Analytical formulas for calculating ranks, dimensions, orthogonal projectors, and ranges of matrices composed of Kronecker products

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Abstract. Kronecker products of matrices have some striking operation properties, one of which is called the mixed-product property $(A\otimes B)(C\otimes D) = AC\otimes BD$. In view of this property, the two-term Kronecker product $A_1 \otimes A_2$ can be rewritten as $A_1 \otimes A_2 = (A_1 \otimes I_{m_2})(I_{m_1} \otimes A_2)$ of dilation forms of A_1 and A_2, and three-term Kronecker product $A_1 \otimes A_2 \otimes A_3$ can be rewritten as the products $A_1 \otimes A_2 \otimes A_3 = (A_1 \otimes I_{m_2} \otimes I_{m_3})(I_{m_1} \otimes A_2 \otimes I_{m_3})(I_{m_1} \otimes I_{m_2} \otimes A_3)$ of the dilation forms of A_1, A_2, and A_3, respectively, where the matrices on the right-hand sides of the two factorizations are commutative. In this note, we approach the commutative Kronecker products on the right-hand sides of the two factorization equalities, and present a variety of new and useful analytical formulas for calculating the ranks, dimensions, orthogonal projectors, and ranges of matrices composed of these Kronecker products.

Keywords: Kronecker product, rank, dimension, orthogonal projector, range

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1 Introduction

Throughout this note, let $\mathbb{C}^{m\times n}$ denote the collection of all $m \times n$ complex matrices, A^*, $r(A)$, and $\mathscr{R}(A)$ denote the conjugate transpose, the rank, and the range of $A \in \mathbb{C}^{m\times n}$, respectively; I_m denote the identity matrix of order m; $[\,A,\, B\,]$ denotes a row block matrix consisting of A and B. The Moore–Penrose inverse of $A \in \mathbb{C}^{m\times n}$, denoted by $A^\dagger$, is the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying the four Penrose equations

$$(1)\ AXA = A,\quad (2)\ XAX = X,\quad (3)\ (AX)^* = AX,\quad (4)\ (XA)^* = XA. \tag{1.1}$$

In addition, we denote by $P_A = AA^\dagger$ and $E_A = I_m - AA^\dagger$ the two orthogonal projectors induced from A. The Kronecker product of any two matrices $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{p\times q}$ is defined to be $A\otimes B = (a_{ij}B) \in \mathbb{C}^{mp\times nq}$.

It is well known that the Kronecker product of matrices is a notion and mathematical technique in matrix theory which have many significant applications in the research areas of both theoretical and applied mathematics. The fundamental theory and formulas for Kronecker products of matrices can be found in most common or specified textbooks and handbooks on matrix analysis or calculus, and have been extensively used by knowledgeable professionals in mathematics and other disciplines. Recall that one of the most important features of Kronecker products is that we rewrite the Kronecker products as conventional matrices of certain factors involving identity matrices, such as, the two-term case

$$A_1 \otimes A_2 = (A_1 \otimes I_{m_2})(I_{m_1} \otimes A_2), \tag{1.2}$$

and the three-term case

$$A_1 \otimes A_2 \otimes A_3 = (A_1 \otimes I_{m_2} \otimes I_{m_3})(I_{m_1} \otimes A_2 \otimes I_{m_3})(I_{m_1} \otimes I_{m_2} \otimes A_3), \tag{1.3}$$

where the five factors in the parentheses on the right-hand sides of (1.2) and (1.3), called the dilation forms of A_1, A_2, and A_3, commute each other by the well-known mixed-product property in (1.11). In addition, the products on the right-hand sides occur in the studies of some linear matrix equations by vectorization operations and Kronecker products of matrices (cf. [1–5, 8, 10]). In order to develop a deeper understanding of these specified Kronecker products and their connections, we provide in this note some new approaches to the the five dilation matrices, including a variety of fundamental and analytical formulas for calculating ranks, dimensions, orthogonal projectors, and ranges of the five dilation matrices and their algebraic operations.

We shall use the following known formulas on ranks of matrices and their consequences on commutativity of two orthogonal projectors (cf. [9, 13]).

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Lemma 1.1. *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$. *Then the following rank equalities*

$$r[A,\ B] = r(A) + r(E_AB) = r(B) + r(E_BA), \tag{1.4}$$
$$r[A,\ B] = r(A) + r(B) - 2r(A^*B) + r[P_AP_B,\ P_BP_A], \tag{1.5}$$
$$r[A,\ B] = r(A) + r(B) - r(A^*B) + 2^{-1}r(P_AP_B - P_BP_A), \tag{1.6}$$
$$r[A,\ B] = r(A) + r(B) - r(A^*B) + r(P_{[A,\ B]} - P_A - P_B + P_AP_B) \tag{1.7}$$

hold. Hence,

$$\begin{aligned} P_AP_B = P_BP_A &\Leftrightarrow P_{[A,\ B]} = P_A + P_B - P_AP_B \\ &\Leftrightarrow r(E_AB) = r(B) - r(A^*B) \\ &\Leftrightarrow r[A,\ B] = r(A) + r(B) - r(A^*B) \\ &\Leftrightarrow \mathscr{R}(P_AP_B) = \mathscr{R}(P_BP_A). \end{aligned} \tag{1.8}$$

Lemma 1.2 ([14]). *Let* $A,\ B,\ C \in \mathbb{C}^{m\times m}$ *be three idempotent matrices. Then the following rank equality holds*

$$\begin{aligned} r[A,\ B,\ C] = {} & r(A) + r(B) + r(C) - r[AB,\ AC] - r[BA,\ BC] - r[CA,\ CB] \\ & + r[AB,\ AC,\ BA,\ BC,\ CA,\ CB]; \end{aligned} \tag{1.9}$$

if $AB = BA$, $AC = CA$, *and* $BC = CB$, *then*

$$r[A,\ B,\ C] = r(A) + r(B) + r(C) - r[AB,\ AC] - r[BA,\ BC] - r[CA,\ CB] + r[AB,\ AC,\ BC]. \tag{1.10}$$

The following basic results on Kronecker products of matrices are well known (cf. [6, 7, 16]).

Lemma 1.3. *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times q}$, $C \in \mathbb{C}^{n\times s}$, *and* $D \in \mathbb{C}^{q\times t}$. *Then, the following equalities hold*

$$(A\otimes B)(C\otimes D) = AC\otimes BD, \tag{1.11}$$
$$(A\otimes B)^* = A^*\otimes B^*, \tag{1.12}$$
$$(A\otimes B)^\dagger = A^\dagger\otimes B^\dagger, \tag{1.13}$$
$$P_{A\otimes B} = P_A\otimes P_B, \tag{1.14}$$
$$r(A\otimes B) = r(A)r(B). \tag{1.15}$$

2 Main results

We first present a group of results associated with the products in (1.2).

Theorem 2.1. *Let* $A_1 \in \mathbb{C}^{m_1\times n_1}$ *and* $A_2 \in \mathbb{C}^{m_2\times n_2}$, *and denote by* $M_1 = A_1\otimes I_{m_2}$ *and* $M_2 = I_{m_1}\otimes A_2$ *the two dilation matrices. Then we have the following results.*

(a) *The following equalities hold*

$$P_{A_1\otimes A_2} = P_{M_1}P_{M_2} = P_{M_2}P_{M_1} = P_{A_1}\otimes P_{A_2}. \tag{2.1}$$

(b) *The following rank equalities hold*

$$r[A_1\otimes I_{m_2},\ I_{m_1}\otimes A_2] = m_1m_2 - (m_1 - r(A_1))(m_2 - r(A_2)), \tag{2.2}$$
$$r[A_1\otimes I_{m_2},\ I_{m_1}\otimes E_{A_2}] = m_1m_2 - (m_1 - r(A_1))(A_2), \tag{2.3}$$
$$r[E_{A_1}\otimes I_{m_2},\ I_{m_1}\otimes A_2] = m_1m_2 - r(A_1)(m_2 - r(A_2)), \tag{2.4}$$
$$r[E_{A_1}\otimes I_{m_2},\ I_{m_1}\otimes E_{A_2}] = m_1m_2 - r(A_1)r(A_2), \tag{2.5}$$

and the following dimension equalities hold

$$\dim(\mathscr{R}(M_1)\cap\mathscr{R}(M_2)) = r(A_1)r(A_2), \tag{2.6}$$
$$\dim(\mathscr{R}(M_1)\cap\mathscr{R}^\perp(M_2)) = r(A_1)(m_2 - r(A_2)), \tag{2.7}$$
$$\dim(\mathscr{R}^\perp(M_1)\cap\mathscr{R}^\perp(M_2)) = r(A_1)(m_2 - r(A_2)), \tag{2.8}$$
$$\dim(\mathscr{R}^\perp(M_1)\cap\mathscr{R}^\perp(M_2)) = (m_1 - r(A_1))(m_2 - r(A_2)), \tag{2.9}$$

$$\begin{aligned} & \dim(\mathscr{R}(M_1)\cap\mathscr{R}(M_2)) + \dim(\mathscr{R}(M_1)\cap\mathscr{R}^\perp(M_2)) + \dim(\mathscr{R}^\perp(M_1)\cap\mathscr{R}^\perp(M_2)) \\ & + \dim(\mathscr{R}^\perp(M_1)\cap\mathscr{R}^\perp(M_2)) = m_1m_2. \end{aligned} \tag{2.10}$$

(c) *The following range equalities hold*

$$\mathscr{R}(M_1)\cap\mathscr{R}(M_2) = \mathscr{R}(M_1M_2) = \mathscr{R}(M_2M_1) = \mathscr{R}(A_1\otimes A_2), \tag{2.11}$$
$$\mathscr{R}(M_1)\cap\mathscr{R}^{\perp}(M_2) = \mathscr{R}(M_1E_{M_2}) = \mathscr{R}(E_{M_2}M_1) = \mathscr{R}(A_1\otimes E_{A_2}), \tag{2.12}$$
$$\mathscr{R}^{\perp}(M_1)\cap\mathscr{R}(M_2) = \mathscr{R}(E_{M_1}M_2) = \mathscr{R}(M_2E_{M_1}) = \mathscr{R}(E_{A_1}\otimes A_2), \tag{2.13}$$
$$\mathscr{R}^{\perp}(M_1)\cap\mathscr{R}^{\perp}(M_2) = \mathscr{R}(E_{M_1}E_{M_2}) = \mathscr{R}(E_{M_2}E_{M_1}) = \mathscr{R}(E_{A_1}\otimes E_{A_2}), \tag{2.14}$$

$$(\mathscr{R}(M_1)\cap\mathscr{R}(M_2))\oplus(\mathscr{R}^{\perp}(M_1)\cap\mathscr{R}(M_2))\oplus(\mathscr{R}(M_1)\cap\mathscr{R}^{\perp}(M_2))\oplus(\mathscr{R}^{\perp}(M_1)\cap\mathscr{R}^{\perp}(M_2)) = \mathbb{C}^{m_1m_2}. \tag{2.15}$$

(d) *The following orthogonal projector equalities hold*

$$P_{\mathscr{R}(M_1)\cap\mathscr{R}(M_2)} = P_{M_1}P_{M_2} = P_{A_1}\otimes P_{A_2}, \tag{2.16}$$
$$P_{\mathscr{R}(M_1)\cap\mathscr{R}^{\perp}(M_2)} = P_{M_1}E_{M_2} = P_{A_1}\otimes E_{A_2}, \tag{2.17}$$
$$P_{\mathscr{R}^{\perp}(M_1)\cap\mathscr{R}(M_2)} = E_{M_1}P_{M_2} = E_{A_1}\otimes P_{A_2}, \tag{2.18}$$
$$P_{\mathscr{R}^{\perp}(M_1)\cap\mathscr{R}^{\perp}(M_2)} = E_{M_1}E_{M_2} = E_{A_1}\otimes E_{A_2}, \tag{2.19}$$
$$P_{\mathscr{R}(M_1)\cap\mathscr{R}(M_2)} + P_{\mathscr{R}^{\perp}(M_1)\cap\mathscr{R}(M_2)} + P_{\mathscr{R}(M_1)\cap\mathscr{R}^{\perp}(M_2)} + P_{\mathscr{R}^{\perp}(M_1)\cap\mathscr{R}^{\perp}(M_2)} = I_{m_1m_2}. \tag{2.20}$$

(e) *The following orthogonal projector equalities hold*

$$P_{[A_1\otimes I_{m_2},\, I_{m_1}\otimes A_2]} = P_{A_1}\otimes I_{m_2} + I_{m_1}\otimes P_{A_2} - P_{A_1}\otimes P_{A_2} = I_{m_1m_2} - E_{A_1}\otimes E_{A_2}, \tag{2.21}$$
$$P_{[A_1\otimes I_{m_2},\, I_{m_1}\otimes E_{A_2}]} = P_{A_1}\otimes I_{m_2} + I_{m_1}\otimes E_{A_2} - P_{A_1}\otimes E_{A_2} = I_{m_1m_2} - E_{A_1}\otimes P_{A_2}, \tag{2.22}$$
$$P_{[E_{A_1}\otimes I_{m_2},\, I_{m_1}\otimes A_2]} = E_{A_1}\otimes I_{m_2} + I_{m_1}\otimes P_{A_2} - E_{A_1}\otimes P_{A_2} = I_{m_1m_2} - P_{A_1}\otimes E_{A_2}, \tag{2.23}$$
$$P_{[E_{A_1}\otimes I_{m_2},\, I_{m_1}\otimes E_{A_2}]} = E_{A_1}\otimes I_{m_2} + I_{m_1}\otimes E_{A_2} - E_{A_1}\otimes E_{A_2} = I_{m_1m_2} - P_{A_1}\otimes P_{A_2}. \tag{2.24}$$

Proof. It can be seen from (1.11) and (1.14) that

$$\begin{aligned}
P_{M_1}P_{M_2} &= (A_1\otimes I_{m_2})(A_1\otimes I_{m_2})^{\dagger}(I_{m_1}\otimes A_2)(I_{m_1}\otimes A_2)^{\dagger}\\
&= (A_1\otimes I_{m_2})(A_1^{\dagger}\otimes I_{m_2})(I_{m_1}\otimes A_2)(I_{m_1}\otimes A_2^{\dagger})\\
&= (P_{A_1}\otimes I_{m_2})(I_{m_1}\otimes P_{A_2})\\
&= P_{A_1}\otimes P_{A_2},\\
P_{M_2}P_{M_1} &= (I_{m_1}\otimes A_2)(I_{m_1}\otimes A_2)^{\dagger}(A_1\otimes I_{m_2})(A_1\otimes I_{m_2})^{\dagger}\\
&= (I_{m_1}\otimes A_2)(I_{m_1}\otimes A_2^{\dagger})(A_1\otimes I_{m_2})(A_1^{\dagger}\otimes I_{m_2})\\
&= (I_{m_1}\otimes P_{A_2})(P_{A_1}\otimes I_{m_2})\\
&= P_{A_1}\otimes P_{A_2},
\end{aligned}$$

establishing (2.1).

Applying (1.1) to A and simplifying by (1.11)–(1.15) yields

$$\begin{aligned}
r[A_1\otimes I_{m_2},\, I_{m_1}\otimes A_2] &= r(A_1\otimes I_{m_2}) + r((I_{m_1m_2} - (A_1\otimes I_{m_2})(A_1\otimes I_{m_2})^{\dagger})(I_{m_1}\otimes A_2))\\
&= m_2r(A_1) + r((I_{m_1} - A_1A_1^{\dagger})\otimes I_{m_2})(I_{m_1}\otimes A_2))\\
&= m_2r(A_1) + r((I_{m_1} - A_1A_1^{\dagger})\otimes A_2))\\
&= m_2r(A_1) + r(I_{m_1} - A_1A_1^{\dagger})r(A_2)\\
&= m_2r(A_1) + (m_1 - r(A_1))r(A_2)\\
&= m_1m_2 - (m_1 - r(A_1))(m_2 - r(A_2)),
\end{aligned}$$

as required for (2.2). Besides, (2.2) can directly be established by applying (1.8) to the left-hand side of (2.2). Eqs. (2.3)–(2.5) can be obtained by a similar approach. Subsequently by (2.2),

$$\dim(\mathscr{R}(M_1)\cap\mathscr{R}(M_2)) = r(M_1) + r(M_2) - r[M_1,\, M_2] = r(A_1)r(A_2),$$

as required for (2.6). Eqs. (2.7)–(2.9) can be established by a similar approach. Adding (2.7)–(2.9) leads to (2.10).

The first equalities in (2.11) follow from (2.6), and last two range equalities follow from (2.1).

Eqs. (2.12)–(2.14) can be established by a similar approach. Adding (2.11)–(2.14) and and combining with (2.10) leads to (2.15).

Eqs. (2.16)–(2.19) follow from (2.11)–(2.14). Adding (2.16)–(2.19) leads to (2.20).

Eqs. (2.21)–(2.24) follow from (1.8). □

Eq. (2.2) was first shown in [3], see also [12] for some extended forms of (2.2). Obviously, Theorem 2.1 provides many new tools to deal with various matrix equalities composed of algebraic operations of Kronecker products of matrices. For example, applying the preceding results to the Kronecker sum and difference $A_1 \otimes I_{m_2} \pm I_{m_1} \otimes A_2$ for two idempotent matrices A_1 and A_2, we obtain the following interesting consequences.

Theorem 2.2. *Let $A_1 = A_1^2 \in \mathbb{C}^{m_1 \times m_1}$ and $A_2 = A_2^2 \in \mathbb{C}^{m_2 \times m_2}$. Then,*

$$r(A_1 \otimes I_{m_2} + I_{m_1} \otimes A_2) = m_1 r(A_2) + m_2 r(A_1) - r(A_1) r(A_2), \tag{2.25}$$
$$r(A_1 \otimes I_{m_2} - I_{m_1} \otimes A_2) = m_1 r(A_2) + m_2 r(A_1) - 2r(A_1) r(A_2). \tag{2.26}$$

Proof. It is easy to verify that $(A_1 \otimes I_{m_2})^2 = A_1^2 \otimes I_{m_2} = A_1 \otimes I_{m_2}$ and $(I_{m_1} \otimes A_2)^2 = I_{m_1} \otimes A_2^2 = I_{m_1} \otimes A_2$ under $A_1^2 = A_1$ and $A_2^2 = A_2$. In this case, applying the following two known rank formulas

$$r(A+B) = r\begin{bmatrix} A & B \\ B & 0 \end{bmatrix} - r(B) = r\begin{bmatrix} B & A \\ A & 0 \end{bmatrix} - r(A),$$
$$r(A-B) = r\begin{bmatrix} A \\ B \end{bmatrix} + r[A,\, B] - r(A) - r(B),$$

where A and B are two idempotent matrices of the same size (cf. [13, 15]), to $A_1 \otimes I_{m_2} \pm I_{m_1} \otimes A_2$ and simplifying by (1.4) and (2.2) yields (2.25) and (2.26), respectively. □

The above two theorems reveal some essential mathematical relations among the dilation forms of two matrices by Kronecker products, which demonstrate that there still exist various fundamental problems on the Kronecker product of two matrices with analytical solutions that can be proposed and obtained. As a natural and useful generalization of the preceding formulas, we next give a group of results associated with the three-term Kronecker products of matrices in (1.3) without proofs.

Theorem 2.3. *Let $A_1 \in \mathbb{C}^{m_1 \times n_1}$, $A_2 \in \mathbb{C}^{m_2 \times n_2}$, and $A_3 \in \mathbb{C}^{m_3 \times n_3}$, and denote*

$$X_1 = A_1 \otimes I_{m_2} \otimes I_{m_3}, \quad X_2 = I_{m_1} \otimes A_2 \otimes I_{m_3}, \quad X_3 = I_{m_1} \otimes I_{m_2} \otimes A_3 \tag{2.27}$$

the three dilation matrices. Then we have the following results.

(a) *The following three projector equalities hold*

$$P_{X_1} = P_{A_1} \otimes I_{m_2} \otimes I_{m_3}, \quad P_{X_2} = I_{m_1} \otimes P_{A_2} \otimes I_{m_3}, \quad P_{X_3} = I_{m_1} \otimes I_{m_2} \otimes P_{A_3}. \tag{2.28}$$

(b) *The following projector equalities hold*

$$P_{A_1 \otimes A_2 \otimes A_3} = P_{M_1} P_{M_2} P_{M_3} = P_{A_1} \otimes P_{A_2} \otimes P_{A_3}. \tag{2.29}$$

(c) *The following eight rank equalities hold*

$$r[A_1 \otimes I_{m_2} \otimes I_{m_3},\, I_{m_1} \otimes A_2 \otimes I_{m_3},\, I_{m_1} \otimes I_{m_2} \otimes A_3]$$
$$= m_1 m_2 m_3 - (m_1 - r(A_1))(m_2 - r(A_2))(m_3 - r(A_3)), \tag{2.30}$$
$$r[A_1 \otimes I_{m_2} \otimes I_{m_3},\, I_{m_1} \otimes A_2 \otimes I_{m_3},\, I_{m_1} \otimes I_{m_2} \otimes E_{A_3}]$$
$$= m_1 m_2 m_3 - (m_1 - r(A_1))(m_2 - r(A_2)) r(A_3), \tag{2.31}$$
$$r[A_1 \otimes I_{m_2} \otimes I_{m_3},\, I_{m_1} \otimes E_{A_2} \otimes I_{m_3},\, I_{m_1} \otimes I_{m_2} \otimes A_3]$$
$$= m_1 m_2 m_3 - (m_1 - r(A_1)) r(A_2)(m_3 - r(A_3)), \tag{2.32}$$
$$r[E_{A_1} \otimes I_{m_2} \otimes I_{m_3},\, I_{m_1} \otimes A_2 \otimes I_{m_3},\, I_{m_1} \otimes I_{m_2} \otimes A_3]$$
$$= m_1 m_2 m_3 - r(A_1)(m_2 - r(A_2))(m_3 - r(A_3)), \tag{2.33}$$
$$r[A_1 \otimes I_{m_2} \otimes I_{m_3},\, I_{m_1} \otimes E_{A_2} \otimes I_{m_3},\, I_{m_1} \otimes I_{m_2} \otimes E_{A_3}]$$
$$= m_1 m_2 m_3 - (m_1 - r(A_1)) r(A_2) r(A_3), \tag{2.34}$$
$$r[E_{A_1} \otimes I_{m_2} \otimes I_{m_3},\, I_{m_1} \otimes A_2 \otimes I_{m_3},\, I_{m_1} \otimes I_{m_2} \otimes E_{A_3}]$$
$$= m_1 m_2 m_3 - r(A_1)(m_2 - r(A_2)) r(A_3), \tag{2.35}$$
$$r[E_{A_1} \otimes I_{m_2} \otimes I_{m_3},\, I_{m_1} \otimes E_{A_2} \otimes I_{m_3},\, I_{m_1} \otimes I_{m_2} \otimes A_3]$$
$$= m_1 m_2 m_3 - r(A_1) r(A_2)(m_3 - r(A_3)), \tag{2.36}$$
$$r[E_{A_1} \otimes I_{m_2} \otimes I_{m_3},\, I_{m_1} \otimes E_{A_2} \otimes I_{m_3},\, I_{m_1} \otimes I_{m_2} \otimes E_{A_3}]$$
$$= m_1 m_2 m_3 - r(A_1) r(A_2) r(A_3), \tag{2.37}$$

the following eight dimension equalities hold

$$\dim(\mathscr{R}(X_1)\cap\mathscr{R}(X_2)\cap\mathscr{R}(X_3)) = r(A_1)r(A_2)r(A_3), \tag{2.38}$$

$$\dim(\mathscr{R}(X_1)\cap\mathscr{R}(X_2)\cap\mathscr{R}^{\perp}(X_3)) = r(A_1)r(A_2)(m_3 - r(A_3)), \tag{2.39}$$

$$\dim(\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}(X_3)) = r(A_1)(m_2 - r(A_2))r(A_3), \tag{2.40}$$

$$\dim(\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}(X_3)) = (m_1 - r(A_1))r(A_2)r(A_3), \tag{2.41}$$

$$\dim(\mathscr{R}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}^{\perp}(X_3)) = r(A_1)(m_2 - r(A_2))(m_3 - r(A_3)), \tag{2.42}$$

$$\dim(\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}(X_2)\cap\mathscr{R}^{\perp}(X_3)) = (m_1 - r(A_1))r(A_2)(m_3 - r(A_3)), \tag{2.43}$$

$$\dim(\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}(X_3)) = (m_1 - r(A_1))(m_2 - r(A_2))r(A_3), \tag{2.44}$$

$$\dim(\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}^{\perp}(X_3)) = (m_1 - r(A_1))(m_2 - r(A_2))(m_3 - r(A_3)), \tag{2.45}$$

and the following dimension equality holds

$$\begin{aligned}
&\dim(\mathscr{R}(X_1)\cap\mathscr{R}(X_2)\cap\mathscr{R}(X_3)) + \dim(\mathscr{R}(X_1)\cap\mathscr{R}(X_2)\cap\mathscr{R}^{\perp}(X_3))\\
&\quad + \dim(\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}(X_3)) + \dim(\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}(X_3))\\
&\quad + \dim(\mathscr{R}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}^{\perp}(X_3)) + \dim(\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}(X_2)\cap\mathscr{R}^{\perp}(X_3))\\
&\quad + \dim(\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}(X_3)) + \dim(\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}^{\perp}(X_3)\\
&= m_1m_2m_3.
\end{aligned} \tag{2.46}$$

(d) *The following eight groups of range equalities hold*

$$\mathscr{R}(X_1)\cap\mathscr{R}(X_2)\cap\mathscr{R}(X_3) = \mathscr{R}(X_1X_2X_3) = \mathscr{R}(A_1\otimes A_2\otimes A_3), \tag{2.47}$$

$$\mathscr{R}(X_1)\cap\mathscr{R}(X_2)\cap\mathscr{R}^{\perp}(X_3) = \mathscr{R}(X_1X_2E_{X_3}) = \mathscr{R}(A_1\otimes A_2\otimes E_{A_3}), \tag{2.48}$$

$$\mathscr{R}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}(X_3) = \mathscr{R}(X_1E_{X_2}X_3) = \mathscr{R}(A_1\otimes E_{A_2}\otimes A_3), \tag{2.49}$$

$$\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}(X_2)\cap\mathscr{R}(X_3) = \mathscr{R}(E_{X_1}X_2X_3) = \mathscr{R}(E_{A_1}\otimes A_2\otimes A_3), \tag{2.50}$$

$$\mathscr{R}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}^{\perp}(X_3) = \mathscr{R}(X_1E_{X_2}E_{X_3}) = \mathscr{R}(A_1\otimes E_{A_2}\otimes E_{A_3}), \tag{2.51}$$

$$\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}(X_2)\cap\mathscr{R}^{\perp}(X_3) = \mathscr{R}(E_{X_1}X_2E_{X_3}) = \mathscr{R}(E_{A_1}\otimes A_2\otimes E_{A_3}), \tag{2.52}$$

$$\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}(X_3) = \mathscr{R}(X_1E_{X_2}X_3) = \mathscr{R}(E_{A_1}\otimes E_{A_2}\otimes A_3), \tag{2.53}$$

$$\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}^{\perp}(X_3) = \mathscr{R}(E_{X_1}E_{X_2}E_{X_3}) = \mathscr{R}(E_{A_1}\otimes E_{A_2}\otimes E_{A_3}), \tag{2.54}$$

and the following direct sum equality holds

$$\begin{aligned}
&(\mathscr{R}(X_1)\cap\mathscr{R}(X_2)\cap\mathscr{R}(X_3)) \oplus (\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}(X_2)\cap\mathscr{R}(X_3))\\
&\quad \oplus (\mathscr{R}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}(X_3)) \oplus (\mathscr{R}(X_1)\cap\mathscr{R}(X_2)\cap\mathscr{R}^{\perp}(X_3))\\
&\quad \oplus (\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}(X_3)) \oplus (\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}(X_2)\cap\mathscr{R}^{\perp}(X_3))\\
&\quad \oplus (\mathscr{R}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}^{\perp}(X_3)) \oplus (\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}^{\perp}(X_3))\\
&= \mathbb{C}^{m_1m_2m_3}.
\end{aligned} \tag{2.55}$$

(e) *The following eight groups of orthogonal projector equalities hold*

$$P_{\mathscr{R}(X_1)\cap\mathscr{R}(X_2)\cap\mathscr{R}(X_3)} = P_{A_1}\otimes P_{A_2}\otimes P_{A_3}, \tag{2.56}$$

$$P_{\mathscr{R}(X_1)\cap\mathscr{R}(X_2)\cap\mathscr{R}^{\perp}(X_3)} = P_{A_1}\otimes P_{A_2}\otimes E_{A_3}, \tag{2.57}$$

$$P_{\mathscr{R}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}(X_3)} = P_{A_1}\otimes E_{A_2}\otimes P_{A_3}, \tag{2.58}$$

$$P_{\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}(X_2)\cap\mathscr{R}(X_3)} = E_{A_1}\otimes P_{A_2}\otimes P_{A_3}, \tag{2.59}$$

$$P_{\mathscr{R}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}^{\perp}(X_3)} = P_{A_1}\otimes E_{A_2}\otimes E_{A_3}, \tag{2.60}$$

$$P_{\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}(X_2)\cap\mathscr{R}^{\perp}(X_3)} = E_{A_1}\otimes P_{A_2}\otimes E_{A_3}, \tag{2.61}$$

$$P_{\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}(X_3)} = E_{A_1}\otimes E_{A_2}\otimes P_{A_3}, \tag{2.62}$$

$$P_{\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}^{\perp}(X_3)} = E_{A_1}\otimes E_{A_2}\otimes E_{A_3}, \tag{2.63}$$

and the following orthogonal projector equality hold

$$\begin{aligned}
&P_{\mathscr{R}(X_1)\cap\mathscr{R}(X_2)\cap\mathscr{R}(X_3)} + P_{\mathscr{R}(X_1)\cap\mathscr{R}(X_2)\cap\mathscr{R}^{\perp}(X_3)}\\
&\quad + P_{\mathscr{R}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}(X_3)} + P_{\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}(X_2)\cap\mathscr{R}(X_3)}\\
&\quad + P_{\mathscr{R}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}^{\perp}(X_3)} + P_{\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}(X_2)\cap\mathscr{R}^{\perp}(X_3)}\\
&\quad + P_{\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}(X_3)} + P_{\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}^{\perp}(X_3)} = I_{m_1m_2m_3}.
\end{aligned} \tag{2.64}$$

(f) *The following eight orthogonal projector equalities hold*

$$P_{[A_1 \otimes I_{m_2} \otimes I_{m_3},\, I_{m_1} \otimes A_2 \otimes I_{m_3},\, I_{m_1} \otimes I_{m_2} \otimes A_3]} = I_{m_1 m_2 m_3} - E_{A_1} \otimes E_{A_2} \otimes E_{A_3}, \tag{2.65}$$

$$P_{[A_1 \otimes I_{m_2} \otimes I_{m_3},\, I_{m_1} \otimes A_2 \otimes I_{m_3},\, I_{m_1} \otimes I_{m_2} \otimes E_{A_3}]} = I_{m_1 m_2 m_3} - E_{A_1} \otimes E_{A_2} \otimes P_{A_3}, \tag{2.66}$$

$$P_{[A_1 \otimes I_{m_2} \otimes I_{m_3},\, I_{m_1} \otimes E_{A_2} \otimes I_{m_3},\, I_{m_1} \otimes I_{m_2} \otimes A_3]} = I_{m_1 m_2 m_3} - E_{A_1} \otimes P_{A_2} \otimes E_{A_3}, \tag{2.67}$$

$$P_{[E_{A_1} \otimes I_{m_2} \otimes I_{m_3},\, I_{m_1} \otimes A_2 \otimes I_{m_3},\, I_{m_1} \otimes I_{m_2} \otimes A_3]} = I_{m_1 m_2 m_3} - P_{A_1} \otimes E_{A_2} \otimes E_{A_3}, \tag{2.68}$$

$$P_{[A_1 \otimes I_{m_2} \otimes I_{m_3},\, I_{m_1} \otimes E_{A_2} \otimes I_{m_3},\, I_{m_1} \otimes I_{m_2} \otimes E_{A_3}]} = I_{m_1 m_2 m_3} - E_{A_1} \otimes P_{A_2} \otimes P_{A_3}, \tag{2.69}$$

$$P_{[E_{A_1} \otimes I_{m_2} \otimes I_{m_3},\, I_{m_1} \otimes A_2 \otimes I_{m_3},\, I_{m_1} \otimes I_{m_2} \otimes E_{A_3}]} = I_{m_1 m_2 m_3} - P_{A_1} \otimes E_{A_2} \otimes P_{A_3}, \tag{2.70}$$

$$P_{[E_{A_1} \otimes I_{m_2} \otimes I_{m_3},\, I_{m_1} \otimes E_{A_2} \otimes I_{m_3},\, I_{m_1} \otimes I_{m_2} \otimes A_3]} = I_{m_1 m_2 m_3} - P_{A_1} \otimes P_{A_2} \otimes E_{A_3}, \tag{2.71}$$

$$P_{[E_{A_1} \otimes I_{m_2} \otimes I_{m_3},\, I_{m_1} \otimes E_{A_2} \otimes I_{m_3},\, I_{m_1} \otimes I_{m_2} \otimes E_{A_3}]} = I_{m_1 m_2 m_3} - P_{A_1} \otimes P_{A_2} \otimes P_{A_3}. \tag{2.72}$$

In addition to (2.27), there are the following three dilation matrices

$$Y_1 = I_{m_1} \otimes A_2 \otimes A_3, \quad Y_2 = A_1 \otimes I_{m_2} \otimes A_3, \quad Y_3 = A_1 \otimes A_2 \otimes I_{m_3} \tag{2.73}$$

generated from any three matrices $A_1 \in \mathbb{C}^{m_1 \times n_1}$, $A_2 \in \mathbb{C}^{m_2 \times n_2}$, and $A_3 \in \mathbb{C}^{m_3 \times m_3}$. Some rank problems on the three-term Kronecker products for the vector case were considered in [11]. We next give a group of results concerning these three dilation matrices.

Theorem 2.4. *Let Y_1, Y_1, and Y_1 be as given in* (2.73)*. Then we have the following results.*

(a) *The following three projector equalities hold*

$$P_{Y_1} = I_{m_1} \otimes P_{A_2} \otimes P_{A_3}, \quad P_{Y_2} = P_{A_1} \otimes I_{m_2} \otimes P_{A_3}, \quad P_{Y_3} = P_{A_1} \otimes P_{A_2} \otimes I_{m_3}. \tag{2.74}$$

(b) *The following twelve matrix equalities hold*

$$\begin{aligned} P_{Y_1} P_{Y_2} &= P_{Y_2} P_{Y_1} = P_{Y_1} P_{Y_3} = P_{Y_3} P_{Y_1} = P_{Y_2} P_{Y_3} = P_{Y_3} P_{Y_2} \\ &= P_{Y_1} P_{Y_2} P_{Y_3} = P_{Y_1} P_{Y_3} P_{Y_2} = P_{Y_2} P_{Y_1} P_{Y_3} = P_{Y_2} P_{Y_3} P_{Y_1} \\ &= P_{Y_3} P_{Y_1} P_{Y_2} = P_{Y_3} P_{Y_2} P_{Y_1} = P_{A_1} \otimes P_{A_2} \otimes P_{A_3}. \end{aligned} \tag{2.75}$$

(c) *The following rank equality holds*

$$\begin{aligned} r[Y_1,\, Y_2,\, Y_3] = {} & m_1 r(A_2) r(A_3) + m_2 r(A_1) r(A_3) + m_3 r(A_1) r(A_2) \\ & - 2r(A_1) r(A_2) r(A_3). \end{aligned} \tag{2.76}$$

(d) *The following range equality holds*

$$\mathscr{R}(Y_1) \cap \mathscr{R}(Y_2) \cap \mathscr{R}(Y_3) = \mathscr{R}(A_1 \otimes A_2 \otimes A_3). \tag{2.77}$$

(e) *The following dimension equality holds*

$$\dim(\mathscr{R}(Y_1) \cap \mathscr{R}(Y_2) \cap \mathscr{R}(Y_3)) = r(A_1) r(A_2) r(A_3). \tag{2.78}$$

(f) *The following projector equality holds*

$$P_{\mathscr{R}(Y_1) \cap \mathscr{R}(Y_2) \cap \mathscr{R}(Y_3)} = P_{A_1} \otimes P_{A_2} \otimes P_{A_3}. \tag{2.79}$$

(g) *The following projector equality holds*

$$\begin{aligned} P_{[Y_1,\, Y_2,\, Y_3]} = {} & I_{m_1} \otimes P_{A_2} \otimes P_{A_3} + P_{A_1} \otimes I_{m_2} \otimes P_{A_3} + P_{A_1} \otimes P_{A_2} \otimes I_{m_3} \\ & - 2(P_{A_1} \otimes P_{A_2} \otimes P_{A_3}). \end{aligned} \tag{2.80}$$

Proof. Eq. (2.74) follows directly from (2.73), and (2.75) follows from (2.74). Since P_{Y_1}, P_{Y_2}, and P_{Y_3} are idempotent matrices, we find from (1.10) and (2.75) that

$$\begin{aligned} r[Y_1,\, Y_2,\, Y_3] &= r[P_{Y_1},\, P_{Y_2},\, P_{Y_3}] = r(P_{Y_1}) + r(P_{Y_2}) + r(P_{Y_3}) \\ &\quad - r[P_{Y_1} P_{Y_2},\, P_{Y_1} P_{Y_3}] - r[P_{Y_2} P_{Y_1},\, P_{Y_2} P_{Y_3}] - r[P_{Y_3} P_{Y_1},\, P_{Y_3} P_{Y_1}] \\ &\quad + r[P_{Y_1} P_{Y_2},\, P_{Y_1} P_{Y_3},\, P_{Y_2} P_{Y_2}] \\ &= r(P_{Y_1}) + r(P_{Y_2}) + r(P_{Y_3}) - 2r(P_{A_1} \otimes P_{A_2} \otimes P_{A_3}) \\ &= m_1 r(A_2) r(A_3) + m_2 r(A_1) r(A_3) + m_3 r(A_1) r(A_2) - 2r(A_1) r(A_2) r(A_3), \end{aligned}$$

establishing (2.76). Eqs. (2.76)–(2.80) are left as exercises for the reader. □

To summarize, the author remarks that this is a formula-oriented not problem-oriented work under the circumstance of Kronecker products of matrices, in which we have established many new and non-trivial analytical formulas, results, and facts about ranks, dimensions, orthogonal projectors, and ranges of matrices composed of Kronecker products of matrices. These findings may be helpful for the deeper investigations into the profound performance of Kronecker products of matrices under various assumptions. Hence, the author hopes that the reader is pleased with the preceding analytical formulas and their consequences, learns more useful algebraic tricks of dealing with Kronecker products of matrices from mathematical and computational aspects, and uses them in dealing with symbolical computations and analysis of matrix expressions that involve Kronecker products of matrices. Furthermore, the author claims that the preceding formulas in all this work can be extended to the situations for multiple Kronecker products with dilation forms, which can help explore more impressive outcomes of researches on Kronecker products of matrices and develop other related mathematical techniques applicable to solving practical problems.

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