Local Sharp Vector Variational Type Inequality and Optimization Problems

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Abstract

In this paper, our goal is to establish the relationship between solutions of local sharp vector variational type inequality and sharp efficient solutions of vector optimization problems, also Minty local sharp vector variational type inequality and sharp efficient solutions of vector optimization problems, under generalized approximate η-convexity conditions for locally Lipschitzian functions.

Keywords: Vector variational type inequality problems, vector optimization problems, efficient solutions, approximate η-convexity, Lipschitzian functions.

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1 Introduction

The study of variational inequality problems is a part of development in the theory of optimization theory because optimization problems can often be reduced to the solution of variational inequality problems. It is very important to point out that these theory pertain to more than just optimization problems and there in lies much of their attractiveness. Several authors have presented many fascinating results on variational inequality problems; *see*, cited references here, ([1], [2], [3], [5], [6], [8],[9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [22], [23], [24]).

Loridan [20] studied the concept of epsilon efficient solutions. Later, White [25] extended ϵ-optimality for vector maximization problems. Burke et al. [4] introduced the notion of weak sharp minima for scalar optimization problems. Recently, Zhu [26] suggested necessary optimality conditions for the local weak sharp efficient solutions.

The aim of this paper is to formulate sharp vector variational type inequality problems and establish relations between sharp vector variational type inequality and vector optimization problems involving locally Lipschitzian functions.

2 Preliminaries

Throughout this paper, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with a norm $\|\cdot\|$. Let X be a nonempty convex subset of $\mathbb{R}^n$. The distance function $d(\cdot, X) : X \longrightarrow \mathbb{R}$ is defined by

$$d(x, X) = \inf_{x_0 \in X} \|x - x_0\|, \quad \forall x \in X.$$

A vector valued function $\eta : X \times X \longrightarrow X$ is said to be τ-Lipschitz continuous if there exists a number $\tau > 0$ such that

$$\|\eta(x, y)\| \leq \tau \|x - y\|, \ \forall x, y \in X.$$

Definition 2.1. Let $\eta : X \times X \longrightarrow X$ be a vector valued function. A function $\varphi : X \longrightarrow \mathbb{R}$ is said to be

(i) η-monotone, if for all $x, y \in X$ such that

$$\langle \varphi(x) - \varphi(y), \eta(x, y)\rangle \geq 0$$

and φ is strictly η-monotone, if equality holds for $x = y$;

(ii) strongly η-monotone, if there exists a constant $\zeta > 0$ such that

$$\langle \varphi(x) - \varphi(y), \eta(x, y)\rangle \geq \zeta\|x - y\|^2, \forall x, y \in X;$$

(iii) η-pseudomonotone, if

$$\langle \varphi(x), \eta(x, y)\rangle \geq 0$$

implies that

$$\langle \varphi(y), \eta(x, y)\rangle \geq 0, \forall x, y \in X;$$

(iv) strongly η-pseudomonotone, if

$$\langle \varphi(x), \eta(x, y)\rangle \geq 0$$

implies that

$$\langle \varphi(y), \eta(x, y)\rangle \geq \zeta\|x - y\|^2, \forall x, y \in X \text{ and } \zeta > 0;$$

(v) Lipschitz near $x_0 \in X$ if there exists a positive constant ϱ and $\delta > 0$, such that for all $x, y \in B(x_0, \delta)$, we have

$$\|\varphi(x) - \varphi(y)\| \leq \varrho\|x - y\|.$$

The function φ is locally Lipschitz on X, if it is Lipschitz near x_0, for every $x_0 \in X$.

Definition 2.2. Let $\eta : X \times X \longrightarrow X$ be a function and $\varphi : X \longrightarrow R \cup \{+\infty\}$ be a proper functional. Then φ is said to be η-subdifferentiable at a point $x \in X$ if there exists a point $f^* \in X$ such that

$$\varphi(y) \geq \varphi(x) + \langle f^*, \eta(y, x)\rangle, \forall y \in X.$$

Then f^* is called η-subdifferential of φ at x. The set of all η-subdifferential of φ at x is denoted by $\nabla\varphi(x)$. That is, the mapping $\nabla\varphi : X \longrightarrow 2^X$ is defined by

$$\nabla\varphi(x) = \{f^* \in X : \varphi(y) - \varphi(x) \geq \langle f^*, \eta(y, x)\rangle, \forall y \in X\}.$$

Definition 2.3. Let X be a real Banach space and $K \subset X$. Let $\eta : K \times K :\longrightarrow X$ be a vector valued function and $\varphi : K \longrightarrow \mathbb{R}$ be a differentiable function. Then φ is said to be

(i) η-invex on K, if

$$\varphi(x) - \varphi(x') - \langle \nabla\varphi(x'), \eta(x, x')\rangle \geq 0, \ \forall x, x' \in K;$$

(ii) η-invex at point $x' \in K$, if

$$\varphi(x) - \varphi(x') - \langle \nabla\varphi(x'), \eta(x, x')\rangle \geq 0, \ \forall x \in K;$$

(iii) η-incave on K, if

$$\varphi(x) - \varphi(x') - \langle \nabla\varphi(x'), \eta(x, x')\rangle \leq 0, \ \forall x, x' \in K;$$

(iv) η-incave at point $x' \in K$, if

$$\varphi(x) - \varphi(x') - \langle \nabla\varphi(x'), \eta(x, x')\rangle \leq 0, \ \forall x \in K.$$

Let $\varphi : X \longrightarrow \mathbb{R}$ be a locally Lipschitz at $x_0 \in X$. The Clarke directional derivative of φ at x_0 in the direction of $v \in X$, denoted by $\varphi^0(x_0, v)$, is defined by

$$\varphi^0(x_0, v) = \lim_{\substack{\lambda \downarrow 0 \\ x \longrightarrow x_0}} \sup \frac{\varphi(x + \lambda v) - \varphi(x)}{\lambda},$$

and the Clarke subdifferential of φ at $x_0 \in X$ [7], denoted by $\partial\varphi(x_0)$, is defined by

$$\partial\varphi(x_0) = \{x_0^* \in X : \varphi^0(x_0, v) \geq \langle x_0^*, v\rangle, \ \forall v \in X\},$$

see, [7]

Let φ be a Lipschitz near each point of an open convex subset $\mathcal{U}$ of X. Then φ is convex on $\mathcal{U}$ if and only if $\partial\varphi$ is η-monotone on $\mathcal{U}$, if and only if

$$\langle \zeta - \zeta', \eta(x, x')\rangle \geq 0, \forall x, x' \in \mathcal{U}, \zeta \in \partial\varphi(x), \ \zeta' \in \partial\varphi(x').$$

Definition 2.4. Let $\eta : X \times X \longrightarrow X$ be a function. A lower semicontinuous function $\varphi : X \longrightarrow \mathbb{R}$ is said to be approximate η-convex at $x_0 \in X$ if for any $\tau > 0$, there exists $\delta > 0$, such that, for all $x, y \in B(x_0, \delta) \cap X$,

$$\varphi(y) \geq \varphi(x) + \langle x^\star, \eta(y,x)\rangle - \tau\|y - x\|, \forall x^* \in \partial\varphi(x).$$

Definition 2.5. Let $\eta : X \times X \longrightarrow X$ be a function. A function $\varphi : X \longrightarrow \mathbb{R}$ is said to be

(i) approximate η-pseudoconvex type-I at $x_0 \in X$ if for any $\tau > 0$, there exists $\delta > 0$, such that, whenever $x, y \in B(x_0, \delta) \cap X$ and

$$\langle x^*, \eta(y,x)\rangle \geq 0, \text{ for some } x^* \in \partial\varphi(x),$$

then

$$\varphi(y) - \varphi(x) \geq -\tau\|y - x\|;$$

(ii) approximate η-pseudoconvex type–II (strictly approximate η-pseudoconvex type–II) at $x_0 \in X$ if for any $\tau > 0$, there exists $\delta > 0$, such that, whenever $x, y \in B(x_0, \delta) \cap X$ and

$$\langle x^*, \eta(y,x)\rangle + \tau\|y - x\| \geq 0, \text{ for some } x^* \in \partial\varphi(x),$$

then

$$\varphi(y) \geq (>) \varphi(x);$$

(iii) approximate η-quasiconvex type-I at $x_0 \in X$ if for any $\tau > 0$, there exists $\delta > 0$, such that, whenever $x, y \in B(x_0, \delta) \cap X$ and

$$\varphi(y) \leq \varphi(x),$$

then

$$\langle x^*, \eta(y,x)\rangle - \tau\|y - x\| \leq 0, \ \forall x^* \in \partial\varphi(x);$$

(iv) approximate η-quasiconvex type-II (strictly approximate η-quasiconvex type–II) at $x_0 \in X$ if for any $\tau > 0$, there exists $\delta > 0$, such that, whenever $x, y \in B(x_0, \delta) \cap X$ and

$$\varphi(y) \leq (<) \varphi(x) + \tau\|y - x\|,$$

then

$$\langle x^*, \eta(y,x)\rangle \leq 0, \ \forall x^* \in \partial\varphi(x).$$

A vector optimization problem (VOP) may be formulated as follows:

$$(VOP) \qquad \text{Min } f(x) = (f_1(x), \cdots, f_p(x)),$$

$$\text{Subject to } \; x \in X \subset \mathbb{R}^n.$$

Definition 2.6. [26]

(i) A vector $x_0 \in X$ is said to be local sharp efficient solution of (VOP), if for any $\tau > 0$ there exists a δ-neighborhood of x_0, such that for all $x \in B(x_0, \delta) \cap X$,

$$\max_{1 \leq i \leq p} \left\{ f_i(x) - f_i(x_0) \right\} \geq \tau \|x - x_0\|;$$

(ii) A vector $x_0 \in X$ is said to be weak local sharp efficient solution of (VOP), if for any $\tau > 0$, there exists a δ-neighborhood of x_0, such that for all $x \in B(x_0, \delta) \cap X$,

$$\max_{1 \leq i \leq p} \left\{ f_i(x) - f_i(x_0) \right\} \geq \tau d(x, \bar{X}),$$

where

$$\bar{X} = \left\{ x \in X \mid f(x) = f(x_0) \right\} = X \cap f^{-1}(f(x_0)).$$

3 Local Sharp Vector Variational Type Inequalities

In this section, we consider local sharp and local weak sharp formulations of vector variational type inequality problems as follows:

(LSVVTI): For finding $x_0 \in X$, there exists a δ-neighborhood of x_0 and for any $\tau > 0$, such that $x \in B(x_0, \delta) \cap X$ and

$$\max_{1 \leq i \leq p} \max_{x^*_{0_i} \in \partial f_i(x_0)} \langle x^*_{0_i}, \eta(x, x_0) \rangle \geq \tau \|x - x_0\|, \; \forall x^*_{0_i} \in \varphi f_i(x_0). \tag{3.1}$$

(WLSVVTI): For finding $x_0 \in X$, there exists a δ-neighborhood of x_0 and for any $\tau > 0$, such that $x \in B(x_0, \delta) \cap X$ and

$$\max_{1 \leq i \leq p} \max_{x^*_{0_i} \in \partial f_i(x_0)} \langle x^*_{0_i}, \eta(x, x_0) \rangle \geq \tau d(x, \bar{X}), \forall x^*_{0_i} \in \partial f_i(x_0), \tag{3.2}$$

where

$$\bar{X} = \{x \in X \mid f(x) = f(x_0)\} = X \cap f^{-1}(f(x_0)).$$

We note that, if x_0 is a solution of (LSVVTI), then x_0 is also a solution of (WLSVVTI).

Special Cases:

1. Assume that, if $\eta(x, x_0) = x - x_0$, then (3.1) reduces to local sharp vector variational inequalities (*LSVVI*) for finding $x_0 \in X$, there exists a δ-neighborhood of x_0 and for any $\tau > 0$, such that $x \in B(x_0, \delta) \cap X$ and

$$\max_{1 \leq i \leq p} \max_{x^*_{0_i} \in \partial f_i(x_0)} \langle x^*_{0_i}, x - x_0 \rangle \geq \tau \|x - x_0\|, \forall x^*_{0_i} \in \varphi f_i(x_0). \tag{3.3}$$

2. Also, (3.2) reduces to weak local sharp vector variational inequalities (*WLSVVI*) for finding $x_0 \in X$, there exists a δ-neighborhood of x_0 and for any $\tau > 0$, such that $x \in B(x_0, \delta) \cap X$ and

$$\max_{1 \leq i \leq p} \max_{x^*_{0_i} \in \partial f_i(x_0)} \langle x^*_{0_i}, x - x_0 \rangle \geq \tau d(x, \bar{X}), \ \forall x^*_{0_i} \in \partial f_i(x_0), \tag{3.4}$$

where

$$\bar{X} = \{x \in X \mid f(x) = f(x_0)\} = X \cap f^{-1}(f(x_0)).$$

3. Again, we note that if $\eta(x, x_0) = x - x_0$, then solution of (LSVVI) is also a solution of $(\text{AVVI})_1$ (defined by [21]), but the converse need not be true,
e.g., consider the function

$$f(x) = (f_1(x), f_2(x)), x \in \mathbb{R},$$

where $f_1(x) =\mid x \mid -x^2$ and $f_2(x) = -x^2$.

If we take $x_0 = 0$, then for any $\tau > 0$, there does not exist any $\delta > 0$ such that

$$\langle x^*_{0_i}, x - x_0 \rangle \leq \tau \|x - x_0\|, \forall i \in \{1, 2\}, x^*_{0_i} \in \partial f_i(x_0), x \in B(x_0, \delta) \cap \mathbb{R},$$

that is, x_0 is a solution of $(\text{AVVI})_1$. But when $x < 0$, then for every $\delta > 0$ and $\tau > 0$, we do not have

$$\max_{1 \leq i \leq p} \max_{x^*_{0_i} \in \partial f_i(x_0)} \langle x^*_{0_i}, x - x_0 \rangle \geq \tau \|x - x_0\|,$$

that is, x_0 is not a solution of (LSVVI).

Theorem 3.1. Let $\eta : X \times X \longrightarrow X$ be a function and $f_i : X \longrightarrow \mathbb{R},\ i = 1, \cdots, p$, be locally Lipschitz and approximate η-convex at $x_0 \in X$, and $\langle f_i(x), \eta(x, x)\rangle = 0$ for all $x \in X$. If x_0 solves (LSVVTI), then x_0 is a local sharp efficient solution of (VOP).

Proof. Contrary assume that $x_0 \in X$ is not a local sharp efficient solution of (VOP). Then, for any $\delta_0 > 0$ and for any $\frac{\tau}{2} > 0$, there exists $x \in B(x_0, \delta_0) \cap X$, such that

$$\max_{1\le i\le p} \big\{ f_i(x) - f_i(x_0) \big\} < \frac{\tau}{2}\|x - x_0\|,$$

that is,

$$f_i(x) - f_i(x_0) < \frac{\tau}{2}\|x - x_0\|. \tag{3.5}$$

Since f_i is approximate η-convex at $x_0 \in X$, it follows that, for any $\frac{\tau}{2} > 0$, there exists $\bar{\delta}_i > 0$ such that by setting $\delta = \min\{\delta_0, \bar{\delta}_i : i = 1, \cdots, p\}$, we have

$$f_i(x) \ge f_i(x_0) + \langle x^*_{0_i}, \eta(x, x_0)\rangle - \frac{\tau}{2}\|x - x_0\|,\ \forall x \in B(x_0, \delta) \cap X \ \text{ and } x^*_{0_i} \in \partial f_i(x_0). \tag{3.6}$$

From (3.5) and (3.6), we have

$$\frac{\tau}{2}\|x - x_0\| > \langle x^*_{0_i}, \eta(x, x_0)\rangle - \frac{\tau}{2}\|x - x_0\|,$$

that is,

$$\tau\|x - x_0\| > \langle x^*_{0_i}, \eta(x, x_0)\rangle,$$

implies that

$$\max_{1\le i\le p} \max_{x^*_{0_i}\in\partial f_i(x_0)} \langle x^*_{0_i}, \eta(x, x_0)\rangle < \tau\|x - x_0\|,\ \forall x \in B(x_0, \delta) \cap X \text{ and } x^*_{0_i} \in \partial f_i(x_0),$$

a contradiction to the fact that x_0 solves (LSVVTI). □

Theorem 3.2. Let $\eta : X \times X \longrightarrow X$ be a function, $f_i : X \longrightarrow \mathbb{R}, i = 1, \cdots, p$, be locally Lipschitz, $-f_i$ be approximate η-convex at $x_0 \in X$ and $\langle f_i(x), \eta(x, x)\rangle = 0$ for all $x \in X$. If x_0 is a local sharp efficient solution of (VOP), then x_0 solves (LSVVTI).

Proof. Suppose that $x_0 \in X$ is not a solution of the (LSVVTI). Then, for any $\delta_0 > 0$ and any $\frac{\tau}{2} > 0$, there exists $x \in B(x_0, \delta_0) \cap X$ and $x^*_{0_i} \in \partial f_i(x_0)$, such that

$$\max_{1\le i\le p} \max_{x^*_{0_i}\in\partial f_i(x_0)} \langle x^*_{0_i}, \eta(x, x_0)\rangle < \frac{\tau}{2}\|x - x_0\|,$$

that is,

$$\langle x^*_{0_i}, \eta(x, x_0)\rangle < \frac{\tau}{2}\|x - x_0\|. \tag{3.7}$$

Since $-f_i$ is approximate η-convex at $x_0 \in X$, it follows that, for any $\frac{\tau}{2} > 0$, there exists $\bar{\delta}_i > 0$, such that by setting $\delta = \min\left\{\delta_0, \bar{\delta}_i : i = 1, \cdots p\right\}$, we have

$$-f_i(x) \geq -f_i(x_0) + \langle x^*_{0_i}, \eta(x, x_0)\rangle - \frac{\tau}{2}\|x - x_0\|, \forall x \in B(x_0, \delta) \cap X \text{ and } x^*_{0_i} \in -\partial f_i(x_0),$$

we can write it as

$$\langle x^*_{0_i}, \eta(x, x_0)\rangle \geq f_i(x) - f_i(x_0) - \frac{\tau}{2}\|x - x_0\|. \tag{3.8}$$

From (3.7) and (3.8), we have

$$\frac{\tau}{2}\|x - x_0\| > f_i(x) - f_i(x_0) - \frac{\tau}{2}\|x - x_0\|, \ \forall x \in B(x_0, \delta) \cap X,$$

that is,

$$f_i(x) - f_i(x_0) < \tau\|x - x_0\|,$$

implies that

$$\max_{1 \leq i \leq p} \left\{f_i(x) - f_i(x_0) < \tau\|x - x_0\|, \ \forall x \in B(x_0, \delta) \cap X,\right.$$

a contradiction to the fact that x_0 is a local sharp efficient solution of (VOP). □

Theorem 3.3. Let $\eta : X \times X \longrightarrow X$ be a function, $f_i : X \longrightarrow \mathbb{R}, i = 1, \cdots p$, be a locally Lipschitz and strictly approximate η-quasiconvex type-II at $x_0 \in X$ and $\langle f_i(x), \eta(x, x)\rangle = 0$ for all $x \in X$. If x_0 solves (LSVVTI), then x_0 is a local sharp efficient solution of (VOP).

Proof. Contrary assume that $x_0 \in X$ is not a local sharp efficient solution of (VOP). Then, for any $\delta_0 > 0$ and any $\tau > 0$, there exists $x \in B(x_0, \delta_0) \cap X$, such that

$$\max_{1 \leq i \leq p} \left\{f_i(x) - f_i(x_0)\right\} < \tau\|x - x_0\|,$$

that is,

$$f_i(x) - f_i(x_0) < \tau\|x - x_0\|,$$

Since f_i is a strictly approximate η-quasiconvex type-II at $x_0 \in X$, it follows that, for any $\tau > 0$, there exists $\bar{\delta}_i > 0$, such that by setting $\delta = \min\{\delta_0, \bar{\delta}_i : i = 1, \cdots, p\}$, we have

$$\langle x^*_{0_i}, \eta(x, x_0)\rangle \leq 0 < \tau\|x - x_0\|, \ \forall x \in B(x_0, \delta) \cap X \ \text{ and } x^*_{0_i} \in \partial f_i(x_0),$$

implies that

$$\max_{1\le i\le p}\max_{x^*_{0_i}\in\partial f_i(x_0)}\langle x^*_{0_i},\eta(x,x_0)\rangle<\tau\|x-x_0\|,\forall x\in B(x_0,\delta)\cap X \text{ and } x^*_{0_i}\in\partial f_i(x_0),$$

which is a contradiction to the fact that x_0 solves (LSVVTI). □

Theorem 3.4. Let $\eta : X\times X\longrightarrow X$ be a function, $f_i : X\longrightarrow\mathbb{R}, i=1,\cdots,p$, be locally Lipschitz, $-f_i$ be a strictly approximate η-pseudoconvex type-II at $x_0\in X$ and $\langle f_i(x),\eta(x,x)\rangle=0$ for all $x\in X$. If x_0 is a local weak sharp efficient solution of (VOP), then x_0 solves (LSVVTI).

Proof. Suppose that $x_0\in X$ is not a solution of the (LSVVTI). Then, for any $\delta_0>0$ and any $\tau>0$, there exists $x\in B(x_0,\delta_0)\cap X$ and $x^*_{0_i}\in\partial f_i(x_0)$, such that

$$\max_{1\le i\le p}\max_{x^*_{0_i}\in\partial f_i(x_0)}\langle x^*_{0_i},\eta(x,x_0)\rangle<\tau\|x-x_0\|,$$

that is,

$$\langle x^*_{0_i},\eta(x,x_0)\rangle<\tau\|x-x_0\|,$$

we can rewrite as

$$\langle -x^*_{0_i},\eta(x,x_0)\rangle+\tau\|x-x_0\|>0,$$

Since $-f_i$ is a strictly approximate η-pseudoconvex type-II at $x_0\in X$, it follows that, for any $\tau>0$, there exists $\bar{\delta}_i>0$ such that, by setting $\delta=\min\{\delta_0,\bar{\delta}_i : i=1,\cdots,p\}$, we have

$$-f_i(x)>-f_i(x_0),\forall x\in B(x_0,\delta)\cap X,$$

that is,

$$f_i(x)-f_i(x_0)<0\le\tau d(x,\bar{X}),$$

implies that

$$\max_{1\le i\le p}\left\{f_i(x)-f_i(x_0)\right\}<\tau d(x,\bar{X}),\ \forall x\in B(x_0,\delta)\cap X,$$

which is a contradiction to the fact that x_0 is a local weak sharp efficient solution of (VOP). □

4 Minty Local Sharp Vector Variational Type Inequalities

In this section, we present relationship between the solutions of Minty local sharp vector variational type inequalities and local sharp efficient solutions of vector optimization problem (VOP).

(MLSVVTI): Finding $x_0 \in X$, there exists a δ-neighborhood of x_0 and any $\tau > 0$, such that $x \in B(x_0, \delta) \cap X$ and

$$\max_{1\leq i\leq p} \max_{x_i^*\in\partial f_i(x)} \langle x_i^*, \eta(x, x_0)\rangle \geq \tau\|x - x_0\|, \quad \forall x_i^* \in \partial f_i(x). \tag{4.1}$$

(MWLSVVTI): For finding $x_0 \in X$, there exists a δ-neighborhood of x_0 and any $\tau > 0$, such that $x \in B(x_0, \delta) \cap X$ and

$$\max_{1\leq i\leq p} \max_{x_i^*\in\partial f_i(x)} \langle x_i^*, \eta(x, x_0)\rangle \geq \tau d(x, \bar{X}), \quad \forall x_i^* \in \partial f_i(x), \tag{4.2}$$

where $\bar{X} = \{x \in X \mid f(x) = f(x_0)\} = X \cap f^{-1}(f(x_0))$.

Theorem 4.1. Let $\eta : X \times X \longrightarrow X$ be a function, $f_i : X \longrightarrow \mathbb{R}, i = 1, \cdots, p$, be locally Lipschitz, $-f_i$ be approximate η-convex at $x_0 \in X$ and $\langle f_i(x), \eta(x, x)\rangle = 0$ for all $x \in X$. If x_0 solves (MLSVVTI), then x_0 is a local sharp efficient solution of (VOP).

Proof. Suppose that $x_0 \in X$ is not a local sharp efficient solution of (VOP). Then, for any $\delta_0 > 0$ and any $\frac{\tau}{2} > 0$, there exists $x \in B(x_0, \delta_0) \cap X$, such that

$$\max_{1\leq i\leq p} \{f_i(x) - f_i(x_0)\} < \frac{\tau}{2}\|x - x_0\|,$$

that is,

$$f_i(x) - f_i(x_0) < \frac{\tau}{2}\|x - x_0\|. \tag{4.3}$$

Since $-f_i$ is approximate η-convex at $x_0 \in X$, it follows that, for any $\frac{\tau}{2} > 0$, there exists $\bar{\delta}_i > 0$, such that by setting $\delta = \min\{\delta_0, \bar{\delta}_i : i = 1, \cdots, p\}$, we have

$$-f_i(x_0) \geq -f_i(x) + \langle -x_i^*, \eta(x_0, x)\rangle - \frac{\tau}{2}\|x_0 - x\|, \ \forall x \in B(x_0, \delta) \cap X \text{ and } -x_i^* \in -\partial f_i(x). \tag{4.4}$$

From (4.3) and (4.4), we have

$$\frac{\tau}{2}\|x-x_0\| > \langle -x_i^*, \eta(x_0,x)\rangle - \frac{\tau}{2}\|x_0-x\|, \forall x \in B(x_0,\delta)\cap X \text{ and } -x_i^* \in -\partial f_i(x),$$

that is,

$$\tau\|x-x_0\| > \langle -x_i^*, \eta(x_0,x)\rangle,$$

implies that

$$\max_{1\le i\le p}\max_{x_i^*\in\partial f_i(x)}\langle x_i^*, \eta(x,x_0)\rangle < \tau\|x-x_0\|, \forall x \in B(x_0,\delta)\cap X \text{ and } x_i^* \in \partial f_i(x),$$

which is a contradiction to the fact that x_0 solves (MLSVVTI). □

Theorem 4.2. Let $\eta : X \times X \longrightarrow X$ be a function, $f_i : X \longrightarrow \mathbb{R}, i = 1, \cdots, p$, be locally Lipschitz and approximate η-convex at $x_0 \in X$ and $\langle f_i(x), \eta(x,x)\rangle = 0$ for all $x \in X$. If x_0 is a local sharp efficient solution of (VOP), then x_0 solves (MLSVVTI).

Proof. Suppose that $x_0 \in X$ is not a solution of the (MLSVVTI). Then, for any $\delta_0 > 0$ and any $\frac{\tau}{2} > 0$, there exists $x \in B(x_0,\delta_0)\cap X$ and $x_i^* \in \partial f_i(x)$, such that

$$\max_{1\le i\le p}\max_{x_i^*\in\partial f_i(x)}\langle x_i^*, \eta(x,x_0)\rangle < \frac{\tau}{2}\|x-x_0\|,$$

that is,

$$\langle x_i^*, \eta(x,x_0)\rangle < \frac{\tau}{2}\|x-x_0\|. \tag{4.5}$$

Since f_i is approximate η-convex at $x_0 \in X$, it follows that, for any $\frac{\tau}{2} > 0$, there exists $\bar{\delta}_i > 0$, such that by setting $\delta = \min\{\delta_0, \bar{\delta}_i : i = 1, \cdots, p\}$, we have

$$f_i(x_0) \ge f_i(x) + \langle x_i^*, \eta(x_0,x)\rangle - \frac{\tau}{2}\|x_0-x\|, \forall x \in B(x_0,\delta)\cap X \text{ and } x_i^* \in \partial f_i(x),$$

we can rewrite as

$$\langle x_i^*, \eta(x,x_0)\rangle \ge f_i(x) - f_i(x_0) - \frac{\tau}{2}\|x-x_0\|. \tag{4.6}$$

From (4.5) and (4.6), we have

$$\frac{\tau}{2}\|x-x_0\| > f_i(x) - f_i(x_0) - \frac{\tau}{2}\|x-x_0\|, \forall x \in B(x_0,\delta)\cap X,$$

that is,

$$f_i(x) - f_i(x_0) < \tau\|x-x_0\|,$$

implies that

$$\max_{1\leq i\leq p}\left\{f_i(x)-f_i(x_0)\right\}<\tau\|x-x_0\|,\forall x\in B(x_0,\delta)\cap X,$$

which is a contradiction to the fact that x_0 is a local sharp efficient solution of (VOP). □

Theorem 4.3. Let $\eta : X\times X \longrightarrow X$ be a function, $f_i : X \longrightarrow \mathbb{R}, i = 1,\cdots,p$, be locally Lipschitz, $-f_i$ be a strictly approximate η-quasiconvex type-II at $x_0 \in X$ and $\langle f_i(x), \eta(x,x)\rangle =$ 0 for all $x \in X$. If x_0 solves (MLSVVTI), then x_0 is a local sharp efficient solution of (VOP).

Proof. Suppose that $x_0 \in X$ is not a local sharp efficient solution of (VOP). Then, for any $\delta_0 > 0$ and any $\tau > 0$, there exists $x \in B(x_0,\delta_0)\cap X$, such that

$$\max_{1\leq i\leq p}\left\{f_i(x)-f_i(x_0)\right\}<\tau\|x-x_0\|,$$

that is,

$$f_i(x)-f_i(x_0)<\tau\|x-x_0\|,$$

we can rewrite as

$$-f_i(x_0)-(-f_i(x))<\tau\|x_0-x\|.$$

Since $-f_i$ is a strictly approximate η-quasiconvex type-II at $x_0 \in X$, it follows that, for any $\tau > 0$, there exists $\bar{\delta}_i > 0$ such that, by setting $\delta = \min\{\delta_0, \bar{\delta}_i : i = 1,\cdots,p\}$, we have

$$\langle -x_i^*, \eta(x_0,x)\rangle \leq 0, \forall x\in B(x_0,\delta)\cap X \text{ and } -x_i^*\in -\partial f_i(x),$$

that is

$$\langle x_i^*, \eta(x,x_0)\rangle \leq 0<\tau\|x-x_0\|,$$

implies that

$$\max_{1\leq i\leq p}\max_{x_i^*\in\partial f_i(x)}\langle x_i^*, \eta(x,x_0)\rangle <\tau\|x-x_0\|, \forall x\in B(x_0,\delta)\cap X \text{ and } x_i^*\in\partial f_i(x),$$

which is a contradiction to the fact that x_0 solves (MLSVVTI). □

Theorem 4.4. Let $\eta : X\times X \longrightarrow X$ be a function, $f_i : X \longrightarrow \mathbb{R}, i = 1,\cdots,p$, be a locally Lipschitz and strictly approximate η-pseudoconvex type-II at $x_0 \in X$ and $\langle f_i(x), \eta(x,x)\rangle = 0$ for all $x \in X$. If x_0 is a local weak sharp efficient solution of (VOP), then x_0 solves (MLSVVTI).

Proof. Suppose that $x_0 \in X$ is not a solution of the (MLSVVTI). Then, for any $\delta_0 > 0$ and any $\tau > 0$, there exists $x \in B(x_0, \delta_0) \cap X$ and $x_i^* \in \partial f_i(x)$, such that

$$\max_{1\leq i\leq p} \max_{x_i^* \in \partial f_i(x)} \langle x_i^*, \eta(x, x_0)\rangle < \tau \|x - x_0\|,$$

that is,

$$\langle x_i^*, \eta(x, x_0)\rangle < \tau \|x - x_0\|,$$

we can rewrite as

$$\langle x_i^*, \eta(x_0, x)\rangle + \tau \|x_0 - x\| > 0.$$

Since f_i is a strictly approximate η-pseudoconvex type-II at $x_0 \in X$, it follows that, for any $\tau > 0$, there exists $\bar{\delta}_i > 0$ such that, by setting $\delta = \min\{\delta_0, \bar{\delta}_i : i = 1, \cdots, p\}$, we have

$$f_i(x_0) > f_i(x), \forall x \in B(x_0, \delta) \cap X,$$

that is,

$$f_i(x) - f_i(x_0) < 0 \leq \tau d(x, \bar{X}),$$

implies that

$$\max_{1\leq i\leq p} \left\{f_i(x) - f_i(x_0)\right\} < \tau \ \ d(x, \bar{X}), \ \forall x \in B(x_0, \delta) \cap X,$$

which is a contradiction to the fact that x_0 is a local weak sharp efficient solution of (VOP). □

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