Weak deflection angle of black-bounce traversable wormholes using Gauss-Bonnet theorem in the dark matter medium

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Abstract: In this paper, first we use the optical metrics of black-bounce traversable wormholes to calculate the Gaussian curvature. Then we use the Gauss-Bonnet theorem to obtain the weak deflection angle of light from the black-bounce traversable wormholes. Then we investigate the effect of dark matter medium on weak deflection angle using the Gauss-Bonnet theorem. We show how weak deflection angle of wormhole is affected by the bounce parameter a. Using the Gauss-bonnet theorem for calculating weak deflection angle shows us that light bending can be thought as a global and topological effect.

Key words: Relativity and gravitation; Gravitational lensing; Deflection angle; Wormholes; Gauss-Bonnet theorem.

1. Introduction

Root of gravitational lensing is the deflection of light by gravitational fields such as a planet, a black hole, or dark matter predicted by Einstein's general relativity, in the weak-field limit [1, 2]. Weak deflection is used to detect dark matter filaments, and it is important topic because it helps to understand the large-scale structure of the universe [1, 3].

One of the important method to calculate the weak deflection angle is proposed by Gibbons and Werner (GW), which is known as Gauss-Bonnet theorem (GBT), using optical geometry [4, 5]. In this method of GW, deflection angle is considered as a partially topological effect and can be found integrating the Gaussian optical curvature of the black hole space outside from the light rays using: [4]

$$\hat{\alpha} = - \int \int_{D_\infty} K \mathrm{d}S, \tag{1.1}$$

Since the GW method provides a unique perspective, it has been applied in various cases [6]-[55].

In this paper, our main motivation is to explore weak deflection angle of black-bounce traversable wormholes [56, 57] using the GBT and then extend our motivation of this research is to shed light on the effect of dark matter medium on the weak deflection angle of black-bounce traversable wormhole using the GBT. Note that the refractive index of the medium is supposed that it is spatially non-uniform but one can consider it uniform at large distances [58]-[66]. To do so, the photons are thought that may be deflected through

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dark matter due to the dispersive effects, where the index of refractive $n(\omega)$ which is for the scattering amplitude of the light and dark-matter in the forward.

2. Calculation of weak deflection angle from black-bounce traversable wormholes using the Gauss-Bonnet Theorem

The "black-bounce form" for the spacetime metric of traversable wormhole is: [56]

$$ds^2 = -\left(1 - \frac{2M}{\sqrt{r^2 + a^2}}\right) dt^2 + \left(1 - \frac{2M}{\sqrt{r^2 + a^2}}\right)^{-1} dr^2 + \left(r^2 + a^2\right)\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{2.1}$$

It is noted that the parameter a stands for the bounce length scale and when $a = 0$, it reduces to the Schwarzschild solution.

We restrict ourselves to the equatorial coordinate plane ($\theta = \frac{\pi}{2}$), so that the black-bounce traversable wormhole spacetime becomes

$$ds^2 = -\left(1 - \frac{2M}{\sqrt{r^2 + a^2}}\right) dt^2 + + \left(1 - \frac{2M}{\sqrt{r^2 + a^2}}\right)^{-1} dr^2 + \left(r^2 + a^2\right) d\varphi^2. \tag{2.2}$$

Then the optical geometry of the black-bounce traversable wormhole spacetime is found by using

$$g^{opt}{}_{\alpha\beta} = \frac{g_{\alpha\beta}}{-g_{00}} \tag{2.3}$$

$$dt^2 = \frac{dr^2}{\left(1 - \frac{2M}{\sqrt{r^2+a^2}}\right)^2} + \frac{(r^2 + a^2)d\varphi^2}{\left(1 - \frac{2M}{\sqrt{r^2+a^2}}\right)}. \tag{2.4}$$

The Gaussian optical curvature K for the black-bounce traversable wormhole optical space is calculated as follows:

$$K \simeq 3\frac{m^2}{r^4} - 2\frac{m}{r^3} - 15\frac{a^2 m^2}{r^6} + 10\frac{a^2 m}{r^5} - \frac{a^2}{r^4}. \tag{2.5}$$

Then we should use the Gaussian optical curvature in GBT to find deflection angle because the GBT is a theory which links the intrinsic geometry of the 2 dimensional space with its topology (D_R in M, with boundary $\partial D_R = \gamma_{\tilde{g}} \cup C_R$) [4]:

$$\int_{D_R} K\, dS + \oint_{\partial D_R} \kappa\, dt + \sum_i \epsilon_i = 2\pi\chi(D_R), \tag{2.6}$$

in which κ is defined as the geodesic curvature ($\kappa = \tilde{g}\left(\nabla_{\dot\gamma}\dot\gamma, \ddot\gamma\right)$), so that $\tilde{g}(\dot\gamma, \dot\gamma) = 1$. It is noted that $\ddot\gamma$ is unit acceleration vector, and ϵ_i is to the exterior angle at the ith vertex. Jump angles is obtained as $\pi/2$ for $r \to \infty$. Then we find that $\theta_O + \theta_S \to \pi$. If D_R is non-singular, the Euler characteristic becomes $\chi(D_R) = 1$, hence GBT becomes

$$\iint_{D_R} K\, dS + \oint_{\partial D_R} \kappa\, dt + \theta_i = 2\pi\chi(D_R). \tag{2.7}$$

The Euler characteristic number χ is 1, then the remaining part yields $\kappa(C_R) = |\nabla_{\dot{C}_R} \dot{C}_R|$ as $r \to \infty$. The radial component of the geodesic curvature is given by

$$\left(\nabla_{\dot{C}_R} \dot{C}_R\right)^r = \dot{C}_R^\varphi \, \partial_\varphi \dot{C}_R^r + \Gamma_{\varphi\varphi}^r \left(\dot{C}_R^\varphi\right)^2. \tag{2.8}$$

For large limits of R, $C_R := r(\varphi) = r = const.$ we obtain

$$\left(\nabla_{\dot{C}_R^r} \dot{C}_R^r\right)^r \to -\frac{1}{r}, \tag{2.9}$$

so that $\kappa(C_R) \to r^{-1}$. After that it is not hard to see that $\mathrm{d}t = r \, \mathrm{d}\varphi$, where $\kappa(C_R)\mathrm{d}t = \mathrm{d}\varphi$. The GBT reduces to this form

$$\iint\limits_{D_R} K \, \mathrm{d}S + \oint\limits_{C_R} \kappa \, \mathrm{d}t \overset{r \to \infty}{=} \iint\limits_{S_\infty} K \, \mathrm{d}S + \int\limits_0^{\pi+\hat{\alpha}} \mathrm{d}\varphi. \tag{2.10}$$

The light ray follows the straight line so that, we can assume that $r_\gamma = b/\sin\varphi$ at zeroth order. The weak deflection angle can be calculated using the formula:

$$\hat{\alpha} = -\int\limits_0^\pi \int\limits_{r_\gamma}^\infty K \, r \, \mathrm{d}r \, \mathrm{d}\varphi. \tag{2.11}$$

Using the Gaussian optical curvature (2.5), we calculate the weak deflection angle of black-bounce traversable wormholes up to second order terms:

$$\hat{\alpha} \approx 4\frac{m}{b} + \frac{a^2\pi}{4b^2}. \tag{2.12}$$

Note that it is in well agreement with the [57] in leading order terms.

3. Deflection angles of photon through dark matter medium from black-bounce traversable wormholes

In this section, we investigate the effect of dark matter medium on the weak deflection angle. To do so, we use he refractive index for the dark matter medium [58]:

$$n(\omega) = 1 + \beta A_0 + A_2 \omega^2. \tag{3.1}$$

Note that $\beta = \frac{\rho_0}{4m^2\omega^2}$ and ρ_0 the mass density of the scattered dark matter particles and $A_0 = -2\varepsilon^2 e^2$ and $A_{2j} \geq 0$.

The terms in $\mathcal{O}\left(\omega^2\right)$ and higher are related to the polarizability of the dark-matter candidate.

The order of ω^{-2} is for the charged dark matter candidate and ω^2 is for a neutral dark matter candidate. In addition, the linear term in ω occurs when parity and charge-parity asymmetries are present.

The 2 dimensional optical geometry of the wormhole is:

$$d\sigma^2 = n^2 \left(\frac{dr^2}{\left(1 - \frac{2M}{\sqrt{r^2+a^2}}\right)^2} + \frac{(r^2+a^2)}{\left(1 - \frac{2M}{\sqrt{r^2+a^2}}\right)} d\varphi^2 \right). \tag{3.2}$$

and

$$\left. \frac{d\sigma}{d\varphi} \right|_{C_R} = n \left(\frac{r^2+a^2}{\left(1 - \frac{2M}{\sqrt{r^2+a^2}}\right)} \right)^{1/2}. \tag{3.3}$$

Using the GBT within optical geometry of black-bounce traversable wormhole, we obtain the weak deflection angle in a dark matter medium:

$$\lim_{R\to\infty} \int_0^{\pi+\alpha} \left[\kappa_g \frac{d\sigma}{d\varphi} \right] \Big|_{C_R} d\varphi = \pi - \lim_{R\to\infty} \int\int_{D_R} \mathcal{K} dS. \tag{3.4}$$

First we calculate the Gaussian optical curvature at linear order of M:

$$\mathcal{K} \approx 10 \frac{csgn(r) a^2 M}{\left(A_2 \omega^2 + \beta A_0 + 1\right)^2 r^5} - 2 \frac{csgn(r) M}{\left(A_2 \omega^2 + \beta A_0 + 1\right)^2 r^3} - \frac{a^2}{\left(A_2 \omega^2 + \beta A_0 + 1\right)^2 r^4} \tag{3.5}$$

After that we find

$$\lim_{R\to\infty} \kappa_g \frac{d\sigma}{d\varphi} \Big|_{C_R} = 1. \tag{3.6}$$

Then for the limit of $R \to \infty$, the deflection angle in dark matter medium can be calculated using the GBT as follows:

$$\alpha = -\lim_{R\to\infty} \int_0^\pi \int_{\frac{b}{\sin\varphi}}^R \mathcal{K} dS. \tag{3.7}$$

Hence, we obtain the weak deflection angle in dark matter medium as follows:

$$\alpha = 4 \frac{M}{b\Psi} + \frac{a^2\pi}{4b^2\Psi} \tag{3.8}$$

$$\Psi = A_2{}^2\omega^4 + 2 A_0 A_2 \beta \omega^2 + A_0{}^2\beta^2 + 2 A_2 \omega^2 + 2\beta A_0 + 1 \tag{3.9}$$

which agrees with the known expression found using another method. Of course, in the absence of the dark matter medium ($\Psi = 0$), this expression reduces to the known vacuum formula $\alpha \approx 4\frac{m}{b} + \frac{a^2\pi}{4b^2}$.

We find that the deflected photon through the dark matter around the black-bounce traversable wormhole has large deflection angle compared to the standard case.

4. Conclusion

In this paper, we have studied weak deflection angle of black-bounce traversable wormholes using the GBT. Then we have investigated the effect of dark matter medium on the weak deflection angle of black-bounce

traversable wormholes. Note that refractive index is taken spatially non-uniform, and it is uniform at large distances.

Hence it is concluded that the deflection angle by black-bounce traversable wormholes increases with increasing the bounce parameter a, on the other hand the deflection angle decreases in a increasing medium of dark matter. It is showed how weak deflection angle of wormhole is affected by the bounce parameter a. Moreover we use the Gauss-bonnet theorem for calculating weak deflection angle which proves us that light bending can be thought as a global and topological effect.

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