Article

Mathematical Description of Elastic Phenomena which Uses Caputo or Riemann-Liouville Fractional Order Partial Derivatives is Nonobjective

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Abstract: In this paper it is shown that mathematical description of strain, constitutive law and dynamics obtained by direct replacement of integer order derivatives with Caputo or Riemann-Liouville fractional order partial derivatives, having integral representation on finite interval, in case of a guitar string, is nonobjective. The basic idea is that different observers, using this type of descriptions, obtain different results which cannot be reconciled, i.e. transformed into each other using only formulas that link the coordinates of the same point in two fixed orthogonal reference frames and formulas that link the numbers representing the same moment of time in two different choices of the origin of time measuring. This is not an academic curiosity! It is rather a problem: which one of the obtained results is correct?

Keywords: objectivity of a mathematical description; elastic phenomena description; fractional order partial derivative

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1. Introduction

The mathematical description of a real world phenomenon is objective if it is independent on the observer. That is, it is possible to reconcile observation of the phenomenon into a single coherent description of it. This requirement was pointed out by Galileo Galilee (1564-1642), Isaac Newton (1643-1727), Albert Einstein (1879-1955) in the context of mathematical description of mechanical movement: "The mechanical event is independent on the observer ". A possible and elementary understanding of the independence of the mechanical event on the observer is the independence of the event of the choice of the reference frame and of the choice of the moment considered origin for time measuring. What this means precisely in this paper is presented in the following. To describe mathematically the evolution of a mechanical event, an observer chooses a fixed orthogonal reference frame in the affine Euclidian space, a fixed moment of time (called origin for time measuring), and a unit for time measuring [second]. For different observers this choice can be different. In this paper the objectivity of a mathematical description means that the description is independent of the choice of the fixed orthogonal reference frame and of the choice of origin for time measuring. This means that the results obtained by two different observers can be reconciled, i.e. transformed into each other using only formulas that link the coordinates of a point in two fixed orthogonal reference frames and formulas that link the numbers representing a moment of time in two different choices of the origin of time measuring. This kind of understanding "objectivity of a mathematical description" is different from the concept of "objectivity in physics" presented in [1]. The advantage of our kind of understanding of the "objectivity of a mathematical description" used in this paper, is that it is less

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restrictive than Galilean invariance, Lorentz invariance, Einstein covariance, General covariance, it can be easily applied in a specific case and the reader does not need prior knowledge of Galilean invariance, Lorentz invariance, Einstein covariance, General covariance and fractional order-deformation gradients. Mathematical descriptions which depend on the choice of the fixed orthogonal reference frame or on the choice of the origin of time measuring are nonobjective. In case of descriptions which are nonobjective two observers who describe the same mechanical event obtain two different results that cannot be reconciled, i.e. cannot be transformed into each other using only formulas that link the coordinates of a point in two fixed orthogonal reference frames and formulas that link the numbers representing the same moment of time in two different choices of the origin of time measuring. The advantage of our kind of understanding of the "no objectivity of a mathematical description" used in this paper, is that the reader does not need prior knowledge of Galilean invariance, Lorentz invariance, Einstein covariance or General covariance and fractional-order deformation gradients. The majority of mathematical descriptions, formulated in terms of integer order derivatives or integer order partial derivatives, reported in the literature (books of Differential Equations of Mathematical Physics), are objectives in the sense of this manuscript. In the following the objectivity of the descriptions of some elastic phenomena, formulated in terms of integer order derivatives, is illustrated.

In classical theory of elasticity [2] a material particle Q of a material body B is represented by a point P of the affine Euclidian space E_3. At any moment of time M the material body B is represented by a connected subset S_M of points of the affine Euclidian space E_3. A point P of S_M represents a material particle Q of the material body. To describe the position of the material particle Q of B, observer O chooses a fixed orthogonal reference frame $R_O=(O;\vec{e}_1,\vec{e}_2,\vec{e}_3)$ in E_3 and describes the position using the coordinates of P (which represent the particle Q), with respect to the reference frame $R_O=(O;\vec{e}_1,\vec{e}_2,\vec{e}_3)$. To describe the time evolution, observer O chooses a moment of time M_O for fixing the origin for time measuring (the moment, when his stopwatch for measuring time, starts) and a unit for time measuring [second]. A moment of time M which is earlier than M_O is represented by a negative real number $t_M<0$ (representing the units of time between moment M and moment M_O), a moment of time M which is later than M_O is represented by a positive real number $t_M>0$ (representing the units of time between moment M_O and moment M) and the moment of time M_O is represented by the real number $t_{M_O}=0$. Observer O describes the movement of the material particle Q of B with functions of the form:

$$Y_k=Y_k(t_M,X_1,X_2,X_3)\quad for\quad k=1,2,3 \text{ and } (X_1,X_2,X_3)\in S^O{}_{M_O} \tag{1}$$

where: (X_1,X_2,X_3) are the coordinates, with respect to R_O, of the point P (which represents the material particle Q of B) at the moment of time M_O i.e. $t_M=t_{M_O}=0$; $(Y_1(t_M,X_1,X_2,X_3),Y_2(t_M,X_1,X_2,X_3),Y_3(t_M,X_1,X_2,X_3))$ are the coordinates, with respect to R_O, of the point P (which represents the same material particle Q) at the moment of time M ; $S^O{}_{M_O}$ is the set of coordinates (X_1,X_2,X_3), with respect to R_O, of the points P from S_{M_O}.

To describe the position of the same material particle Q of B, observer O^* chooses a fixed orthogonal reference frame $R_{O^*}=(O^*;\vec{e}^*{}_1,\vec{e}^*{}_2,\vec{e}^*{}_3)$ in E_3 and describes the position using the coordinates of P (which represents the particle Q), with respect to the reference frame $R_{O^*}=(O^*;\vec{e}^*{}_1,\vec{e}^*{}_2,\vec{e}^*{}_3)$. To describe the time evolution, observer O^* chooses a moment of time

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M_{O*} for fixing the origin for measuring time (the moment, when his stopwatch for measuring time, starts) and the same unit for time measuring [second]. A moment of time M which is earlier than M_{O*} is represented by a negative real number $t*_M < 0$ (representing the units of time between moment M and moment M_{O*}), a moment of time M which is later than M_{O*} is represented by a positive real number $t*_M > 0$ (representing the units of time between moment M_{O*} and moment M) and the moment of time M_{O*} is represented by the real number $t_{M_{O*}} = 0$.

Observer $O*$ describes the movement of the same material particle Q (as the observer O), with functions of the form:

$$Y*_k = Y*_k\,(t*_M, X*_1, X*_2, X*_3) \quad \text{for } k = 1,2,3 \quad \text{and} \quad (X*_1, X*_2, X*_3) \in S^{O*}{}_{M_{O*}} \quad (2)$$

where: $(X*_1, X*_2, X*_3)$ are the coordinates, with respect to R_{O*}, of the point P (which represents the particle Q of B) at the moment of time M_{O*} (i.e. $t*_M = t*_{M_{O*}} = 0$);

$$(Y*_1\,(t*_M, X*_1, X*_2, X*_3), Y*_2\,(t*_M, X*_1, X*_2, X*_3), Y*_3\,(t*_M, X*_1, X*_2, X*_3))$$

are the coordinates, with respect to R_{O*}, of the point P (which represents the same material particle Q of B) at the moment of time M ; $S^{O*}{}_{M_{O*}}$ is the set of coordinates $(X*_1, X*_2, X*_3)$ with respect to R_{O*}, of points P from $S_{M_{O*}}$.

Because t_M and $t*_M$ represent the same moment of time M the following relations hold:

$$t_M = t*_M + t_{M_{O*}} \;;$$
$$t*_M = t_M + t*_{M_O} \quad (3)$$

Because (1) and (2) describe the movement of the same material particle Q the following relations hold:

$$X*_k = X*_{k0} + \sum_{i=1}^{i=3} a_{ki} \cdot Y_i(t_{M_{O*}}, X_1, X_2, X_3) \quad \text{for} \quad k = 1,2,3 \quad (4)$$

$$Y*_k\,(t*_M, X*_1, X*_2, X*_3) = X*_{k0} + \sum_{i=1}^{i=3} a_{ki} \cdot Y_i(t_M, X_1, X_2, X_3) \quad \text{for } k = 1,2,3$$

$$(X*_1, X*_2, X*_3) \in S^{O*}{}_{M_{O*}} \;;\; (X_1, X_2, X_3) \in S^{O}{}_{M_O};$$

$$X_k = X_{k0*} + \sum_{i=1}^{i=3} a_{ik} \cdot Y*_i\,(t*_{M_O}, X*_1, X*_2, X*_3) \quad \text{for } k = 1,2,3$$

$$Y_k(t_M, X_1, X_2, X_3) = X_{k0*} + \sum_{i=1}^{i=3} a_{ik} \cdot Y*_i\,(t*_M, X*_1, X*_2, X*_3) \quad \text{for } k = 1,2,3 \quad (5)$$

$$(X_1, X_2, X_3) \in S^{O}{}_{M_O};\; (X*_1, X*_2, X*_3) \in S^{O*}{}_{M_{O*}}$$

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The significance of the quantities appearing in the above relations are:

$a_{ij} = \langle \vec{e}*_i, \vec{e}_j \rangle =$ constant = scalar product of the unit vectors $\vec{e}*_i$ and $\vec{e}_j$ in E_3 .

$$\vec{e}_j = \sum_{k=1}^{3} a_{kj} \cdot \vec{e}*_k$$

$$\vec{e}*_j = \sum_{k=1}^{3} a_{jk} \cdot \vec{e}_k$$

$(X_{1O*}, X_{2O*}, X_{3O*})$ are the coordinates of the point $O*$ in the reference frame R_O

$(X*_{1O}, X*_{2O}, X*_{3O})$ are the coordinates of the point O in the reference frame R_{O*}

Relations (3), (4), (5) reconcile the descriptions (1) and (2) and make possible the description by one of them. This means that the description (1) of the material particles movement in elasticity is objective. Two observers who describe the material particles movement of an elastic body with (1), obtain results that can be reconciled, i.e. transformed into each other using only formulas (3) -(5).

Observer O describes the displacement of the particle Q of B at the moment of time M by the vector valued function :

$$\vec{U}(t_M, X_1, X_2, X_3) = \sum_{j=1}^{3} (Y_j(t_M, X_1, X_2, X_3) - X_j) \cdot \vec{e}_j \qquad (6)$$

Observer $O*$ describes the displacement of the of the same particle Q of B at the moment of time M by the vector valued function :

$$\vec{U}*(t*_M, X*_1, X*_2, X*_3) = \sum_{j=1}^{3} (Y*_j(t*_M, X*_1, X*_2, X*_3) - X*_j) \cdot \vec{e}*_j \qquad (7)$$

Relations which reconcile the displacement description made by (6) with that made by (7) and make possible the description of the displacement by one of them, are the following:

$$\vec{U}*(t*_M, X*_1, X*_2, X*_3) = \vec{U}(t_M, X_1, X_2, X_3) - \vec{U}(t_{M_{O*}}, X_1, X_2, X_3) \qquad (8)$$

$$\vec{U}(t_M, X_1, X_2, X_3) = \vec{U}*(t*_M, X*_1, X*_2, X*_3) - \vec{U}*(t*_{M_O}, X*_1, X*_2, X*_3)$$

Therefore, description (6) is objective. Two observers who describe small displacement of the particles of an elastic body with (6) obtain results that can be reconciled, i.e. transformed into each other using only formulas (3) -(5).

Observer O describes the small deformation of the material body B at the particle Q at the moment of time M with the vector valued function $\vec{\Delta}$

$$\vec{\Delta}=\sum_{i=1}^{i=3}[\sum_{j=1}^{j=3}\frac{1}{2}\cdot\left(\frac{\partial U_i}{\partial X_j}+\frac{\partial U_j}{\partial X_i}\right)\cdot(X'_j-X_j)]\vec{e}_i \quad (9)$$

Description (9) is objective. Two observers who describe the small deformation of an elastic body with (9) obtain results that can be reconciled, i.e. transformed into each other using only formulas (3) -(5).

Observer O describes the strain of the material body B at the particle Q at the moment of time M with the functions:

$$\varepsilon_{jk}(t_M, X_1, X_2, X_3)=\frac{1}{2}\cdot\left(\frac{\partial U_j}{\partial X_k}+\frac{\partial U_k}{\partial X_j}\right) \quad j,k=1,2,3 \quad (10)$$

called the components of the strain tensor $\Gamma_M(Q)$.

Description (10) is objective. Two observers who describe the strain tensor of an elastic body with (10) obtain results that can be reconciled, i.e. transformed into each other using only formulas (3) -(5).

Observer O describes the principal directions of strains and the principal strains with the solution of the equations

$$\sum_{j=1}^{j=3}(\varepsilon_{ij}-\lambda\cdot\delta_{ij})\cdot V_i=0,\ i=1,2,3 \quad (11)$$

and $$\det(\varepsilon_{ij}-\lambda\cdot\delta_{ij})=0 \quad (12)$$

Descriptions (11) and (12) are objective. Two observers who describe the principal strains and principal direction of strains of an elastic body with (12) and (11), respectively, obtain results that can be reconciled, i.e. transformed into each other using formulas (3) -(5).

For a homogeneous and isotropic material body B, observer O describes the relationship between the components of the stress tensor and strain tensor with the constitutive law of Hooke :

$$\sigma_{ij}(t_M, X_1, X_2, X_3)=\lambda\cdot\theta(t_M, X_1, X_2, X_3)\cdot\delta_{ij}+2\mu\cdot\varepsilon_{ij}(t_M, X_1, X_2, X_3) \quad (13)$$

where λ and μ are the Lame constants δ_{ij} are the Kronecker coefficients and $\theta=\sum_{i=1}^{3}\varepsilon_{ii}$.

Description (13) is objective. Two observers who describe the relationship between the components of the stress tensor and strain tensor with the constitutive law of Hooke (13) obtain results that can be reconciled, i.e. transformed into each other using formulas (3) -(5).

Observer O describes the dynamics of an isotropic elastic solid with the equation:

$$\rho\cdot\frac{\partial^2\vec{U}}{\partial t_M^{\ 2}}(t_M, X_1, X_2, X_3)=\mu\cdot\Delta_X\vec{U}+(\lambda+\mu)\cdot grad_X(div_X\vec{U})+\vec{F}_O(X_1, X_2, X_3) \quad (14)$$

In equation (14) $\rho=\rho(X_1, X_2, X_3)$ is the density of the material body, $\vec{F}_O=\vec{F}_O(X_1, X_2, X_3)$ is the body force and $\vec{U}(t_M, X_1, X_2, X_3)$ is the displacement with respect to the reference frame $R_O=(O;\vec{e}_1,\vec{e}_2,\vec{e}_3)$.

Description (14) is objective. Two observers who describe the dynamic of one elastic body with (14) obtain results that can be reconciled, i.e. transformed into each other using formulas (3) -(5).

The objectivity of the above presented descriptions implies that, different observers describing the same phenomenon using integer order partial derivatives, obtain results which can be reconciled.

Beside the above presented objective mathematical descriptions there are mathematical descriptions of the presented elastic phenomena which use fractional order temporal or spatial partial derivatives. For instance in reference [3] the authors define the fractional thermal strain applying directly the temporal Caputo fractional ordrer derivative to the classical strain (see formula (18) in the section 2 of the present paper).In reference [4] the authors use constitutive law applying directly the Riemann-Liouville temporal fractional ordrer derivative to the classical strain (see formula (37) in secion 5 of the present paper). In [3] and [4] the analysis of the objectivity (in the sense of our understanding) is missing. In the paper [5] the authors present a significant number of constitutive laws in mechanics and thermodynamics.Among the constitutive laws presented there are constitutive laws in which the temporal Caputo fractional order derivative is applied directly to the classical strain without an analysis of the objectivity. In [6] the authors present fractional order strain and stress combining forward and backward fractional Caputo derivatives without an analysis of the objectivity . In [7] the authors use fractional order strain and stress combining forward and backward fractional Caputo derivatives for describing the static and kinematic in elasticity without an analysis of the objectivity. In [8] application of the fractional continuum mechanics to thermoelasticity is analyzed. Acording to the abstract " Contrary to classical theory, the obtained description is non-local, which is inherently the consequence of the fractional derivative definition based on the interval. So, all fields obtained in the framework of this new formulation, such as temperature, thermal stresses, total stresses, displacements, etc., at the specific point of interest, depend on the information from its surroundings. The dimensions of these surroundings and the ways of influencing the results are governed by the fractional differential operator applied". Although this paper contains many interesting ideas and information including a certain kind of objectivity, we only limit ourselves to the presentation of the abstract because the purpose of our paper is different. Namely we want to prove that direct replacement of integer-order derivatives with Caputo or Riemann-Liouville fractional order derivatives is not appropriate for describing strain, constitutive law and dynamics in the case of a guitar string. During the review process we found out (from one of Reviewers) that "integer-order derivatives cannot be simply replaced by fractional-order derivatives to develop the fractional-order theories". Taking into account that the above statement concerns all the equations of mathematical physics the curiosity pushed us to search in the scientific literature the formal proof of the statement. We have not find such a demonstration, not even in the case of elastic phenomena. That is why in our work we consider that this statement is only a conjecture or a belief based on professional experience, (For details concerning the difference between "what we know and what we imagine to know "see C. Foias; Is Mathematics a human creation? Conference with the occasion of awarding the DOCTOR HONORIS CAUSA title of the University of the West Timisoara; 1999) and we follow our purpose to demonstrate that: direct replacement of integer-order derivatives with Caputo or Riemann-Liouville fractional order derivatives is not appropriate for describing strain, constitutive law and dynamics in the case of a guitar string. These results can be interesting for the authors of the works [11], [12], [13],[14],[15] who use forward and backward Caputo or Riemann-Liouville fractional order derivatives for describing strain ,stress, constitutive equation and dynamics in case of 1D solids. These papers are relevant to the study we are developing further because they do not clearly underline why integer-order derivatives cannot be simply replaced by fractional-order derivatives to develop the fractional-order theories .During the development of our paper and in the section Conclusions and Comments we will refer these papers showing in which kind our results can help the authors of [11]-[15] to understand why "integer-order derivatives cannot be simply replaced by fractional-order derivatives to develop the fractional-order theories".

Remember that for a continuously differentiable function $f:[0,\infty)\times[0,\infty)\rightarrow R$ the Caputo spatial and temporal fractional partial derivative of order α, $0<\alpha$, is defined with the following first and second integral representation on a finite interval, respectively (see[17]) :

$$ {}^{C}{}_{0}D_{x}^{\alpha} f(x,t) = \frac{1}{\Gamma(n-\alpha)} \cdot \int_{0}^{x} \frac{\frac{\partial^{n} f}{\partial \xi^{n}}(\xi,t)}{(x-\xi)^{\alpha+1-n}} d\xi $$

$$ {}^{C}{}_{0}D_{t}^{\alpha} f(x,t) = \frac{1}{\Gamma(n-\alpha)} \cdot \int_{0}^{t} \frac{\frac{\partial^{n} f}{\partial \tau^{n}}(x,\tau)}{(t-\tau)^{\alpha+1-n}} d\tau \quad (15) $$

Remark that the derivative defined with (15) was considered by other people before Caputo, like Gherasimov (see [14]). So, the name of Caputo, used in this paper, may be is not appropriate.

For a continuously differentiable function $f:[0,\infty)\times[0,\infty)\to R$ the Riemann-Liouville spatial and temporal fractional partial derivative of order $\alpha, 0<\alpha$, is defined with the following first and second integral representation on a finite interval, respectively (see[17]) :

$$ {}^{R-L}{}_{0}D_{x}^{\alpha} f(x,t) = \frac{1}{\Gamma(n-\alpha)} \cdot \frac{\partial^{n}}{\partial x^{n}} \int_{0}^{x} \frac{f(\xi,t)}{(x-\xi)^{\alpha+1-n}} d\xi $$

$$ {}^{R-L}{}_{0}D_{t}^{\alpha} f(x,t) = \frac{1}{\Gamma(n-\alpha)} \cdot \frac{\partial^{n}}{\partial t^{n}} \int_{0}^{t} \frac{f(x,\tau)}{(t-\tau)^{\alpha+1-n}} d\tau \quad (16) $$

In formulas (15)-(16), Γ is the Euler gamma function and $n=[\alpha]+1$, $[\alpha]$ being the integer part of α.

2. Strain description of the guitar string, which uses Caputo fractional spatial partial derivative, having integral representation on finite interval, is nonobjective

In [3] authors use fractional order strains for describing dipolar thermo-elastic phenomena. The analysis of the objectivity of the mathematical description presented in [3] is completely missing. At first, we thought that also in the case of the use of fractional derivatives, the objectivity of the description is fulfilled and therefore it is ignored. But the curiosity pushed us to see how the fulfillment of the objectivity condition (in sense of our manuscript) can be proven mathematically. We chose for the special issue Mathematical Modelling in Applied Sciences the very simple case that of the guitar string. Thus was "born" sections 2 , 3 and 4 of the manuscript in which we analyzed the objectivity of the description of guitar string strain defined instead of integer order partial derivative (formula (10)) with spatial Caputo fractional order partial derivative having integral representation on finite interval . That is:

$$ {}^{C,\alpha}\varepsilon_{jk}(t_M, X_1, X_2, X_3) = \frac{1}{2} \cdot \left({}^{C}{}_{0}D_{X_k}{}^{\alpha}U_j + {}^{C}{}_{0}D_{X_j}{}^{\alpha}U_k \right) \quad (17) $$

The strain considered by us is not the strain considered in [3] which is:

$$ \overline{\varepsilon_{jk}} = (1+\tau^{\beta} \cdot D^{\beta}{}_{t}\varepsilon_{jk})) \quad (18) $$

The similarity consists only in the fact that in both cases Caputo fractional partial derivatives, having integral representation on finite interval, are used. But the result obtained by us can be instructive for the authors of the paper [3] because: in this section it is shown that, in case of a guitar string the strain tensor description which uses spatial Caputo fractional partial derivative, having integral representation on a finite interval, is nonobjective. In other word the direct use of Caputo fractional partial derivative, having integral representation on a finite interval affect the objectivity of a mathematical description.

In [18] the phisycs of guitar string vibration is presented. A guitar string of length L, fixed at both ends, is considered and the transverse displacement of the string particles, generated by its initial shape, is analyzed. To describe the transverse displacement of the string particles, observer O represents the string particles with points in the 2-D affine Euclidian space E_2, chooses an orthogonal reference frame $R_O = (O;\vec{e}_1,\vec{e}_2)$, such that one of the string end is represented by the origin O and the other end of the string is represented by a point fixed on the axis $O\vec{e}_1$ at the distance L from O. It is assumed that the points representing the string particles at any moment of time are in the plane determined by the vectors $\vec{e}_1,\vec{e}_2$. Beside that, for describing the time evolution, observer O chooses a moment of time M_O as origin for time measuring (this is the moment of start of his stopwatch) and a unit for time measuring [second]. With these elements observer O describes an arbitrary moment of time M with a real number t_M representing the number of time units between the moment M and moment M_O: $t_M < 0$ if M is earlier than M_O, $t_M > 0$ if M is later than M_O, $t_M = 0$ if $M = M_O$.

Assume that observer $O*$ chooses the same moment of time as origin for time measuring, i.e. $M_O = M_{O*}$ and the same unit for time measuring [second]. So, $t*_M = t_M = t$. Concerning the string particles, observer $O*$ represents this particles with points in the 2-D affine Euclidian space E_2. The orthogonal reference frame of observer $O*$, $R_{O*} = (O*;\vec{e}*_1,\vec{e}*_2)$ is chosen such that the following relations are verified: $a_{ij} = \langle \vec{e}*_i, \vec{e}_j \rangle = \delta_{ij}$; $0 < X_{1O*} < L$, $X_{2O*} > 0$.

The fact that (X_1, X_2) and $(X*_1, X_2*)$ are the coordinates of the point which represents the same particle of the string (in the two reference frames) is assured by the relations: $X*_1 = X*_{1O} + X_1, X_2* = X*_{2O} + X_2$ or $X_1 = X_{1O*} + X*_1, X_2 = X_{2O*} + X*_2$. When at the initial moment of time $t*_M = t_M = t = 0$ the shape of the string is the first harmonic (see [18]), observers O and $O*$ represent the whole string in the 2-D afine Euclidian space with the following sets of points:

$$S^O{}_{M_O} = \{(X_1, X_2) : X_1 \in [0, L], X_2 = \varphi(X_1) = \sin\frac{\pi X_1}{L}\} \tag{19}$$

$$S^{O*}{}_{M_{O*}} = \{(X*_1, X_2*) : X*_1 \in [X*_{1O}, X*_{1O} + L], X_2* = \varphi*(X*_1) = X*_{2O} + \sin\frac{\pi(X_{1O*} + X*_1)}{L}\}$$

The functions φ and $\varphi*$ appearing in (17) represent the initial shape of the string with respect to the frames $R_O = (O;\vec{e}_1,\vec{e}_2)$ and $R_{O*} = (O*;\vec{e}*_1,\vec{e}*_2)$ respectively, i.e. the first harmonic in the two reference frames (see [18]).

Observer O describes a movement of the point P from $S^O{}_{M_O}$, which represents the particle Q of the string, with the functios $Y_1(t, X_1, X_2), Y_2(t, X_1, X_2)$ given by:

$$Y_1(t, X_1, X_2) = Y_1(t, X_1, \varphi(X_1)) = X_1, \quad \text{for } t \ge 0, \; X_1 \in [0, L]; \tag{20}$$

$$Y_2(t, X_1, X_2) = \sin\frac{\pi \cdot X_1}{L} \cdot \cos\frac{\pi \cdot \nu \cdot t}{L} \quad \text{for } t \ge 0, \; X_1 \in [0, L]$$

The constant ν which appears in (20) is the frequency of vibration (see[18]).

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Observer $O*$ describes the movement of the point P from $S^{O*}{}_{M_{O*}}$, which represents the same particle Q of the string , with the functios $Y*_1(t,X*_1,X*_2), Y*_2(t,X*_1,X*_2)$ given by :

$$Y*_1(t,X*_1,X*_2) = Y*_1(t,X*_1,\varphi*(X*_1)) = X*_1 \text{ for } t \ge 0 \tag{21}$$

$$Y*_2(t,X*_1,X*_2) = Y*_2(t,X*_1,\varphi*(X*_1)) = X*_{2O} + \sin\frac{\pi(X_{1O*}+X*_1)}{L}\cdot\cos\frac{\pi\cdot\nu\cdot t}{L} \text{ for } t \ge 0$$

Observer O describes the components U_1, U_2 of the displacement vector with the funcions:

$$U_1(t,X_1,\varphi(X_1)) = Y_1(t,X_1,\varphi(X_1)) - X_1 = 0; \tag{22}$$

$$U_2(t,X_1,\varphi(X_1)) = Y_2(t,X_1,\varphi(X_1)) - \varphi(X_1) = \sin\frac{\pi\cdot X_1}{L}\cdot(\cos\frac{\pi\cdot\nu\cdot t}{L} - 1)$$

Observer $O*$ describes the components $U*_1$, $U*_2$ of the displacement vector with functions:

$$U*_1(t,X*_1,\varphi*(X*_1)) = Y*_1(t,X*_1,\varphi*(X*_1)) - X*_1 = 0 \tag{23}$$

$$U*_2(t,X*_1,\varphi*(X*_1)) = Y*_2(t,X*_1,\varphi*(X*_1)) - \varphi*(X*_1) = \sin\frac{\pi(X_{1O*}+X*_1)}{L}\cdot(\cos\frac{\pi\cdot\nu\cdot t}{L} - 1)$$

Using Caputo fractional spatial partial derivative of order α $(0<\alpha<1)$ defined on finite interval with formula (17) , observers O and $O*$ obtain the following components for the strain

$${}^{C,\alpha}\varepsilon_{11}(t,X_1,\varphi(X_1)) = {}^{C,\alpha}\varepsilon_{22}(t,X_1,\varphi(X_1)) = 0$$

$${}^{C,\alpha}\varepsilon_{12}(t,X_1,\varphi(X_1)) = {}^{C,\alpha}\varepsilon_{21}(t,X_1,\varphi(X_1)) = \frac{\pi}{2\cdot L}\cdot(\cos\frac{\pi\cdot\nu\cdot t}{L} - 1)\cdot\frac{1}{\Gamma(1-\alpha)}\cdot\int_0^{X_1}\frac{\cos\frac{\pi\xi}{L}}{(X_1-\xi)^{\alpha}}d\xi \tag{24}$$

$${}^{C,\alpha}\varepsilon*_{11}(t,X*_1,\varphi(X*_1)) = {}^{C,\alpha}\varepsilon*_{22}(t,X*_1,\varphi(X*_1)) = 0$$

$${}^{C,\alpha}\varepsilon*_{12}(t,X*_1,\varphi*(X*_1)) = {}^{C,\alpha}\varepsilon*_{21}(t,X*_1,\varphi*(X*_1)) = \frac{\pi}{2\cdot L}\cdot(\cos\frac{\pi\cdot\nu\cdot t}{L} - 1)\cdot\frac{1}{\Gamma(1-\alpha)}\cdot\int_0^{X*_1}\frac{\cos\frac{\pi(X_{1O*}+\eta)}{L}}{(X*_1-\eta)^{\alpha}}d\eta \tag{25}$$

If the considered description is objective, then for $i, j = 1,2$ the following equalities hold:

$${}^{C,\alpha}\varepsilon_{ij}(t,X_1,\varphi(X_1)) = {}^{C,\alpha}\varepsilon*_{ij}(t,X*_1,\varphi(X*_1)) \tag{26}$$

In particular if the description is objective, then

$${}^{C,\alpha}{}_{0X_{12}}\varepsilon_{12}(t,X_1,\varphi(X_1)) = {}^{C,\alpha}{}_{0X*_{12}}\varepsilon*_{12}(t,X*_1,\varphi*(X*_1))$$

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On the other hand, equality $X_1 = X_{1O*} + X*_1$ implies that the following equalities hold:

$$
\begin{aligned}
&{}^{C,\alpha}\varepsilon_{12}(t, X_1, \varphi(X_1)) = \frac{\pi}{2\cdot L}\cdot(\cos\frac{\pi\cdot\nu\cdot t}{L}-1)\cdot\frac{1}{\Gamma(1-\alpha)}\cdot\int_0^{X_1}\frac{\cos\frac{\pi\xi}{L}}{(X_1-\xi)^\alpha}d\xi = \\
&\frac{\pi}{2\cdot L}\cdot(\cos\frac{\pi\cdot\nu\cdot t}{L}-1)\cdot\frac{1}{\Gamma(1-\alpha)}\cdot\int_0^{X_{1O*}}\frac{\cos\frac{\pi\xi}{L}}{(X_1-\xi)^\alpha}d\xi + \frac{\pi}{2\cdot L}\cdot(\cos\frac{\pi\cdot\nu\cdot t}{L}-1)\cdot\frac{1}{\Gamma(1-\alpha)}\cdot\int_{X_{1O*}}^{X_1}\frac{\cos\frac{\pi\xi}{L}}{(X_1-\xi)^\alpha}d\xi = \\
&\frac{\pi}{2\cdot L}\cdot(\cos\frac{\pi\cdot\nu\cdot t}{L}-1)\cdot\frac{1}{\Gamma(1-\alpha)}\cdot(\int_0^{X_{1O*}}\frac{\cos\frac{\pi\xi}{L}}{(X_1-\xi)^\alpha}d\xi + \int_0^{X_1-X_{1O*}}\frac{\cos\frac{\pi(X_{1O*}+\eta)}{L}}{(X_1-X_{1O*}-\eta)^\alpha}d\eta) = \\
&\frac{\pi}{2\cdot L}\cdot(\cos\frac{\pi\cdot\nu\cdot t}{L}-1)\cdot\frac{1}{\Gamma(1-\alpha)}\cdot(\int_0^{X_{1O*}}\frac{\cos\frac{\pi\xi}{L}}{(X_1-\xi)^\alpha}d\xi + \int_0^{X*_1}\frac{\cos\frac{\pi(X_{1O*}+\eta)}{L}}{(X*_1-\eta)^\alpha}d\eta) = \\
&\frac{\pi}{2\cdot L}\cdot(\cos\frac{\pi\cdot\nu\cdot t}{L}-1)\cdot\frac{1}{\Gamma(1-\alpha)}\cdot\int_0^{X_{1O*}}\frac{\cos\frac{\pi\xi}{L}}{(X_1-\xi)^\alpha}d\xi + {}^{C,\alpha}\varepsilon*_{12}(t, X*_1, \varphi*(X*_1)).
\end{aligned}
\tag{27}
$$

It follows that : if the strain description is objective, then the next identity holds:

$$
\frac{\pi}{2\cdot L}\cdot(\cos\frac{\pi\cdot\nu\cdot t}{L}-1)\cdot\frac{1}{\Gamma(1-\alpha)}\cdot\int_0^{X_{1O*}}\frac{\cos\frac{\pi\xi}{L}}{(X_1-\xi)^\alpha}d\xi = 0 \tag{28}
$$

for any $t > 0$ and for any X_1 with $0 < X_1 < X_{1O*} < L$.

For $\nu \neq 0$, identity (28) in general is not valid. So, the strain tensor description of the guitar string with (24), is nonobjective. Observers O and $O*$ describing the strain with (24) and (25) respectively, obtain different results, which cannot be reconciled, i.e. ${}^{C,\alpha}\varepsilon_{12}(t, X_1, \varphi(X_1)) \neq {}^{C,\alpha}\varepsilon*_{12}(t, X*_1, \varphi*(X*_1))$. The result obtained by us can be interesting also for those researchers (authors of the papers [11]-[15]) who want to have a formal argument for the statement "in case of the guitar string the fractional order strains cannot be defined by replacing directly the integer order derivatives with spatial Caputo fractional order partial derivatives".

3. Strain description of the guitar string, which uses Riemann-Liouville fractional order spatial partial derivative having integral representation on finite interval, is nonobjective

When for the strain tensor description of the guitar string Riemann-Liouville fractional order spatial partial derivative (having integral representation on finite interval) is used then, by a similar procedure as is described in section 2 , observers O and $O*$ obtain the following components for the strain tensor:

$$
\begin{aligned}
&{}^{R-L,\alpha}\varepsilon_{11}(t, X_1, \varphi(X_1)) = {}^{R-L,\alpha}\varepsilon_{22}(t, X_1, \varphi(X_1)) = 0 \\
&{}^{R-L,\alpha}\varepsilon_{12}(t, X_1, \varphi(X_1)) = {}^{R-L,\alpha}\varepsilon_{21}(t, X_1, \varphi(X_1)) = \\
&\frac{1}{2}(\cos\frac{\pi\cdot\nu\cdot t}{L}-1)\cdot\frac{1}{\Gamma(1-\alpha)}\cdot\frac{\partial}{\partial X_1}\int_0^{X_1}\frac{\sin\frac{\pi\cdot\xi}{L}}{(X_1-\xi)^\alpha}d\xi
\end{aligned}
\tag{29}
$$

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$$^{R-L,\alpha}\varepsilon^*_{11}(t, X^*_1, \varphi(X^*_1)) = {}^{R-L,\alpha}\varepsilon^*_{22}(t, X^*_1, \varphi(X^*_1)) = 0$$

$$^{R-L,\alpha}\varepsilon^*_{12}(t, X^*_1, \varphi^*(X^*_1)) = {}^{R-L,\alpha}\varepsilon^*_{21}(t, X^*_1, \varphi^*(X^*_1)) =$$

$$\frac{1}{2}\cdot(\cos\frac{\pi\cdot\nu\cdot t}{L}-1)\frac{1}{\Gamma(1-\alpha)}\cdot\frac{\partial}{\partial X^*_1}\int_0^{X^*_1}\frac{\sin\frac{\pi(X_{1O^*}+X^*_1)}{L}}{(X^*_1-\eta)^{\alpha}}d\eta \qquad (30)$$

If the considered description is obiective, then: for $i, j = 1,2$ the following equalities hold:

$$^{R-L,\alpha}\varepsilon_{ij}(t, X_1, \varphi(X_1)) = {}^{R-L,\alpha}\varepsilon^*_{ij}(t, X^*_1, \varphi(X^*_1)). \qquad (31)$$

In particular, if the description is objective, then

$$^{R-L,\alpha}{}_{0X_{12}}\varepsilon_{12}(t, X_1, \varphi(X_1)) = {}^{R-L,\alpha}{}_{0X^*_{12}}\varepsilon^*_{12}(t, X^*_1, \varphi^*(X^*_1)) \qquad (32)$$

On the other hand, equality $X^*_1 = X^*_{1O} + X_1$ involves that the following equalities hold:

$$^{R-L,\alpha}\varepsilon_{12}(t, X_1, \varphi(X_1)) = \frac{1}{2}\cdot(\cos\frac{\pi\cdot\nu\cdot t}{L}-1)\cdot\frac{1}{\Gamma(1-\alpha)}\cdot\frac{\partial}{\partial X_1}\int_0^{X_1}\frac{\sin\frac{\pi\xi}{L}}{(X_1-\xi)^{\alpha}}d\xi =$$

$$\frac{1}{2}\cdot(\cos\frac{\pi\cdot\nu\cdot t}{L}-1)\cdot\frac{1}{\Gamma(1-\alpha)}\cdot\frac{\partial}{\partial X_1}\int_0^{X_{1O^*}}\frac{\sin\frac{\pi\xi}{L}}{(X_1-\xi)^{\alpha}}d\xi + \frac{1}{2}\cdot(\cos\frac{\pi\cdot\nu\cdot t}{L}-1)\cdot\frac{1}{\Gamma(1-\alpha)}\cdot\frac{\partial}{\partial X_1}\int_{X_{1O^*}}^{X_1}\frac{\sin\frac{\pi\xi}{L}}{(X_1-\xi)^{\alpha}}d\xi =$$

$$\frac{1}{2}\cdot(\cos\frac{\pi\cdot\nu\cdot t}{L}-1)\cdot\frac{1}{\Gamma(1-\alpha)}\cdot(\frac{\partial}{\partial X_1}\int_0^{X_{1O^*}}\frac{\sin\frac{\pi\xi}{L}}{(X_1-\xi)^{\alpha}}d\xi + \frac{\partial}{\partial X_1}\int_0^{X_1-X_{1O^*}}\frac{\sin\frac{\pi(X_{1O^*}+\eta)}{L}}{(X_1-X_{1O^*}-\eta)^{\alpha}}d\eta) =$$

$$\frac{1}{2}\cdot(\cos\frac{\pi\cdot\nu\cdot t}{L}-1)\cdot\frac{1}{\Gamma(1-\alpha)}\cdot(\frac{\partial}{\partial X_1}\int_0^{X_{1O^*}}\frac{\sin\frac{\pi\xi}{L}}{(X_1-\xi)^{\alpha}}d\xi + \frac{\partial}{\partial X^*_1}\int_0^{X^*_1}\frac{\cos\frac{\pi(X_{1O^*}+\eta)}{L}}{(X^*_1-\eta)^{\alpha}}d\eta) =$$

$$\frac{1}{2}\cdot(\cos\frac{\pi\cdot\nu\cdot t}{L}-1)\cdot\frac{1}{\Gamma(1-\alpha)}\cdot\frac{\partial}{\partial X_1}\int_0^{X_{1O^*}}\frac{\sin\frac{\pi\xi}{L}}{(X_1-\xi)^{\alpha}}d\xi + {}^{R-L,\alpha}\varepsilon^*_{12}(t, X^*_1, \varphi^*(X^*_1))$$

$$(33)$$

It follows that : if the considered description is objective, then the next identity holds:

$$\frac{1}{2}\cdot(\cos\frac{\pi\cdot\nu\cdot t}{L}-1)\cdot\frac{1}{\Gamma(1-\alpha)}\cdot\frac{\partial}{\partial X_1}\int_0^{X_{1O^*}}\frac{\sin\frac{\pi\xi}{L}}{(X_1-\xi)^{\alpha}}d\xi = 0 \qquad (34)$$

for any $t > 0$ and for any X_1 with $0 < X_1 < X_{1O^*} < L$.

For $\nu \neq 0$, identity (34) in general is not valid. So, the strain tensor description with (29) is nonobjective. Observers O and O^* describing the strain tensor compoments with (29) and (30) respectively, obtain different results which cannot be reconciled, i.e. $^{R-L,\alpha}\varepsilon_{12}(t, X_1, \varphi(X_1)) \neq {}^{R-L,\alpha}\varepsilon^*_{12}(t, X^*_1, \varphi^*(X^*_1))$. The problem is : which one of the obtained

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results is correct? This result can be interesting also for those researchers (authors of the papers [11]-[15]) who want to have a formal argument for the statement "in case of the guitar string the fractional order strains cannot be defined by replacing directly the integer order derivatives with spatial Riemann-Liouville fractional order partial derivatives".

4. Principal strain description of the guitar string which uses Caputo or Riemann-Liouville fractional order spatial partial derivatives, having integral representation on finite interval, is nonobjective

In case of the observer O description ,with integer order derivatives, the principal strains of the guitar string are the roots λ_1, λ_2 of the equation

$$\det(\varepsilon_{ij} - \lambda \cdot \delta_{ij}) = 0 \qquad (35)$$

In case of the observer O^* description ,with integer order derivatives, the principal strains of the guitar string are the roots λ^*_1 , λ^*_2 of the equation

$$\det(\varepsilon^*_{ij} - \lambda \cdot \delta_{ij}) = 0 \qquad (36)$$

It is easy to see that $\lambda_1 \cdot \lambda_2 = \varepsilon_{12}$ and $\lambda^*_1 \cdot \lambda^*_2 = \varepsilon^*_{12}$. When the strain tensor is described using Caputo or Riemann-Liouville fractional order spatial partial derivatives, having integral representation on finite interval, then $\varepsilon_{12} \neq \varepsilon^*_{12}$ (section 2 and 3). Therefore, the roots of the equations (35) and (36) are different. So, the principal strain description with these tools is nonobjective. Observers O and O^* describing the principal strains with (35) and (36) respectively, obtain different results which cannot be reconciled. The problem is : which one of the obtained results is correct? This result can be interesting also for those researchers (authors of the papers [11]-[15]) who want to have a formal argument for the statement "in case of the guitar string the fractional order principal strain cannot be obtained by replacing directly the integer order derivatives with spatial Caputo or Riemann-Liouville fractional order partial derivatives in the formula of integer order strain".

5. The Hooke constitutive law description for a guitar string which use Riemann-Liouville or Caputo fractional temporal partial derivative of order α $(0 < \alpha < 1)$, having integral representation on finite interval, is nonobjective

In [4] ,instead of the description of the constitutive law of Hooke given by (11) (in terms of the observer O), the authors describe the constitutive law of Hooke using Riemann-Liouville fractional order temporal partial derivatives, having integral representation on finite interval. According to [4], in terms of the observer O the constitutive law is described by:$\sigma(t) = E_0 E(t) + E_1 \cdot D^{\alpha}[E(t)]$. For $E_0 = 0$ and $E_1 = 1$ this law become:

$$\sigma_{ij}(t_M, X_1, X_2, X_3) = \lambda \cdot \theta(t_M, X_1, X_2, X_3) \cdot \delta_{ij} + 2\mu \cdot {}_0^{R-L} D_{t_M}^{\alpha} \varepsilon_{ij}(t_M, X_1, X_2, X_3) \qquad (37)$$

In terms of the observer O^* this description becomes:

$$\sigma^*_{ij}(t^*_M, X^*_1, X^*_2, X^*_3) = \lambda \cdot \theta^*(t^*_M, X^*_1, X^*_2, X^*_3) \cdot \delta_{ij} + 2\mu_0 \cdot {}^{R-L}_0 D_{t^*_M}^{\alpha} \varepsilon^*_{ij}(t^*_M, X^*_1, X^*_2, X^*_3) \qquad (38)$$

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In order to see that description (37) is nonobjective, consider again the guitar string, under the same hypothesis as in the Section 3, except the moments M_O, M_{O*}, chosen as origin for time measuring, which now are assumed to be different and the reference frames $R_O = (O; \vec{e}_1, \vec{e}_2)$, $R_{O*} = (O*; \vec{e}*_1, \vec{e}*_2)$ which now are assumed to be the same, i.e. $O = O*$, $\vec{e}_1 = \vec{e}*_1$, $\vec{e}_2 = \vec{e}*_2$. Assume that $t_{M_{O*}} > 0$ i.e. the moment M_{O*} is later than the moment M_O. With this choice $t*_M < t_M$ and at the moment of time M_O, M_{O*} observers O and $O*$ represent the string particles in the afine Euclidian space with the following sets of points:

$$S^O{}_{M_O} = \{(X_1, X_2) : X_1 \in [0, L], X_2 = \varphi(X_1) = \sin\frac{\pi \cdot X_1}{L}\} \tag{39}$$

$$S^{O*}{}_{M_{O*}} = \{(X_1, X*_2) : X_1 \in [0, L], X_2* = \varphi*(X_1) = \sin\frac{\pi \cdot X_1}{L} \cdot \cos\frac{\pi \cdot v \cdot t_{M_{O*}}}{L}\}$$

Observer O describes a movement of a point P from $S^O{}_{M_O}$, which represents a particle Q of the string, with the functios $Y_1(t_M, X_1, X_2), Y_2(t_M, X_1, X_2)$ given by:

$$Y_1(t_M, X_1, X_2) = Y_1(t_M, X_1, \varphi(X_1)) = X_1, \text{ for } t_M \geq 0,\ X_1 \in [0, L]; \tag{40}$$

$$Y_2(t_M, X_1, X_2) = \sin\frac{\pi \cdot X_1}{L} \cdot \cos\frac{\pi \cdot v \cdot t_M}{L} \quad \text{for } t_M \geq 0,\ X_1 \in [0, L]$$

Observer $O*$ describes the movement of the point P from $S^{O*}{}_{M_{O*}}$, which represents the same particle Q of the string, with the functios $Y*_1(t*_M, X_1, X*_2)$, $Y*_2(t*_M, X_1, X*_2)$ given by :

$$Y*_1(t*_M, X_1, X*_2) = Y*_1(t*_M, X_1, \varphi*(X_1)) = X_1 \quad \text{for } t*_M \geq 0 \tag{41}$$

$$Y*_2(t*_M, X_1, X*_2) = Y*_2(t*_M, X_1, \varphi*(X_1)) = \sin\frac{\pi \cdot X_1}{L} \cdot \cos\frac{\pi \cdot v \cdot (t_{M_{O*}} + t*_M)}{L} \quad \text{for } t*_M \geq 0$$

For observer O the components U_1, U_2 of the displacement vector are :

$$U_1(t_M, X_1, \varphi(X_1)) = Y_1(t_M, X_1, \varphi(X_1)) - X_1 = 0; \tag{42}$$

$$U_2(t_M, X_1, \varphi(X_1)) = Y_2(t_M, X_1, \varphi(X_1)) - \varphi(X_1) = \sin\frac{\pi \cdot X_1}{L} \cdot (\cos\frac{\pi \cdot v \cdot t_M}{L} - 1)$$

For observer $O*$ the components $U*_1$, $U*_2$ of the displacement vector are:

$$U*_1(t*_M, X_1, \varphi*(X_1)) = Y*_1(t*_M, X_1, \varphi*(X_1)) - X_1 = 0 \quad ; \tag{43}$$

$$U*_2(t*_M, X_1, \varphi*(X_1)) = Y*_2(t*_M, X_1, \varphi*(X_1)) - \varphi*(X_1) =$$
$$= \sin\frac{\pi \cdot X_1}{L} \cdot (\cos\frac{\pi \cdot v \cdot (t_{M_{O*}} + t*_M)}{L} - \cos\frac{\pi \cdot v \cdot t_{M_{O*}}}{L})$$

For observer O the strain tensor components $\varepsilon_{ij}(t, X_1, \varphi(X_1))$, computed using integer order derivatives, are:

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$$\varepsilon_{11}(t_M, X_1, \varphi(X_1)) = 0$$
$$\varepsilon_{22}(t_M, X_1, \varphi(X_1)) = 0 \tag{44}$$
$$\varepsilon_{12}(t_M, X_1, \varphi(X_1)) = \varepsilon_{21}(t_M, X_1, \varphi(X_1)) = \frac{\pi}{2L}\cos\frac{\pi \cdot X_1}{L}\cdot(\cos\frac{\pi\cdot\nu\cdot t_M}{L} - 1)$$

For observer O^* the strain tensor components $\varepsilon^*_{ij}(t^*_M, X^*_1, \varphi^*(X_1))$, computed using integer order derivatives, are:

$$\varepsilon^*_{11}(t^*_M, X_1, \varphi^*(X_1)) = 0 \qquad \varepsilon^*_{22}(t^*_M, X_1, \varphi^*(X_1)) = 0$$
$$\varepsilon^*_{12}(t^*_M, X_1, \varphi^*(X_1)) = \varepsilon_{21}(t^*_M, X_1, \varphi^*(X_1)) =$$
$$= \frac{\pi}{2L}\cos\frac{\pi\cdot X_1}{L}\cdot(\cos\frac{\pi\cdot\nu\cdot(t_{M_{O^*}} + t^*_M)}{L} - \cos\frac{\pi\cdot\nu\cdot t_{M_{O^*}}}{L}) \tag{45}$$

For observer O, the stress tensor components $\sigma_{ij}(t_M, X_1, \varphi(X_1))$, according to (37) (the modified constitutive law of Hooke [4]), are:

$$\sigma_{11}(t_M, X_1, \varphi(X_1)) = 0$$
$$\sigma_{22}(t_M, X_1, \varphi(X_1)) = 0 \tag{46}$$
$$\sigma_{12}(t_M, X_1, \varphi(X_1)) = \sigma_{21}(t_M, X_1, \varphi(X_1)) = 2\mu\cdot {}_0^{R-L}D_{t_M}^{\alpha}\varepsilon_{12}(t_M, X_{1,}, \varphi(X_1)) =$$
$$= 2\mu\cdot\frac{\pi}{2L}\cdot\cos\frac{\pi\cdot X_1}{L}\cdot\frac{1}{\Gamma(1-\alpha)}\cdot\frac{\partial}{\partial t_M}\int_0^{t_M}\frac{\cos\frac{\pi\cdot\nu\cdot\tau}{L} - 1}{(t_M - \tau)^{\alpha}}d\tau$$

For observer O^*, the stress tensor components $\sigma^*_{ij}(t^*_M, X_1, \varphi^*(X_1))$, according to (38) (the modified constitutive law of Hooke [4]), are:

$$\sigma^*_{11}(t^*_M, X_1, \varphi^*(X_1)) = 0 \qquad \sigma^*_{22}(t^*_M, X_1, \varphi^*(X_1)) = 0$$
$$\sigma^*_{12}(t_M, X_1, \varphi^*(X_1)) = \sigma^*_{21}(t^*_M, X_1, \varphi^*(X_1)) = 2\mu\cdot {}_0^{R-L}D_{t^*_M}^{\alpha}\varepsilon_{12}(t^*_M, X_{1,}, \varphi^*(X_1)) =$$
$$= 2\mu\cdot\frac{\pi}{2L}\cdot\cos\frac{\pi\cdot X_1}{L}\cdot\frac{1}{\Gamma(1-\alpha)}\cdot\frac{\partial}{\partial t^*_M}\int_0^{t^*_M}\frac{(\cos\frac{\pi\cdot\nu\cdot(t_{M_{O^*}} + \tau^*)}{L} - \cos\frac{\pi\cdot\nu\cdot t_{M_{O^*}}}{L})}{(t^*_M - \tau^*)^{\alpha}}d\tau^* \tag{47}$$

Because the reference frames $R_O = (O; \vec{e}_1, \vec{e}_2)$, $R_{O^*} = (O^*; \vec{e}^*_1, \vec{e}^*_2)$ are the same, if the considered description is objective, then the following equalities hold:

$$\sigma_{ij}(t_M, X_1, \varphi(X_1)) = \sigma^*_{ij}(t^*_M, X_1, \varphi^*(X_1)) \quad \text{for } i, j = 1,2$$

In particular if the description is objective, then $\sigma_{12}(t_M, X_1, \varphi(X_1)) = \sigma^*_{12}(t^*_M, X_1, \varphi^*(X_1))$. On the other hand, the following equalities hold:

$$\sigma_{12}(t_M, X_1, \varphi(X_1)) = 2\mu\cdot\frac{\pi}{2L}\cdot\cos\frac{\pi\cdot X_1}{L}\cdot\frac{1}{\Gamma(1-\alpha)}\cdot\frac{\partial}{\partial t_M}\int_0^{t_M}\frac{\cos\frac{\pi\cdot\nu\cdot\tau}{L}-1}{(t_M-\tau)^{\alpha}}d\tau =$$

$$2\mu\cdot\frac{\pi}{2L}\cdot\cos\frac{\pi\cdot X_1}{L}\cdot\frac{1}{\Gamma(1-\alpha)}\cdot\left(\frac{\partial}{\partial t_M}\int_0^{t_{M_{O^*}}}\frac{\cos\frac{\pi\cdot\nu\cdot\tau}{L}-1}{(t_M-\tau)^{\alpha}}d\tau+\frac{\partial}{\partial t_M}\int_{t_{M_{O^*}}}^{t_M}\frac{\cos\frac{\pi\cdot\nu\cdot\tau}{L}-1}{(t_M-\tau)^{\alpha}}d\tau\right) =$$

$$2\mu\cdot\frac{\pi}{2L}\cdot\cos\frac{\pi\cdot X_1}{L}\cdot\frac{1}{\Gamma(1-\alpha)}\cdot\left(\frac{\partial}{\partial t_M}\int_0^{t_{M_{O^*}}}\frac{\cos\frac{\pi\cdot\nu\cdot\tau}{L}-1}{(t_M-\tau)^{\alpha}}d\tau+\frac{\partial}{\partial t_M}\int_0^{t_M-t_{M_{O^*}}}\frac{\cos\frac{\pi\cdot\nu\cdot(t_{M_{O^*}}+\tau^*)}{L}-1}{(t^*_M-\tau^*)^{\alpha}}d\tau^*\right) =$$

$$2\mu\cdot\frac{\pi}{2L}\cdot\cos\frac{\pi\cdot X_1}{L}\cdot\frac{1}{\Gamma(1-\alpha)}\cdot\frac{\partial}{\partial t_M}\int_0^{t_{M_{O^*}}}\frac{\cos\frac{\pi\cdot\nu\cdot\tau}{L}-1}{(t_M-\tau)^{\alpha}}d\tau +$$

$$2\mu\cdot\frac{\pi}{2L}\cdot\cos\frac{\pi\cdot X_1}{L}\cdot\frac{1}{\Gamma(1-\alpha)}\cdot\frac{\partial}{\partial t^*_M}\int_0^{t^*_M}\frac{\cos\frac{\pi\cdot\nu\cdot(t_{M_{O^*}}+\tau^*)}{L}-\cos\frac{\pi\cdot\nu\cdot t_{M_{O^*}}}{L}+\cos\frac{\pi\cdot\nu\cdot t_{M_{O^*}}}{L}-1}{(t^*_M-\tau^*)^{\alpha}}d\tau^* =$$

$$2\mu\cdot\frac{\pi}{2L}\cdot\cos\frac{\pi\cdot X_1}{L}\cdot\frac{1}{\Gamma(1-\alpha)}\cdot\left(\frac{\partial}{\partial t_M}\int_0^{t_{M_{O^*}}}\frac{\cos\frac{\pi\cdot\nu\cdot\tau}{L}-1}{(t_M-\tau)^{\alpha}}d\tau+\frac{\partial}{\partial t^*_M}\int_0^{t^*_M}\frac{\cos\frac{\pi\cdot\nu\cdot t_{M_{O^*}}}{L}-1}{(t^*_M-\tau^*)^{\alpha}}d\tau^*\right)+$$

$$+\sigma^*_{12}(t^*_M, X_1, \varphi^*(X_1))$$

(48)

It follows that if the description is objective, then the next identity holds:

$$2\mu\cdot\frac{\pi}{2L}\cdot\cos\frac{\pi\cdot X_1}{L}\cdot\frac{1}{\Gamma(1-\alpha)}\cdot\left(\frac{\partial}{\partial t_M}\int_0^{t_{M_{O^*}}}\frac{\cos\frac{\pi\cdot\nu\cdot\tau}{L}-1}{(t_M-\tau)^{\alpha}}d\tau+\frac{\partial}{\partial t^*_M}\int_0^{t^*_M}\frac{\cos\frac{\pi\cdot\nu\cdot t_{M_{O^*}}}{L}-1}{(t^*_M-\tau^*)^{\alpha}}d\tau^*\right)=0$$

(49)

for any $t^*_M < t_M$ and $0 \le X_1 \le L$

In general, identity (49) is not valid. So, the constitutive law description with (37) is nonobjective. Observers O and O^* describing the constitutive law with (37) and (38) respectively, obtain different values for the stress tensor components namely: $\sigma_{12}(t_M, X_1, \varphi(X_1)) \neq \sigma^*_{12}(t^*_M, X_1, \varphi^*(X_1))$. These results cannot be reconciled. The problem is: which one of the obtained component is correct? This result can be instructive for the authors of the paper [4] because it shows that the direct introduction of the Riemann-Liouville fractional order temporal partial derivative in Hooke law affects the objectivity of the description of the constitutive law. It can be also interesting for those researchers (authors of the papers [11]-[15]) who want to have a formal argument for the statement "in case of the guitar string the fractional order stress-strain relation cannot be defined by introducing directly the Riemann-Liouville fractional order temporal partial derivative in Hooke law".

In any case, the analysis of objectivity of the description proposed in [4] is necessary.

In the paper [5] the authors present a significant number of constitutive laws in mechanics and thermodynamics. Among the constitutive laws presented there are constitutive laws in which the temporal Caputo fractional order derivative is applied directly to the classical strain without an analysis of the objectivity. Concerning these constitutive laws it has to be mentioned that replacing in (37), (38) the Riemann-Liouville fractional order temporal partial derivatives, having integral representation on finite interval, with Caputo fractional order temporal partial derivatives, having

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integral representation on finite interval, the Hooke constitutive law, in terms of observer O, becomes :

$$\sigma_{ij}(t_M, X_1, X_2, X_3) = \lambda \cdot \theta(t_M, X_1, X_{2,}, X_3) \cdot \delta_{ij} + 2\mu \cdot {}_0^C D_t^\alpha \varepsilon_{ij}(t_M, X_{1,}, X_2, X_3) \tag{50}$$

and in terms of the observer O^* this description becomes :

$$\sigma^*_{ij}(t^*_M, X^*_1, X^*_2, X^*_3) = \lambda \cdot \theta^*(t^*_M, X^*_1, X^*_{2,}, X^*_3) \cdot \delta_{ij} + 2\mu \cdot {}_0^C D_t^\alpha \varepsilon^*_{ij}(t^*_M, X^*_{1,}, X^*_2, X^*_3) \tag{51}$$

In order to see that description (50) is nonobjective, use (44), (45) and compute the stress tensor components

$\sigma_{ij}(t_M, X_1, \varphi(X_1))$ and $\sigma^*_{ij}(t^*_M, X_1, \varphi^*(X_1))$ corresponding to (50) and (51) (the modified Hooke law according to [5])respectively, obtaining the following results:

$$\sigma_{11}(t_M, X_1, \varphi(X_1)) = 0 \qquad \sigma_{22}(t_M, X_1, \varphi(X_1)) = 0$$

$$\sigma_{12}(t_M, X_1, \varphi(X_1)) = \sigma_{21}(t_M, X_1, \varphi(X_1)) = 2\mu \cdot {}_0^C D_{t_M}^\alpha \varepsilon_{12}(t_M, X_{1,}, \varphi(X_1)) = \\ = -2\mu \cdot \frac{\pi}{2L} \cdot \cos\frac{\pi \cdot X_1}{L} \cdot \frac{1}{\Gamma(1-\alpha)} \cdot \frac{\pi \cdot \nu}{L} \cdot \int_0^{t_M} \frac{\sin\dfrac{\pi \cdot \nu \cdot \tau}{L}}{(t_M - \tau)^\alpha} d\tau$$

$$\sigma^*_{11}(t^*_M, X_1, \varphi^*(X_1)) = 0 \qquad \sigma^*_{22}(t^*_M, X_1, \varphi^*(X_1)) = 0$$

$$\sigma^*_{12}(t_M, X_1, \varphi^*(X_1)) = \sigma^*_{21}(t^*_M, X_1, \varphi^*(X_1)) = 2\mu \cdot {}_0^C D_{t^*_M}^\alpha \varepsilon_{12}(t^*_M, X_{1,}, \varphi^*(X_1)) = \\ = -2\mu \cdot \frac{\pi}{2L} \cdot \cos\frac{\pi \cdot X_1}{L} \cdot \frac{1}{\Gamma(1-\alpha)} \cdot \frac{\pi \nu}{L} \cdot \int_0^{t^*_M} \frac{\sin\dfrac{\pi \cdot \nu \cdot (t_{M_{O^*}} + \tau^*)}{L}}{(t^*_M - \tau^*)^\alpha} d\tau^*$$

Using equality

$$\sigma_{12}(t_M, X_1, \varphi(X_1)) = \sigma^*_{12}(t_M, X_1, \varphi^*(X_1)) - \\ - \frac{\pi \cdot \nu}{L} 2 \cdot \mu \cdot \frac{\pi}{2L} \cdot \cos\frac{\pi \cdot X_1}{L} \cdot \frac{1}{\Gamma(1-\alpha)} \cdot \int_0^{t_{M_{O^*}}} \frac{\sin\dfrac{\pi \nu \tau}{L}}{(t_M - \tau)^\alpha} d\tau$$

and objectivity of the description (50) ($\sigma_{12}(t_M, X_1, \varphi(X_1)) = \sigma^*_{12}(t_M, X_1, \varphi^*(X_1))$) the following identity is obtained:

$$-\frac{\pi \cdot \nu}{L} 2 \cdot \mu \cdot \frac{\pi}{2L} \cdot \cos\frac{\pi \cdot X_1}{L} \cdot \frac{1}{\Gamma(1-\alpha)} \cdot \int_0^{t_{M_{O^*}}} \frac{\sin\dfrac{\pi \nu \tau}{L}}{(t_M - \tau)^\alpha} d\tau = 0 \tag{52}$$

for any $t^*_M < t_M$ and $0 \le X_1 \le L$

In general identity (52) is not valid. So, the description (50) is nonobjective. Observers O and O^* describing the stress tensor componets with (50) and (51) respectively, obtain different values for the stress components $\sigma_{12}(t_M, X_1, \varphi(X_1)) \neq \sigma^*_{12}(t^*_M, X_1, \varphi^*(X_1))$. These results cannot be reconciled. The problem is : which one of the obtained results is correct? This result can be instructive for the authors of the paper [5] because it shows that the direct introduction of the temporal Caputo fractional partial derivative in the Hooke law affects the objectivity of the description of the constitutive law. It can be also interesting for those researchers (authors of the papers [11]-[15]) who want to have a formal argument for the statement "in case of the guitar string the fractional order stress- strains relation cannot be obtained by introducing directly the temporal Caputo fractional order partial derivative in the Hooke law".

In any case, the analysis of the objectivity of the mathematical description of those constitutive law for which the stress-strain relation is obtained introducing directly the temporal Caputo fractional partial derivative in Hooke law, proposed in [5] is necessary.

6. The dynamics description of a guitar string, which uses Caputo or Riemann-Liouville fractional order temporal partial derivatives with integral representation on a finite interval, is nonobjective

In case of a homogeneous guitar string, neglecting the mass forces equation (14) describing the string vibration becomes a scalar equation. In terms of observer O this equation is given by

$$\rho_0 \cdot \frac{\partial^2 U_2}{\partial t_M^{\,2}}(t_M, X_1) = \mu \cdot \frac{\partial^2 U_2}{\partial X_1^{\,2}}(t_M, X_1) \quad (53)$$

and in terms of observer O^* this equation is given by

$$\rho_0 \cdot \frac{\partial^2 U^*_2}{\partial t^{*\,2}_M}(t^*_M, X^*_1,) = \mu \cdot \frac{\partial^2 U^*_2}{\partial X^{*\,2}_1}(t^*_M, X^*_1) \quad (54)$$

here ρ_0 is constant and represents the string linear density.

There are several papers which apply in dynamics description of the elastic solid (for example [13]) fractional order Caputo or Riemann-Liouville temporal partial derivatives, represented with integral on finite interval, ignoring the condition of the objectivity of such a description. For this reason, in this section we present a mathematical description of the dynamics of an elastic guitar string, using Caputo or Riemann-Liouville temporal partial derivatives, which have integral representation on a finite interval, showing that the description is nonobjective.

Consider first the case when the reference frames $R_O = (O; \vec{e}_1, \vec{e}_2)$, $R_{O^*} = (O^*; \vec{e}^*_1, \vec{e}^*_2)$ coincide and the Caputo temporal partial derivatives are used, assuming that $1 < \alpha < 2$.

After the substitution, for the observers O and O^* equations (53) and (54) become:

$$\rho_0 \cdot {}^C_0 D_{t_M}^{\alpha} U_2(t_M, X_1) = \mu \cdot \frac{\partial^2 U_2}{\partial X_1^{\,2}}(t_M, X_1) \quad (55)$$

$$\rho_0 \cdot {}^C_0 D_{t^*_M}^{\alpha} U^*_2(t^*_M, X_1) = \mu \cdot \frac{\partial^2 U^*_2}{\partial X_1^{\,2}}(t^*_M, X_1) \quad (56)$$

If this description is objective, then:

for any solution $U_2(t_M, X_1)$ of (55) the function $U^*_2(t^*_M, X_1)$ defined by :

$$U^*_2(t^*_M, X_1) = U_2(t^*_M + t_{M_{O^*}}, X_1) \quad (57)$$

is a solution of (56) and

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for any solution $U^*_2(t^*_M, X_1)$ of (56) the function $U_2(t_M, X_1)$ defined by

$$U_2(t_M, X_1) = U^*_2(t_M + t^*_{M_O}, X_1) \tag{58}$$

is a solution of (55).

Assume that the description (55) is objective and start with a solution $U_2(t_M, X_1)$ of (55).Consider the function $U^*_2(t^*_M, X_1)$ defined by (57).

For $t_M > t_{M_O^*} > 0$ equalities

$$\rho_0 \cdot {}^C_0 D_{t_M}^{\alpha} U_2(t_M, X_1) = \rho_0 \cdot {}^C_0 D_{t^*_M}^{\alpha} U^*_2(t^*_M, X_1) + \frac{1}{\Gamma(2-\alpha)} \cdot \int_0^{t_{M_O^*}} \frac{\dfrac{\partial^2 U_2}{\partial \tau^2}(\tau, X_1)}{(t_M - \tau)^{\alpha-1}} d\tau \tag{59}$$

$$\mu \cdot \frac{\partial^2 U_2}{\partial X_1^2}(t_M, X_1) = \mu \cdot \frac{\partial^2 U^*_2}{\partial X_1^2}(t^*_M, X_1) \tag{60}$$

replaced in (55) and the assumption that (55) is objective implies that the following identity holds:

$$\frac{1}{\Gamma(2-\alpha)} \cdot \int_0^{t_{M_O^*}} \frac{\dfrac{\partial^2 U_2}{\partial \tau^2}(\tau, X_1)}{(t_M - \tau)^{\alpha-1}} d\tau = 0 \tag{61}$$

Identity (61) in general is not verified. So, the description (55), is nonobjective. Observers O and O^* describing the same dynamics with (55) and (56) respectively, obtain different results which cannot be reconciled.

When temporal Riemann-Liouville fractional partial derivative is used, then in a similar way the following objectivity condition is obtained:

$$\frac{1}{\Gamma(2-\alpha)} \cdot \frac{\partial^2}{\partial t^2_M} \int_0^{t_{M_O^*}} \frac{U_2(\tau, X_1)}{(t_M - \tau)^{\alpha-1}} d\tau = 0 \tag{62}$$

Identity (62) in general is not verified. So, the description which uses Riemann-Liouville fractional order temporal partial derivatives, having integral representation on finite interval, is nonobjective. Observers O and O^* describing the same dynamics obtain different results which cannot be reconciled. The problem is: which one of the obtained results is correct? This result can be instructive for the authors of the paper [16] because it shows that the direct introduction of the temporal Caputo or Riemann-Liouville fractional order partial derivatives in the classical dynamic equation affects the objectivity of the description of the dynamics of guitar string. It can be also interesting for those researchers (authors of the papers [11]-[15]) who want to have a formal argument for the statement" in case of the guitar string the fractional order dynamics equation cannot be defined by replacing directly the integer order derivatives appearing in classical equation with temporal Caputo or Riemann-Liouville fractional order partial derivatives".

7. Conclusions

1. Mathematical descriptions of the small deformations, strain , principal directions of strain and principal strains, constitutive law, dynamics of an isotropic elastic solid, using integer order partial derivatives, are objective. This means that the results obtained by two different observers can be reconciled, i.e. transformed into each other using formulas that link the coordinates of a point in two fixed orthogonal reference frames and formulas that link the numbers representing a moment of time in two different choices of the origin of time measuring.

2. Mathematical descriptions of strain, principal strain, constitutive law, dynamics, obtained replacing directly the integer order derivatives with Caputo or Riemann-Liouwille fractional order spatial or temporal partial derivatives, having integral representation on finite interval, in case of an isotropic elastic guitar string, are nonobjective, i.e.depend on the choice of the fixed orthogonal reference frame or on the choice of the origin of time measuring. Due to that, observers describing the same elastic phenomenon with these tools, obtain different results which cannot be reconciled, i.e. transformed into each other using formulas that link the coordinates of a point in two fixed orthogonal reference frames and formulas that link the numbers representing a moment of time in two different choices of the origin of time measuring. This is not an academic curiosity! It is rather a problem: which one of the reported results is correct?

3. The fractional order strain used by us in sections 2,3 and 4 is different from those which appear in the existing literature, accessible to us for free. For instance, in [11], in 1D case, the so called "strain measure" defined with formula (2.7), is considered for this purpose. This not coincide with the fractional order strain considered in our manuscript, because formula (2.7) use the so called left and right Caputo fractional order derivative defined with formulas (2.3) and (2.4), respectively. The left Caputo fractional derivative depends on a parameter "a "and the right Caputo fractional derivative depends on a parameter "L". The parameter "a" represents the coordinate of the left end and the parameter "L "represents the coordinate of the right end of the 1D rod respectively. This make, that the so called left and right Caputo fractional order derivatives depend on the choice of the system of coordinates and the size of the 1D rod. Therefore, they are mathematical tools dependent on the mechanical event which has to be described. Moreover, the right and left Caputo fractional derivatives concept due to the parameters "a" and "L" become fuzzy and can lead to the question "Which derivative?" for detail see [17]. The Caputo and the Riemann-Liouville fractional order derivatives used by us are independent on the mechanical event which has to be described. (see formulas (15) and (16)). As far as we understand the fractional order strain defined in [12] with formula (4.18) in [13] with formula (47) in [14] with formula (47) is very similar to that considered in [11]. In other words, the authors of the papers [11], [12], [13], [14] do not directly replace fractional derivatives in the classical expression of the strain. They modify them because they have the "conviction" that direct replacement does not lead to a "good" description. We used the word "conviction" because we did not find in the scientific literature the demonstration of the general statement "integer-order derivatives cannot be simply replaced by fractional-order derivatives to develop the fractional-order theories" . What we do in sections 2, 3 and 4 is the formal demonstration that the direct replacement of integer order derivatives with Caputo or Riemann-Lioville fractional order derivatives leads to the loss of the objectivity of the description of strain and the principal strain in case of the guitar string. In this perspective, our contribution in sections 2,3, 4 appears as an argument that supports the general statement "integer-order derivatives cannot be simply replaced by fractional-order derivatives to develop the fractional-order theories".

4. In [12] subsection 3.3. the authors use a certain concept of objectivity for fractional kinetics. We reproduce here what the authors say about this concept: "This new concept of fractional continua should not of course violate the objectivity requirements. It is clear that under the change of the observer the distances between arbitrary pairs of points in the space and time intervals between events should be preserved. As common, the change of the observer may equivalently be viewed as a certain rigid-body motions superimposed on the current configuration. Thoroughly we will use this concept to prove that the proposed fractional kinematics leads to the same results (in the sense of objectivity) as the classical ones. It should be emphasised that it is crucial to observe how fractional deformation gradients transform under isomorphism (superimposed rigid-bodymotions)". As far as we understand the concept of objectivity used in [12] is different from that we use in our manuscript and is proven in case when the fractional order strain and stress is not obtained by direct replacement of the integer order derivatives with fractional order derivatives. In this perspective, our contribution in section 6. appears as an argument that supports the general statement "integer-order derivatives cannot be simply replaced by fractional-order derivatives to develop the fractional-order

theories" . Namely direct replacement leads to the loss of the objectivity (in our sense) of the description of kinematics.

5. In [15] the authors describe the relation between the stress and strain with formula (2). This is in fact a constitutive law. In formula (2) beside the Young’s modulus the fractional order operators are affected by two multiplicative parameters. One of them is a material constant the other one is a material length scale parameter, i.e. ,the integer order derivatives are not replaced directly with fractional order derivatives. So we have not found neither the concept of fractional order constitutive law nor the method of demonstrating the lack of objectivity used by us in section 5. In this perspective, our contribution in section 5 appears as an argument that supports the general statement "integer-order derivatives cannot be simply replaced by fractional-order derivatives to develop the fractional-order theories" . Namely direct replacement leads to the loss of the objectivity (in our sense) of the description of constitutive law.

6. The loss of objectivity in the case of the spatial fractional order description of the strain analyzed in section 2. 3.and 4. is due to the fact that instead of the equalities

$$\frac{\partial U_{iO*}}{\partial X_j *} = \frac{\partial U_{iO}}{\partial X_j} \quad \varepsilon_{jk}(t, X_1, \varphi(X_1) = \frac{1}{2}\cdot\left(\frac{\partial U_j}{\partial X_k} + \frac{\partial U_k}{\partial X_j}\right) = \varepsilon *_{jk}(t, X *_1, \varphi *(X *_1)$$

valid in case of integer order spatial partial derivatives

in case of Caputo fractional order spatial partial derivatives according to (27) the following equality holds:

$$ {}^{C,\alpha}\varepsilon_{12}(t, X_1, \varphi(X_1)) = \frac{\pi}{2\cdot L}\cdot(\cos\frac{\pi\cdot v\cdot t}{L} - 1)\cdot\frac{1}{\Gamma(1-\alpha)}\cdot\int_0^{X_{1O*}}\frac{\cos\frac{\pi\xi}{L}}{(X_1-\xi)^{\alpha}}d\xi + {}^{C,\alpha}\varepsilon *_{12}(t, X *_1, \varphi *(X *_1))$$

in case of Riemann-Liouville fractional order spatial partial derivatives according to (33) the following equality holds :

$$ {}^{R-L,\alpha}\varepsilon_{12}(t, X_1, \varphi(X_1)) = \frac{1}{2}\cdot(\cos\frac{\pi\cdot v\cdot t}{L} - 1)\cdot\frac{1}{\Gamma(1-\alpha)}\cdot\frac{\partial}{\partial X_1}\int_0^{X_{1O*}}\frac{\sin\frac{\pi\xi}{L}}{(X_1-\xi)^{\alpha}}d\xi + {}^{R-L,\alpha}\varepsilon *_{12}(t, X *_1, \varphi *(X *_1))$$

For objectivity the additional terms which appear in case of Caputo or Riemann-Liouville fractional order spatial derivative has to be equal to zero. At the end of section 2.and 3. there is a short discusion about the situation when the additional terms are equal to zero.But even if the additional term is equal to zero, the objectivity of the description does not result because the condition is only necessary. This means that following this way we cannot find an answer to the question: which is the suitable choices of fractional-order assuring the objectivity?

7. The loss of objectivity in the case of the description of the constitutive law with temporal fractional order derivative discussed in sections 5. is due to the fact that in case of Caputo fractional order temporal partial derivatives the following equality holds:

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$$\sigma_{12}(t_M, X_1, \varphi(X_1)) = \sigma*_{12}(t_M, X_1, \varphi*(X_1)) -$$

$$-\frac{\pi\cdot\nu}{L}2\cdot\mu\cdot\frac{\pi}{2L}\cdot\cos\frac{\pi\cdot X_1}{L}\cdot\frac{1}{\Gamma(1-\alpha)}\cdot\int_0^{t_{MO*}}\frac{\sin\frac{\pi\nu\tau}{L}}{(t_M-\tau)^\alpha}d\tau$$

In case of Riemann-Liouville fractional order temporal partial derivatives the following equality holds:

$$\sigma_{12}(t_M, X_1, \varphi(X_1)) = 2\mu\cdot\frac{\pi}{2L}\cdot\cos\frac{\pi\cdot X_1}{L}\cdot\frac{1}{\Gamma(1-\alpha)}\cdot\left(\frac{\partial}{\partial t_M}\int_0^{t_{MO*}}\frac{\cos\frac{\pi\cdot\nu\cdot\tau}{L}-1}{(t_M-\tau)^\alpha}d\tau + \frac{\partial}{\partial t*_M}\int_0^{t*_M}\frac{\cos\frac{\pi\cdot\nu\cdot t_{MO*}}{L}-1}{(t*_M-\tau^*)^\alpha}d\tau^*\right) -$$

$$+ \sigma*_{12}(t*_M, X_1, \varphi*(X_1))$$

For objectivity the additional terms which appear in case of Caputo or Riemann-Liouville fractional order temporal derivative has to be equal to zero. But even if the additional term is zero, the objectivity of the description does not result because the condition is only necessary. This means that following this way we cannot find an answer to the question: which is the suitable choices of fractional-order assuring the objectivity?

8. The loss of objectivity in the case of the description of the dynamics of the guitar string with temporal fractional order derivative discussed in sections 6. is due to the fact that in case of Caputo fractional order temporal partial derivatives the following equality holds:

$$\rho_0\cdot{}^C{}_0D_{t_M}{}^\alpha U_2(t_M, X_1) = \rho_0\cdot{}^C{}_0D_{t*_M}{}^\alpha U*_2(t*_M, X_1) + \frac{1}{\Gamma(2-\alpha)}\cdot\int_0^{t_{MO*}}\frac{\frac{\partial^2 U_2}{\partial\tau^2}(\tau, X_1)}{(t_M-\tau)^{\alpha-1}}d\tau$$

In case of Riemann-Liouville fractional order temporal partial derivatives the following equality holds:

$$\rho_0\cdot{}^C{}_0D_{t_M}{}^\alpha U_2(t_M, X_1) = \rho_0\cdot{}^C{}_0D_{t*_M}{}^\alpha U*_2(t*_M, X_1) + \frac{1}{\Gamma(2-\alpha)}\cdot\frac{\partial^2}{\partial t^2_M}\int_0^{t_{MO*}}\frac{U_2(\tau, X_1)}{(t_M-\tau)^{\alpha-1}}d\tau$$

For objectivity the additional terms which appear in case of Caputo or Riemann-Liouville fractional order temporal derivative has to be equal to zero. But even if the additional term is zero, the objectivity of the description does not result because the condition is only necessary This means that following this way we cannot find an answer to the question: which is the suitable choices of fractional-order assuring the objectivity?

9.Direct replacement of integer-order derivatives with Caputo or Riemann-Liouville fractional order derivatives is not appropriate for describing stress,constitutive law and dynamics in the case of a guitar string.

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