Legendre-Jacobi criterium). □

2.1. Homothetic flat extremals

The extremals g of $I(g)$ are Euler-Lagrange prolongations (the best approximations) of the flat solutions $g_{ij}(x) = \delta_{ij}$. Let us show that the Euclidean metrics extremals are stable with respect to conformal changing.

To simplify the problem, we consider a 2-dimensional manifold with the Riemannian metric $g_{11} = f, g_{22} = h, g_{12} = 0$. Then the least squares Lagrangian is

$$\mathcal{L}(g) = g_{ip}g^{jq}g^{kr}\Gamma^i_{jk}\Gamma^p_{qr}\sqrt{det(g_{ij})} = L\sqrt{det(g_{ij})},$$

and the Euler-Lagrange PDEs are

$$\frac{\partial\mathcal{L}}{\partial g_{mn}} - D_{x^l}\frac{\partial\mathcal{L}}{\partial(\partial_{x^l}g_{mn})} = 0.$$

We find

$$\frac{\partial\mathcal{L}}{\partial g_{mn}} = \sqrt{det(g_{ij})}\left(\frac{\partial L}{\partial g_{mn}} + \frac{1}{2}Lg^{nm}\right).$$

The Lagrangian density

$$L = 2g_{11}g^{11}g^{22}(\Gamma^1_{12})^2 + g_{11}(g^{11})^2(\Gamma^1_{11})^2 + g_{11}(g^{22})^2(\Gamma^1_{22})^2$$

$$+g_{22}(g^{11})^2(\Gamma^2_{11})^2 + 2g_{22}g^{11}g^{22}(\Gamma^2_{12})^2 + g_{22}(g^{22})^2(\Gamma^2_{22})^2$$

becomes

$$L = \frac{3}{4f^2h}g^2_{11,2} + \frac{3}{4fh^2}g^2_{22,1} + \frac{1}{4f^3}g^2_{11,1} + \frac{1}{4h^3}g^2_{22,2}.$$

We get

$$\frac{\partial L}{\partial g_{11}} = -\frac{3f_2^2}{2f^3h} - \frac{3h_1^2}{4f^2h^2} - \frac{3f_1^2}{4f^4}, \frac{\partial L}{\partial g_{22}} = -\frac{3f_2^2}{4f^2h^2} - \frac{3h_1^2}{2fh^3} - \frac{3h_2^2}{4h^4},$$

$$\frac{\partial\mathcal{L}}{\partial g_{11,1}} = \sqrt{fh}\,\frac{f_1}{2f^3}, \frac{\partial\mathcal{L}}{\partial g_{11,2}} = \sqrt{fh}\,\frac{3f_2}{2f^2h},$$

$$\frac{\partial\mathcal{L}}{\partial g_{22,1}} = \sqrt{fh}\,\frac{3h_1}{2fh^2}, \frac{\partial\mathcal{L}}{\partial g_{22,2}} = \sqrt{fh}\,\frac{h_2}{2h^3}.$$

 doi:10.20944/preprints202009.0234.v1

It follows the Euler-Lagrange PDEs system

$$\sqrt{fh}\left(-\frac{9f_2^2}{8f^3h}-\frac{3h_1^2}{8f^2h^2}-\frac{5f_1^2}{8f^4}+\frac{h_2^2}{8fh^3}\right)$$

$$-D_{x^1}\left(\sqrt{fh}\frac{f_1}{2f^3}\right)-D_{x^2}\left(\sqrt{fh}\frac{3f_2}{2f^2h}\right)=0.$$

$$\sqrt{fh}\left(-\frac{3f_2^2}{8f^2h^2}-\frac{9h_1^2}{8fh^3}-\frac{5h_2^2}{8h^4}+\frac{f_1^2}{8f^3h}\right)$$

$$-D_{x^1}\left(\sqrt{fh}\frac{3h_1}{2fh^2}\right)-D_{x^2}\left(\sqrt{fh}\frac{h_2}{2h^3}\right)=0.$$

Remark 2. *If $f=f(x^1), h=h(x^2)$, then the previous PDEs system is reduced to*

$$\sqrt{fh}\left(-\frac{5f_1^2}{8f^4}+\frac{f_1^2}{8fh^3}\right)-D_{x^1}\left(\sqrt{fh}\,\frac{f_1}{2f^3}\right)=0.$$

$$\sqrt{fh}\left(-\frac{5h_2^2}{8h^4}+\frac{h_2^2}{8fh^3}\right)-D_{x^2}\left(\sqrt{fh}\,\frac{h_2}{2h^3}\right)=0,$$

Remark 3. *If $f=h$ (conformal case), then one gets the PDEs system*

$$2f(f_1^2+f_2^2)+(ff_{11}-2f_1^2)+3(ff_{22}-2f_2^2)=0,$$

$$2f(f_1^2+f_2^2)+3(ff_{11}-2f_1^2)+(ff_{22}-2f_2^2)=0$$

equivalent to

$$f(f_1^2+f_2^2)+2(ff_{22}-2f_2^2)=0,\ \ f(f_1^2+f_2^2)+2(ff_{11}-2f_1^2)=0.$$

Since f must be positive throughout, this system of PDEs has only solutions of the form $f(x)=c>0$ (see Maple (pde, pdsolve(pde)). The metrics with $c>0$, $c\neq 1$ are homothetic to δ_{ij}. Consequently, the Euclidean metrics extremals are stable with respect to conformal changing.

For comparison we use $\mathcal{L}(g^{-1})=g_{ip}g^{jq}g^{kr}\Gamma^i_{jk}\Gamma^p_{qr}\sqrt{det(g_{ij})}$, the general form of Euler-Lagrange PDEs $\frac{\partial\mathcal{L}}{\partial g^{mn}}=0$ and we formulate the following

Theorem 2. *The extremals $g=(g_{ij})$ of $\mathcal{L}(g^{-1})$ are solutions of PDEs system*

$$g^{kr}\left(-2g_{mp}g_{ni}g^{jq}+g_{mi}g_{np}g^{jq}-2g_{ip}\delta^j_m\delta^q_n+\frac{1}{2}g_{ip}g^{jq}g_{mn}\right)\Gamma^i_{jk}\Gamma^p_{qr}=0.$$

For calculus of the matrix $H_{(ab);(mn)}=\frac{\partial^2\mathcal{L}}{\partial g^{ab}\partial g^{mn}}$, we need $\frac{\partial g^{kr}}{\partial g^{ab}}=\delta^k_a\delta^r_b$ and $\frac{\partial g_{mj}}{\partial g^{ab}}=-g_{ma}g_{bj}$. We find

$$H_{(ab);(mn)}=2g_{ip}\Gamma^i_{ma}\Gamma^p_{nb}$$

$$+g^{kr}\Big[2g_{ap}g_{bi}\Gamma^i_{mk}\Gamma^p_{nr}+(4g_{mp}g_{ni}-g_{ip}g_{mn}-2g_{mi}g_{np})\Gamma^i_{ak}\Gamma^p_{br}$$

$$+\frac{1}{2}g^{jq}(g_{ip}g_{ma}g_{nb}-g_{ap}g_{mn}g_{ib})\Gamma^i_{jk}\Gamma^p_{qr}\Big].$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 10 September 2020 doi:10.20944/preprints202009.0234.v1

This matrix is not (neither positive nor negative) definite since it vanishes in the center of normal coordinates. This is why this matrix is of no help in determining that extremals could be extremum points.

2.2. Homothetic flatness extremals

General case There are extremals of the type $g_{ij}(x) = f(x)\delta_{ij}$, $f(x) > 0$, on a Riemannian manifold (M, g_{ij}) of dimension n? Since

$$\Gamma^i_{jk} = \frac{1}{2f}(f_j\delta^i_k + f_k\delta^i_j - f_i\delta_{jk}),\ i,j,k = 1,...,n,$$

the Euler-Lagrange PDEs reduced to

$$\sum_r f_r^2\delta_{ij}\left(\frac{3n}{2} - 7\right) + f_i f_j(6 - 3n) = 0.$$

It follows $f_k = 0, \forall k = 1,...,n$. Therefore $f(x) = c$ and $g_{ij} = c\delta_{ij}$. The metrics with $c > 0, c \neq 1$ are homothetic to δ_{ij}. Consequently, the Euclidean metric extremals are stable with respect to conformal changing.

Bidimensional case Let us consider a 2-dimensional Riemannian manifold with the metric $g_{11} = f, g_{22} = h, g_{12} = 0$. Let us show again that the Euclidean metric extremals are stable with respect to conformal changing.

In this case

$$\mathcal{L}(g^{-1}) = g_{ip}g^{jq}g^{kr}\Gamma^i_{jk}\Gamma^p_{qr}\sqrt{det(g_{ij})} = L\sqrt{det(g_{ij})}$$

and the general form of Euler-Lagrange PDEs system is

$$\frac{\partial\mathcal{L}}{\partial g^{mn}} = \sqrt{det(g_{ij})}\left(\frac{\partial L}{\partial g^{mn}} - \frac{1}{2}Lg_{nm}\right) = 0.$$

Since

$$L = \frac{3}{4f^2h}g^2_{11,2} + \frac{3}{4fh^2}g^2_{22,1} + \frac{1}{4f^3}g^2_{11,1} + \frac{1}{4h^3}g^2_{22,2},$$

$$\frac{\partial L}{\partial g^{11}} = \frac{3f_2^2}{2fh} + \frac{3h_1^2}{4h^2} + \frac{3f_1^1}{4f^2},\ \frac{\partial L}{\partial g^{22}} = \frac{3f_2^2}{4f^2} + \frac{3h_1^2}{2fh} + \frac{3h_2^2}{4h^2},$$

the Euler-Lagrange PDEs system becomes

$$12fh^2f_2^2 + 6f^2hh_1^2 + 6h^3f_1^2 - 3fh^2f_2^2 - 3f^2hh_1^2 - h^3f_1^2 - f^3f_2^2 = 0,$$

$$6fh^2f_2^2 + 12f^2hh_1^2 + 6f^3h_2^2 - 3fh^2f_2^2 - 3f^2hh_1^2 - h^3f_1^2 - f^3f_2^2 = 0.$$

Remark 4. *If $f = f(x^1), h = h(x^2)$, then Euler-Lagrange PDEs are reduced to $h^3f_1^2 = 0, f^3h_2^2 = 0$, i.e., $f_1 = 0, h_2 = 0$ (Euclidean case).*

Remark 5. *The conformal case $f = h$ leads to $f_1^2 + f_2^2 = 0$, and we get $f_1 = f_2 = 0$, i.e., f is constant (confirming the general case).*

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 10 September 2020 doi:10.20944/preprints202009.0234.v1

3. Least squares Lagrangian density attached to Riemann-flatness

Let ∇ be a symmetric connection of components Γ^i_{jk} and g be a Riemannian metric of components g_{ij}. We use the operator $P^{ps}_{jk} = \frac{1}{2}(\delta^p_j \delta^s_k - \delta^p_k \delta^s_j)$ which is a projection, i.e., $P^2 = P$, and is covariant constant. The Riemann-flatness PDEs system is either $Riem^\nabla = 0$ or

$$R^l{}_{ijk} = \frac{\partial}{\partial x^j}\Gamma^l_{ik} - \frac{\partial}{\partial x^k}\Gamma^l_{ij} + \Gamma^l_{js}\Gamma^s_{ik} - \Gamma^l_{ks}\Gamma^s_{ij}$$

$$= 2P^{ps}_{jk}\left(\frac{\partial}{\partial x^p}\Gamma^l_{is} + \Gamma^l_{pn}\Gamma^n_{is}\right) = 2P^{ps}_{jk}\left(\frac{\partial}{\partial x^p}\Gamma^l_{is} - \Gamma^l_{sn}\Gamma^n_{ip}\right) = 0,$$

where $i,j,k,l,... = \overline{1,n}$, and has the general solution $\Gamma^i_{jk} = 0$.

Each of the Riemann - flatness PDEs systems $R^l{}_{ijk} = 0$ is a system of $\frac{n^2(n^2-1)}{12}$ distinct first order linear quadratic PDEs whose unknowns are $\frac{n^2(n+1)}{2}$ functions Γ^i_{jk}; for $n > 7$, overdetermined system; for $n < 7$, undetermined system; for $n = 7$, determined system.

The curvature flatness was discussed in [3], [4], [5], [6], [7], [21] based on the idea of finding an adapted coordinate system. We bring up another point of view, looking for suitable metrics and connections, and not for adapted coordinate systems.

On the smooth oriented manifold (M, ∇, g), we introduce the square of the norm $L = \|Riem^\nabla\|^2 = g_{ip}g^{jq}g^{kr}g^{ls}R^i{}_{jkl}R^p{}_{qrs}$, which is a Lagrangian density of first order in Γ^i_{jk}. It determines a functional (Riemann - flatness deviation) similar to the Yang-Mills functional, namely $I(\nabla) = \int_M \|Riem^\nabla\|^2 d\mu$. The extremals ∇ of $\mathcal{L}(\nabla, \partial\nabla) = \|Riem^\nabla\|^2\sqrt{\det(g_{ij})}$ are solutions of Euler-Lagrange PDEs $\frac{\partial \mathcal{L}}{\partial \Gamma^u_{vw}} - D_{x^t}\frac{\partial \mathcal{L}}{\partial(\partial_{x^t}\Gamma^u_{vw})} = 0$.

Theorem 3. *The explicit form of Euler-Lagrange PDEs attached to the Lagrangian $\mathcal{L}(\nabla, \partial\nabla)$ is*

$$(\delta^i_u\delta^v_{[k}\delta^b_{l]}\Gamma^w_{bj} + \delta^w_j\delta^a_{[k}\delta^v_{l]}\Gamma^i_{au})R^p{}_{qrs}g^{jq}g^{kr}g^{ls}g_{ip}\sqrt{det(g_{ij})}$$

$$-D_{x^t}\left[\delta^t_{[k}\delta^v_{l]}R^p{}_{qrs}g^{wq}g^{kr}g^{ls}g_{up}\sqrt{det(g_{ij})}\right] = 0.$$

The Riemann-flatness solutions $\Gamma^i_{jk} = 0$ are global minimum points. The others solutions are best approximation of flatness PDEs solutions.

Let $(M, g = (g_{ij}))$ be a Riemannian manifold. The Riemannian metric (g_{ij}) determines the Riemannian curvature tensor field $Riem^g$ of components

$$R_{ijkl} = -\frac{1}{2}\left(\frac{\partial^2 g_{ik}}{\partial x^j \partial x^l} + \frac{\partial^2 g_{jl}}{\partial x^i \partial x^k} - \frac{\partial^2 g_{jk}}{\partial x^i \partial x^l} - \frac{\partial^2 g_{il}}{\partial x^j \partial x^k}\right) + g_{mn}(\Gamma^m_{jk}\Gamma^n_{il} - \Gamma^m_{jl}\Gamma^n_{ik})$$

$$= \frac{1}{2}\delta^p_{[i}\delta^q_{j]}\delta^r_{[l}\delta^s_{k]}\frac{\partial^2 g_{pr}}{\partial x^q \partial x^s} - g_{mn}\delta^q_j\delta^p_i\delta^r_{[l}\delta^s_{k]}\Gamma^m_{qr}\Gamma^n_{ps},$$

where

$$\delta^p_{[i}\delta^q_{j]} = \delta^p_i\delta^q_j - \delta^p_j\delta^q_i, \; \Gamma^i_{jk} = \frac{1}{2}g^{il}(\delta^r_l\delta^s_j\delta^t_k + \delta^r_l\delta^s_k\delta^t_j - \delta^t_l\delta^r_j\delta^s_k)\frac{\partial g_{rs}}{\partial x^t}.$$

In this case Riemannian curvature flatness condition means the tensorial PDEs system

$$\frac{1}{2}\delta^p_{[i}\delta^q_{j]}\delta^r_{[l}\delta^s_{k]}\frac{\partial^2 g_{pr}}{\partial x^q \partial x^s} - g_{mn}\delta^q_j\delta^p_i\delta^r_{[l}\delta^s_{k]}\Gamma^m_{qr}\Gamma^n_{ps} = 0,$$

on $S^2_+T^*M$, with $\frac{n^2(n^2-1)}{12}$ distinct second order linear non-homogeneous PDEs whose unknowns are $\frac{n(n+1)}{2}$ functions g_{ij} (positive definite tensor); for $n < 3$, undetermined system; for $n > 3$, overdetermined system; for $n = 3$, determined system. This PDEs system is parabolic since the set of eigenvalues of the matrix $T^{pqrs}_{ijlk} = \delta^p_{[i}\delta^q_{j]}\delta^r_{[l}\delta^s_{k]}$ (tensorial product of a matrix by itself) contains the eigenvalue 0. Indeed all eigenvectors, respectively eigenvalues are: X^{ijlk}-symmetric in (i,j) or in (l,k), with $\lambda = 0$; X^{ijlk}-skewsymmetric in (i,j) and in (l,k) with $\lambda = 2$. Of course, this PDEs system has all properties of curvature tensor field.

On the Riemannian manifold $(M, g = (g_{ij}))$, we introduce the square of the norm $L = \|Riem^g\|^2 = g^{ip}g^{jq}g^{kr}g^{ls}R_{ijkl}R_{pqrs}$ which is of second order with respect to g_{ij} and of order zero with respect to g^{ij}. In this way the Riemann - flatness deviation is written either $I(g) = \int_M \|Riem^g\|^2 d\mu$ or $I(g^{-1}) = \int_M \|Riem^g\|^2 d\mu$. For $I(g)$ the extremals g are solutions of fourth order Euler-Lagrange PDEs

$$\frac{\partial \mathcal{L}}{\partial g_{mn}} - D_{x^l}\frac{\partial \mathcal{L}}{\partial(\partial_{x^l} g_{mn})} + D_{x^k}D_{x^l}\frac{\partial \mathcal{L}}{\partial(\partial_{x^k}\partial_{x^l} g_{mn})} = 0,$$

while for $I(g^{-1})$ the Euler-Lagrange PDEs are reduced to $\frac{\partial \mathcal{L}}{\partial g^{mn}} = 0$.

Theorem 4. *The extremals $g = (g_{ij})$ of the Lagrangian $\mathcal{L}(g^{-1})$ are solutions of the PDEs system*

$$-2\delta^c_{[k}\delta^d_{l]}R_{pqrs}g^{ap}g^{bq}g^{kr}g^{ls}g_{nv}g_{wm}\Gamma^v_{bc}\Gamma^w_{ad}$$

$$+2R_{ijkl}R_{pqrs}(\delta^i_m\delta^p_n g^{jq} + \delta^j_m\delta^q_n g^{ip})g^{kr}g^{ls} - \frac{1}{2}R_{ijkl}R_{pqrs}g^{ip}g^{jq}g^{kr}g^{ls}\ g_{mn} = 0.$$

The Riemann-flat solutions $g_{ij}(x) = \delta_{ij}$ are global minimum points. So are the metrics obtained from δ_{ij} by changing variables, such as

$$g(x) = \mathrm{diag}\left(\frac{1}{h_1(x^1)^2}, \dots, \frac{1}{h_n(x^n)^2}\right).$$

The others solutions are best approximation of flatness PDEs solutions.

3.1. Non-flat extremals

We consider a 2-dimensional Riemannian manifold (M, g), where $g_{11} = f, g_{22} = h, g_{12} = 0$. In this case $\mathcal{L} = (g^{11}g^{22}R_{1212})^2\sqrt{det(g_{ij})}$, and

$$R_{1212} = -\frac{1}{2}(g_{11,22} + g_{22,11}) + g_{ab}(\Gamma^a_{21}\Gamma^b_{12} - \Gamma^a_{22}\Gamma^b_{11})$$

$$= -\frac{1}{2}(g_{11,22} + g_{22,11}) + g_{11}(\Gamma^1_{21}\Gamma^1_{12} - \Gamma^1_{22}\Gamma^1_{11}) + g_{22}(\Gamma^2_{21}\Gamma^2_{12} - \Gamma^2_{22}\Gamma^2_{11})$$

or

$$R_{1212} = -\frac{1}{2}(h_{11} + f_{22}) + \frac{1}{4f}(f_2^2 + f_1h_1) + \frac{1}{4h}(h_1^2 + f_2h_2).$$

We get

$$\frac{\partial R_{1212}}{\partial g^{11}} = \frac{1}{4}(f_2^2 + f_1h_1), \frac{\partial R_{1212}}{\partial g^{22}} = \frac{1}{4}(h_1^2 + f_2h_2), \frac{\partial R_{1212}}{\partial g^{12}} = 0.$$

The Euler-Lagrange PDEs $\frac{\partial \mathcal{L}}{\partial g^{mn}} = 0$ become the following system of equations

$$2g^{11}(g^{22})^2R^2_{1212} + 2(g^{11}g^{22})^2R_{1212}\frac{\partial R_{1212}}{\partial g^{11}} - \frac{1}{2}(g^{11}g^{22}R_{1212})^2 g_{11} = 0,$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 10 September 2020 doi:10.20944/preprints202009.0234.v1

$$2g^{22}(g^{11})^2 R_{1212}^2 + 2(g^{11}g^{22})^2 R_{1212}\frac{\partial R_{1212}}{\partial g^{22}} - \frac{1}{2}(g^{11}g^{22}R_{1212})^2 g_{22} = 0.$$

Explicitly,

$$\frac{R_{1212}}{2f^2h^2}[3fR_{1212} + f_2^2 + f_1h_1] = 0, \ \frac{R_{1212}}{2f^2h^2}[3fR_{1212} + h_1^2 + f_2h_2] = 0,$$

Case 1: $R_{1212} = -\frac{1}{2}(h_{11} + f_{22}) + \frac{1}{4f}(f_2^2 + f_1h_1) + \frac{1}{4h}(h_1^2 + f_2h_2) = 0$ produces the Euclidean metric.

Case 2: $3fR_{1212} + f_2^2 + f_1h_1 = 0, \ 3hR_{1212} + h_1^2 + f_2h_2 = 0$. Equivalently

$$3fR_{1212} + f_2^2 + f_1h_1 = 0, \ h(f_2^2 + f_1h_1) - f(h_1^2 + f_2h_2) = 0.$$

Case 3: The conformal case $f = h$ becomes:

(1) $R_{1212} = -\frac{1}{2}(f_{11} + f_{22}) + \frac{1}{2f}(f_1^2 + f_2^2) = 0$, i.e., Euclidean space.

(2) $3fR_{1212} + f_1^2 + f_2^2 = 0$. Therefore $3f(f_{11} + f_{22}) = 5(f_1^2 + f_2^2)$ or $3f\Delta f = 5\|\text{grad } f\|^2$ (Poisson PDE). Maple answer (pde, sol := pdsolve(pde)): this PDE has solutions of the form $f(x^1, x^2) = \varphi_1(x^1)\varphi_2(x^2)$, where

$$\frac{d^2\varphi_1}{dx^2}(x) = \frac{1}{3}c_1\varphi_1(x) + \frac{5}{3}\frac{(\frac{d\varphi_1}{dx}(x))^2}{\varphi_1(x)}, \ \frac{d^2\varphi_2}{dy^2}(y) = \frac{1}{3}c_1\varphi_2(y) + \frac{5}{3}\frac{(\frac{d\varphi_2}{dy}(y))^2}{\varphi_2(y)}$$

or

$$C_1\sin((x/3)\sqrt{2c_1}) + C_2\cos((x/3)\sqrt{2c_1}) - \frac{\sqrt{2c_1}}{\varphi_1(x)^{2/3}} = 0,$$

$$C_3\sin((y/3)\sqrt{2c_1}) + C_4\cos((y/3)\sqrt{2c_1}) - \frac{\sqrt{2c_1}}{\varphi_2(y)^{2/3}} = 0.$$

Globally, these solutions are not convenient since they are not strictly positive.

We have two particular cases: a) If $f_1 = 0$, then $f(x^1, x^2) = f(x^2)$ and hence $-3ff_{22} + 5f_2^2 = 0$, a Liouville equation with the general solution

$$\frac{3}{2f(x^2)^{2/3}} + C_1x^2 + C_2 = 0.$$

This function is strictly positive only locally. b) If $f_2 = 0$, then $f(x^1, x^2) = f(x^1)$ and hence $-3ff_{11} + 5f_1^2 = 0$, a Liouville equation with the general solution

$$\frac{3}{2f(x^1)^{2/3}} + c_1x^1 + c_2 = 0.$$

This function is strictly positive only locally.

Theorem 5. *The extremals $g = (g_{ij})$ of the Lagrangian*

$$\mathcal{L}(g, \partial g, \partial^2 g) = ||Riem^g||^2\sqrt{det(g_{ij})}$$

are solutions of the PDEs system

$$R_{pqrs}g^{kr}g^{ls}\Big[2(\Gamma^n_{jk}\Gamma^m_{il} - \Gamma^n_{jl}\Gamma^m_{ik})g^{ip}g^{jq}$$

$$-R_{ijkl}\left(2(g^{mi}g^{np}g^{jq} + g^{ip}g^{mj}g^{nq}) - \frac{1}{2}g^{nm}g^{ip}g^{jq}\right)\Big]\sqrt{det(g_{ij})}$$

$$+D_{x^h}\left[\delta^a_i\delta^b_j\delta^c_{[l}\delta^d_{k]}\left[\Gamma^m_{ad}(\delta^h_c\delta^n_b + \delta^h_b\delta^n_c) - \Gamma^h_{ad}\delta^m_b\delta^n_c\right.\right.$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 10 September 2020 doi:10.20944/preprints202009.0234.v1

$$+\Gamma^m_{bc}(\delta^h_a\delta^n_d+\delta^h_d\delta^n_a)-\Gamma^h_{bc}\delta^m_a\delta^n_d\Big]R_{pqrs}g^{ip}g^{jq}g^{kr}g^{ls}\sqrt{det(g_{ij})}\Big]$$

$$+D^2_{x^hx^a}\left[g^{ip}g^{jq}g^{kr}g^{ls}\delta^m_{[i}\delta^h_{j]}\delta^n_{[l}\delta^a_{k]}R_{pqrs}\sqrt{det(g_{ij})}\right]=0.$$

The Riemann-flat solutions $g_{ij}(x)=\delta_{ij}$ are global minimum points.

3.2. Non-flat extremals

We consider a 2-dimensional Riemannian manifold (M,g), $g_{11}=f$, $g_{22}=h$, $g_{12}=0$. Then $\mathcal{L}=(g^{11}g^{22}R_{1212})^2\sqrt{det(g_{ij})}$, where

$$R_{1212}=-\frac{1}{2}(g_{11,22}+g_{22,11})+g_{ab}(\Gamma^a_{21}\Gamma^b_{12}-\Gamma^a_{22}\Gamma^b_{11})$$

$$-\frac{1}{2}(h_{11}+f_{22})+\frac{1}{4f}(f_2^2+f_1h_1)+\frac{1}{4h}(h_1^2+f_2h_2).$$

Since

$$\frac{\partial\mathcal{L}}{\partial g_{mn}}=2(g^{11}g^{22})^2R_{1212}\sqrt{det(g_{ij})}\frac{\partial R_{1212}}{\partial g_{mn}}+\frac{1}{2}\mathcal{L}g^{mn}$$

$$-2g^{11}g^{22}(g^{22}g^{m1}g^{n1}+g^{11}g^{m2}g^{n2})R^2_{1212}\sqrt{det(g_{ij})},$$

$$\frac{\partial R_{1212}}{\partial g_{11}}=-\frac{1}{4f^2}(f_2^2+f_1h_1),\ \frac{\partial R_{1212}}{\partial g_{22}}=-\frac{1}{4h^2}(h_1^2+f_2h_2),$$

$$\frac{\partial\mathcal{L}}{\partial g_{11}}=R_{1212}\sqrt{det(g_{ij})}\left[-\frac{3}{2f^3h^2}R_{1212}-\frac{1}{2f^4h^2}(f_2^2+f_1h_1)\right].$$

$$\frac{\partial\mathcal{L}}{\partial g_{22}}=R_{1212}\sqrt{det(g_{ij})}\left[-\frac{3}{2f^2h^3}R_{1212}-\frac{1}{2f^2h^4}(h_1^2+f_2h_2)\right],$$

$$\frac{\partial\mathcal{L}}{\partial g_{11,1}}=\frac{h_1}{2f^3h^2}R_{1212}\sqrt{det(g_{ij})}.$$

$$\frac{\partial\mathcal{L}}{\partial g_{11,2}}=\frac{1}{2f^2h^2}\left(\frac{2f_2}{f}+\frac{h_2}{h}\right)R_{1212}\sqrt{det(g_{ij})}.$$

$$\frac{\partial\mathcal{L}}{\partial g_{22,1}}=\frac{1}{2f^2h^2}\left(\frac{f_1}{f}+\frac{2h_1}{h}\right)R_{1212}\sqrt{det(g_{ij})}.$$

$$\frac{\partial\mathcal{L}}{\partial g_{22,2}}=\frac{f_2}{2f^2h^3}R_{1212}\sqrt{det(g_{ij})}.$$

$$\frac{\partial\mathcal{L}}{\partial g_{11,11}}=\frac{\partial\mathcal{L}}{\partial g_{11,12}}=\frac{\partial\mathcal{L}}{\partial g_{22,12}}=\frac{\partial\mathcal{L}}{\partial g_{22,22}}=0.$$

$$\frac{\partial\mathcal{L}}{\partial g_{11,22}}=\frac{\partial\mathcal{L}}{\partial g_{22,11}}=-(g^{11}g^{22})^2R_{1212}\sqrt{det(g_{ij})}$$

it follows the Euler-Lagrange PDEs

$$\frac{\partial\mathcal{L}}{\partial g_{11}}-D_{x^1}\frac{\partial\mathcal{L}}{\partial(\partial g_{11,1})}-D_{x^2}\frac{\partial\mathcal{L}}{\partial(\partial g_{11,2})}$$

$$+D_{x^1}D_{x^1}\frac{\partial\mathcal{L}}{\partial g_{11,11}}+2D_{x^1}D_{x^2}\frac{\partial\mathcal{L}}{\partial g_{11,12}}+D_{x^2}D_{x^2}\frac{\partial\mathcal{L}}{\partial g_{11,22}}=0,$$

$$\frac{\partial\mathcal{L}}{\partial g_{22}}-D_{x^1}\frac{\partial\mathcal{L}}{\partial(\partial g_{22,1})}-D_{x^2}\frac{\partial\mathcal{L}}{\partial(\partial g_{22,2})}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 10 September 2020 doi:10.20944/preprints202009.0234.v1

$$+D_{x^1}D_{x^1}\frac{\partial\mathcal{L}}{\partial g_{22,11}}+2D_{x^1}D_{x^2}\frac{\partial\mathcal{L}}{\partial g_{22,12}}+D_{x^2}D_{x^2}\frac{\partial\mathcal{L}}{\partial g_{22,22}}=0.$$

Equivalently,

$$-\left(\frac{3}{2f^3h^2}R_{1212}+\frac{1}{2f^4h^2}(f_2^2+f_1h_1)\right)R_{1212}\sqrt{fh}-D_{x^1}\left(\frac{h_1}{2f^3h^2}R_{1212}\sqrt{fh}\right)$$

$$-D_{x^2}\left[\frac{1}{2f^2h^2}\left(\frac{2f_2}{f}+\frac{h_2}{h}\right)R_{1212}\sqrt{fh}\right]+D_{x^2x^2}\left(-\frac{1}{f^2h^2}R_{1212}\sqrt{fh}\right)=0.$$

$$-\left(\frac{3}{2f^2h^3}R_{1212}+\frac{1}{2f^2h^4}\left(h_1^2+f_2h_2\right)\right)R_{1212}\sqrt{fh}$$

$$-D_{x^1}\left[\frac{1}{f^2h^2}\left(\frac{f_1}{f}+\frac{2h_1}{h}\right)R_{1212}\sqrt{fh}\right]-D_{x^2}\left(\frac{f_2}{2f^2h^3}R_{1212}\sqrt{fh}\right)$$

$$+D_{x^1x^1}\left(-\frac{1}{f^2h^2}R_{1212}\sqrt{fh}\right)=0.$$

The conformal case $f=h$. The PDE system becomes

$$-\frac{3}{2f^4}R_{1212}^2-\frac{1}{2f^5}(f_2^2+f_1^2)R_{1212}-D_{x^1}\left(\frac{f_1}{2f^4}R_{1212}\right)$$

$$-D_{x^2}\left(\frac{3f_2}{2f^4}R_{1212}\right)+D_{x^2x^2}\left(-\frac{1}{f^3}R_{1212}\right)=0.$$

$$-\frac{3}{2f^4}R_{1212}^2-\frac{1}{2f^5}(f_2^2+f_1^2)R_{1212}-D_{x^1}\left(\frac{3f_1}{2f^4}R_{1212}\right)$$

$$-D_{x^2}\left(\frac{f_2}{2f^4}R_{1212}\right)+D_{x^1x^1}\left(-\frac{1}{f^3}R_{1212}\right)=0.$$

The case $R_{1212}=-\frac{1}{2}(f_{11}+f_{22})+\frac{1}{2f}(f_1^2+f_2^2)=0$ produce trivial solution $g_{11}=1=g_{22},\ g_{12}=0$. We subtract the second equation from the first one and we get

$$D_{x^1}\left(\frac{f_1}{f^4}R_{1212}\right)-D_{x^2}\left(\frac{f_2}{f^4}R_{1212}\right)+(D_{x^2x^2}-D_{x^1x^1})\left(-\frac{1}{f^3}R_{1212}\right)=0.$$

Particular cases a) $f_1=0$ and

$$D_{x^2}\left[\frac{f_2}{f^4}(-ff_{22}+f_2^2)\right]+D_{x^2x^2}\left[\frac{1}{f^4}(-ff_{22}+f_2^2)\right]=0.$$

The second PDE is equivalent to $\frac{f_2}{f^4}(-ff_{22}+f_2^2)+D_{x^2}\left[\frac{1}{f^4}(-ff_{22}+f_2^2)\right]=c_2$ or to

$$\frac{1}{f^4}(-ff_{22}+f_2^2)=e^{-f}\left(c+c_2\int e^f\,dx^2\right).$$

The Liouville equation $-ff_{22}+f_2^2=0$ has the solution $f(x^2)=ae^{bx^2},a>0$.

b) $f_2=0$ and

$$D_{x^1}\left[\frac{f_1}{f^4}(-ff_{11}+f_1^2)\right]+D_{x^1x^1}\left[\frac{1}{f^4}(-ff_{11}+f_1^2)\right]=0.$$

The second PDE is equivalent to $\frac{f_1}{f^4}(-ff_{11}+f_1^2)+D_{x^1}\left[\frac{1}{f^4}(-ff_{11}+f_1^2)\right]=c_1$ or to

$$\frac{1}{f^4}(-ff_{11}+f_1^2)=e^{-f}\left(c+c_1\int e^f\,dx^1\right).$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 10 September 2020 doi:10.20944/preprints202009.0234.v1

The Liouville equation $-ff_{11}+f_1^2=0$ has the solution $f(x^1)=ce^{dx^1}, c>0$.

4. Least squares Lagrangian density attached to Ricci - flatness

Let (M,∇) be an equiaffine manifold. The components R_{ik} of the Ricci tensor field Ric^{∇} are obtained by the contraction of the first and third indices of the curvature tensor field $R^l{}_{ijk}$, i.e.,

$$R_{ik}=R^l{}_{ilk}=\frac{\partial}{\partial x^l}\Gamma^l_{ik}-\frac{\partial}{\partial x^k}\Gamma^l_{il}+\Gamma^l_{ls}\Gamma^s_{ik}-\Gamma^l_{ks}\Gamma^s_{il}$$

$$=\mathcal{P}^{ps}_{qk}\left(\frac{\partial}{\partial x^p}\Gamma^q_{is}+\Gamma^q_{pn}\Gamma^n_{is}\right)=\mathcal{P}^{ps}_{qk}\left(\frac{\partial}{\partial x^p}\Gamma^q_{is}-\Gamma^q_{sn}\Gamma^n_{ip}\right),$$

$i,j,k,...=\overline{1,n}$. Each of the Ricci - flatness PDEs systems $R_{ik}=0$ is a system of $\frac{n(n+1)}{2}$ distinct first order divergence quadratic tensorial PDEs with $\frac{n^2(n+1)}{2}$ unknown functions Γ^i_{jk}; for $n>1$, undetermined system; for $n=1$, determined system. Here $\mathcal{P}^{ps}_{qk}=\delta^p_q\delta^s_k-\delta^p_k\delta^s_q$ works like a trace between p and q, in order to produce a divergence term. This operator is associated to the projection P. Any divergence PDE represents a conservation law.

Ricci flatness was described in the papers [8], [10], [11], [22], [23], [13] underlining locally the difference between an "Euclidean ball" and a "geodesic ball".

In Physics, Ricci-flat manifolds represent vacuum solutions to the analogues of Einstein's equations for Riemannian manifolds of any dimension, with vanishing cosmological constant.

Let $g=(g_{ij})$ be a Riemannian metric. On the smooth oriented manifold (M,∇,g), let us consider the Lagrangian density $L=\|Ric^{\nabla}\|^2=g^{ik}g^{jl}R_{ij}R_{kl}$ (square of the norm, first order in Γ^i_{jk}) and the functional (Ricci - flatness deviation) $I(\nabla)=\int_M\|Ric^{\nabla}\|^2d\mu$. The Euler-Lagrange PDEs are $\frac{\partial\mathcal{L}}{\partial\Gamma^l_{mn}}-D_{x^r}\frac{\partial\mathcal{L}}{\partial(\partial_{x^r}\Gamma^l_{mn})}=0$.

Theorem 6. *Let* $R_{ij}=\mathcal{P}^{ps}_{qj}\left(\frac{\partial}{\partial x^p}\Gamma^q_{is}+\Gamma^q_{pn}\Gamma^n_{is}\right)$. *The extremals* Γ^i_{jk} *of the Lagrangian* $\mathcal{L}(\nabla,\partial\nabla)=g^{ik}g^{jl}R_{ij}R_{kl}\sqrt{\det(g_{ij})}$ *are solutions of PDEs system*

$$[\delta^v_u\Gamma^w_{ij}-\delta^v_j\Gamma^w_{iu}+\delta^v_i(\delta^w_j\Gamma^c_{cu}-\Gamma^w_{ju})]R_{kl}g^{ik}g^{jl}\sqrt{det(g_{ij})}$$

$$-D_{x^t}\left(\delta^t_{[u}\delta^w_{j]}R_{kl}g^{vk}g^{jl}\sqrt{det(g_{ij})}\right)=0.$$

The Ricci-flat solutions Γ^i_{jk} are global minimum points. The other solutions are best approximation of flatness PDEs solutions.

In case that $(M,g=(g_{ij}))$ is a Riemannian manifold, the Ricci tensor field Ric^g has the components

$$R_{ik}=\frac{\partial\Gamma^l_{ik}}{\partial x^l}-\Gamma^m_{il}\Gamma^l_{km}-\nabla_k\left(\frac{\partial}{\partial x^i}\left(\ln\sqrt{\det(g_{mn})}\right)\right),$$

where

$$\Gamma^i_{jk}=\frac{1}{2}g^{il}(\delta^r_l\delta^s_j\delta^t_k+\delta^r_l\delta^s_k\delta^t_j-\delta^t_l\delta^r_j\delta^s_k)\frac{\partial g_{rs}}{\partial x^t}.$$

The Ricci tensor field of a connection derived from a metric is always symmetric. In this case, the Ricci-flat manifold

$$\frac{\partial\Gamma^l_{ik}}{\partial x^l}-\Gamma^m_{il}\Gamma^l_{km}-\nabla_k\left(\frac{\partial}{\partial x^i}\left(\ln\sqrt{\det(g_{mn})}\right)\right)=0$$

means $\frac{n(n+1)}{2}$ distinct PDEs with $\frac{n(n+1)}{2}$ unknown functions g_{ij}, on $S^2_+T^*M$. They are special cases of Einstein manifolds, where the cosmological constant vanishes.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 10 September 2020 doi:10.20944/preprints202009.0234.v1

On the Riemannian manifold $(M, g = (g_{ij}))$, let us consider the Lagrangian density $L = \|Ric^g\|^2 = g^{ik}g^{jl}R_{ij}R_{kl}$ (square of the norm) which is of second order in g_{ij} and order zero in g^{ij}. The Ricci - flatness deviation is described either by $I(g) = \int_M \|Ric^g\|^2 d\mu$ or by $I(g^{-1}) = \int_M \|Ric^g\|^2 d\mu$.

For $I(g)$ the extremals g are solutions of fourth order Euler-Lagrange PDEs $\frac{\partial \mathcal{L}}{\partial g_{mn}} - D_{x^l} \frac{\partial \mathcal{L}}{\partial(\partial_{x^l} g_{mn})} + D_{x^k}D_{x^l} \frac{\partial \mathcal{L}}{\partial(\partial_{x^k}\partial_{x^l} g_{mn})} = 0$. To simplify, we work first with $I(g^{-1})$ since the Euler-Lagrange PDEs determined by $\mathcal{L} = g^{ik}g^{jl}R_{ij}R_{kl} \sqrt{\det(g_{ij})}$ are reduced to $\frac{\partial \mathcal{L}}{\partial g^{mn}} = 0$.

Theorem 7. *We fix a harmonic coordinate system [13]. The extremals $g = (g_{ij})$ of the functional $I(g^{-1})$ are solutions of PDEs system*

$$2g^{ik}R_{im}R_{kn} + 2g^{ik}g^{jl}R_{kl}\Big(-\frac{1}{2}\frac{\partial^2 g_{ij}}{\partial x^m \partial x^n} + g_{cd}\Gamma^c_{im}\Gamma^d_{nj} + g^{ab}g_{md}g_{nc}\Gamma^c_{ia}\Gamma^d_{bj}\Big)$$

$$-\frac{1}{2}g^{ik}g^{jl}R_{ij}R_{kl}\; g_{mn} = 0.$$

The Ricci-flat solutions $g_{ij}(x)$ are global minimum points. The other solutions are best approximation of flatness PDEs solutions.

Theorem 8. *We fix a harmonic coordinate system [13]. The extremals $g = (g_{ij})$ of the functional $I(g)$ are solutions of PDEs system*

$$\sqrt{det(g_{ij})}\Big[R_{ij}R_{kl}(-2g^{mi}g^{nk}g^{jl} + \frac{1}{2}g^{ik}g^{jl}g^{mn}) + 2g^{ik}g^{jl}$$

$$\times R_{ij}\Big[- g^{mp}g^{nq}\left(-\frac{1}{2}\frac{\partial^2 g_{kl}}{\partial x^p \partial x^q} + g_{rs}\Gamma^r_{kp}\Gamma^s_{ql}\right) - g^{pq}\Gamma^n_{kp}\Gamma^m_{ql}\Big]\Big]$$

$$-D_{x^h}\Big[\sqrt{det(g_{ij})}R_{kl}g^{ab}g^{ik}g^{jl}\Big[\Big(\delta^n_d(\delta^h_a\delta^m_i + \delta^h_i\delta^m_a) - \delta^h_d\delta^m_i\delta^n_a\Big)\Gamma^d_{bj}$$

$$+\Big(\delta^n_d\left(\delta^h_j\delta^m_b + \delta^h_b\delta^m_j\right) - \delta^h_d\delta^m_b\delta^n_j\Big)\Gamma^d_{ia}\Big]\Big] + D^2_{x^h x^t}\Big[\sqrt{det(g_{ij})}g^{ht}g^{mk}g^{nl}R_{kl}\Big] = 0.$$

The Ricci-flat solutions $g_{ij}(x)$ are global minimum points. The other solutions are best approximation of flatness PDEs solutions.

5. Least squares Lagrangian density attached to scalar curvature - flatness

Let ∇ be an equiaffine connection of components Γ^i_{jk} and $g = (g_{ij})$ be a Riemannian metric, where $i, j, k, ... = \overline{1, n}$. On the manifold (M, ∇, g), we introduce the functional (total scalar curvature) $I(\nabla) = \int_M \mathcal{R}^\nabla \, d\mu$, where $\mathcal{R}^\nabla = g^{ij}R_{ij}$, and the Lagrangian $\mathcal{L} = \mathcal{R}^\nabla \sqrt{\det(g_{ij})}$ is of first order with respect to Γ^i_{jk}. The general Euler-Lagrange PDEs are $\frac{\partial \mathcal{L}}{\partial \Gamma^l_{mn}} - D_{x^r}\frac{\partial \mathcal{L}}{\partial(\partial_{x^r}\Gamma^l_{mn})} = 0$.

Theorem 9. *The Euler-Lagrange PDEs attached to the functional $I(\nabla)$, i.e., to the Lagrangian $\mathcal{L} = g^{ik}R_{ik} \sqrt{\det(g_{ik})}$, are*

$$\mathcal{P}^{ps}_{qk}\left[g^{ik}\left(\delta^q_l\delta^m_p\Gamma^n_{is} + \delta^m_i\delta^n_s\Gamma^q_{pl}\right)\sqrt{\det(g_{ab})} - \delta^q_l\delta^n_s D_{x^p}\left(g^{mk}\sqrt{\det(g_{ab})}\right)\right] = 0.$$

Proof. Since $\mathcal{L} = g^{ik}R_{ik} \sqrt{\det(g_{ik})}$, $R_{ik} = \mathcal{P}^{ps}_{qk}\left(\frac{\partial}{\partial x^p}\Gamma^q_{is} + \Gamma^q_{pr}\Gamma^r_{is}\right)$, and

$$\frac{\partial \Gamma^i_{jk}}{\partial \Gamma^l_{mn}} = \delta^i_l\delta^m_j\delta^n_k, \quad \frac{\partial\left(\partial_{x^p}\Gamma^q_{is}\right)}{\partial\left(\partial_{x^r}\Gamma^l_{mn}\right)} = \delta^r_p\delta^q_l\delta^m_i\delta^n_s,$$

 doi:10.20944/preprints202009.0234.v1

$$\frac{\partial R_{ik}}{\partial \Gamma^l_{mn}} = \mathcal{P}^{ps}_{qk}\left(\delta^q_l \delta^m_p \Gamma^n_{is} + \delta^m_i \delta^n_s \Gamma^q_{pl}\right); \frac{\partial R_{ik}}{\partial_{x^r} \Gamma^l_{mn}} = \mathcal{P}^{ps}_{qk}\ \delta^r_p \delta^q_l \delta^m_i \delta^n_s$$

we obtain the PDEs in the Theorem. □

On a smooth oriented Riemannian manifold $(M, g = (g_{ij}))$, we attach the functional (total scalar curvature) $I(g) = \int_M R^g\ d\mu$, $R^g = g^{ij} R_{ij}$. Here the Lagrangian $\mathcal{L} = g^{ij} R_{ij} \sqrt{\det(g_{ij})}$ is of the second order with respect to g_{ij}, and of order zero with respect to g^{ij}. In dimension two, this is a topological quantity, namely the Euler characteristic of the Riemann surface by the Gauss-Bonnet formula. In $n \geq 3$ dimension we prefer to write the functional in the form $I(g^{-1}) = \int_M R^g\ d\mu$.

Theorem 10. *The Euler-Lagrange PDEs attached to the functional $I(g^{-1})$, $n \geq 3$, i.e., to the Lagrangian $\mathcal{L} = g^{ij} R_{ij} \sqrt{\det(g_{ij})}$, are Einstein PDEs $R_{ij} = 0$.*

Proof. The Euler-Lagrange PDEs are $\frac{\partial \mathcal{L}}{\partial g^{mn}} = 0$, where $\mathcal{L} = g^{ij} R_{ij} \sqrt{\det(g_{ij})}$. On the other hand, we have

$$\frac{\partial \det(g_{ij})}{\partial g^{mn}} = -\det(g_{ij}) g_{mn}, \quad \frac{\partial g_{jl}}{\partial g^{mn}} = -g_{mj} g_{nl}, \quad \frac{\partial g^{jl}}{\partial g^{mn}} = \delta^j_m \delta^l_n.$$

The term $g^{ij} \frac{\partial R_{ij}}{\partial g^{mn}} \sqrt{\det(g_{ij})}$ is of divergence type, and it has no contribution to the Euler-Lagrange equations. Consequently $\frac{\partial R}{\partial g^{mn}} = \frac{\partial (g^{ij} R_{ij})}{\partial g^{mn}} = R_{mn}$. Finally, we obtain the explicit Euler–Lagrange PDEs as $R_{ij} = 0$. □

Theorem 11. [16] *The solutions of the problem "$\min_{g_{ij}} \int_M R^g\ d\mu$ subject to $\int_M d\mu = 1, n \geq 3$", are solutions of nD Einstein PDEs $R_{ij} = \dfrac{R}{n}\ g_{ij}$.*

Proof. We use the Lagrangian $\mathcal{L} = g^{ij} R_{ij} \sqrt{\det(g_{ij})} - \lambda \sqrt{\det(g_{ij})}$, where λ is a constant multiplier. Taking the variations with respect to g^{ij}, we obtain

$$R_{ij} - \frac{R - \lambda}{2}\ g_{ij} = 0.$$

The hypothesis $n \geq 3$ and $\lambda = c$ implies that R is constant. We replace R, respectively R_{ij}, in $\int_M R^g\ d\mu$ and we obtain $R\ vol(M) = \int_M R^g\ d\mu = \frac{R-\lambda}{2}\ n\ vol(M)$. Consequently, $\lambda = \frac{(n-2)R}{n}$ and $R_{ij} = \frac{R}{n}\ g_{ij}$. □

The exact solutions of Einstein PDEs were discussed many times. In dimension four, there are topological obstructions to the existence of Einstein metrics.

On a smooth oriented Riemannian manifold $(M, g = (g_{ij}))$, we attach a scalar curvature - flatness deviation either by the action $I(g) = \int_M (R^g)^2\, d\mu$ or as the functional $I(g^{-1}) = \int_M (R^g)^2\, d\mu$.

Theorem 12. *The Euler-Lagrange PDEs attached to functional $I(g^{-1})$, i.e. to the Lagrangian $\mathcal{L} = (g^{ij} R_{ij})^2 \sqrt{\det(g_{ij})}$ (zero order with respect to g^{ij}), are either $R = 0$ or $R_{ij} = 0$.*

Corollary 1. *The solutions $g_{ij}(x)$ of PDEs $R = 0$ or $R_{ij} = 0$ are Euler-Lagrange prolongations of Euclidean metrics $g_{ij}(x) = \delta_{ij}$.*

6. Conclusions and future work

In this paper were studied least squares Lagrangian densities attached to flatness PDEs on Riemannian manifolds. The index form technique facilitates the understanding of the significance of the geometric PDEs and of the Lagrangian densities attached to them using the Riemannian metrics. This paper is the continuation of some ideas in the papers [2], [20].

Some of our results are proved for a wider class of manifolds and as a special case we also reprove well-known results for the Einstein PDEs. Also, the proposed approach can be used for each least squares Lagrangian on Riemannian setting and is able to give intermediate results, which can be seen as the "best approximation" of solutions of geometric PDEs.

In the Riemannian case we have two facilities: (i) the most important Lagrange-type densities are the squares of the norms of important geometric objects: connection, curvature tensor field, Ricci tensor field, scalar curvature field; (ii) to obtain the Euler-Lagrange PDEs, we can select alternatively either the variations with respect to the metric g or the variations with respect to the inverse metric g^{-1}.

In light of the above discussion, if one is able to say something about the solution of a PDEs system whose solution is a Riemannian metric or an affine connection, one could perhaps say something interesting about the behaviour of the manifold and its structure. Further research into the nature of the geometric extremals (metrics or connections) may yield strong theoretic results for finite dimensional Riemannian manifolds.

Acknowledgements: Thanks to referees for pertinent remarks.

Funding: This research received funding from Balkan Society of Geometers, Bucharest, Romania.

Data Availability: The data used to support the findings of this paper are included within the article and are available from the corresponding author upon request.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. (1984) Anderson, I. M. Natural variational principles on Riemannian manifolds. *Ann. Math.*, **1984** (2) *120*, 329-370, DOI
2. (2019) Bejenaru, A., Udriste, C. Multivariate optimal control in Riemannian setting. *Symmetry, special issue: Geometry of Submanifolds and Homogeneous Spaces*, mdpi, **2019**, *11*(7), 893
3. (1965) Charlap, L. Compact flat Riemannian manifolds, I. *Ann. Math.*, **1965**, (2) 81, 15-30
4. (1965) Charlap, L., Vasquez, A. T. Compact flat Riemannian manifolds, II. *Amer. J. Math.*, **1965** *87*, 551-563
5. (1973) Charlap, L., Vasquez, A. T.. Compact flat Riemannian manifolds III. The group of affinities. *Amer. J. Math.*, **1973** *95*, 3, 471-494
6. (1981) DeTurck, D., Kazdan, J. L., Jerry, L.. Some regularity theorems in Riemannian geometry. *Ann. Sci. Ec. Norm. Super.*, **1981**, 4, *14* (3), 249-260
7. (1985) DeTurk, D., Yang, D.. Local existence of smooth metrics with prescribed curvature. *Nonlinear problems in geometry* (Mobile, Ala., **1985**, pp. 37-43), *Contemp. Math.*, *51*, Amer. Math. Soc., Providence, RI, **1986**
8. (2014) Diallo, A. S., Hassirou, M., Katambe, I. Affine Osserman connections which are Ricci flat but not flat. *Int. J. Pure Appl. Math.*, **2014**, *91*, 3, 305-312
9. (1998) Evans, L. C.. Partial Differential Equations. *American Mathematical Society*, Providence, **1998**.
10. (1975) Fischer, A. E., Wolf, J. A. The structure of compact Ricci-flat Riemannian Manifolds. *J. Diff. Geom.*, **1975**, *10*, 277-288
11. (2007) Gray, A.. The volume of a small geodesic ball. *Michigan Math. J.*, **2007**, *20*, 329-344
12. (2003) Gurski, M. J., Viaclovsky, J. A.. Fully nonlinear equations on Riemannian manifolds with negative curvature. *Indiana U. Math. J.*, **2003**, *52*, 2, 399-419
13. (1997) Hebey, E., Herzlich, M. Harmonic coordinates, harmonic radius and convergence of Riemannian manifolds, *Rend. Mat.*, VII, **1997**, *17*, 569-605
14. (1982) Hamilton, R.: Three manifolds with positive Ricci curvature. *J. Diff. Geom.*, **1982**, *17*, 255-306
15. (2012) Hirica, I. E., Udriste, C. Basic evolution PDEs in Riemannian geometry. *Balkan J. Geom. Appl.*, **2012**, *17*, 1, 30-40
16. (2015) Krupka, D. D. Introduction to Global Variational Geometry. *Springer*, **2015**.
17. (1976) Sabitov, I. H., Sefel, S. Z. Connections between the order of smoothness of a surface and that of its metric. *Akademija Nauk SSSR. Sibirskoe Otdelenie. Sibirskii Matematiceskii Zurnal*, **1976**, *17* (4), 916-925

 doi:10.20944/preprints202009.0234.v1

18. (1969) Spencer, D. Overdetermined systems of linear partial differential equations. *Bull. Am. Math. Soc.*, **1969**, *75*, 179-239
19. (1965) Szarski, J. Differential Inequalities. *Krakow*, **1965**.
20. (2020) Udriste, C., Tevy, I. Geometric dynamics on Riemannian manifolds. *Mathematics*, mdpi, **2020**, 8, 79; doi:10.3390/math8010079
21. (1970) Vasquez, A. T. Flat Riemannian manifolds. *J. Diff. Geom.*, **1970**, *4*, 3, 367-382
22. (1956) Willmore, T. J.: On compact Riemannian manifolds with zero Ricci curvature. *Proc. Edinburgh Math. Soc.*, **1956**, *10*, 131-133
23. (2016) Zdenek, D., Kowalski, O. How many Ricci flat affine connections are there with arbitrary torsion? *Publ. Math., Debrecen*, **2016**, *88*, 3-4, 511-516
24. (1978) Yau, S. -T. The role of Partial Differential Equations in Differential Geometry. *Proceedings of the International Congress of Mathematicians*, Helsinki, **1978**.