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Article

Fixed Circle and Fixed Disc Results for New Type of Θ_c-Contractive Mappings in Metric Spaces

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Abstract: This paper aims to introduce the notions of various types of Θ_c-contractions for which we establish some fixed circle and fixed disc theorems in the setting of metric spaces. Some illustrative examples are also provided to support our results. Moreover, we present some fixed circle and fixed disc results of integral type contractive self-mappings.

Keywords: fixed point; fixed circle; fixed disc; Θ_c-contractions

1. Introduction and Preliminaries

Metric fixed point theory is the branch of mathematical analysis which study the existence and uniqueness of the fixed point of the mappings defined on a metric space (Υ, ξ). The most important theorem in this theory is the classical Banach contraction principle due to Banach [1]. Many authors extended and generalized this principle in various directions (see[2–10]). In this regard, Jleli and Samet [11], introduced the auxiliary functions $\Theta:(0,\infty) \to (1,\infty)$ and utilized the same to prove a fixed point theorem concerning a new type of contraction, called Θ-contraction (or JS contraction).

Definition 1. *[11] Let $\Theta:(0,\infty) \to (1,\infty)$ be a function such that the following conditions are hold:*

$(JS1)$ *Θ is non-decreasing;*
$(JS2)$ *for every sequence $\{\alpha_n\} \subset (0,\infty)$,*

$$\lim_{n\to\infty} \Theta(\alpha_n) = 1 \quad \Leftrightarrow \quad \lim_{n\to\infty} \alpha_n = 0^+;$$

$(JS3)$ *there exists $\lambda \in (0,1)$ and $k \in (0,\infty)$ such that $\lim_{\alpha\to 0^+} \frac{\Theta(\alpha)-1}{\alpha^\lambda} = k$.*

In the sequel, we adopt the following notations:

- Ω_1 the class of all functions Θ which satisfy $(JS1)$.
- $\Omega_{1,2,3}$ the class of all functions Θ which satisfy $(JS1)$-$(JS3)$.

From now on, the mapping S is a self-mapping defined on a metric space (Υ, ξ). Utilizing the above auxiliary functions Jleli and Samet [11] defined Θ-contraction as follows:

Definition 2. *[11] Let $S : Y \to Y$ and $\Theta \in \Omega_{1,2,3}$. The mapping S is said to be Θ-contraction, if there exists a constant $\lambda \in (0,1)$ such that*

$$\xi(Sz, Sw) > 0 \;\Rightarrow\; \Theta(\xi(Sz, Sw)) \leq [\Theta(\xi(z,w))]^{\lambda},\ \forall\, z, w \in Y \tag{1}$$

By considering the notion of Θ-contractions, the authors in [11] proved that every Θ-contraction mapping defined on a generalized metric space possesses a unique fixed point. On the other hand, in the case that the mapping S has more than one fixed point, there exist some mappings that fixes all the points of circle, such circle is called a fixed circle.

For a metric space (Y, ξ), the two sets $C_{z_0,r} = \{z \in Y : \xi(z_0, z) = r\}$ and $D_{z_0,r} = \{z \in Y : \xi(z_0, z) \leq r\}$ are called circle and disc, respectively, with center z_0 and radius r. The notion of fixed circle was introduced recently in [12] as under:

Definition 3. *[12] Let $S : Y \to Y$ be a mapping and $C_{z_0,r}$ a circle on Y. Then $C_{z_0,r}$ is said to be a fixed circle of S if $Sz = z$, for all $z \in C_{z_0,r}$.*

Example 1. *[13] let $\mathbb{C}$ be the set of all complex numbers and consider the mapping $S : \mathbb{C} \to \mathbb{C}$ defined by*

$$Sz = \begin{cases} \frac{1}{\bar{z}}, & if z \neq 0; \\ 0, & if z = 0, \end{cases}$$

where $\bar{z}$ is the conjugate of z. Then $C_{0,1}$ is the fixed circle of S.

These kind of mappings have some applications to neural networks (see [14]). For more details of such kind of mappings and fixed circle results we refer the reader to [12,13,15–19].

Here we would like to point out that there exist some mappings which map the circle $C_{z_0,r}$ to it self but $C_{z_0,r}$ is not a fixed circle, that is, the mapping does not fix the all point of the circle as we will see in the following example:

Example 2. *[13] let $\mathbb{C}$ be the set of all complex numbers and consider the mapping $T : \mathbb{C} \to \mathbb{C}$ defined by*

$$Tz = \begin{cases} \frac{1}{z}, & if z \neq 0; \\ 0, & if z = 0. \end{cases}$$

Then $T(C_{0,1}) = C_{0,1}$, but $C_{0,1}$ is not a fixed circle of T. In fact, the mapping T fixes only two point of the unit circle.

Now, we introduce the notion of fixed disc as follows:

Definition 4. *Let $S : Y \to Y$ be a mapping and $D_{z_0,r}$ a disc on Y. Then $D_{z_0,r}$ is said to be a fixed disc of S if $Sz = z$, for all $z \in D_{z_0,r}$.*

This paper aims to present some fixed circle (disc) results for many types of contraction self-mappings namely: Θ_c-contractions, Θ_c-weak contractions, Ćirić type Θ_c-contractions, Reich type Θ_c-contractions, Chatterjea type Θ_c-contractions, Hardy-Rogers type Θ_c-contractions and Khan type Θ_c-contractions in the setting of metric spaces by using JS technique. Furthermore, we establish some fixed circle (disc) results of integral type contractive self-mappings.

2. Fixed circle (disc) results

First, we introduce the notions of Θ_c-contractions, Ćirić type Θ_c-contractions and Θ_c-weak contractions as follows.

Definition 5. *Let $S : Y \to Y$ and $\Theta \in \Omega_1$. The mapping S is called Θ_c-contraction, if there exists $\lambda \in (0,1)$, and $z_0 \in Y$ such that*

$$\xi(z, Sz) > 0 \ \Rightarrow \ \Theta(\xi(z, Sz)) \leq [\Theta(\xi(z, z_0))]^{\lambda}, \ \forall\, z \in Y.$$

Definition 6. *Let $S : Y \to Y$ and $\Theta \in \Omega_1$. The mapping S is said to be Ćirić type Θ_c-contraction if there exist $\lambda \in (0,1)$ and $z_0 \in Y$ such that*

$$\xi(z, Sz) > 0 \ \Rightarrow \ \Theta(\xi(z, Sz)) \leq [\Theta(m(z, z_0))]^{\lambda}, \ \forall\, z \in Y,$$

where $m(z, z_0) = \max\{\xi(z, z_0), \xi(z, Sz), \xi(z_0, Sz_0), \frac{1}{2}[\xi(z, Sz_0) + \xi(z_0, Sz)]\}$.

Definition 7. *Let $S : Y \to Y$ and $\Theta \in \Omega_1$. The mapping S is said to be a Θ_c-weak contraction if there exist $\lambda \in (0,1)$ and $z_0 \in Y$ such that*

$$\xi(z, Sz) > 0 \ \Rightarrow \ \Theta(\xi(z, Sz)) \leq [\Theta(M(z, z_0))]^{\lambda}, \ \forall\, z \in Y,$$

where

$$M(z, z_0) = \max\left\{ \begin{array}{c} \xi(z, z_0), a\xi(z, Sz) + (1-a)\xi(z_0, Sz_0), \\ (1-a)\xi(z, Sz) + a\xi(z_0, Sz_0), \frac{1}{2}[\xi(z, Sz_0) + \xi(z_0, Sz)] \end{array} \right\}, \ 0 \leq a < 1.$$

Remark 1. *(1) Every Θ_c-contraction is Θ_c-weak contraction.*
(2) Taking $a = 0$, then Definition 7 coincides with Definition 6.

The following proposition follows from Definition 7.

Proposition 1. *Let (Y, ξ) be a metric space and $S : Y \to Y$ a Θ_c-weak contraction with $z_0 \in Y$, then $Sz_0 = z_0$.*

Proof. Assume that $Sz_0 \neq z_0$. From Definition 7, we have

$$\begin{aligned} \Theta(\xi(z_0, Sz_0)) &\leq [\Theta(M(z_0, z_0))]^{\lambda} \\ &= \left[\Theta\left(\max\left\{ \begin{array}{r} \xi(z_0, z_0), a\xi(z_0, Sz_0) + (1-a)\xi(z_0, Sz_0), \\ (1-a)\xi(z_0, Sz_0) + a\xi(z_0, Sz_0), \\ \frac{1}{2}[\xi(z_0, Sz_0) + \xi(z_0, Sz_0)] \end{array} \right\}\right)\right]^{\lambda} \\ &= [\Theta\left(\max\left\{0, \xi(z_0, Sz_0)\right\}\right)]^{\lambda} \\ &= [\Theta(\xi(z_0, Sz_0))]^{\lambda}, \end{aligned}$$

a contradiction as $\lambda \in (0,1)$. Therefore, we must have $Sz_0 = z_0$. □

Using Θ_c-weak contraction condition, we present the following fixed circle results.

Theorem 1. *Let (Y, ξ) be a metric space, $S : Y \to Y$ and $r = \inf\{\xi(z, Sz) : z \neq Sz\}$. If S is a Θ_c-weak contraction with $z_0 \in Y$ and $\xi(z_0, Sz) = r$, for all $z \in C_{z_0,r}$, then $C_{z_0,r}$ is a fixed circle of S.*

Proof. Let $z \in C_{z_0,r}$ and assume on contrary that $Sz \neq z$. From the definition of r, we have $\xi(z, Sz) \geq r$. As S is Θ_c-weak contraction, using Proposition 1, we have

$$\begin{aligned}\Theta(\xi(z,Sz)) &\leq [\Theta(M(z,z_0))]^{\lambda} \\ &= \left[\Theta\left(\max\left\{\begin{array}{r}\xi(z,z_0), a\xi(z,Sz)+(1-a)\xi(z_0,Sz_0), \\ (1-a)\xi(z,Sz)+a\xi(z_0,Sz_0), \\ \frac{1}{2}[\xi(z,Sz_0)+\xi(z_0,Sz)]\end{array}\right\}\right)\right]^{\lambda} \\ &= [\Theta(\max\{r, a\xi(z,Sz),(1-a)\xi(z,Sz)\})]^{\lambda} \\ &< \Theta(\max\{r, a\xi(z,Sz),(1-a)\xi(z,Sz)\}). \end{aligned} \tag{2}$$

Now, we have the following possibilities:
Case 1: If $\max\{r, a\xi(z,Sz),(1-a)\xi(z,Sz)\} = r$, then from (2), the definition of r and the fact that the function Θ is nondecreasing, we have

$$\Theta(r) \leq \Theta(\xi(z,Sz)) < \Theta(r),$$

a contradiction.
Case 2: If $\max\{r, a\xi(z,Sz),(1-a)\xi(z,Sz)\} = a\xi(z,Sz)$, then we have two possibilities, $a = 0$ or $0 < a < 1$. Assume that $0 < a < 1$, from (2) and the fact that the function Θ is nondecreasing, we have

$$\Theta(\xi(z,Sz)) < \Theta(a\xi(z,Sz)) \leq \Theta(\xi(z,Sz)),$$

a contradiction. If $a = 0$, then from (2), we get

$$\Theta(\xi(z,Sz)) < \Theta(0)$$

this inequality contradicts with the definition of Θ (as $\Theta : (0,\infty) \to (1,\infty)$).
Case 3: If $\max\{r, a\xi(z,Sz),(1-a)\xi(z,Sz)\} = (1-a)\xi(z,Sz)$, then from (2) and the fact that the function Θ is nondecreasing, we have

$$\Theta(\xi(z,Sz)) < \Theta((1-a)\xi(z,Sz)) \leq \Theta(\xi(z,Sz)),$$

a contradiction. Therefore, $Sz = z$ for all $z \in C_{z_0,r}$. Consequently, $C_{z_0,r}$ is a fixed circle of S. □

Next we prove the following fixed disc as follows.

Theorem 2. *Let (Y,ξ) be a metric space, $S : Y \to Y$ and $r = \inf\{\xi(z,Sz) : z \neq Sz\}$. If S is a Θ_c-weak contraction with $z_0 \in Y$ and $\xi(z_0,Sz) = r$, for all $z \in D_{z_0,r}$, then $D_{z_0,r}$ is a fixed disc of S.*

Proof. The mapping S fixes the circle $C_{z_0,r}$ (in view of Theorem 1). Now, in order to show that $D_{z_0,r}$ is a fixed disc of S it is sufficient to show that S fixes any circle $C_{z_0,\rho}$ with $\rho < r$. Let $z \in C_{z_0,\rho}$ and for contrary let us assume that $z \neq Sz$, for some $z \in C_{z_0,\rho}$. Since S is Θ_c-weak contraction, by using Proposition 1, we have

$$\begin{aligned}\Theta(\xi(z,Sz)) &\leq [\Theta(M(z,z_0))]^{\lambda} \\ &= \left[\Theta\left(\max\left\{\rho, a\xi(z,Sz),(1-a)\xi(z,Sz), \frac{\rho+r}{2}\right\}\right)\right]^{\lambda} \\ &< \Theta\left(\max\left\{\rho, a\xi(z,Sz),(1-a)\xi(z,Sz), \frac{\rho+r}{2}\right\}\right). \end{aligned} \tag{3}$$

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Now, we have the following possibilities:
Case 1: If $\max\left\{\rho, a\xi(z,Sz), (1-a)\xi(z,Sz), \frac{\rho+r}{2}\right\} = \rho$, then from (3), the definition of r and the fact that the function Θ is nondecreasing, we have

$$\Theta(\rho) < \Theta(r) \leq \Theta(\xi(z,Sz)) < \Theta(\rho)$$

a contradiction.
Case 2: If $\max\left\{\rho, a\xi(z,Sz), (1-a)\xi(z,Sz), \frac{\rho+r}{2}\right\} = a\xi(z,Sz)$, then we have two possibilities, $a = 0$ or $0 < a < 1$. Assume that $0 < a < 1$, from (3) and the fact that the function Θ is nondecreasing, we have

$$\Theta(\xi(z,Sz)) < \Theta(a\xi(z,Sz)) \leq \Theta(\xi(z,Sz))$$

a contradiction. If $a = 0$, then from (2), we get

$$\Theta(\xi(z,Sz)) < \Theta(0)$$

this inequality contradicts with the definition of Θ (as $\Theta : (0,\infty) \to (1,\infty)$).
Case 3: If $\max\left\{\rho, a\xi(z,Sz), (1-a)\xi(z,Sz), \frac{\rho+r}{2}\right\} = (1-a)\xi(z,Sz)$, then from (3) and the fact that the function Θ is nondecreasing, we have

$$\Theta(\xi(z,Sz)) < \Theta((1-a)\xi(z,Sz)) \leq \Theta(\xi(z,Sz)),$$

a contradiction.
Case 4: If $\max\left\{\rho, a\xi(z,Sz), (1-a)\xi(z,Sz), \frac{\rho+r}{2}\right\} = \frac{\rho+r}{2}$, then by the definition of r, the inequality (3) and the fact that the function Θ is nondecreasing, we have

$$\Theta(r) \leq \Theta(\xi(z,Sz)) < \Theta\left(\frac{\rho+r}{2}\right) \leq \Theta(r),$$

a contradiction. Therefore, $Sz = z$ for all $z \in D_{z_0,r}$. Consequently, $D_{z_0,r}$ is a fixed disc of S. □

By Theorems 1 and 2, and in view of Remark 1 , we have the following results:

Corollary 1. *Let (Y,ξ) be a metric space, $S : Y \to Y$ and $r = \inf\{\xi(z,Sz) : z \neq Sz\}$. If S is a Θ_c-contraction with $z_0 \in Y$ and $\xi(z_0,Sz) = r$, for all $z \in C_{z_0,r}(or D_{z_0,r})$, then $C_{z_0,r}(or D_{z_0,r})$ is a fixed circle (or disc) of S.*

Corollary 2. *Let (Y,ξ) be a metric space, $S : Y \to Y$ and $r = \inf\{\xi(z,Sz) : z \neq Sz\}$. If S is a Ćirić type Θ_c-contraction with $z_0 \in Y$ and $\xi(z_0,Sz) = r$, for all $z \in C_{z_0,r}(or D_{z_0,r})$, then $C_{z_0,r}(or D_{z_0,r})$ is a fixed circle (or disc) of S.*

The following example exhibit the utility of Theorems 1 and 2.

Example 3. *Let $Y = [-4,\infty)$ be a metric space endowed with the usual metric ξ. Define $S : Y \to Y$ as*

$$Sz = \begin{cases} z \;, & if \; -4 \leq z < 4 \\ z+3, & if \quad z \geq 4. \end{cases}$$

Then S is a Θ-weak contraction. To show this, let $\Theta(t) = e^t$, $z_0 = 0$, $a = \frac{1}{2}$ and $\lambda = \frac{6}{11}$.
Observe that for all $z \in [-4,\infty)$ such that $z \geq 4$, we have

$$\xi(z,Sz) = 3 > 0,$$

$$
\begin{aligned}
M(z,z_0) &= \max\left\{\xi(z,0), \frac{1}{2}\xi(z,Sz), \frac{1}{2}\xi(z,Sz), \frac{\xi(z,0)+\xi(0,Sz)}{2}\right\} \\
&= \max\left\{|z|, \frac{3}{2}, \frac{3}{2}, \frac{|z|+|z+3|}{2}\right\} \\
&= \frac{|z|+|z+3|}{2}
\end{aligned}
$$

and

$$
\begin{aligned}
\Theta(\xi(z,Sz)) = e^3 &\leq e^{\frac{3}{11}(|z|+|z+3|)} \\
&= [e^{\frac{|z|+|z+3|}{2}}]^{\frac{6}{11}} = [\Theta(M(z,z_0))]^{\lambda}.
\end{aligned}
$$

Also, we have

$$r = \inf\{\xi(z,Sz) : z \neq Sz\} = 3.$$

Therefore, all the conditions of Theorems 1 and 2 are satisfied. Observe that $C_{0,3} = \{-3,3\}$ is a fixed circle and $D_{0,3} = [-3,3]$ is a fixed disc of S.

Remark 2. *1. Taking $\Theta(t) = e^t$, $z_0 = 0$, and $k = \frac{3}{4}$ in Example 3, we can easily show that the mapping S is θ_c-contraction.*
2. Putting $\Theta(t) = e^t$, $z_0 = 0$, and $k = \frac{6}{11}$ in Example 3, we can easily show that the mapping S is Ćirić type Θ_c-contraction.

Next, we introduce the concepts of Reich type Θ_c-contractions, Chatterjea type Θ_c-contractions and Hardy-Rogers type Θ_c-contractions as follows:

Definition 8. *Let $S : Y \to Y$ and $\Theta \in \Omega_1$. The mapping S is said to be a Reich type Θ_c-contraction if there exist $\lambda \in (0,1)$ and $z_0 \in Y$ such that*

$$\xi(z,Sz) > 0 \;\Rightarrow\; \Theta(\xi(z,Sz)) \leq [\Theta(\alpha\xi(z,z_0) + \beta\xi(z,Sz) + \gamma\xi(z_0,Sz_0))]^{\lambda},\ \forall\, z \in Y,$$

where $\alpha + \beta + \gamma < 1$ and $\alpha, \beta, \gamma \geq 0$.

Definition 9. *Let $S : Y \to Y$ and $\Theta \in \Omega_1$. The mapping S is said to be a Chatterjea type Θ_c-contraction if there exist $\lambda \in (0,1)$ and $z_0 \in Y$ such that*

$$\xi(z,Sz) > 0 \;\Rightarrow\; \Theta(\xi(z,Sz)) \leq [\Theta(\eta(\xi(z,Sz_0) + \xi(z_0,Sz)))]^{\lambda},\ \forall\, z \in Y,$$

where $\eta \in (0, \frac{1}{2})$.

Definition 10. *Let $S : Y \to Y$ and $\Theta \in \Omega_1$. The mapping S is called a Hardy-Rogers type Θ_c-contraction if there exist $\lambda \in (0,1)$ and $z_0 \in Y$ such that*

$$\xi(z,Sz) > 0 \;\Rightarrow\; \Theta(\xi(z,Sz)) \leq [\Theta(M^*(z,z_0))]^{\lambda},\ \forall\, z \in Y,$$

where $M^*(z,z_0) = \alpha\xi(z,z_0) + \beta\xi(z,Sz) + \gamma\xi(z_0,Sz_0) + \delta\xi(z,Sz_0) + \eta\xi(z_0,Sz)$ with $\alpha+\beta+\gamma+\delta+\eta < 1$ and $\alpha, \beta, \gamma, \delta, \eta \geq 0$.

Remark 3. *(1) Taking $\alpha = \beta = \gamma = 0$ and $\delta = \eta$, then Definition 10 coincides with Definition 9.*
(2) Putting $\eta = \delta = 0$ in Definition 10, we obtain Definition 8.

The following proposition follows from Definition 10.

Proposition 2. *Let (Y, ξ) be a metric space. If $S : Y \to Y$ is a Hardy-Rogers type Θ_c-contraction with $z_0 \in Y$, then $Sz_0 = z_0$.*

Proof. Assume that $Sz_0 \neq z_0$. From Definition 10 and the fact that Θ is non-decreasing, we obtain

$$\begin{aligned}
\Theta(\xi(z_0, Sz_0)) &\leq [\Theta(M^*(z_0, z_0))]^\lambda \\
&< \Theta(M^*(z_0, z_0)) \\
&= \Theta(\alpha\xi(z_0, z_0) + \beta\xi(z_0, Sz_0) + \gamma\xi(z_0, Sz_0) + \delta\xi(z_0, Sz_0) + \eta\xi(z_0, Sz_0)) \\
&= \Theta((\beta + \gamma + \delta + \eta)\xi(z_0, Sz_0)) \leq \Theta(\xi(z_0, Sz_0)),
\end{aligned}$$

a contradiction. Therefore, we must have $Sz_0 = z_0$. □

Using a Hardy-Rogers type Θ_c-contraction, we present the following fixed circle results.

Theorem 3. *Let (Y, ξ) be a metric space, $S : Y \to Y$ a Hardy-Rogers type Θ_c-contraction with $z_0 \in Y$ and $r = \inf\{\xi(z, Sz) : z \neq Sz\}$. If $\xi(z_0, Sz) = r$ for all $z \in C_{z_0,r}$ then $C_{z_0,r}$ is a fixed circle of S.*

Proof. Let $z \in C_{z_0,r}$ and assume on contrary that $Sz \neq z$. By the definition of r, we have $\xi(z, Sz) \geq r$. Since S is Hardy-Rogers type Θ_c-contraction, by using Proposition 2 and the fact that Θ is non-decreasing, we have

$$\begin{aligned}
\Theta(\xi(z, Sz)) &\leq [\Theta(M^*(z, z_0))]^\lambda \\
&< \Theta(\alpha\xi(z, z_0) + \beta\xi(z, Sz) + \gamma\xi(z_0, Sz_0) + \delta\xi(z, Sz_0) + \eta\xi(z_0, Sz)) \\
&= \Theta(\alpha r + \beta\xi(z, Sz) + \delta r + \eta r) \\
&\leq \Theta((\alpha + \beta + \delta + \eta)\xi(z, Sz)) \leq \Theta(\xi(z, Sz)),
\end{aligned}$$

a contradiction. Therefore, $Sz = z$ for all $z \in C_{z_0,r}$. Consequently, $C_{z_0,r}$ is a fixed circle of S. □

Next, we prove the following fixed disc result as follows:

Theorem 4. *Let (Y, ξ) be a metric space, $S : Y \to Y$ a Hardy-Rogers type Θ_c-contraction with $z_0 \in Y$ and $r = \inf\{\xi(z, Sz) : z \neq Sz\}$. If $\xi(z_0, Sz) = r$ for all $x \in D_{z_0,r}$ then $D_{z_0,r}$ is a fixed disc of S.*

Proof. In view of Theorem 3, S fixes the circle $C_{z_0,r}$. Now, in order to show that $D_{z_0,r}$ is a fixed disc of the mapping S it is sufficient to show that S fixes any circle $C_{z_0,\rho}$ with $\rho < r$. Let $z \in C_{z_0,\rho}$ and assume that $\xi(z, Sz) > 0$. Since S is Hardy-Rogers type Θ_c-contraction, by using Proposition 2 and the fact that Θ is non-decreasing, we have

$$\begin{aligned}
\Theta(\xi(z, Sz)) &\leq [\Theta(M^*(z, z_0))]^\lambda \\
&< \Theta(\alpha\xi(z, z_0) + \beta\xi(z, Sz) + \gamma\xi(z_0, Sz_0) + \delta\xi(z, Sz_0) + \eta\xi(z_0, Sz)) \\
&= \Theta(\alpha\rho + \beta\xi(z, Sz) + \delta\rho + \eta\rho) \\
&\leq \Theta((\alpha + \beta + \delta + \eta)\xi(z, Sz)) \leq \Theta(\xi(z, Sz)),
\end{aligned}$$

a contradiction. Thus, we obtain $Sz = z$. So, $D_{z_0,\rho}$ is a fixed disc of S. □

By Theorem 3 and 4 and in view of Remark 3 , we deduce the following results.

Corollary 3. *Let (Y, ξ) be a metric space, $S : Y \to Y$ and $r = \inf\{\xi(z, Sz) : z \neq Sz\}$. If S is a Reich type Θ_c-contraction with $z_0 \in Y$ and $\xi(z_0, Sz) = r$ for all $z \in C_{z_0,r}$(or $D_{z_0,r}$), then $C_{z_0,r}$(or $D_{z_0,r}$) is a fixed circle (or disc) of S.*

Corollary 4. *Let (Y, ξ) be a metric space, $S : Y \to Y$ and $r = \inf\{\xi(z, Sz) : z \neq Sz\}$. If S is a Chatterjea type Θ_c-contraction with $z_0 \in Y$ and $\xi(z_0, Sz) = r$ for all $z \in C_{z_0,r}$(or $D_{z_0,r}$) then $C_{z_0,r}$(or $D_{z_0,r}$) is a fixed circle (or disc) of S.*

The following example exhibit the utility of Theorems 3 and 4.

Example 4. *Let $Y = \{3, 4, \ln(\frac{3}{e}), \ln(3), \ln(3e)\}$ be endowed with the usual metric ξ. Define $S : Y \to Y$ as*

$$Sz = \begin{cases} 4, & if\ z = 3 \\ z, & otherwise. \end{cases}$$

Let $\Theta(t) = e^t$, $z_0 = \ln 3$, $k = \frac{3}{10-3\ln 3}$, $\alpha = \delta = \eta = \frac{1}{3}$ and $\beta = 0$. Then S is a Hardy-Rogers type Θ_c-contraction. Indeed, for $z = 3$

$$\xi(z, Sz) = \xi(3, T3) = 1 > 0,$$

$$\begin{aligned} M(z, z_0) &= \alpha\xi(z, z_0) + \beta\xi(z, Sz) + \delta\xi(z, Sz_0) + \eta\xi(z_0, Sz) \\ &= \frac{1}{3}[\xi(3, \ln 3) + \xi(3, \ln 3) + \xi(\ln 3, 4)] \\ &= \frac{10}{3} - \ln 3 \end{aligned}$$

and

$$\begin{aligned} \Theta(\xi(z, Sz)) = \Theta(\xi(3, 4)) = e &\leq [e^{\frac{10}{3} - \ln 3}]^{\frac{3}{10-3\ln 3}} \\ &= [\Theta(M(z, z_0))]^{\lambda}. \end{aligned}$$

Also, we have

$$r = \inf\{\xi(z, Sz) : z \neq Sz\} = \{\xi(3, 4)\} = 1.$$

Hence, all the conditions of Theorems 3 and 4 are satisfied. Observe that S fixes the circle $C_{\ln 3,1} = \{\ln(\frac{3}{e}), \ln(3e)\}$ and the disc $D_{\ln 3,1} = \{\ln(\frac{3}{e}), \ln 3, \ln(3e)\}$.

Remark 4.
1. *Taking $\Theta(t) = e^t$, $z_0 = \ln 3$, $k = \frac{3}{4}$, $\alpha = \frac{3}{4}$ and $\beta = \frac{1}{5}$ in Example 4, we can easily show that the mapping S is Reich type Θ_c-contraction.*
2. *Puting $\Theta(t) = e^t$, $z_0 = \ln 3$, $k = \frac{3}{7-2\ln 3}$ and $\delta = \eta = \frac{1}{3}$ in Example 4, we can easily show that the mapping S is Chatterjea type Θ_c-contraction.*

We close this section by introducing the concept of Khan type Θ_c-contraction followed by related fixed circle (disc) results.

Definition 11. *Let $S : Y \to Y$ and $\Theta \in \Omega_1$. The mapping S is called Khan type Θ_c-contraction if there exist $\lambda \in (0, 1)$ and $z_0 \in Y$ such that for all $z \in Y$, if $\max\{\xi(Sz_0, z_0), \xi(Sz, z)\} \neq 0$, then*

$$\Theta(\xi(Sz, z)) \leq \left[\Theta\left(h\frac{\xi(Sz, z)\xi(Sz_0, z) + \xi(Sz_0, z_0)\xi(Sz, z_0)}{\max\{\xi(Sz_0, z_0), \xi(Sz, z)\}}\right)\right]^{\lambda},$$

where $h \in (0, \frac{1}{2})$ and if $\max\{\xi(Sz_0, z_0), \xi(Sz, z)\} = 0$, then $Sz = z$.

The following proposition is a direct consequence of Definition 11.

Proposition 3. *Let (Y, ξ) be a metric space. If $S : Y \to Y$ is Khan type Θ_c-contraction with $z_0 \in Y$, then $Sz_0 = z_0$.*

Proof. Assume that $Sz_0 \neq z_0$, then $\max\{\xi(Sz_0, z_0), \xi(Sz, z)\} \neq 0$. As S is Khan type Θ_c-contraction, we have

$$\begin{aligned}\Theta(\xi(Sz_0, z_0)) &\leq \left[\Theta\left(h\frac{\xi(Sz_0, z_0)\xi(Sz_0, z_0) + \xi(Sz_0, z_0)\xi(Sz_0, z_0)}{\xi(Sz_0, z_0)}\right)\right]^{\lambda} \\ &= [\Theta(2h\xi(Sz_0, z_0))]^{\lambda} \\ &< \Theta(2h\xi(Sz_0, z_0)) \leq \Theta(\xi(Sz_0, z_0)),\end{aligned}$$

a contradiction. Therefore, we must have $Sz_0 = z_0$. □

Now, utilizing the definition of Khan type Θ_c-contraction, we prove the following fixed circle and fixed disc results.

Theorem 5. *Let (Y, ξ) be a metric space, $S : Y \to Y$ a Khan type Θ_c-contraction with $z_0 \in Y$ and $r = \inf\{\xi(z, Sz) : z \neq Sz\}$. Then $C_{z_0,r}$ is a fixed circle of S.*

Proof. Let $z \in C_{z_0,r}$ and assume on contrary that $Sz \neq z$, then $\max\{\xi(Sz_0, z_0), \xi(Sz, z)\} \neq 0$. From the definition of r, we have $\xi(z, Sz) \geq r$. As S is Khan type Θ_c-contraction, by using Proposition 3 and the fact that Θ is non-decreasing, we have

$$\begin{aligned}\Theta(\xi(Sz, z)) &\leq \left[\Theta\left(h\frac{\xi(Sz, z)\xi(Sz_0, x) + \xi(Sz_0, z_0)\xi(Sz, z_0)}{\xi(Sz, z)}\right)\right]^{\lambda} \\ &= [\Theta(hr)]^{\lambda} < \Theta(hr) \leq \Theta(h\xi(Sz, z)) \leq \Theta(\xi(Sz, z)),\end{aligned}$$

a contradiction. Therefore, $Sz = z$ for all $z \in C_{z_0,r}$. Consequently, $C_{z_0,r}$ is a fixed circle of S. □

Theorem 6. *Let (Y, ξ) be a metric space, $S : Y \to Y$ a Khan type Θ_c-contraction with $z_0 \in Y$ and $r = \inf\{\xi(z, Sz) : z \neq Sz\}$. Then $D_{z_0,r}$ is a fixed disc of S.*

Proof. In view of Theorem 5, S fixes the circle $C_{z_0,r}$. Now, in order to show that $D_{z_0,r}$ is a fixed disc of the mapping S it is sufficient to show that S fixes any circle $C_{z_0,\rho}$ with $\rho < r$. Let $z \in C_{z_0,\rho}$ and assume that $\xi(z, Sz) > 0$. By the Khan type Θ_c-contractive condition, we have

$$\Theta(\xi(z, Sz)) \leq [\Theta(h\rho)]^{\lambda} < \Theta(h\rho) \leq \Theta(\rho).$$

As Θ is non-decreasing function, we get

$$\xi(z, Sz) < \rho < r,$$

a contradiction (as $r \leq \xi(z, Sz)$). Thus, we obtain $Sz = z$ for all $z \in C_{z_0,\rho}$ with $\rho < r$. Therefore, $D_{z_0,r}$ is a fixed disc of S. □

The following example shows the utility of Theorems 5 and 6.

Example 5. *Let $X = \mathbb{R}$ be endowed with the usual metric ξ. Define $S : \mathbb{R} \to \mathbb{R}$ by*

$$Sz = \begin{cases} z \;, & if\ |z| < 4.5, \\ z + 1, & if\ |z| \geq 4.5. \end{cases}$$

Then S is Khan type Θ-contraction with $\Theta(t) = e^t$, $z_0 = 0$, $k = \frac{2}{3}$ and $h = \frac{1}{3}$. In fact

$$\max\{\xi(Sz_0, z_0), \xi(Sz, z)\} = 1 \neq 0,$$

for all $z \in \mathbb{R}$ such that $|z| \geq 4.5$. Now, we have

$$h\frac{\xi(Sz,z)\xi(Sz_0,x) + \xi(Sz_0,z_0)\xi(Sz,z_0)}{\xi(Sz,z)} = h\xi(0,z) = \frac{1}{3}|z|$$

and

$$\begin{aligned}\Theta(\xi(Sz,z)) &= e \leq e^{\frac{2}{9}|z|} \\ &= [e^{\frac{1}{3}|z|}]^{\frac{2}{3}} = \left[\Theta(h\frac{\xi(Sz,z)\xi(Sz_0,z) + \xi(Sz_0,z_0)\xi(Sz,z_0)}{\xi(Sz,z)})\right]^{\lambda}.\end{aligned}$$

Also, we have

$$r = \inf\{\xi(z, Sz) : z \neq Sz\} = 1.$$

Therefore, all the conditions of Theorems 5 and 6 are satisfied. Observe that S fixes the circle $C_{0,1} = \{-1, 1\}$ and the disc $D_{0,1} = [-1, 1]$.

3. Fixed circle (disc) results of integral type

In this section, we establish some fixed circle and disc results of integral type.

Let $\varphi : [0, \infty) \to [0, \infty)$ be a locally integrable function such that, for each $t > 0$

$$\int_0^t \varphi(s)ds > 0. \tag{4}$$

Definition 12. *Let $S : Y \to Y$ and $\Theta \in \Omega_1$. The mapping S is called an integral type Θ_c-contraction if there exist $\lambda \in (0, 1)$ and $z_0 \in Y$ such that, for all $z \in Y$*

$$\xi(z, Sz) > 0 \quad \Rightarrow \quad \int_0^{\Theta(\xi(z,Sz))} \varphi(t)dt \leq \int_0^{[\Theta(\xi(z,z_0))]^{\lambda}} \varphi(t)dt.$$

where $\varphi : [0, \infty) \to [0, \infty)$ be a function defined as in (4).

The following proposition is useful in the proof of the main results of this section.

Proposition 4. *Let (Y, ξ) be a metric space and $S : Y \to Y$. If S is an integral type Θ_c-contraction with $z_0 \in Y$, then $Sz_0 = z_0$.*

Proof. Assume that $Sz_0 \neq z_0$. From Definition 12, we have

$$\int_0^{\Theta(\xi(z_0,Sz_0))} \varphi(t)dt \leq \int_0^{[\Theta(\xi(z_0,z_0))]^{\lambda}} \varphi(t)dt,$$

which contradicts the definition of Θ, as Θ:$(0, \infty) \to (1, \infty)$ and $\xi(z_0, z_0) = 0$. Hence, we must have $Sz_0 = z_0$. $\square$

In the following theorem we present fixed circle result for integral type Θ_c-contraction.

Theorem 7. *Let (Y, ξ) be a metric space and $S : Y \to Y$. If S is an integral type Θ_c-contraction with $z_0 \in Y$ and $r = \inf\{\xi(z, Sz) : z \neq Sz\}$. Then $C_{z_0,r}$ is a fixed circle of S.*

Proof. Assume that $z \neq Sz$ for some $z \in C_{z_0,r}$. Making use of the definition of r, we have

$$r \leq \xi(z, Sz).$$

Since Θ is non-decreasing function, we have

$$\Theta(r) \leq \Theta(\xi(z, Sz))$$

and

$$\int_0^{\Theta(r)} \varphi(t)dt \leq \int_0^{\Theta(\xi(z,Sz))} \varphi(t)dt. \tag{5}$$

As S is integral type Θ_c-contraction, by using (5), we obtain

$$\begin{aligned} \int_0^{\Theta(r)} \varphi(t)dt \leq \int_0^{\Theta(\xi(z,Sz))} \varphi(t)dt &\leq \int_0^{[\Theta(\xi(z,z_0))]^\lambda} \varphi(t)dt \\ &< \int_0^{\Theta(\xi(z,z_0))} \varphi(t)dt = \int_0^{\Theta(r)} \varphi(t)dt, \end{aligned}$$

a contradiction. Therefore, we have $Sz = z$. Consequently, $C_{z_0,r}$ is a fixed circle of S. □

Next, we prove the following fixed disc results.

Theorem 8. *Let (Y, ξ) be a metric space, $\varphi : [0, \infty) \to [0, \infty)$ be defined as in (4), S an integral type Θ_c-contraction with $z_0 \in Y$ and $r = \inf\{\xi(z, Sz) : x \neq Sz\}$. Then $D_{z_0,r}$ is a fixed disc of S.*

Proof. In view of Theorem 7, S fixes the circle $C_{z_0,r}$. Now, in order to show that $D_{z_0,r}$ is a fixed disc of the mapping S it is sufficient to show that S fixes any circle $C_{z_0,\rho}$ with $\rho < r$. Let $z \in C_{z_0,\rho}$ and assume that $z \neq Sz$. Making use of the definition of r, we have

$$\rho < r \leq \xi(z, Sz).$$

Since Θ is non-decreasing function, we have

$$\Theta(\rho) \leq \Theta(\xi(z, Sz))$$

and

$$\int_0^{\Theta(\rho)} \varphi(t)dt \leq \int_0^{\Theta(\xi(z,Sz))} \varphi(t)dt. \tag{6}$$

As S is integral type Θ_c-contraction, by using (6), we obtain

$$\begin{aligned} \int_0^{\Theta(\rho)} \varphi(t)dt \leq \int_0^{\Theta(\xi(z,Sz))} \varphi(t)dt &\leq \int_0^{[\Theta(\xi(z,z_0))]^\lambda} \varphi(t)dt \\ &< \int_0^{\Theta(\xi(z,z_0))} \varphi(t)dt = \int_0^{\Theta(\rho)} \varphi(t)dt, \end{aligned}$$

a contradiction. Therefore, we must have $Sz = z$, $\forall\, z \in C_{z_0,\rho}$. Consequently, $D_{z_0,r}$ is a fixed disc of S. □

Remark 5. *Using similar arguments as in Definition 12, we can define the notions of an integral Ćirić type Θ_c-contraction mapping, an integral type Θ_c-weak contraction mapping, an integral Hardy-Rogers type Θ_c-contraction mapping, an integral Reich type Θ_c-contraction mapping, an integral Chatterjea type Θ_c-contraction mapping and an integral Khan type Θ_c-contraction and obtain corresponding fixed circle and fixed disc theorems.*

4. Conclusion

In closing, we would like to bring to the readers' attention the following open questions:

Question 1 Under what extra condition we have the uniqueness of a fixed circle/disc?

Question 2 Can we prove the same results in partial metric spaces?

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