


Article

Study on Anisotropic Strange Stars in $f(\mathbb{T}, \mathcal{T})$ Gravity

Ines G. Salako¹, M. Khlopov², Saibal Ray³^{*}, M.Z. Arouko⁴, Pameli Saha⁵ and Ujjal Debnath⁵

¹ Ecole de Génie Rural (EGR), 01 BP 55 Kétou, Benin & Institut de Mathématiques et de Sciences Physiques (IMSP), 01 BP 613 Porto-Novo, Benin; inessalako@gmail.com

² National Research Nuclear University, MEPHI (Moscow Engineering Physics Institute), Moscow 115409, Russia; Université de Paris, CNRS, Astroparticule et Cosmologie, F-75013 Paris, France & Institute of Physics, Southern Federal University, Rostov on Don 344090, Russia; khlopov@apc.in2p3.fr

³ Department of Physics, Government College of Engineering and Ceramic Technology, Kolkata 700010, West Bengal, India; saibal@iucaa.ernet.in

⁴ Département de Physique, Université d'Abomey-Calavi, BP 526 Calavi, Benin; maximearouko55@gmail.com

⁵ Department of Mathematics, Indian Institute of Engineering Science and Technology, Shibpur, Howrah 711103, West Bengal, India; pameli.saha15@gmail.com

⁵ Department of Mathematics, Indian Institute of Engineering Science and Technology, Shibpur, Howrah 711103, West Bengal, India; ujjaldebnath@gmail.com

* Correspondence: saibal@associates.iucaa.in

‡ These authors contributed equally to this work.

Abstract: In this work, we study the existence of strange star in the background of $f(\mathbb{T}, \mathcal{T})$ gravity in the Einstein spacetime geometry, where \mathbb{T} is the torsion tensor and \mathcal{T} is the trace of the energy-momentum tensor. The equations of motion are derived for anisotropic pressure within the spherically symmetric strange star. We explore the physical features like energy conditions, mass-radius relations, modified TOV equations, principal of causality, adiabatic index, redshift and stability analysis of our model. These features are realistic and appealing to further investigation of properties of compact objects in $f(\mathbb{T}, \mathcal{T})$ gravity as well as their observational signatures.

Keywords: General relativity; alternative gravity; compact stars; anisotropic fluid.

1. Introduction

In relativistic astrophysics, compact stars are more compact (i.e., possess larger mass and smaller radius) than ordinary stars, has received much attention to the researchers to study their ages, structures and evolutions. Neutron star is the final stage of a gravitationally collapsed star which, after exhausting all its thermo-nuclear fuel, gets stabilized by degenerate neutron pressure. A massive neutron star may be again collapsed into black hole but lower mass neutron star may be converted into quark star. The strange quark matter [1–3] consists of u , d , s quarks and some electrons, which may be more stable than the ordinary nuclear matter. Actually, neutron star consists of neutrons [4] but strange star consists of quarks or strange matters.

In 1916, Schwarzschild et al. [5,6] have first given the interior stellar solution for isotropic fluid (having equal radial pressure (p_r) and transversal pressure (p_t)). But much later in 1972, Ruderman [7] first observed that the interior geometry of the nuclear matter of stellar system with density of order 10^{15} gm/cc possesses anisotropic behavior (i.e., $p_r \neq p_t$). After that Herrera et al. [8] have analyzed the local anisotropic nature for self gravitating systems. It is interesting to note that in the experimental area Hewish et al. [9] explained the observed compactness of many astrophysical objects (rotating neutron stars) like Her X-1, 4U 1820 – 30, RXJ 1856 – 37 and SAX J 1808.4 – 3658. Thus the theoretical modeling gradually became complimentary to the observational results.

The study of anisotropic stars in the background of General Relativity (GR) is important in the current research. Hossein et al. [10] have studied the stable structure of stellar objects for anisotropic system. In the context of the now Standard cosmology, leading beyond the standard model of particle theory, such analysis inevitably involves additional elements in the standard model of gravity. Consequently, Kalam et al. [11] have studied the anisotropic neutron star model by considering quintessence dark energy. To study compact stellar models, the researchers have used well known Krori-Barua (KB) metric [12,13]. Paul et al. [14] have analyzed a new exact solution for compact star. Several astrophysical natures of neutron star as well as quark star have been investigated [15,16]. Bhar [17] has proposed a stable anisotropic quintessence strange star model whereas Abbas et al. [18] have studied anisotropic strange star for quintessence dark energy model. Using MIT bag model, Rahaman et al. [19] have observed the existence of strange star. Also, by considering MIT bag model, the stability of strange star with anisotropic fluid has been analyzed by Arbañil and Malheiro [20]. Furthermore, for MIT bag model, Murad [21] has studied the anisotropic charged strange star.

The study of anisotropic stars in the background of modified gravity theory is of great interest to many researchers [22–27]. The compact stars in $f(R)$, $f(G)$, $f(\mathbb{T})$ and $f(R, G)$ gravity theories have been studied by several authors [18,28–33], where R , G and \mathbb{T} are Ricci scalar, Gauss-Bonnet scalar and the torsion scalar, respectively. Strange stars in $f(\mathbb{T})$ gravity with MIT bag model have been studied in [34]. Compact stars, in the form of neutron star as well as quark star, in Rastall theory of gravity model have been studied by several researches [35–40] whereas strange stars with MIT bag model in the Rastall gravity have been studied in Ref. [41]. Charged anisotropic collapsing stars with heat flux in $f(R)$ gravity has been investigated by Nazar and Abbas [42].

Studies on anisotropic stars in different types of modified gravity theories are available in literature [43–48]. Saha and Debnath [49] have investigated anisotropic stars in $f(\mathbb{T})$ gravity with modified Chaplygin gas. The anisotropic compact stars with quintessence field and modified Chaplygin gas in the framework of $f(\mathbb{T})$ gravity model has been studied by Saha and Debnath [50]. On the other hand, strange stars as well as gravastars in $f(R, \mathcal{T})$ gravity have been studied in [51–58], where \mathcal{T} is the trace of the energy-momentum tensor.

We note from literature survey that almost all the works under the background of $f(\mathbb{T}, \mathcal{T})$ gravity have been performed in different cosmological context [59–70]. However, there are a few astrophysical applications of $f(\mathbb{T}, \mathcal{T})$ gravity as can be found in the works of Pace and Said [71,72]. Pace and Said have studied quark star incorporating the MIT bag model [71] and neutron star models using a perturbative approach [72] in $f(\mathbb{T}, \mathcal{T})$ gravity theory whereas gravastar has been investigated by Ghosh et al. [70] in the same alternative theory of gravity.

Our motivation to study anisotropic strange stars in $f(\mathbb{T}, \mathcal{T})$ gravity lies in the above background in the presence of Einstein spacetime within the spherically symmetric geometry. Thus we are investigating in the present work strange quark star with MIT bag model [71] assuming anisotropy in the matter distribution and would like to note any appreciable effects due to anisotropy in addition to GR. The organization of the paper is as follows: In Section 2, the mathematical formulation of $f(\mathbb{T}, \mathcal{T})$ gravity in Einstein spacetime is given. In Section 3, we investigate the basic stellar equations in the framework of $f(\mathbb{T}, \mathcal{T})$ gravity. In Section 4, we find out the solutions to the Einstein field equations due to anisotropic fluid source. In Section 5, we explore the physical features like energy conditions, mass-radius relations, modified TOV equations, principal of causality, adiabatic index, redshift and stability analysis of our model whereas in Section 6 we perform a comparison with the standard results of GR, in order to show what is the effect of our model, and why it justifies our analysis and do relevant discussion on the results. In Section 7, we deliver the conclusions of the work.

2. Basic Mathematical Formalism of $f(\mathbb{T}, \mathcal{T})$ Gravity and the Einstein Field Equations

The modified theories of teleparallel gravity are those for which the scalar torsion of teleparallel action is substituted by an arbitrary function of this latter. As it is done in teleparallel, the modified

versions of this theory are also described by the orthonormal tetrads and its components are defined on the tangent space of each point of the manifold. The line element is written as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \eta_{ij} \theta^i \theta^j, \quad (1)$$

with the following definitions

$$dx^\mu = e_i^\mu \theta^i; \quad \theta^i = e^i_\mu dx^\mu. \quad (2)$$

Note that $\eta_{ij} = \text{diag}(1, -1, -1, -1)$ is the Minkowskian metric and the $\{e^i_\mu\}$ are the components of the tetrad which satisfy the following identity:

$$e_i^\mu e^i_\nu = \delta^\mu_\nu, \quad e_\mu^i e^j_\mu = \delta^i_j. \quad (3)$$

In General Relativity, one use the following Levi-Civita's connection which preserves the curvature whereas the torsion vanishes, such as

$$\overset{\circ}{\Gamma}{}^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} (\partial_\nu g_{\sigma\mu} + \partial_\mu g_{\sigma\nu} - \partial_\sigma g_{\mu\nu}). \quad (4)$$

But in the teleparallel theory and its modified version, one keeps the scalar torsion by using Weizenbock's connection defined as:

$$\Gamma^\lambda_{\mu\nu} = e_i^\lambda \partial_\mu e^i_\nu = -e^i_\mu \partial_\nu e_i^\lambda. \quad (5)$$

From this connection, one obtains the geometric objects. The first is the torsion as defined by

$$T^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}, \quad (6)$$

from which we define the contorsion as

$$K^\lambda_{\mu\nu} \equiv \Gamma^\lambda_{\mu\nu} - \overset{\circ}{\Gamma}{}^\lambda_{\mu\nu} = \frac{1}{2} (T^\lambda_{\mu\nu} + T^\lambda_{\nu\mu} - T^\lambda_{\nu\mu}), \quad (7)$$

where the expression $\overset{\circ}{\Gamma}{}^\lambda_{\mu\nu}$ designs the above defined connection. Then we can write

$$K^{\mu\nu}{}_\lambda = -\frac{1}{2} (T^{\mu\nu}{}_\lambda - T^{\nu\mu}{}_\lambda + T^\lambda{}_{\nu\mu}). \quad (8)$$

The two previous geometric objects (the torsion and the contorsion) are used to define another tensor by

$$S_\lambda{}^{\mu\nu} = \frac{1}{2} (K^{\mu\nu}{}_\lambda + \delta_\lambda^\mu T^{\alpha\nu}{}_\alpha - \delta_\lambda^\nu T^{\alpha\mu}{}_\alpha). \quad (9)$$

The torsion scalar is usually constructed from torsion and contorsion as follows:

$$\mathbb{T} = S_\sigma{}^{\mu\nu} T^{\sigma}{}_{\mu\nu}. \quad (10)$$

In the modified versions of teleparallel gravity, one can use a general algebraic function of scalar torsion instead of the scalar torsion only as it is done in the initial theory. So, the modified action can be given by

$$\mathbb{S} = \int d^4x \, e \left[\frac{\mathbb{T} + f(\mathbb{T}, \mathcal{T})}{16\pi} + \mathcal{L}_m \right], \quad (11)$$

where $T_{\mu\nu}$ is the energy-momentum tensor of the strange quark matter (SQM) distribution and \mathcal{L}_m represents the Lagrangian for the matter distribution.

We define $T_{\mu\nu}$ as follows

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}, \quad (12)$$

and also define the trace of $T_{\mu\nu}$ as $\mathcal{T} = g^{\mu\nu}T_{\mu\nu}$. As \mathcal{L}_m depends only on the metric tensor components $g_{\mu\nu}$ and not on their derivatives, so we find

$$T_{\mu\nu} = g_{\mu\nu}\mathcal{L}_m - 2\frac{\partial\mathcal{L}_m}{\partial g^{\mu\nu}}. \quad (13)$$

Varying the action with respect to the tetrad, one obtains the equations of motion [59] as

$$\begin{aligned} & [\partial_{\xi}(ee_a^{\rho}S_{\rho}^{\sigma\xi}) - ee_a^{\lambda}S^{\rho\xi\sigma}T_{\rho\xi\lambda}](1 + f_{\mathbb{T}}) + ee_a^{\rho}(\partial_{\xi}\mathbb{T})S_{\rho}^{\sigma\xi}f_{\mathbb{T}\mathbb{T}} \\ & + \frac{1}{4}ee_a^{\sigma}(\mathbb{T}) = -\frac{1}{4}ee_a^{\sigma}f - ee_a^{\rho}(\partial_{\xi}\mathcal{T})S_{\rho}^{\sigma\xi}f_{\mathbb{T}\mathcal{T}} \\ & + f_{\mathcal{T}} \left(\frac{eT_a^{\sigma} + ee_a^{\sigma}p}{2} \right) + 4\pi eT_a^{\sigma}, \end{aligned} \quad (14)$$

with $f_{\mathcal{T}} = \partial f/\partial\mathcal{T}$, $f_{\mathbb{T}} = \partial f/\partial\mathbb{T}$, $f_{\mathbb{T}\mathcal{T}} = \partial^2 f/\partial\mathbb{T}\partial\mathcal{T}$, $f_{\mathbb{T}\mathbb{T}} = \partial^2 f/\partial\mathbb{T}^2$ and T_a^{σ} represents the stress-energy tensor for the anisotropic fluid distribution defined as

$$T_{\mu\nu} = (p_r - p_t)u_{\mu}u_{\nu} - p_t g_{\mu\nu} + (\rho + p_t)u_{\mu}u_{\nu}, \quad (15)$$

where p_r and p_t represent the radial and tangential pressures of the SQM distribution, u_{μ} is the four velocity which satisfies the conditions $u_{\mu}u^{\mu} = 1$ and $u^{\mu}\nabla_{\nu}u_{\mu} = 0$.

By using some transformations, we can establish the following relations:

$$e_v^a e^{-1} \partial_{\xi}(ee_a^{\rho}S_{\rho}^{\sigma\xi}) - S^{\rho\xi\sigma}T_{\rho\xi\nu} = -\nabla^{\xi}S_{\nu\xi}^{\sigma} - S^{\xi\rho\sigma}K_{\rho\xi\nu}, \quad (16)$$

$$G_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathbb{T} = -\nabla^{\rho}S_{\nu\rho\mu} - S^{\sigma\rho}_{\mu}K_{\rho\sigma\nu}. \quad (17)$$

Hence from the combination of Equations (16) and (17), the field equation (14) can be written as (under the relativistic geometrized units where Newton's gravitational constant G and the speed of light c are fixed to be the unity, i.e, $G = c = 1$):

$$\begin{aligned} \frac{1}{2}(1 + f_{\mathbb{T}})G_{\mu\nu} &= \frac{1}{4}g_{\mu\nu}\mathbb{T}(1 + f_{\mathbb{T}}) - S_{\nu\mu}^{\lambda}(f_{\mathbb{T}\mathbb{T}}\partial_{\lambda}\mathbb{T} + f_{\mathbb{T}\mathcal{T}}\partial_{\lambda}\mathcal{T}) - \frac{1}{4}g_{\mu\nu}(f + \mathbb{T}) \\ &+ \frac{1}{2}f_{\mathcal{T}}(T_{\nu\mu} + g_{\mu\nu}p_t) + 4\pi T_{\nu\mu}, \end{aligned} \quad (18)$$

$$G_{\mu\nu} = 8\pi T_{\mu\nu}^{eff}, \quad (19)$$

where

$$T_{\mu\nu}^{eff} = \frac{1}{(1+f_{\mathbb{T}})} \left\{ \frac{g_{\mu\nu}\mathbb{T}}{16\pi} (1+f_{\mathbb{T}}) - \frac{S_{\nu\mu}{}^{\lambda}}{4\pi} (f_{\mathbb{T}\mathbb{T}}\partial_{\lambda}\mathbb{T} + f_{\mathbb{T}\mathcal{T}}\partial_{\lambda}\mathcal{T}) - \frac{g_{\mu\nu}}{16\pi} (f + \mathbb{T}) + \frac{f_{\mathcal{T}}}{8\pi} (T_{\nu\mu} + g_{\mu\nu}p_t) + T_{\nu\mu} \right\}. \quad (20)$$

The covariant derivative of Equation (19) reads as

$$\begin{aligned} \nabla_{\mu}T_{\nu}{}^{\mu} &= \frac{1}{(4\pi + (1/2)f_{\mathcal{T}})} \left\{ (f_{\mathbb{T}\mathbb{T}}\partial_{\sigma}\mathbb{T} + f_{\mathbb{T}\mathcal{T}}\partial_{\sigma}\mathcal{T}) e^a{}_{\nu}[e^{-1}\partial_{\lambda}(ee_a{}^{\alpha}S_{\alpha}{}^{\sigma\lambda}) - e_a{}^{\alpha}T^{\lambda}{}_{\gamma\alpha}S_{\lambda}{}^{\gamma\sigma}] - \frac{1}{4}(1+f_{\mathbb{T}})\partial_{\nu}\mathbb{T} \right. \\ &+ \nabla_{\mu}S_{\nu}{}^{\mu\lambda} (f_{\mathbb{T}\mathbb{T}}\partial_{\lambda}\mathbb{T} + f_{\mathbb{T}\mathcal{T}}\partial_{\lambda}\mathcal{T}) + S_{\nu}{}^{\mu\lambda} (f_{\mathbb{T}\mathbb{T}\mathbb{T}}\partial_{\mu}\mathbb{T}\partial_{\lambda}\mathbb{T} + f_{\mathbb{T}\mathbb{T}\mathcal{T}}\partial_{\mu}\mathcal{T}\partial_{\lambda}\mathbb{T} + f_{\mathbb{T}\mathbb{T}}\nabla_{\mu}\partial_{\lambda}\mathbb{T} + f_{\mathbb{T}\mathcal{T}\mathbb{T}}\partial_{\mu}\mathbb{T}\partial_{\lambda}\mathcal{T} \\ &+ f_{\mathbb{T}\mathcal{T}\mathcal{T}}\partial_{\mu}\mathcal{T}\partial_{\lambda}\mathcal{T} + f_{\mathbb{T}\mathcal{T}}\nabla_{\mu}\partial_{\lambda}\mathcal{T}) + \frac{1}{4} (f_{\mathbb{T}}\partial_{\nu}\mathbb{T} + f_{\mathcal{T}}\partial_{\nu}\mathcal{T} + \partial_{\nu}\mathbb{T}) - \frac{1}{2} (f_{\mathcal{T}\mathbb{T}}\partial_{\mu}\mathbb{T} + f_{\mathcal{T}\mathcal{T}}\partial_{\mu}\mathcal{T}) (T_{\nu}{}^{\mu} + \delta_{\nu}^{\mu}p_t) \\ &\left. - \frac{1}{2}f_{\mathcal{T}}\partial_{\nu}p_t \right\}. \quad (21) \end{aligned}$$

In the current work, we are specifically focused on the existence of anisotropic strange stars in the extended teleparallel gravity and for this purpose we use an algebraic function following the prescription by Harko et al. [59] to comply with the observational data, such that $f(\mathbb{T}, \mathcal{T}) = \omega \mathbb{T}^n \mathcal{T} - 2\Lambda$, where ω , n and Λ are arbitrary constants, specifically ω represents the matter-geometry coupling constant, n is a pure number (for simplicity assumed to be unity here) and Λ can be recognized as the Cosmological constant as has been treated in their above mentioned work by Harko et al. [59].

For the above mentioned functional form of $f(\mathbb{T}, \mathcal{T})$ the Equation (19) reduces to

$$G_{\mu\nu} = 8\pi T_{\mu\nu}^{eff}, \quad (22)$$

where

$$T_{\mu\nu}^{eff} = g_{\mu\nu} \left[\frac{(-\omega(\rho - p_r - 2p_t) + 2\Lambda)}{16\pi} + \frac{\omega p_t}{8\pi} \right] + T_{\nu\mu} \left(1 + \frac{\omega}{8\pi} \right) \quad (23)$$

and

$$\nabla_{\mu}T_{\nu}{}^{\mu} = \frac{1}{(4\pi + (1/2)\omega)} \left\{ \frac{\omega}{4} (\partial_{\nu}\mathcal{T}) - \frac{\omega}{2} \partial_{\nu}p_t \right\}. \quad (24)$$

3. Explicit Stellar Equations in $f(\mathbb{T}, \mathcal{T})$ Gravity under the Einsteinian Spacetime

To describe the interior spacetime of the spherically symmetric static stellar system, we take metric as follows:

$$ds^2 = e^{v(r)} dt^2 - e^{\lambda(r)} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (25)$$

where the metric potentials ν and λ are the functions of the radial coordinate r only.

In order to preserve the invariance under local Lorentz transformations of line element and to get the form (2), the choice of tetrad matrices will not unique. Following the refs. [73–77] one can get $e = \det \left[e^i_{\mu} \right] = e^{(\nu+\lambda)/2} r^2 \sin(\theta)$ and with Eqs. (3)-(10), the torsion scalar in terms of r becomes

$$T(r) = \frac{2e^{-\lambda}}{r^2} \left(e^{\lambda/2} - 1 \right) \left(e^{\lambda/2} - 1 - r\nu' \right), \quad (26)$$

where the prime (') denotes the derivative with respect to the radial coordinate r .

Now the nonzero components of the Einstein tensors can be provided as follows:

$$G_0^0 = \frac{e^{-\lambda}}{r^2} (-1 + e^{\lambda} + \lambda' r), \quad (27)$$

$$G_1^1 = \frac{e^{-\lambda}}{r^2} (-1 + e^{\lambda} - \nu' r), \quad (28)$$

$$G_2^2 = G_3^3 = \frac{e^{-\lambda}}{4r} [2(\lambda' - \nu') - (2\nu'' + \nu'^2 - \nu'\lambda')r], \quad (29)$$

where primes stand for derivative with respect to the radial coordinate r only.

Substituting Equation (15) into Equation (22), we find the explicit form of the Einstein field equations for the interior metric (25) as follows:

$$e^{-\lambda} \left(\frac{\lambda'}{r} - \frac{1}{r^2} \right) + \frac{1}{r^2} = 8\pi \left\{ \left[\frac{(-\omega(\rho - p_r - 2p_t) + 2\Lambda)}{16\pi} + \frac{\omega p_t}{8\pi} \right] + \rho \left(1 + \frac{\omega}{8\pi} \right) \right\} = 8\pi \rho^{eff}, \quad (30)$$

$$e^{-\lambda} \left(\frac{\nu'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2} = -8\pi \left\{ \left[\frac{(-\omega(\rho - p_r - 2p_t) + 2\Lambda)}{16\pi} + \frac{\omega p_t}{8\pi} \right] - p_r \left(1 + \frac{\omega}{8\pi} \right) \right\} = 8\pi p_r^{eff}, \quad (31)$$

$$\begin{aligned} \frac{e^{-\lambda}}{4r} [2(\lambda' - \nu') - (2\nu'' + \nu'^2 - \nu'\lambda')r] &= -8\pi \left\{ \left[\frac{(-\omega(\rho - p_r - 2p_t) + 2\Lambda)}{16\pi} + \frac{\omega p_t}{8\pi} \right] - p_t \left(1 + \frac{\omega}{8\pi} \right) \right\} \\ &= 8\pi p_t^{eff}. \end{aligned} \quad (32)$$

Here ρ^{eff} , p_r^{eff} and p_t^{eff} represent the effective energy density, radial pressure and tangential pressure for our system and given as

$$\rho^{eff} = \rho + \frac{\omega}{16\pi} (\rho + p_r + 4p_t + 2\Lambda), \quad (33)$$

$$p_r^{eff} = p_r + \frac{\omega}{16\pi} (\rho + p_r - 4p_t - 2\Lambda), \quad (34)$$

$$p_t^{eff} = p_t + \frac{\omega}{16\pi} (\rho - p_r + 2p_t - 2\Lambda). \quad (35)$$

We assume that the SQM distribution inside the strange stars is governed by the simple phenomenological MIT Bag model equation of state (EOS) [78]. In the Bag model, by introducing *ad hoc* Bag function all the corrections of the energy and pressure functions of SQM have been maintained. We also consider that the quarks are non-interacting and massless in a simplified Bag model. The quark pressure therefore can be defined as

$$p_r = \sum_{f=u,d,s} p^f - B, \quad (36)$$

where p^f is the individual pressure of the up (u), down (d) and strange (s) quark flavors and B is the vacuum energy density (also well known as the 'Bag' constant) which is a constant quantity within a numerical range. In the present paper we consider the value of Bag constant as $B = 83 \text{ MeV} / \text{fm}^3$ [79].

Now the individual quark pressure (p^f) can be defined as $p^f = \frac{1}{3}\rho^f$, where ρ^f is the energy density of the individual quark flavor. Hence, the energy density of the de-confined quarks inside the Bag is given by

$$\rho = \sum_{f=u,d,s} \rho^f + B. \quad (37)$$

Using Equations (36) and (37), we have the EOS for SQM given as

$$p_r = \frac{1}{3}(\rho - 4B). \quad (38)$$

It is observed that ignoring critical aspects of the quantum particle physics in the framework of GR several authors [56,57,80–87] successfully have introduced this simplified form of the MIT Bag EOS to study stellar systems made of SQM.

To have non-singular monotonically decreasing matter density inside the spherically symmetric stellar system, following Mak and Harko [88], we assume simplified form of ρ given as

$$\rho(r) = \rho_c \left[1 - \left(1 - \frac{\rho_0}{\rho_c} \right) \frac{r^2}{R^2} \right], \quad (39)$$

where ρ_c and ρ_0 are two specific constants and denote the maximum and minimum values of ρ at the center and on the surface, respectively.

Now following Moraes et al. [89] we consider p_t is related to ρ by a relation given as

$$p_t = c_1\rho + c_2, \quad (40)$$

where c_1 and c_2 are purely constants.

We define the mass function of the spherically symmetric stellar system as [52]

$$m(r) = 4\pi \int_0^r \rho_{eff}(r) r^2 dr. \quad (41)$$

At this juncture we consider the Schwarzschild metric to represent the exterior spacetime of our system given by

$$ds^2 = \left(1 - \frac{2M}{r} \right) dt^2 - \frac{dr^2}{\left(1 - \frac{2M}{r} \right)} - r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (42)$$

where M is the total mass of the stellar system.

Now, substituting Equation (41) into Equation (30) we find

$$e^{-\lambda(r)} = 1 - \frac{2m(r)}{r}, \quad (43)$$

where one can note that the metric potential e^λ has the form of Schwarzschild type.

In this case, the conservation equation (24) in $f(T, \mathcal{T})$ gravity takes the form as follows

$$-p'_r + \frac{v'}{2}(p_r + \rho) + \frac{2}{r}(p_t - p_r) = \frac{1}{(4\pi + (1/2)\omega)} \left\{ \frac{\omega\rho'}{4} - \frac{\omega p'_r}{4} - \omega p'_t \right\}. \quad (44)$$

The essential stellar structure equations required to describe static spherically symmetric sphere in $f(R, \mathcal{T})$ gravity theory are given by

$$\begin{aligned} \frac{dm}{dr} &= 4\pi r^2 \rho^{eff}, \\ \frac{dp_r}{dr} &= \frac{1}{\left[-1 + \frac{\omega}{16\pi+2\omega} \left(1 - \frac{d\rho}{dp_r} + 4\frac{dp_t}{dp_r}\right)\right]} \left\{ -(\rho + p_r) \left[\frac{4\pi r p_r^{eff} + \frac{m}{r^2}}{\left(1 - \frac{2m}{r}\right)} \right] \right\} \\ &+ \frac{2}{r}(p_t - p_r). \end{aligned} \quad (45)$$

4. Solution to the Einstein Field Equations in $f(\mathbb{T}, \mathcal{T})$ Gravity for Stellar Modeling

Substituting Equation (38) in Equations (39) and (40), we have the following pressures

$$p_r = \frac{1}{3}\rho_c \left[1 - \left(1 - \frac{\rho_0}{\rho_c}\right) \frac{r^2}{R^2} \right] - \frac{4}{3}B, \quad (46)$$

$$p_t = c_1\rho_c \left[1 - \left(1 - \frac{\rho_0}{\rho_c}\right) \frac{r^2}{R^2} \right] + c_2. \quad (47)$$

Again, putting Equations (38), (46) and (47) in Equations (33), (34) and (35), we get the following physical parameters:

$$\rho^{eff} = 10\left(1 - \frac{7r^2}{250}\right) + \frac{\omega \left[10\left(1 - \frac{7r^2}{250}\right) + \frac{1}{3} \left\{ -4 + 10\left(1 - \frac{7r^2}{250}\right) \right\} + 4 \left\{ 5 + 20\left(1 - \frac{7r^2}{250}\right) \right\} + 2\Lambda \right]}{16\pi}, \quad (48)$$

$$p_r^{eff} = \frac{1}{3} \left\{ -4 + 10\left(1 - \frac{7r^2}{250}\right) \right\} + \frac{\omega \left[10\left(1 - \frac{7r^2}{250}\right) + \frac{1}{3} \left\{ -4 + 10\left(1 - \frac{7r^2}{250}\right) \right\} - 4 \left\{ 5 + 20\left(1 - \frac{7r^2}{250}\right) \right\} - 2\Lambda \right]}{16\pi}, \quad (49)$$

$$p_t^{eff} = 5 + 20\left(1 - \frac{7r^2}{250}\right) + \frac{\omega \left[10\left(1 - \frac{7r^2}{250}\right) + \frac{1}{3} \left\{ 4 - 10\left(1 - \frac{7r^2}{250}\right) \right\} + 2 \left\{ 5 + 20\left(1 - \frac{7r^2}{250}\right) \right\} - 2\Lambda \right]}{16\pi}. \quad (50)$$

Using Equation (48), we integrate Equation (45) which yields the mass function as

$$m(r) = \frac{r^5(16\pi + \omega + 4c_1\omega)(\rho_0 - \rho_c)}{20R^2} + \frac{\rho_c r^3(16\pi + \omega + 4c_1\omega)}{12} - \frac{r^3\omega(2B - 6c_2 - 3\Lambda)}{18} + d, \quad (51)$$

where d is integrating constant.

With the help of Equations (31), (39), (43), (46), (47) and (51), we have the two unknown functions of r of the given metric (25) as

$$\lambda(r) = -\ln \left(1 - \frac{r^4(16\pi + \omega + 4c_1\omega)(\rho_0 - \rho_c)}{20R^2} - \frac{\rho_c r^2(16\pi + \omega + 4c_1\omega)}{6} + \frac{r^2\omega(2B - 6c_2 - 3\Lambda)}{9} - \frac{2d}{r} \right) \quad (52)$$

and

$$\nu(r) = \int \left\{ \frac{r}{1 - \frac{2m(r)}{r}} \left[-8\pi \left\{ \left[\frac{-\omega(\rho - p_r - 2p_t) + 2\Lambda}{16\pi} + \frac{\omega p_t}{8\pi} \right] - p_r \left(1 + \frac{\omega}{8\pi} \right) \right\} + \frac{1}{r^2} \right] - \frac{1}{r} \right\} dr \quad (53)$$

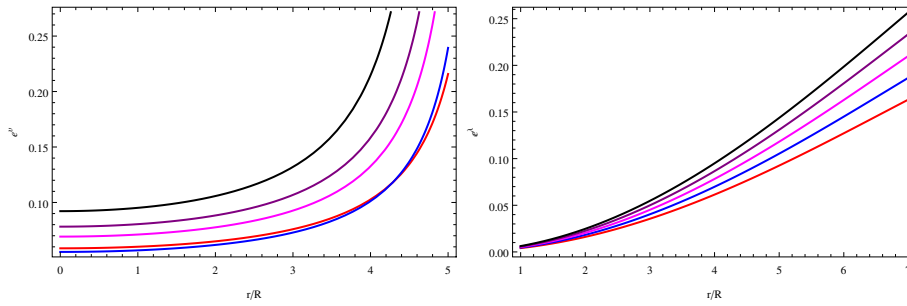


Figure 1. Plot of $e^{\nu(r)}$ and $e^{\lambda(r)}$ versus r/R for the strange star candidate *LMC X-4*. The red, purple, magenta, blue and black colors represent respectively cases $\omega = 0$, $\omega = 0.5$, $\omega = 1$, $\omega = 1.5$ and $\omega = 2$.

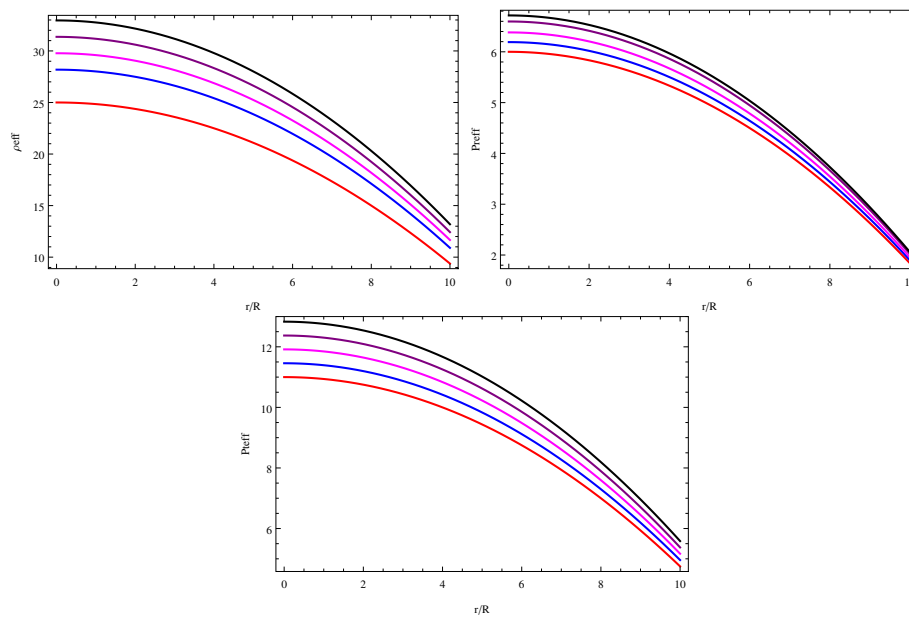


Figure 2. Plot of ρ^{eff} , p_r^{eff} and p_t^{eff} versus r/R for the strange star candidate *LMC X-4*. The red, purple, magenta, blue and black colors represent respectively cases $\omega = 0$, $\omega = 0.5$, $\omega = 1$, $\omega = 1.5$ and $\omega = 2$.

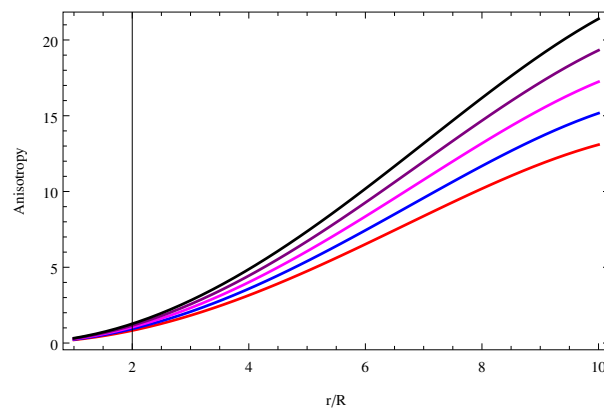


Figure 3. Plot of anisotropy versus r/R for the strange star candidate *LMC X-4*. The red, purple, magenta, blue and black colors represent respectively cases $\omega = 0$, $\omega = 0.5$, $\omega = 1$, $\omega = 1.5$ and $\omega = 2$.

5. Physical Features of Strange (Quark) Stars in $f(\mathbb{T}, \mathcal{T})$ Gravity

In this Section, we study some physical features of the compact star, in order to examine the physical validity and stability of the system in the $f(\mathbb{T}, \mathcal{T})$ gravity.

5.1. Energy Conditions

According to relativistic classical field theories of gravitation, anisotropic fluid model must obey the energy conditions. There is often a linear relationship between the energy density and pressure of the matter by the name (i) Null Energy Condition (NEC), (ii) Weak Energy Condition (WEC), (iii) Dominant Energy Condition (DEC) and (iv) Strong Energy Condition (SEC) [90,91]. In the $f(\mathbb{T}, \mathcal{T})$ gravity theory these are as follows:

$$NEC : \rho_{eff} + p_{effr} \geq 0, \rho_{eff} + p_{efft} \geq 0, \quad (54)$$

$$WEC : \rho_{eff} + p_{effr} \geq 0, \rho_{eff} \geq 0, \rho_{eff} + p_{efft} \geq 0,$$

$$SEC : \rho_{eff} + p_{effr} \geq 0, \rho_{eff} + p_{efft} \geq 0, \rho_{eff} + p_{effr} + 2p_{efft} \geq 0, \quad (55)$$

$$DEC : \rho_{eff} \geq 0, \rho_{eff} - p_{effr} \geq 0, \rho_{eff} - p_{efft} \geq 0. \quad (56)$$

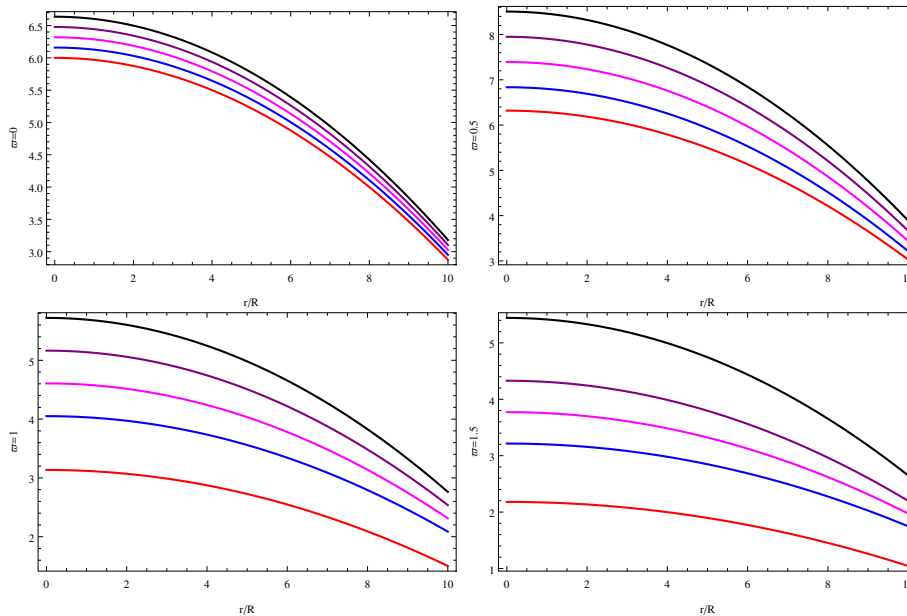


Figure 4. Plot of NEC, WEC, DEC and SEC versus r/R for the strange star candidate LMC X – 4 due to different chosen values of ω . The red, purple, magenta, blue and black colors represent respectively $\rho_{eff} - p_{effr}$, $\rho_{eff} - p_{efft}$, $\rho_{eff} + p_{effr}$, $\rho_{eff} + p_{efft}$, $\rho_{eff} + p_{effr} + 2p_{efft}$ with $\omega = 0$, $\omega = 0.5$, $\omega = 1$, $\omega = 1.5$ and $\omega = 2$ respectively.

5.2. Mass-Radius Relation

From Equation (51), we can rewrite the mass function for our stellar system in $f(\mathbb{T}, \mathcal{T})$ gravity as follows:

$$m(r) = m_{GR} + \omega m_{MG}, \quad (57)$$

where

$$m_{GR} = -\frac{4\pi r^3 (-5R^2 \rho_c + 3r^2 (-\rho_0 + \rho_c))}{15R^2}, \quad (58)$$

$$\begin{aligned}
m_{MG} &= \frac{r^3 (5R^2(-2B + 3\Lambda) + 30R^2c_2 + 6r^2(1 + 3c_1)\rho_0 + 2(5R^2 - 3r^2)(1 + 3c_1)\rho_c)}{90R^2} \\
&= \frac{r^3}{12} \frac{2(5R^2(-2B + 3\Lambda) + 30R^2c_2)}{15R^2} + \frac{(1 + 3c_1)}{12\pi} m_{GR},
\end{aligned} \tag{59}$$

where m_{GR} represents the mass function of the SQM distribution in the Einstein frame whereas in the expression of $m(r)$ the second term, i.e. m_{MG} , represents the mass of the matter distribution and originates due to coupling between the matter and geometry in the $f(\mathbb{T}, \mathcal{T})$ gravity.

In order to plot the Mass (M/M_\odot) versus Radius (R) curve in Fig. 5 for the strange stars due to the different values of ω , we can rewrite (57), the mass function for our stellar system in $f(\mathbb{T}, \mathcal{T})$ gravity, as follows:

$$M(R) = M_{GR} + \omega M_{MG}, \tag{60}$$

where

$$M_{GR} = \frac{4}{15} \pi R^3 (3\rho_0 + 2\rho_c), \tag{61}$$

$$M_{MG} = \frac{1}{18} R^3 (-2B + 3\Lambda + 6c_2) + \frac{(1 + 3c_1)}{12\pi} M_{GR}. \tag{62}$$

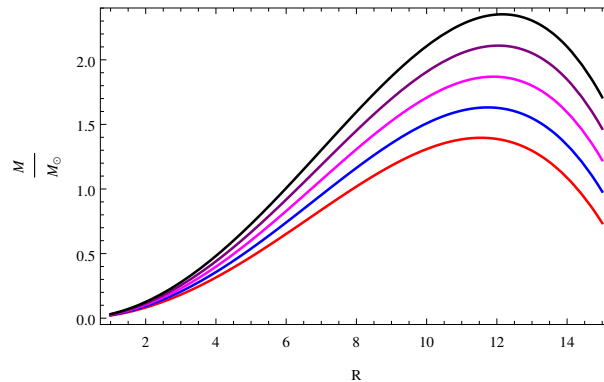


Figure 5. Plot of Mass (M/M_\odot) versus Radius (R in km) curve for the strange star candidate *LMC X – 4* due to different values of ω . The red, purple, magenta, blue and black colors represent respectively cases $\omega = 0$, $\omega = 0.5$, $\omega = 1$, $\omega = 1.5$ and $\omega = 2$.

5.3. Stability of the Stellar Model

In order to study the stability of the stellar model in a more formal manner in $f(\mathbb{T}, \mathcal{T})$ gravity, the following three issues will be discussed: (i) Modified form of the TOV equation, (ii) Herrera cracking concept and (iii) Adiabatic index in the following sub-subsections.

5.3.1. Modified TOV Equation in $f(\mathbb{T}, \mathcal{T})$ Gravity Theory

To check the hydrostatic equilibrium of our proposed model, the Tolman-Oppenheimer-Volkoff (TOV) [92,93] condition is very useful which is followed by the Equation (44). There exist four different forces, such as the gravitational force (F_g), hydrostatic force (F_h), anisotropic force (F_a) and an extra force in connection to $f(\mathbb{T}, \mathcal{T})$ gravity (F_e) provided in Equation (44) and can be rewritten as

$$F_g + F_h + F_a + F_e = 0,$$

where

$$\left. \begin{aligned} F_g &= \frac{v'}{r}(\rho + p_r), \\ F_h &= -p'_r, \\ F_a &= \frac{2}{r}(p_t - p_r), \\ F_e &= -\frac{1}{4\pi + \frac{\omega}{2}} \left(\frac{\omega \rho'}{4} - \frac{\omega p'_r}{4} - \omega p'_t \right). \end{aligned} \right\} \quad (63)$$

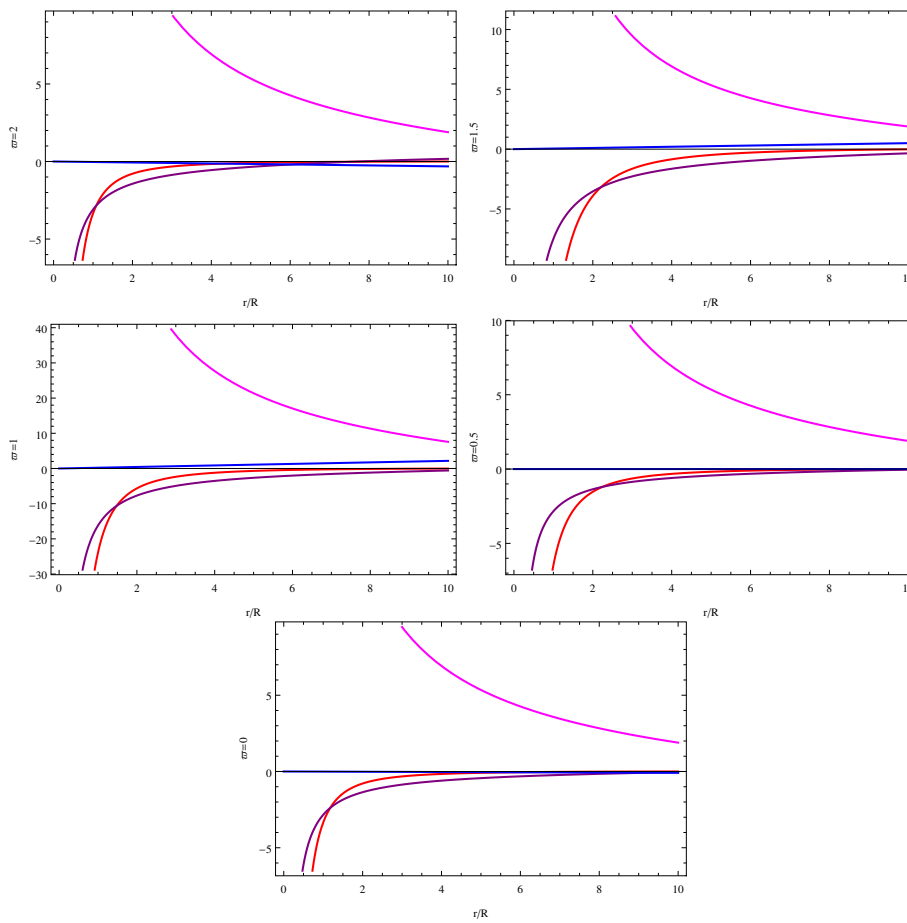


Figure 6. Plot of the different forces versus r/R for the strange star candidate $LMC X - 4$ due to different chosen values of ω . The red, purple, magenta, blue and black colors represent respectively f_g, f_h, f_a, f_e with $\omega = 0, \omega = 0.5, \omega = 1, \omega = 1.5$ and $\omega = 2$ respectively.

5.3.2. Principle of Causality

Anisotropic fluid stellar model will be physically admissible if we pay special attention in making of boundaries of the radial and the transversal speeds of sound in $(0, 1)$ within the matter distribution, i.e., they are less than the speed of light c (in relativistic geometrized units, the speed of light c becomes 1). This is known as the Causality condition which is based on the concept of Cracking condition as provided by Herrera [94]. Here, the radial and transversal speeds of sound of our model are

$$v_{sr}^2 = \frac{dp_r^{eff}}{d\rho^{eff}} = -1 + \frac{2(8\pi + \omega)}{12\pi + \omega + 3\omega c_1}, \quad (64)$$

$$v_{st}^2 = \frac{dp_t^{eff}}{d\rho^{eff}} = \frac{\omega + 3(8\pi + \omega)c_1}{2(12\pi + \omega + 3\omega c_1)}. \quad (65)$$

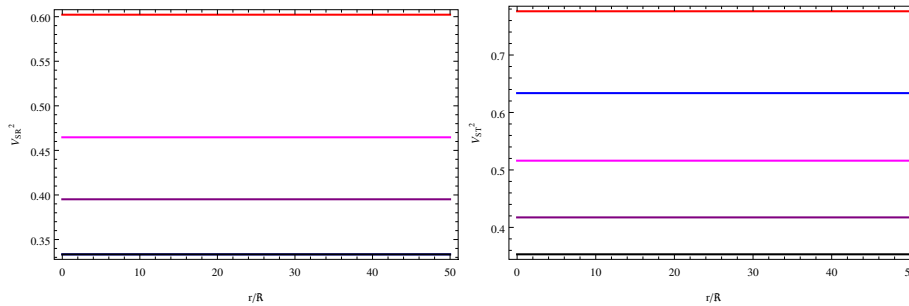


Figure 7. Plot of v_{st}^2 and v_{sr}^2 versus r/R for the strange star candidate *LMC X-4*. The red, purple, magenta, blue and black colors represent respectively cases $\omega = 0$, $\omega = 0.5$, $\omega = 1$, $\omega = 1.5$ and $\omega = 2$.

5.4. Cracking Condition

There is another way to check the stability of the system through Herrera cracking condition [94, 95]. Here, we check whether the square of radial speed of sound exceeds the square of transverse speed of sound or not for this anisotropic fluid star model. So, this criterion sets up

$$-1 \leq v_{st}^2 - v_{sr}^2 \leq 1 = \begin{cases} -1 \leq v_{st}^2 - v_{sr}^2 \leq 0 & \text{for potentially stable model} \\ 0 \leq v_{st}^2 - v_{sr}^2 \leq 1 & \text{for potentially unstable model} \end{cases} \quad (66)$$

Using Eq. (66), we calculate the difference of two speeds and they are

$$v_{st}^2 - v_{sr}^2 = -\frac{8\pi + \omega - 3(8\pi + 3\omega)c_1}{2(12\pi + \omega + 3\omega c_1)}, \quad (67)$$

$$|v_{st}^2 - v_{sr}^2| = \left| -\frac{8\pi + \omega - 3(8\pi + 3\omega)c_1}{2(12\pi + \omega + 3\omega c_1)} \right|. \quad (68)$$

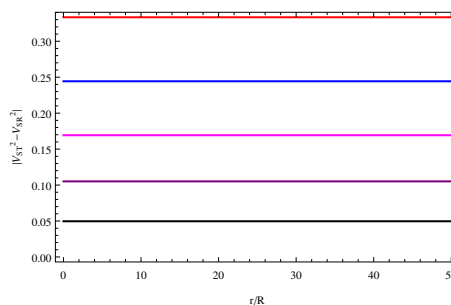


Figure 8. Plot of $|v_{st}^2 - v_{sr}^2|$ versus r/R for the strange star candidate *LMC X-4*. The red, purple, magenta, blue and black colors represent respectively cases $\omega = 0$, $\omega = 0.5$, $\omega = 1$, $\omega = 1.5$ and $\omega = 2$.

5.4.1. Adiabatic Index

Adiabatic index is a crucial feature to investigate regarding the stable status of an anisotropic fluid star model. It has been proposed that the radial adiabatic index should be greater than $\frac{4}{3}$ which

represents the rigidity of the EOS parameter of the model [96–99]. The radial adiabatic index is defined as

$$\Gamma_r = \frac{4(8\pi + \omega)(4\pi + \omega - 3\omega c_1)(R^2(B - \rho_c) + r^2(-\rho_0 + \rho_c))}{(12\pi + \omega + 3\omega c_1)(R^2(3\Lambda\omega + 2B(16\pi + \omega)) + 6R^2\omega c_2 - 2(4\pi + \omega - 3\omega c_1)(r^2\rho_0 + (R - x)(R + r)\rho_c))}, \quad (69)$$

$$\Gamma_t = \frac{3(\omega + 3(8\pi + \omega)c_1)(R^2(8\pi + 3\omega)c_2 + (8\pi + \omega + (8\pi + 3\omega)c_1)(r^2\rho_0 + (R - r)(R + r)\rho_c))}{2(12\pi + \omega + 3\omega c_1)(R^2(2B - 3\Lambda)\omega + 3R^2(8\pi + \omega)c_2 + (\omega + 3(8\pi + \omega)c_1)(r^2\rho_0 + (R - r)(R + r)\rho_c))}. \quad (70)$$

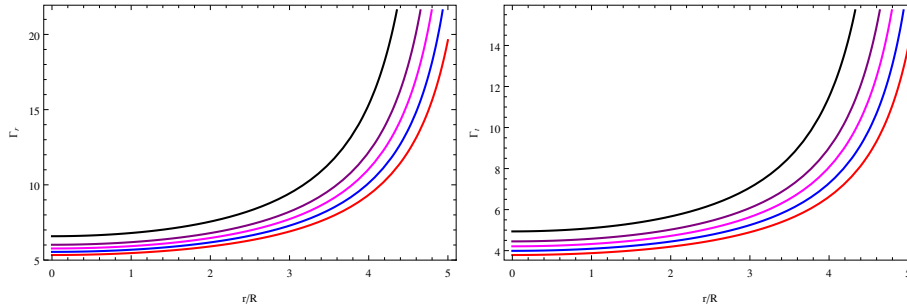


Figure 9. Plot of Γ_r and Γ_t versus r/R for the strange star candidate LMC X - 4. The red, purple, magenta, blue and black colors represent respectively cases $\omega = 0$, $\omega = 0.5$, $\omega = 1$, $\omega = 1.5$ and $\omega = 2$.

5.5. Compactification Factor and Redshift

According to Biswas et al. [58], the compactness of the anisotropic fluid model is given by the compactification factor

$$u(r) = \frac{m(r)}{r} \quad (71)$$

$$= \frac{r^4(16\pi + \omega + 4c_1\omega)(\rho_0 - \rho_c)}{20R^2} + \frac{\rho_c r^2(16\pi + \omega + 4c_1\omega)}{12} - \frac{r^2\omega(2B - 6c_2 - 3\Lambda)}{18} + \frac{d}{r}$$

and the twice of the compactification factor [100] is followed by

$$2u(r) = \frac{r^4(16\pi + \omega + 4c_1\omega)(\rho_0 - \rho_c)}{10R^2} + \frac{\rho_c r^2(16\pi + \omega + 4c_1\omega)}{6} - \frac{r^2\omega(2B - 6c_2 - 3\Lambda)}{9} + \frac{2d}{r}. \quad (72)$$

The surface redshift can be defined as

$$z_s = \frac{1}{\sqrt{1 - \frac{2m(r)}{r}}} - 1 \quad (73)$$

$$= -1 + \frac{1}{\sqrt{1 - \frac{5R^2(r^3(-2B + 3\Lambda)\omega + 18c_3) + 2r^3(15R^2\omega c_2 + (12\pi + \omega + 3\omega c_1)(3r^2(\rho_0 - \rho_c) + 5R^2\rho_c))}{45R^2r}}}}$$

6. A Comparison Study Between $f(\mathbb{T}, \mathcal{T})$ gravity and Standard GR

In this Section we perform a comparative study with the standard results of GR, in order to show what is the effect of our model in the framework of $f(\mathbb{T}, \mathcal{T})$ gravity and why it justifies our analysis and thereafter do relevant discussion on the results. These aspects can be discussed as follows:

(i) We have plotted in Fig. 1 the behavior of the metric potentials e^ν and e^λ with respect to the fractional radial coordinate r/R respectively both of which exhibit normal features as are expected, i.e. they should exhibit regular behavior at the center as well as on the surface of the stellar system. On

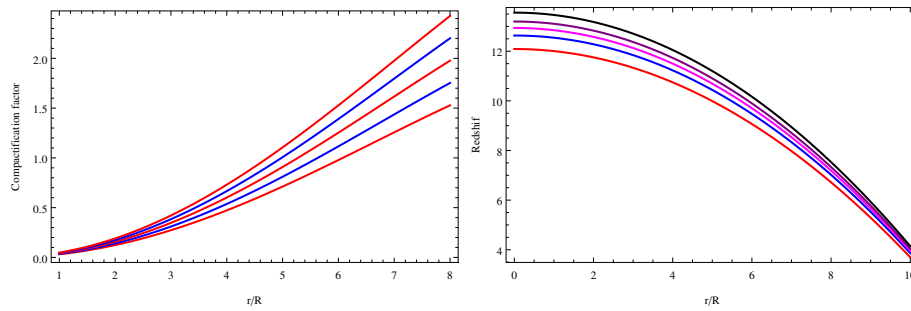


Figure 10. Plot of the compactification factor and redshift versus r/R for the strange star candidate LMC X – 4. The red, purple, magenta, blue and black colors represent respectively cases $\omega = 0$, $\omega = 0.5$, $\omega = 1$, $\omega = 1.5$ and $\omega = 2$.

the other hand, in Fig. 2 the variation of the effective energy density ρ^{eff} and the effective pressures p_r^{eff} and p_t^{eff} are shown with respect to r/R respectively whereas the anisotropic feature is shown in Fig. 3. We observe from all the plots of Fig. 2 that a kind of monotony is evolving from the center to a minimum value and hence indicates the physically viable character of the $f(\mathbb{T}, \mathcal{T})$ gravity model under consideration. One can note that the standard solutions to the Einstein field equations for stellar modeling can be recovered for $\omega = 0$ and $\Lambda = 0$.

(ii) From Fig. 4, we can conclude that the conditions NEC, WEC, DEC and SEC are satisfied for different values of coupling matter geometry parameter ω for our anisotropic fluid model absence of which GR features can be retrieved.

(iii) Since $r \rightarrow 0$, $m(r) \rightarrow 0$ in Fig. 5 so one can notice that the mass function is regular at the origin. From this mass-radius relationship we also note a striking feature which depicts that with increasing value of radius the mass of the physical system also increases and beyond $R \sim 12$ km mass starts gradually decreasing. Thus we observe a stable configuration of the strange star around 12 km below and beyond which it suffers from instability. Moreover, we observe that in contrast to GR case the curve allows for a massive quark star at the same radius.

(iv) Based on Equation (63), i.e. the modified TOV equation, we have drawn Fig. 6 which indicates that our propounded anisotropic fluid model for different values of coupling matter-geometry parameter ω is in hydrostatic equilibrium state under the combination of considered forces.

(v) From Equations (64) and (65), we observe that the Causality condition depend on the coupling matter-geometry parameter ω and the constant c_1 . The related Fig. 7 exhibits that Principle of Causality is preserved inside the anisotropic strange star in $f(\mathbb{T}, \mathcal{T})$ as here $0 \leq v_{sr} \leq 1$ and $0 \leq v_{st}^2 \leq 1$ if $0 \leq c_1 \leq 1$.

(vi) In a similar way, Equations (67) and (68) indicate that the Cracking condition depend on the coupling matter-geometry parameter ω and c_1 . If $0 \leq c_1 \leq \frac{1}{3}$ then we have $-1 \leq v_{st}^2 - v_{sr}^2 \leq 0$ and $0 \leq |v_{st}^2 - v_{sr}^2| \leq 1$ which is evident from Fig. 8 and inconformity with the criteria imposed on the system [94,101]. It means that we have potentially stable anisotropic star fluid model.

(vii) From Fig. 9, we note that the variation of both Γ_r and Γ_t is a monotonically increasing function at all the interior points of our model with $\Gamma_r > \frac{4}{3}$ and hence provides confirmation on stability of our model as predicted for the inside of an anisotropic, relativistic and dynamically stable stellar system [102–104].

(viii) Fig. 10 (left panel) indicates that $u(r)$ is lying between the range $\frac{1}{4}$ and $\frac{1}{2}$. Also, the $2u(r)$ follows the maximum allowed value $\frac{8}{9}$ for our proposed model. On the other hand, we have plotted figure for the surface redshift with the help of Equation (73) in Fig. 10 (right panel) which shows that it is a decreasing function starting from the core with higher value and on the surface of the star acquires a finite non-zero value. In this context it is to be noted that Barraco and Hamity [105] proved that for an isotropic star and in absence of the cosmological constant $Z_s < 2$. According to Böhmer and Harko [106] the surface redshift for an anisotropic star can reach maximum higher value $Z_s \leq 5$, in

presence of the cosmological constant. Though according to Ivanov [107] maximum acceptable value of the surface redshift will be restricted upto 5.211. As our result for the surface redshift maintains the limit, i.e. $z_s \leq 5$ always, so proposed model is quite realistic.

In a nutshell, in all the above figures one can note that we get back the GR result for $\omega = 0$ whereas all the other non-zero values of ω correspond to $f(\mathbb{T}, \mathcal{T})$ gravity contribution, e.g. in Fig. 6 the null effect due to modified gravity exhibits through the flat curve which is being coincided with the x -axis and hence shows the standard GR result. However, the effect of ω becomes gradually prominent as we go on increasing its value towards higher magnitude.

7. Conclusion

In this work, we present mathematical modeling of strange star in the background of $f(\mathbb{T}, \mathcal{T})$ gravity in Einstein-Maxwell spacetime in order to obtain viable physical properties. We develop the equations of motion using anisotropic property within the spherically symmetric strange star and thereafter explore the physical features, like the energy conditions, mass-radius relations, modified TOV equations, principle of causality, adiabatic index, redshift and stability analysis of our model.

The salient features of the presented model can be put forward as follows:

(1) For the matter-geometry coupling parameter $\omega = 0$ the usual results as available in GR can be retrieved from the results due to modified gravity $f(\mathbb{T}, \mathcal{T})$. The above statement obviously also follows from the Figs. 1 - 10, e.g. for $\omega = 0$ the usual form of TOV equation as in GR can easily be retrieved because in this situation the force due to modified gravity vanishes, i.e., the effect of coupling also becomes ineffective. In order to verify model parameters we have used the data for the strange star candidate LMC X - 4 and have drawn all the Figs. 1 - 10 for different values of the matter-geometry coupling parameter ω .

(2) In the present work we consider tetrad formalism to correspond to the metric (25), however it is known that in spherically symmetric solutions the diagonal tetrad is not the consistent choice. In this connection it is to note that in literature this type of approach is not at all unavailable in the $f(\mathbb{T}, \mathcal{T})$ gravity theory [59,70–72]. Harko et al. [59] have employed the methodology in cosmology whereas Pace and Said [71,72] as well as Ghosh et al. [70] have used this technique in astrophysics. Nevertheless, irrespective of the background mathematical formalism, eventually one can easily find out the modified Einstein equations in $f(\mathbb{T}, \mathcal{T})$ gravity which have eventually the same form whatever be their background mathematical root or origin.

(3) In this work, as a first step, we considered a very specific ansatz, viz. $f(\mathbb{T}, \mathcal{T}) = \omega T^n T - 2\Lambda$, to present a simple mathematical model following the prescription by Harko et al. [59] to comply with the observational data. However, after successful execution of this specification there are ample scope to opt for other specified functional forms of $f(\mathbb{T}, \mathcal{T})$ gravity in the future projects following other ansatz, e.g. $f(\mathbb{T}, \mathcal{T}) = \alpha \mathcal{T} + \gamma T^2$ which has been also proposed by Harko et al. [59] or linear function in the forms (i) $f = \alpha T(r) + \beta \mathcal{T}(r) + \varphi$ as proposed by Pace and Said [71] and (ii) $f = T(r) + \mathcal{T}(r) + \omega h(\mathbb{T}, \mathcal{T}) + \mathcal{O}(\omega^2)$ as proposed by Pace and Said [72]. In the above ansatz, which has been employed in the present work, the value of n is assumed to be unity, i.e. $n = 1$, however $n \neq 1$ case also can be considered for further study.

(4) We would also like to point out here that in the present investigation, following Rahaman et al. [79], we have considered the value for Bag constant as $B = 83 \text{ MeV} / fm^3$. However, it seems that this value is quite arbitrary and needs specific range with lower and upper values [108]. This means that further extension of the present work will provide more realistic stellar model to validate on the physical ground.

So, based on the features as shown by the Figs. 1 - 10 and other aspects, we can overall conclude that our propounded model is quite realistic and appeals to further study of theoretically observable features as well as observational signatures of its predicted compact objects.

Funding: This research received no external funding.

Acknowledgments: The work by MK on astrophysical effects of modified gravity was supported by grant of the Russian Science Foundation (Project No-18-12-00213). SR and UD are thankful to the authority of Inter-University Centre for Astronomy and Astrophysics, Pune, India for providing them Visiting Associateship under which a part of this work was carried out.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Bodmer, A.R. Collapsed nuclei. *Phys. Rev. D* **1971**, *4*, 1601-1606.
2. Terazawa, H. Prospect of Elementary-Particle Physics and Nuclear Physics. INS-Report-336, Tokyo University, 1979.
3. Witten, E. Cosmic separation of phases. *Phys. Rev. D* **1984**, *30*, 272-285.
4. Baade, W.; Zwicky, F. Remarks on super-novae and cosmic rays. *Phys. Rev.* **1934**, *46*, 76-77.
5. Schwarzschild, K. Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie. *Sitzungsber. Preuß. Akad. Wissenschaften Berlin Phys. Math. Kl* **1916**, 189-196 [On the gravitational field of a mass point according to Einstein's theory, arXiv:physics/9905030].
6. Schwarzschild, K. Über das Gravitationsfeld einer Kugel aus inkompressibler Flüssigkeit nach der Einsteinschen Theorie. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin, Phys.-Math. Klasse* **1916**, 424-434 [On the gravitational field of a sphere of incompressible fluid according to Einstein's theory, arXiv: physics/9912033].
7. Ruderman, R. Pulsars: Structures and dynamics. *Ann. Rev. Astron. Astrophys.* **1972**, *10*, 427-476.
8. Herrera, L.; Santos, N.O. Local anisotropy in self-gravitating systems. *Phys. Rep.* **1997**, *286*, 53-130.
9. Hewish, A.; Bell, S.J.; Pilkington, J.D.H.; Scott, P.F.; Collins, R.A. Observation of a rapidly pulsating radio source. *Nature* **1968**, *217*, 709-713.
10. Hossein, S.M.; Rahaman, F.; Naskar, J.; Kalam, M.; Ray, S. Anisotropic compact stars with variable cosmological constant. *Int. J. Mod. Phys. D* **2012**, *21*, 1250088.
11. Kalam, M.; Rahaman, F.; Molla, S.; Hossein, S.M. Anisotropic quintessence stars. *Astrophys. Space Sci.* **2014**, *349*, 865-871.
12. Krori, K.D.; Barua, J. A singularity-free solution for a charged fluid sphere in general relativity. *J. Phys. A: Math. Gen.* **1975**, *8*, 508-511.
13. Kalam, M.; Rahaman, F.; Ray, S.; Hossein, M.; Karar, I.; Naskar, J. Anisotropic strange star with de Sitter spacetime. *Eur. Phys. J. C* **2012**, *72*, 2248.
14. Paul, B.C.; Deb, R. Relativistic solutions of anisotropic compact objects. *Astrophys. Space Sci.* **2014**, *354*, 421-430.
15. Witten, E. Cosmic separation of phases. *Phys. Rev. D* **1984**, *30*, 272-285.
16. Cheng, K.S.; Dai, Z.G.; Lu, T. Strange stars and related astrophysical phenomena. *Int. J. Mod. Phys. D* **1998**, *7*, 139-176.
17. Bhar, P. Singularity-free anisotropic strange quintessence star. *Astrophys. Space Sci.* **2015**, *356*, 309-318.
18. Abbas, G.; Zubair, M.; Mustafa, G. Anisotropic strange quintessence stars in $f(R)$ gravity. *Astrophys. Space Sci.* **2015**, *358*, 26.
19. Rahaman, F.; Chakraborty, K.; Kuhfittig, P.K.F.; Shit, G.C.; Rahman, M. A new deterministic model of strange stars. *Eur. Phys. J. C* **2014**, *74*, 3126.
20. Arbañil, J.D.V.; Malheiro, M. Radial stability of anisotropic strange quark stars. *J. Cosmol. Astropart. Phys.* **2016**, *11*, 012.
21. Murad, M.H. Some analytical models of anisotropic strange stars. *Astrophys. Space Sci.* **2016**, *361*, 20.
22. Mak, M.K.; Harko, T. Quark stars admitting a one-parameter group of conformal motions. *Int. J. Mod. Phys. D* **2004**, *13*, 149-156.
23. Abbas, G. Cardy-Verlinde Formula of Non-Commutative Schwarzschild Black Hole. *Adv. High Energy Phys.* **2014**, *2014*, 306256.
24. Abbas, G. Collapse and expansion of anisotropic plane symmetric source. *Astrophys. Space Sci.* **2014**, *350*, 307-311.
25. Abbas, G.; Sabiullah, U. Geodesic study of regular Hayward black hole Geodesic study of regular Hayward black hole. *Astrophys. Space Sci.* **2014**, *352*, 769-774.

26. Abbas, G. Phantom Energy Accretion onto a Black hole in Horava-Lifshitz Gravity. *Sci. China Phys. Mech. Astron.* **2014**, *57*, 604-607.
27. Abbas, G. Effects of electromagnetic field on the collapse and expansion of anisotropic gravitating source. *Astrophys. Space Sci.* **2014**, *352*, 955-961.
28. Abbas, G.; Kanwal, A.; Zubair, M. Anisotropic Compact Stars in $f(T)$ Gravity. *Astrophys. Space Sci.* **2015**, *357*, 109.
29. Abbas, G.; Nazeer, S.; Meraj, M.A. Cylindrically Symmetric Models of Anisotropic Compact Objects. *Astrophys. Space Sci.* **2014**, *354*, 449-455.
30. Abbas, G.; Momeni, D.; Ali, M.A.; Myrzakulov, R.; Qaisar, S. Anisotropic Compact Stars in $f(G)$ Gravity. *Astrophys. Space Sci.* **2015**, *357*, 158.
31. Deb, D.; Ghosh, S.; Maurya, S.K.; Khlopov, M.; Ray, S. Anisotropic compact stars in $f(T)$ gravity under Karmarkar condition. *Tech Vistas* **2018**, *1*, 1- 20.
32. Das, A.; Ghosh, S.; Deb, D.; Rahaman, F.; Ray, S. Study of gravastars under $f(T)$ gravity. *Nucl. Phys. B* **2020**, *954*, 114986.
33. Nojiri, S.; Odintsov, S.D. Modified Gauss–Bonnet theory as gravitational alternative for dark energy. *Phys. Lett. B* **2005**, *631*, 1-6.
34. Abbas, G.; Qaisar, S.; Jawad, A. Strange stars in $f(T)$ gravity with MIT bag model. *Astrophys. Space Sci.* **2015**, *359*, 57.
35. Oliveira, A.M.; Velten, H.E.S.; Fabris, J.C.; Casarini, L. Neutron stars in Rastall gravity. *Phys. Rev. D* **2015**, *92*, 044020.
36. Abbas, G.; Shahzad, M.R. A new model of quintessence compact stars in the Rastall theory of gravity. *Eur. Phys. J. A* **2018**, *54*, 211.
37. Abbas, G.; Shahzad, M.R. Models of anisotropic compact stars in the Rastall theory of gravity. *Astrophys. Space Sci.* **2019**, *364*, 50.
38. Mota, C.E.; Santos, L.C.N.; Grams, G.; da Silva, F.M.; Menezes, D.P. Combined Rastall and rainbow theories of gravity with applications to neutron stars. *Phys. Rev. D* **2019**, *100*, 024043.
39. Mota, C.E.; Santos, L.C.N.; da Silva, F.M.; Flores, C.V.; da Silva, T.J.N.; Menezes, D.P. Anisotropic Compact Stars in Rastall-Rainbow Gravity. arXiv:1911.03208 [astro-ph.HE].
40. Rizaldy, R.; Sulaksono, A. Deformation of a magnetized quark star in Rastall gravity. *J. Phys. Conf. Ser.* **2019**, *1321*, 022016.
41. Shahzad, M.R.; Abbas, G. Strange stars with MIT bag model in the Rastall theory of gravity. *Int. J. Geom. Meth. Mod. Phys.* **2019**, *16*, 1950132.
42. Nazar, H.; Abbas, G. Charged anisotropic collapsing stars with heat flux in $f(R)$ gravity. *Chin. J. Phys.* **2020**, *63*, 436-447.
43. Sharif, M.; Waseem, A. Anisotropic quark stars in $f(R, T)$ gravity. *Eur. Phys. J. C* **2018**, *78*, 868.
44. Maurya, S.K.; Tello-Ortiz, F. Charged anisotropic strange stars in general relativity. *Eur. Phys. J. C* **2019**, *79*, 33.
45. Maurya, S.K.; Banerjee, A.; Jasim, M.K.; Kumar, J.; Prasad, A.K.; Pradhan, A. Anisotropic compact stars in the Buchdahl model: A comprehensive study. *Phys. Rev. D* **2019**, *99*, 044029.
46. Maurya, S.K.; Tello-Ortiz, F. Generalized relativistic anisotropic compact star models by gravitational decoupling. *Eur. Phys. J. C* **2019**, *79*, 85.
47. Prasad, A.K.; Kumar, J.; Maurya, S.K.; Dayanandan, B. Relativistic model for anisotropic compact stars using Karmarkar condition. *Astrophys. Space Sci.* **2019**, *364*, 66.
48. Abbas, G.; Shahzad, M.R. Models of anisotropic compact stars in the Rastall theory of gravity. *Astrophys. Space Sci.* **2019**, *364*, 50.
49. Saha, P.; Debnath, U. Anisotropic Quintessence Strange Stars in $f(T)$ Gravity with Modified Chaplygin Gas. *Adv. High Energy Phys.*, **2018**, 3901790.
50. Saha, P.; Debnath, U. Study of anisotropic compact stars with quintessence field and modified chaplygin gas in $f(T)$ gravity. *Eur. Phys. J. C* **2019**, *79*, 919.
51. Deb, D.; Rahaman, F.; Ray, S.; Guha, B.K. Anisotropic strange stars under simplest minimal matter-geometry coupling in the $f(R, T)$ gravity. *Phys. Rev. D* **2018**, *97*, 084026.
52. Deb, D.; Rahaman, F.; Ray, S.; Guha, B.K. Strange stars in $f(R, T)$ gravity. *J. Cosmol. Astropart. Phys.* **2018**, *03*, 044.

53. Biswas, S.; Ghosh, S.; Ray, S.; Rahaman, F.; Guha, B.K. Strange stars in Krori-Barua space-time under $f(R, T)$ gravity. *Ann. Phys.* **2019**, *401*, 1.
54. Moraes, P.H.R.S.; Arbañil, J.D.V.; Malheiro, M. Stellar equilibrium configurations of compact stars in $f(R, T)$ theory of gravity. *J. Cosmol. Astropart. Phys.* **2016**, *06*, 005.
55. Das, A.; Ghosh, S.; Guha, B.K.; Das, S.; Rahaman, F.; Ray, S. Gravastars in $f(R, T)$ gravity. *Phys. Rev. D* **2017**, *95*, 124011.
56. Deb, D.; Ketov, S.V.; Khlopov, M.; Ray, S. Study on charged strange stars in $f(R, T)$ gravity. *J. Cosmol. Astropart. Phys.* **2019**, *10*, 070.
57. Deb, D.; Ketov, S.V.; Maurya, S.K.; Khlopov, M.; Moraes, P.H.R.S.; Ray, S. Exploring physical features of anisotropic strange stars beyond standard maximum mass limit in gravity. *Mon. Not. R. Astron. Soc.* **2019**, *485*, 5652-5665.
58. Biswas, S.; Shee, D.; Ray, S.; Guha, B.K. Anisotropic strange star with Tolman–Kuchowicz metric under $f(R, T)$ gravity. *Eur. Phys. J C* **2020**, *80*, 175.
59. Harko, T.; Lobo, F.S.N.; Otalora, G.; Saridakis, E.N. $f(T, T)$ gravity and cosmology. *J. Cosmol. Astropart. Phys.* **2014**, *12*, 021.
60. Momeni, D.; Myrzakulov, R. Cosmological reconstruction of $f(T, T)$ gravity. *Int. J. Geom. Methods Mod. Phys.* **2014**, *11*, 1450077.
61. Nassur, S.B.; Houndjo, M.J.S.; Rodrigues, M.E.; Kpadonou, A.V.; Tossa, J. From the early to the late time universe within $f(T, T)$ gravity. *Astrophys. Space Sci.* **2015**, *360*, 60.
62. Salako, I.G.; Jawad, A.; Chattopadhyay, S. Holographic dark energy reconstruction in $f(T, T)$ gravity. *Astrophys. Space Sci.* **2015**, *358*, 13.
63. Ganiou, M.G.; Salako, I.G.; Houndjo, M.J.S.; Tossa, J. $f(T, T)$ cosmological models in phase space. *Astrophys. Space Sci.* **2016**, *361*, 57.
64. Ganiou, M.G.; Salako, I.G.; Houndjo, M.J.S.; Tossa, J. Geodesic Deviation Equation in CDM $f(T, T)$ Gravity. *Int. J. Theor. Phys.* **2016**, *55*, 3954-3972.
65. Junior, E.L.B.; Rodrigues, M.E.; Salako, I.G.; Houndjo, M.J.S. Reconstruction, thermodynamics and stability of the CDM model in $f(T, T)$ gravity. *Class. Quantum Gravit.* **2016**, *33*, 125006.
66. Sáez-Gómez, D.; Carvalho, C.S.; Lobo, F.S.N.; Tereno, I. Constraining $f(T, T)$ gravity models using type Ia supernovae. *Phys. Rev. D* **2016**, *94*, 024034.
67. Farrugia, G.; Said, J.L. Growth factor in $f(T, T)$ gravity. *Phys. Rev. D* **2016**, *94*, 124004.
68. Rezaei, T.M.; Amani, A. Stability and interacting $f(T, T)$ gravity with modified Chaplygin gas. *Can. J. Phys.* **2017**, *95*, 1068-1073.
69. Arouko, M.Z.; Salako, I.G.; Kanfon, A.D.; Houndjo, M.J.S.; Baffou, E. Rip cosmologies, Wormhole Solutions and Big Trip in the $f(T, T)$ theory of gravity. arXiv: 2004.06000 [gr-qc].
70. Ghosh, S.; Kanfon, A.D.; Das, A.; Houndjo, M.J.S.; Salako, I.G.; Ray, S. Gravastars in $f(T, T)$ gravity. *Int. J. Mod. Phys. A* **2020**, *35*, 2050017.
71. Pace, M.; Said, J.L. Quark stars in $f(T, T)$ -gravity. *Eur. Phys. J. C* **2017**, *77*, 62.
72. Pace, M.; Said, J.L. A perturbative approach to neutron stars in $f(T, T)$ -gravity. *Eur. Phys. J. C* **2017**, *77*, 283.
73. Maluf, J.W.; Ulhoa, S.C. On the gravitational angular momentum of rotating sources. *Gen. Rel. Gravit.* **2009**, *41*, 1233-1247.
74. Maluf, J.W.; Faria, F.F.; Ulhoa, S.C. On reference frames in spacetime and gravitational energy in freely falling frames. *Class. Quantum Gravit.* **2007**, *24*, 2743-2754.
75. Maluf, J.W.; Ulhoa, S.C.; da Rocha-Neto, J.F. Gravitational pressure on event horizons and thermodynamics in the teleparallel framework. *Phys. Rev. D* **2012**, *85*, 044050.
76. Daouda, M.H.; Rodrigues, M.E.; Houndjo, M.J.S. New static solutions in $f(T)$ theory. *Eur. Phys. J. C* **2011**, *71*, 1817.
77. Daouda, M.H.; Rodrigues, M.E.; Houndjo, M.J.S. Static anisotropic solutions in $f(T)$ theory. *Eur. Phys. J. C* **2012**, *72*, 1890.
78. Chodos, A.; Jaffe, R.L.; Johnson, K.; Thorn, C.B.; Weisskopf, V.F. New extended model of hadrons. *Phys. Rev. D* **1974**, *9*, 3471-3495.
79. Rahaman, F.; Chakraborty, K.; Kuhfittig, P.K.F.; Shit, G.C.; Rahman, M. A new deterministic model of strange stars. *Eur. Phys. J. C* **2014**, *74*, 3126.

80. Zdunik, J.L.; Bulik, T.; Kluzniak, W.; Haensel, P.; Gondek-Rosinska, D. On the mass of moderately rotating strange stars in the MIT bag model and LMXBs. *Astron. Astrophys.* **2000**, *359*, 143-147.
81. Maieron, C.; Baldo, M.; Burgio, G.F.; Schulze, H.-J. Hybrid stars with the color dielectric and the MIT bag models. *Phys. Rev. D* **2004**, *70*, 043010.
82. Nicotra, O.E.; Baldo, M.; Burgio, G.F.; Schulze, H.-J. Hybrid protoneutron stars with the MIT bag mode. *Phys. Rev. D* **2006**, *74*, 123001.
83. Bao, T.; Liu, G.-Z.; Zhu, M.-F. Properties of hybrid stars in an extended MIT bag model. *Chin. Phys. C* **2009**, *33*, 340-344.
84. Uechi, S.T.; Uechi, H. Hardon-quark hybrid stars constructed by the nonlinear σ - ω - ρ mean-field model and MIT-bag model. arXiv:1003.4815 [nucl-th].
85. Isayev, A.A. Stability of magnetized strange quark matter in the MIT bag model with a density dependent bag pressure. *Phys. Rev. C* **2015**, *91*, 015208.
86. Cardoso, P.H.G.; da Silva, T.N.; Deppman, A.; Menezes, D.P. Quark matter revisited with non-extensive MIT bag model. *Eur. Phys. J. A* **2017**, *53*, 191.
87. Joshi, S.; Sau, S.; Sanyal, S. Quark cores in extensions of the MIT Bag model. arXiv:2002.07647 [nucl-th].
88. Mak, M.K.; Harko, T. Anisotropic stars in general relativity. *Proc. Roy. Soc. Lond. A* **2003**, *459*, 393-408.
89. Moraes, P.H.R.S.; Correa, R.A.C.; Lobato, R.V. Analytical general solutions for static wormholes in $f(R, T)$ gravity. *J. Cosmol. Astropart. Phys.* **2017**, *1707*, 029.
90. Maurya, S.K.; Gupta, Y.K.; Ray, S.; Deb, D. Generalised model for anisotropic compact stars. *Eur. Phys. J. C* **2016**, *76*, 693.
91. de León, J.P. Limiting configurations allowed by the energy conditions. *Gen. Rel. Gravit.* **1993**, *25*, 1123-1137.
92. Tolman, R.C. Static solutions of Einstein's field equations for spheres of fluid. *Phys. Rev.* **1939**, *55*, 364-373.
93. Oppenheimer, J.R.; Volkoff, G.M. On massive neutron cores. *Phys. Rev.* **1939**, *55*, 374-381.
94. Herrera, L. Cracking of self-gravitating compact objects. *Phys. Lett. A* **1992**, *165*, 206-210.
95. Abreu, H.; Hernández, H.; L.A. Núñez, L.A. Sound speeds, cracking and stability of self-gravitating anisotropic compact objects. *Class. Quantum Gravit.* **2007**, *24*, 4631-4645.
96. Chandrasekhar, S. The Dynamical Instability of Gaseous Masses Approaching the Schwarzschild Limit in General Relativity. *Astrophys. J.* **1964**, *12*, 114-116.
97. Heintzmann, H.; Hillebrandt, W. Neutron stars with an anisotropic equation of state - mass, redshift and stability. *Astron. Astrophys.* **1975**, *38*, 51-55.
98. Hillebrandt, W.; Steinmetz, K.O. Anisotropic neutron star models-Stability against radial and nonradial pulsations. *Astron. Astrophys.* **1976**, *53*, 283-287.
99. Bombaci, I. The maximum mass of a neutron star. *Astron. Astrophys.* **1996**, *305*, 871-877.
100. Buchdahl, H.A. General relativistic fluid spheres. *Phys. Rev.* **1959**, *116*, 1027-1034.
101. Andréasson, H. Sharp bounds on the critical stability radius for relativistic charged spheres. *Commun. Math. Phys.* **2009**, *288*, 715-730.
102. Bondi, H. The contraction of gravitating spheres. *Proc. R. Soc. Lond. A* **1964**, *281*, 39-48.
103. Heintzmann, H.; Hillebrandt, W. Neutron stars with an anisotropic equation of state: Mass, redshift and stability. *Astron. Astrophys.* **1975**, *24*, 51-55.
104. Chan, R.; Herrera, L.; Santos, N.O. Dynamical instability for radiating anisotropic collapse. *Mon. Not. R. Astron. Soc.* **1993**, *265*, 533-544.
105. Barraco, D.E.; Hamity, V.H. Maximum mass of a spherically symmetric isotropic star. *Phys. Rev. D* **2002**, *65*, 124028.
106. Böhmer, C.G.; Harko, T. Bounds on the basic physical parameters for anisotropic compact general relativistic objects. *Class. Quantum Gravit.* **2006**, *23*, 6479-6491.
107. Ivanov, B.V. Maximum bounds on the surface redshift of anisotropic stars. *Phys. Rev. D* **2002**, *65*, 104011.
108. Aziz, A.; Ray, S.; Rahaman, F.; Khlopov, M.; Guha, B.K. Constraining values of bag constant for strange star candidates. *Int. J. Mod. Phys. D* **2019**, *28*, 1941006.