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equivalence relation to neighborhood relation, Lin et al. proposed neighborhood multigranulation rough sets (NMGRS). Lots of studies focus on deriving models by the same decision strategy. Huang et al. proposed intuitionistic fuzzy multigranulation rough sets[9]. Feng et al. proposed variable precision multigranulation decision-theoretic fuzzy rough sets[5]. Li et al. proposed three-way cognitive concept learning via multi-granularity[15]. More over, there are researches about multigranulation rough sets and their relative models, such as multigranulation rough set theory over two universes[31], a comparative study of multigranulation rough sets and concept lattices via rule acquisition[16].

In an information explosion era, approximation computing becomes more and more difficult: the size of data sometimes is too huge to handle, the structure of data is changing to be more complex, and the granular structures often increase or decrease. The issue of computing and updating approximations in MGRS and their derived models attracts much research interest. The studies are often categorized into four classes, namely, how to update approximations while varying attributes[37,42], how to update approximations while varying attribute values[3,40], how to update approximations while varying decision attribute values[37,21], how to update approximations while varying object set[7,19].

No matter what variation is, there always exist two means to determine the relation between two sets: set operation or matrix product. Therefore, we can classify those studies into two categories. One is based on set operation. Scholars use set operation to determine whether a set is contained in another set or not, or whether their intersection is empty or not (see Chuan Luo[20,22], Wenhao Shu[29], Guangming Lang[13], Mingjie Cai[2], Wei Wei[34] and etc). When granules sizes are generally big, set operation is a time consuming way because of its searching strategy: when we compute the intersection of two sets, we must confirm whether every object in a set is in another set or not. In the extreme case, when the two sets are both $U$(all samples are in a data set), then the time complexity of computing the intersection of them is $|U|^2$. The other category is based on matrix. These studies are mostly based on matrix product or other operations. Scholars often change a set into binary matrix, and then design algorithms based on properties of binary matrix (see Jingqian Wang[33], Chengxiang Hu[6], Yanyong Huang[10] and etc). Although the time complexity of determining the relation between two sets via matrix approach is a constant, they often consider all the objects in the universe without filtering.

We attempt to combine the two approaches and derive new methods to overcome their defects. In other words, we concentrate on which part of the universe does not need to be considered while computing and updating approximations in NMGRS. At the same time, we determine the relation between two sets by matrix product.

Lin et al. proposed NMGRS[17] in 2012. NMGRS is a powerful mathematical tool in representing some real life situations and it extends the application of MGRS into a broader area. When we apply NMGRS in real life situations, approximation computing is an essential part. Approximations must be computed before making decision. Positive region must be calculated so that attributes can be selected in some attribute reduction process. Calculating approximations is necessary when applying rough set theory, thus how to compute approximations is an important issue in NMGRS. Granulation and approximation are two important issues in all rough sets models that can not be avoided. In NMGRS theory, different granule sizes have a big influence in its applications. Decreasing or increasing granules changes the approximations directly. Thus choosing an appropriate size of granules makes a great difference in decision making process[27]. Similar phenomenon exists in attribute reduction process because decreasing or increasing granules changes the positive region directly.

Neighborhood classes often decrease or increase. So it is important to update approximations in NMGRS based on approximations we have computed. We design algorithms for updating approximations while decreasing or increasing neighborhood classes. First, searching region while updating approximations in NMGRS need to be shrunk. A shrunk searching region can reduce the executing time of the algorithms. Second, matrix-based approaches need to be proposed to make the algorithms more efficient.

The rest of this paper is organized as follows. Section 2 introduces several basic concepts of NMGRS and

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matrix-based static algorithms to calculate approximations in NMGRS. In Section 3, dynamic approaches for updating approximations in NMGRS while decreasing or increasing neighborhood classes are proposed. Several algorithms designed using approaches that we proposed are given in Section 4. Experimental evaluations are conducted in Section 5 to verify the efficiency and validity of these algorithms. Finally, some conclusion and future work are given in Section 6.

## 2. Preliminaries

In this section, we review several basic concepts of NMGRS and propose matrix-based static approaches for computing approximations in NMGRS.

### 2.1. Neighborhood Multigranulation rough sets

In this subsection, several basic concepts of NMGRS are reviewed.

**Definition 1.** [17]Let $NIS = (U, AT, \delta)$ be a neighborhood information system, where $A_k \in AT$ for any $k \in \{1,2,\cdots,m\}$, and $\delta$ is a nonnegative number. $\forall k \in \{1,2,\cdots,m\}, A_k^C, A_k^N \subseteq A_k$ are categorical and numerical attributes, respectively. The neighborhood classes of objects $x$ induced by $A_k^C$, $A_k^N$ and $A_k^C \cup A_k^N$ are defined as

- $n_{A_k^C}(x) = \left\{x_i \in U | d_{A_k^C}(x,x_i) = 0\right\}$;
- $n_{A_k^N}(x) = \left\{x_i \in U | d_{A_k^N}(x,x_i) \leqslant \delta\right\}$;
- $n_{A_k^C \cup A_k^N}(x) = \left\{x_i \in U | d_{A_k^C}(x,x_i) = 0 \wedge d_{A_k^N}(x,x_i) \leqslant \delta\right\}$;

where $d$ is a distance [28] between $x$ and $x_i$.

Accordingly, we say $(U,\delta)$ is a neighborhood approximation space, if there is an attribute subset in the system generating a neighborhood relation on the universe. We consider this system as a neighborhood information system, and denote it as $NIS = (U, AT, \delta)$, where $U$ is a nonempty finite set and $AT$ is a group of attribute set.

In this paper we consider distance between two objects as Euclid distance, which is defined as follows.

**Definition 2.** [1] The Euclidean distance between two points $p$ and $q$ is the length of the line segment connecting them in Cartesian coordinates. If $p = (p_1, p_2, \cdots, p_n)$ and $q = (q_1, q_2, \cdots, q_n)$ are two points in Euclidean n-space, then the distance (d) between p and q is given by the Pythagorean formula:

$$d(p,q) = d(q,p) = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2 + \cdots + (q_n - p_n)^2}.$$

**Definition 3.** [17](ONMGRS)Let $NIS = (U, AT, \delta)$ be a neighborhood information system, where $A_k \in AT$ for any $k \in \{1,2,\cdots,m\}$, and $\delta$ is a nonnegative number. For any $X \subseteq U$, the optimistic neighborhood multigranulation lower and upper approximations of $X$ are denoted by $\underline{\sum_{k=1}^{m} \delta_{A_k}^O}(X)$ and $\overline{\sum_{k=1}^{m} \delta_{A_k}^O}(X)$, respectively,

$$\underline{\sum_{k=1}^{m} \delta_{A_k}^O}(X) = \left\{x \in U | n_{A_1^C \cup A_1^N}(x) \subseteq X \vee n_{A_2^C \cup A_2^N}(x) \subseteq X \vee \cdots \vee n_{A_m^C \cup A_m^N}(x) \subseteq X\right\},$$

$$\overline{\sum_{k=1}^{m} \delta_{A_k}^O}(X) = \sim \underline{\sum_{k=1}^{m} \delta_{A_k}^O}(\sim X).$$

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where $n_{A_1^C \cup A_k^N}(x)$ is the neighborhood class of $x$ in terms of the attribute set $A_k$ and neighborhood radius $\delta$, $\sim X$ is the complement of the set $X$.

**Theorem 1.** [17] *Let $NIS = (U, AT, \delta)$ be a neighborhood information system, where $A_k \in AT$ for any $k \in \{1, 2, \cdots, m\}$, and $\delta$ is a nonnegative number. For any $X \subseteq U$, the optimistic neighborhood multigranulation upper approximation of $X$ is denoted by $\overline{\sum_{k=1}^{m} \delta_{A_k}^O}(X)$. We have*

$$\overline{\sum_{k=1}^{m}} \delta_{A_k}^O(X) = \left\{ x \in U | n_{A_1^C \cup A_1^N}(x) \cap X \neq \emptyset \wedge n_{A_2^C \cup A_2^N}(x) \cap X \neq \emptyset \wedge \cdots \wedge n_{A_m^C \cup A_m^N}(x) \cap X \neq \emptyset \right\}.$$

**Definition 4.** [17](PNMGRS)Let $NIS = (U, AT, \delta)$ be a neighborhood information system, where $A_k \in AT$ for any $k \in \{1, 2, \cdots, m\}$, and $\delta$ is a nonnegative number. For any $X \subseteq U$, the pessimistic neighborhood multigranulation lower and upper approximations of X are denoted by $\underline{\sum_{k=1}^{m} \delta_{A_k}^P}(X)$ and $\overline{\sum_{k=1}^{m} \delta_{A_k}^P}(X)$, respectively,

$$\underline{\sum_{k=1}^{m}} \delta_{A_k}^P(X) = \left\{ x \in U | n_{A_1^C \cup A_1^N}(x) \subseteq X \wedge n_{A_2^C \cup A_2^N}(x) \subseteq X \wedge \cdots \wedge n_{A_m^C \cup A_m^N}(x) \subseteq X \right\},$$

$$\overline{\sum_{k=1}^{m}} \delta_{A_k}^P(X) = \sim \underline{\sum_{k=1}^{m}} \delta_{A_k}^P(\sim X).$$

**Theorem 2.** [17] *Let $NIS = (U, AT, \delta)$ be a neighborhood information system, where $A_k \in AT$ for any $k \in \{1, 2, \cdots, m\}$, and $\delta$ is a nonnegative number. For any $X \subseteq U$, the pessimistic neighborhood multigranulation upper approximation of $X$ is denoted by $\overline{\sum_{k=1}^{m} \delta_{A_k}^P}(X)$. We have*

$$\overline{\sum_{k=1}^{m}} \delta_{A_k}^P(X) = \left\{ x \in U | n_{A_1^C \cup A_1^N}(x) \cap X \neq \emptyset \vee n_{A_2^C \cup A_2^N}(x) \cap X \neq \emptyset \vee \cdots \vee n_{A_m^C \cup A_m^N}(x) \cap X \neq \emptyset \right\}.$$

### *2.2. Matrix-based algorithm for computing approximations in NMGRS*

**Definition 5.** [6]Let $U = \{x_1, x_2, \cdots, x_n\}$, for any $X \subseteq U$, the matrix representation of $X$ is denoted by $V(X) = [v_1(X), \cdots, v_n(X)]$, where

$$v_i(X) = \begin{cases} 1 & x_i \in X \\ 0 & x_i \notin X \end{cases} \qquad i \in \{1, 2, \cdots, n\}$$

**Lemma 3.** [4] *Let $U = \{x_1, x_2, \cdots, x_n\}$, for any $X, Y \subseteq U$, if $Y \subseteq X$, then*

$$V(Y) \cdot V^T(\sim X) = 0.$$

*where "$T$" denotes the transpose operation and "$\cdot$" is matrix product.*

**Lemma 4.** [4] *Let $U = \{x_1, x_2, \cdots, x_n\}$. For any $X, Y \subseteq U$, if $Y \cap X \neq \emptyset$, then*

$$V(Y) \cdot V^T(X) > 0.$$

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**Table 1.** A neighborhood information system.

| $U$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ |
|---|---|---|---|---|
| $x_1$ | 1 | 3 | 1.5 | 0.5 |
| $x_2$ | 2 | 2 | 0.5 | 1 |
| $x_3$ | 3 | 1 | 2 | 2.5 |
| $x_4$ | 2 | 3 | 1.5 | 1 |
| $x_5$ | 1 | 1 | 2 | 0.5 |
| $x_6$ | 3 | 2 | 0.5 | 2 |

**Example 1.** Let $NIS=(U,AT,\delta)$ be a neighborhood information system, as shown in *Table* 1, where $U=\{x_1,x_2,x_3,x_4,x_5,x_6\}$. $A_1=A_1^C\cup A_1^N$, $A_1^C=\{a_1\}$ and $A_1^N=\{a_3\}$, $A_2=A_2^C\cup A_2^N$, $A_2^C=\{a_2\}$ and $A_2^N=\{a_4\}$. Let $X=\{x_2,x_4,x_5\}$, $\delta=1$. According to Definition 5, we have $V(X)=[0,1,0,1,1,0]^T$. Suppose $Y=\{x_2,x_4\}$, $Z=\{x_2,x_6\}$, then $V(Y)=[0,1,0,1,0,0]$, $V(Z)=[0,1,0,0,0,1]$. Obviously, $Y\subseteq X$, $Z\cap X\neq\emptyset$. $V(\sim X)=[1,0,1,0,0,1]$, $V(Y)\cdot V^T(\sim X)=[0,1,0,1,0,0]\cdot[1,0,1,0,0,1]^T=0$, $V(Z)\cdot V^T(X)=[0,1,0,0,0,1]\cdot[0,1,0,1,1,0]^T=1>0$.

**Definition 6.** Let $NIS=(U,AT,\delta)$ be a neighborhood information system, where $A_k\in AT$ for any $k\in\{1,2,\cdots,m\}$, and $\delta$ is a nonnegative number. For any $X\subseteq U$, the upper approximation character sets of X in NMGRS can be defined as:

$$\delta^L_{A_k}(X)=\left\{x\in U|n_{A_k^C\cup A_k^N}(x)\subseteq X\right\},\forall k=1,2,\cdots,m.$$

**Theorem 5.** *Let $NIS=(U,AT,\delta)$ be a neighborhood information system, where $A_k\in AT$ for any $k\in\{1,2,\cdots,m\}$. For any $X\subseteq U$, the pessimistic and optimistic lower approximations in NMGRS can be calculated by:*

$$\underline{\sum_{k=1}^{m}\delta^P_{A_k}(X)}=\cap_{k=1}^m\delta^L_{A_k}(X);\underline{\sum_{k=1}^{m}\delta^O_{A_k}(X)}=\cup_{k=1}^m\delta^L_{A_k}(X).$$

**Proof.** *This theorem can be easily obtained by Definition 3 and Definition 4.* □

**Example 2.** (Continuation of Example 1) By Theorem 3, since

$$\begin{aligned}
&V(n_{A_1^C\cup A_1^N}(x_1))\cdot V^T(\sim X)=V(n_{A_1^C\cup A_1^N}(x_5))\cdot V^T(\sim X)=[1,0,0,0,1,0]\cdot[1,0,1,0,0,1]^T=1\neq 0,\\
&V(n_{A_1^C\cup A_1^N}(x_2))\cdot V^T(\sim X)=V(n_{A_1^C\cup A_1^N}(x_4))\cdot V^T(\sim X)=[0,1,0,1,0,0]\cdot[1,0,1,0,0,1]^T=0,\\
&V(n_{A_1^C\cup A_1^N}(x_3)\cdot V^T(\sim X)=[0,0,1,0,0,0]\cdot[1,0,1,0,0,1]^T=1\neq 0,\\
&V(n_{A_1^C\cup A_1^N}(x_6)\cdot V^T(\sim X)=[0,0,0,0,0,1]\cdot[1,0,1,0,0,1]^T=1\neq 0;
\end{aligned}$$

$$\begin{aligned}
&V(n_{A_2^C\cup A_2^N}(x_1))\cdot V^T(\sim X)=V(n_{A_2^C\cup A_2^N}(x_4))\cdot V(\sim X)^T=[1,0,0,1,0,0]\cdot[1,0,1,0,0,1]^T=1\neq 0,\\
&V(n_{A_2^C\cup A_2^N}(x_2))\cdot V^T(\sim X)=V(n_{A_2^C\cup A_2^N}(x_6))\cdot V(\sim X)^T=[0,1,0,0,0,1]\cdot[1,0,1,0,0,1]^T=1\neq 0,\\
&V(n_{A_2^C\cup A_2^N}(x_3))\cdot V^T(\sim X)=[0,0,1,0,0,0]\cdot[1,0,1,0,0,1]^T=1\neq 0,\\
&V(n_{A_2^C\cup A_2^N}(x_5))\cdot V^T(\sim X)=[0,0,0,0,1,0]\cdot[1,0,1,0,0,1]^T=0,
\end{aligned}$$

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then by Definition 6,

$$V(\delta^L_{A_1}(X)) = V(n_{A_1^C \cup A_1^N}(x_2)) \vee V(n_{A_1^C \cup A_1^N}(x_4)) = [0,1,0,1,0,0],$$
$$V(\delta^L_{A_2}(X)) = V(n_{A_1^C \cup A_1^N}(x_5)) = [0,1,0,0,1,0].$$

By Theorem 5,

$$\underline{\sum_{k=1}^{m} \delta^P_{A_k}}(X) = V(\delta^U_{A_1}(X)) \wedge V(\delta^U_{A_2}(X)) = [0,1,0,1,0,0] \wedge [0,1,0,0,1,0] = [0,1,0,0,0,0],$$
$$\underline{\sum_{k=1}^{m} \delta^O_{A_k}}(X) = V(\delta^U_{A_1}(X)) \vee V(\delta^U_{A_2}(X)) = [0,1,0,1,0,0] \vee [0,1,0,0,1,0] = [0,1,0,1,1,0].$$

By Definition 5,

$$\underline{\sum_{k=1}^{m} \delta^P_{A_k}}(X) = \emptyset, \underline{\sum_{k=1}^{m} \delta^O_{A_k}}(X) = \{x_2, x_4, x_5\}.$$

**Definition 7.** Let $NIS = (U, AT, \delta)$ be a neighborhood information system, where $A_k \in AT$ for any $k \in \{1,2,\cdots,m\}$, and $\delta$ is a nonnegative number. For any $X \subseteq U$, the upper approximation character sets of X in NMGRS can be defined as:

$$\delta^U_{A_k}(X) = \left\{ x \in U | n_{A_k^C \cup A_k^N}(x) \cap X \neq \emptyset \right\}, \forall k = 1,2,\cdots,m.$$

**Theorem 6.** *Let $NIS = (U, AT, \delta)$ be a neighborhood information system, where $A_k \in AT$ for any $k \in \{1,2,\cdots,m\}$. For any $X \subseteq U$, the optimistic and pessimistic upper approximations can be calculated by:*

$$\overline{\sum_{k=1}^{m} \delta^P_{A_k}}(X) = \cup_{k=1}^{m} \delta^U_{A_k}(X); \overline{\sum_{k=1}^{m} \delta^O_{A_k}}(X) = \cap_{k=1}^{m} \delta^U_{A_k}(X).$$

**Proof.** *This theorem can be easily obtained by Theorem 1 and Theorem 2.* □

**Example 3.** (Continuation of Example 1.) From Tabel 1, by Lemma 4, since

$$V(n_{A_1^C \cup A_1^N}(x_1)) \cdot V^T(X) = V(n_{A_1^C \cup A_1^N}(x_5)) \cdot V^T(X) = [1,0,0,0,1,0] \cdot [0,1,0,1,1,0]^T = 1 > 0,$$
$$V(n_{A_1^C \cup A_1^N}(x_2)) \cdot V^T(X) = V(n_{A_1^C \cup A_1^N}(x_4)) \cdot V^T(X) = [0,1,0,1,0,0] \cdot [0,1,0,1,1,0]^T = 1 > 0,$$
$$V(n_{A_1^C \cup A_1^N}(x_3) \cdot V^T(X) = [0,0,1,0,0,0] \cdot [0,1,0,1,1,0]^T = 0,$$
$$V(n_{A_1^C \cup A_1^N}(x_6) \cdot V^T(X) = [0,0,0,0,0,1] \cdot [0,1,0,1,1,0]^T = 0;$$

$$V(n_{A_2^C \cup A_2^N}(x_1)) \cdot V^T(X) = V(n_{A_1^C \cup A_2^N}(x_4)) \cdot V(X) = [1,0,0,1,0,0] \cdot [0,1,0,1,1,0]^T = 1 > 0,$$
$$V(n_{A_2^C \cup A_2^N}(x_2)) \cdot V^T(X) = V(n_{A_1^C \cup A_2^N}(x_6)) \cdot V(X) = [0,1,0,0,0,1] \cdot [0,1,0,1,1,0]^T = 1 > 0,$$
$$V(n_{A_2^C \cup A_2^N}(x_3)) \cdot V^T(X) = [0,0,1,0,0,1] \cdot [0,1,0,1,1,0]^T = 0,$$
$$V(n_{A_2^C \cup A_2^N}(x_5)) \cdot V^T(X) = [0,0,0,0,1,0] \cdot [0,1,0,1,1,0]^T = 1 > 0,$$

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**Algorithm 1** Matrix-based algorithm for computing approximations in NMGRS

**Require:** (1)A neighborhood information system $NIS = (U, AT, \delta)$, the granular structures $A_1, A_2, \cdots, A_m$.(2)A target concept $X \subseteq U$ (3)neighborhood classes $n_{A_k^C \cup A_k^N}(x), \forall x \in U, \forall k \in \{1, 2, \cdots, m\}$.

**Ensure:** Approximations in NMGRS.

$m \longleftarrow |AT|$
$n \longleftarrow |U|$
**for** $k = 1 \rightarrow m$ **do**
  **for** $i = 1 \rightarrow n$ **do**
    **for** $j = 1 \rightarrow n$ **do**
      **if** $v_i(X) = 1 \wedge V(n_{A_1^C \cup A_1^N}(x_i)) \cdot V^T(\sim(X)) = 0$ **then** $v_i(\delta^L_{A_k}) = 1$
      **end if**
      **if** $v_i(X) = 1 \wedge v_i(n_{A_1^C \cup A_1^N}(x_j)) = 1$ **then** $v_j(\delta^U_{A_k}) = 1$
      **end if**
    **end for**
  **end for**
**end for**
$V(\overline{\sum_{k=1}^m \delta^O_{A_k}}(X)) \longleftarrow V(\delta^U_{A_1})$
$V(\overline{\sum_{k=1}^m \delta^P_{A_k}}(X)) \longleftarrow V(\delta^U_{A_1})$
$V(\overline{\sum_{k=1}^m \delta^P_{A_k}}(X)) \longleftarrow V(\delta^L_{A_1})$
$V(\overline{\sum_{k=1}^m \delta^P_{A_k}}(X)) \longleftarrow V(\delta^L_{A_1})$
**for** $k = 2 \rightarrow m$ **do**
  $V(\overline{\sum_{k=1}^m \delta^O_{A_k}}(X)) \longleftarrow V(\overline{\sum_{k=1}^m \delta^O_{A_k}}(X)) \vee V(\delta^L_{A_k})$
  $V(\overline{\sum_{k=1}^m \delta^O_{A_k}}(X)) \longleftarrow V(\overline{\sum_{k=1}^m \delta^O_{A_k}}(X)) \wedge V(\delta^U_{A_k})$
  $V(\overline{\sum_{k=1}^m \delta^P_{A_k}}(X)) \longleftarrow V(\overline{\sum_{k=1}^m \delta^P_{A_k}}(X)) \wedge V(\delta^L_{A_k})$
  $V(\overline{\sum_{k=1}^m \delta^P_{A_k}}(X)) \longleftarrow V(\overline{\sum_{k=1}^m \delta^P_{A_k}}(X)) \vee V(\delta^U_{A_k})$
**end for**
**Return** $\overline{\sum_{k=1}^m \delta^O_{A_k}}(X), \overline{\sum_{k=1}^m \delta^P_{A_k}}(X), \underline{\sum_{k=1}^m \delta^O_{A_k}}(X)$ and $\underline{\sum_{k=1}^m \delta^P_{A_k}}(X)$

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- *If* $(A_k^C)^t \subseteq (A_k^C)^{t+1}$, *we have* $n^{t+1}_{A_k^C}(x) \subseteq n^t_{A_k^C}(x) \Rightarrow n^{t+1}_{A_k^C \cup A_k^N}(x) \subseteq n^t_{A_k^C \cup A_k^N}(x)$;
- *If* $(A_k^N)^t \subseteq (A_k^N)^{t+1}$, *we have* $n^{t+1}_{A_k^N}(x) \subseteq n^t_{A_k^N}(x) \Rightarrow n^{t+1}_{A_k^C \cup A_k^N}(x) \subseteq n^t_{A_k^C \cup A_k^N}(x)$;
- *If* $\delta \leqslant \Delta$, *we have* $n^{t+1}_{A_k^C \cup A_k^N}(x) \subseteq n^t_{A_k^C \cup A_k^N}(x)$.

□

**Theorem 8.** *Let* $NIS^t = (U, AT^t, \Delta)$ *be a neighborhood information system at time t,* $NIS^{t+1} = (U, AT^{t+1}, \delta)$ *be a neighborhood information system at time* $t+1$. *For any* $X \subseteq U$, *the following results hold.*
$(1)\underline{\sum_{k=1}^m \Delta^t_{A_k}}^O(X) \subseteq \underline{\sum_{k=1}^m \delta^{t+1}_{A_k}}^O(X)$;
$(2)\overline{\sum_{k=1}^m \Delta^t_{A_k}}^O(X) \supseteq \overline{\sum_{k=1}^m \delta^{t+1}_{A_k}}^O(X)$.

**Proof.** $(1)\forall x \in \underline{\sum_{k=1}^m \Delta^t_{A_k}}^O(X), x \in \underline{\sum_{k=1}^m \delta^t_{A_k}}^O(X) \Rightarrow \exists k \in \{1,2,\cdots,m\}, n^t_{A_k^C \cup A_k^N}(x) \subseteq X \Rightarrow \exists k \in \{1,2,\cdots,m\}, n^{t+1}_{A_k^C \cup A_k^N}(x) \subseteq X \Rightarrow x \in \underline{\sum_{k=1}^m \delta^{t+1}_{A_k}}^O(X)$, we have $\underline{\sum_{k=1}^m \Delta^t_{A_k}}^O(X) \subseteq \underline{\sum_{k=1}^m \delta^{t+1}_{A_k}}^O(X)$;
$(2)\forall x \in \overline{\sum_{k=1}^m \delta^{t+1}_{A_k}}^O(X), x \in \overline{\sum_{k=1}^m \delta^{t+1}_{A_k}}^O(X) \Rightarrow \forall k \in \{1,2,\cdots,m\}, n^{t+1}_{A_k^C \cup A_k^N}(x) \cap X \neq \emptyset \Rightarrow \forall k \in \{1,2,\cdots,m\}, n^t_{A_k^C \cup A_k^N}(x) \cap X \neq \emptyset \Rightarrow x \in \overline{\sum_{k=1}^m \delta^t_{A_k}}^O(X)$, we have $\overline{\sum_{k=1}^m \Delta^{t+1}_{A_k}}^O(X) \subseteq \overline{\sum_{k=1}^m \delta^t_{A_k}}^O(X)$; □

**Theorem 9.** *Let* $NIS^t = (U, AT^t, \Delta)$ *be a neighborhood information system at time t,* $NIS^{t+1} = (U, AT^{t+1}, \delta)$ *be a neighborhood information system at time* $t+1$. *For any* $X \subseteq U$, *the following results hold.*
$(1)\underline{\sum_{k=1}^m \Delta^t_{A_k}}^P(X) \subseteq \underline{\sum_{k=1}^m \delta^{t+1}_{A_k}}^P(X)$;
$(2)\overline{\sum_{k=1}^m \Delta^t_{A_k}}^P(X) \supseteq \overline{\sum_{k=1}^m \delta^{t+1}_{A_k}}^P(X)$.

**Proof.** *It is similar to that of Theorem 8.* □

Theorems 8 and 9 indicate the relations among lower and upper approximations in NMGRS between time $t$ and time $t+1$. However, they are not clear enough for updating approximations in NMGRS. The following theorem provides accurate approaches for updating approximations in NMGRS from time $t$ to $t+1$.

**Theorem 10.** *Let* $NIS^t = (U, AT^t, \Delta)$ *be a neighborhood information system at time t,* $NIS^{t+1} = (U, AT^{t+1}, \delta)$ *be a neighborhood information system at time* $t+1$. *For any* $X \subseteq U$, *we have*

*(i) if* $\underline{\Delta \sum_{k=1}^m \Delta_{A_k}}^O(X) = \left\{x | \exists k \in \{1,2,\cdots,m\}, n^{t+1}_{A_k^C \cup A_k^N}(x) \subseteq X,\ x \in \overline{\sum_{k=1}^m \Delta^t_{A_k}}^P(X) - \underline{\sum_{k=1}^m \Delta^t_{A_k}}^P(X)\right\}$, *then* $\underline{\sum_{k=1}^m \delta^{t+1}_{A_k}}^O(X) = \underline{\sum_{k=1}^m \Delta^t_{A_k}}^O(X) \cup \underline{\Delta \sum_{k=1}^m \Delta_{A_k}}^O(X)$.

*(ii) if* $\overline{\Delta \sum_{k=1}^m \Delta_{A_k}}^O(X) = \left\{x | \exists k \in \{1,2,\cdots,m\}, n^{t+1}_{A_k^C \cup A_k^N}(x) \cap X = \emptyset,\ x \in \overline{\sum_{k=1}^m \Delta^t_{A_k}}^P(X) - \underline{\sum_{k=1}^m \Delta^t_{A_k}}^P(X)\right\}$, *then* $\overline{\sum_{k=1}^m \delta^{t+1}_{A_k}}^O(X) = \overline{\sum_{k=1}^m \Delta^t_{A_k}}^O(X) - \overline{\Delta \sum_{k=1}^m \Delta_{A_k}}^O(X)$.

*(iii) if* $\underline{\Delta \sum_{k=1}^m \Delta_{A_k}}^P(X) = \left\{x | \forall k \in \{1,2,\cdots,m\}, n^{t+1}_{A_k^C \cup A_k^N}(x) \subseteq X,\ x \in \overline{\sum_{k=1}^m \Delta^t_{A_k}}^P(X) - \underline{\sum_{k=1}^m \Delta^t_{A_k}}^P(X)\right\}$, *then* $\underline{\sum_{k=1}^m \delta^{t+1}_{A_k}}^P(X) = \underline{\sum_{k=1}^m \Delta^t_{A_k}}^P(X) \cup \underline{\Delta \sum_{k=1}^m \Delta_{A_k}}^P(X)$.

*(iv) if* $\overline{\Delta \sum_{k=1}^m \Delta_{A_k}}^P(X) = \left\{x | \forall k \in \{1,2,\cdots,m\}, n^{t+1}_{A_k^C \cup A_k^N}(x) \cap X = \emptyset,\ x \in \overline{\sum_{k=1}^m \Delta^t_{A_k}}^P(X) - \underline{\sum_{k=1}^m \Delta^t_{A_k}}^P(X)\right\}$, *then* $\overline{\sum_{k=1}^m \delta^{t+1}_{A_k}}^P(X) = \overline{\sum_{k=1}^m \Delta^t_{A_k}}^P(X) - \overline{\Delta \sum_{k=1}^m \Delta_{A_k}}^P(X)$.

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**Proof.**

(i) $''\Leftarrow'':$ *Since we have* $\underline{\sum_{k=1}^{m}\Delta_{A_k}^{t}}^{O}(X)\subseteq\underline{\sum_{k=1}^{m}\delta_{A_k}^{t+1}}^{O}(X)$, *and* $\underline{\Delta\sum_{k=1}^{m}\Delta_{A_k}}^{O}(X)\subseteq\underline{\sum_{k=1}^{m}\delta_{A_k}^{t+1}}^{O}(X)$, *we have* $\underline{\sum_{k=1}^{m}\Delta_{A_k}^{t}}^{O}(X)\cup\underline{\Delta\sum_{k=1}^{m}\Delta_{A_k}}^{O}(X)\subseteq\underline{\sum_{k=1}^{m}\delta_{A_k}^{t+1}}^{O}(X)$.
$''\Rightarrow'':$*If* $x\notin\underline{\sum_{k=1}^{m}\Delta_{A_k}^{t}}^{O}(X)\cup\underline{\Delta\sum_{k=1}^{m}\Delta_{A_k}}^{O}(X)$, *since we have* $n^{t+1}_{A_k^C\cup A_k^N}(x)\subseteq n^{t}_{A_k^C\cup A_k^N}(x)$, $n^{t}_{A_k^C\cup A_k^N}(x)\not\subseteq X\Rightarrow n^{t+1}_{A_k^C\cup A_k^N}(x)\not\subseteq X$, $x\notin\underline{\Delta\sum_{k=1}^{m}\Delta_{A_k}}^{O}(X)$ *and* $x\in U-\overline{\sum_{k=1}^{m}\delta_{A_k}^{t}}^{P}(X)\Rightarrow n^{t}_{A_k^C\cup A_k^N}(x)\cap X=\emptyset\Rightarrow n^{t+1}_{A_k^C\cup A_k^N}(x)\cap X=\emptyset\Rightarrow x\notin\underline{\sum_{k=1}^{m}\delta_{A_k}^{t+1}}^{O}(X)$, *we have* $x\notin\underline{\sum_{k=1}^{m}\delta_{A_k}^{t}}^{O}(X)\cup\underline{\Delta\sum_{k=1}^{m}\Delta_{A_k}}^{O}(X)\Rightarrow x\notin\underline{\sum_{k=1}^{m}\delta_{A_k}^{t+1}}^{O}(X)$, $\underline{\sum_{k=1}^{m}\delta_{A_k}^{t+1}}^{O}(X)\subseteq\underline{\sum_{k=1}^{m}\delta_{A_k}^{t}}^{O}(X)\cup\underline{\Delta\sum_{k=1}^{m}\Delta_{A_k}}^{O}(X)$. *This completes the proof.*

(ii) $''\Rightarrow'':\ \forall x\in\overline{\sum_{k=1}^{m}\delta_{A_k}^{t+1}}^{O}(X)\Rightarrow n^{t+1}_{A_k^C\cup A_k^N}(x)\cap X\neq\emptyset\Rightarrow n^{t}_{A_k^C\cup A_k^N}(x)\cap X\neq\emptyset\Rightarrow x\in\overline{\sum_{k=1}^{m}\delta_{A_k}^{t}}^{O}(X)$, $x\in\overline{\sum_{k=1}^{m}\delta_{A_k}^{t+1}}^{O}(X)\Rightarrow x\notin\overline{\Delta\sum_{k=1}^{m}\delta_{A_k}}^{O}(X)$, *thus we have* $\overline{\sum_{k=1}^{m}\delta_{A_k}^{t+1}}^{O}(X)\subseteq\overline{\sum_{k=1}^{m}\delta_{A_k}^{t}}^{O}(X)-\overline{\Delta\sum_{k=1}^{m}\delta_{A_k}}^{O}(X)$.
$''\Leftarrow'':\ \forall x\in\overline{\sum_{k=1}^{m}\Delta_{A_k}^{t}}^{O}(X)$, *if* $n^{t+1}_{A_k^C\cup A_k^N}(x)\cap X\neq\emptyset\Rightarrow x\in\overline{\sum_{k=1}^{m}\Delta_{A_k}^{t+1}}^{O}(X)$, *if* $n^{t+1}_{A_k^C\cup A_k^N}(x)\cap X=\emptyset\Rightarrow x\in\overline{\Delta\sum_{k=1}^{m}\Delta_{A_k}}^{O}(X)$, *since we have* $\overline{\sum_{k=1}^{m}\Delta_{A_k}^{t+1}}^{O}(X)\subseteq\overline{\sum_{k=1}^{m}\Delta_{A_k}^{t}}^{O}(X)$, *we have* $\overline{\sum_{k=1}^{m}\Delta_{A_k}^{t}}^{O}(X)-\overline{\Delta\sum_{k=1}^{m}\Delta_{A_k}}^{O}(X)\subseteq\overline{\sum_{k=1}^{m}\delta_{A_k}^{t+1}}^{O}(X)$. *This completes the proof.*

(iii) *It is similar to i.*

(iv) *It is similar to ii.*

□

**Example 4.** (Continuation of Example 1)Supposed $AT^t=\{A_1^t,A_2^t\}$, $A_1^t=(A_1^C)^t\cup(A_1^N)^t$, $(A_1^C)^t=\{a_1\}$ and $(A_1^N)^t=\emptyset$, $A_2^t=(A_2^C)^t\cup(A_2^N)^t$, $(A_2^C)^t=\emptyset$ and $(A_2^N)^t=\{a_3\}$, $AT^{t+1}=\{A_1^{t+1},A_2^{t+1}\}$, $A_1^{t+1}=(A_1^C)^{t+1}\cup(A_1^N)^{t+1}$, $(A_1^C)^{t+1}=\{a_1\}$ and $(A_1^N)^{t+1}=\{a_4\}$, $A_2^{t+1}=(A_2^C)^{t+1}\cup(A_2^N)^{t+1}$, $(A_2^C)^{t+1}=\{a_1\}$ and $(A_2^N)^{t+1}=\{a_3\}$. Let $X=\{x_2,x_3,x_4,x_6\}$ and $\Delta=0.5,\delta=0.5$.

By Definition 1 we have

$$n^{t}_{A_1^C\cup A_1^N}(x_1)=\{x_1,x_5\},n^{t}_{A_1^C\cup A_1^N}(x_2)=\{x_2,x_4\},n^{t}_{A_1^C\cup A_1^N}(x_3)=\{x_3,x_6\},$$
$$n^{t}_{A_1^C\cup A_1^N}(x_4)=\{x_2,x_4\},n^{t}_{A_1^C\cup A_1^N}(x_5)=\{x_1,x_5\},n^{t}_{A_1^C\cup A_1^N}(x_6)=\{x_3,x_6\}.$$

$$n^{t}_{A_2^C\cup A_2^N}(x_1)=\{x_1,x_3,x_4,x_5\},n^{t}_{A_2^C\cup A_2^N}(x_2)=\{x_2,x_6\},n^{t}_{A_2^C\cup A_2^N}(x_3)=\{x_1,x_3,x_4,x_5\},$$
$$n^{t}_{A_2^C\cup A_2^N}(x_4)=\{x_1,x_3,x_4,x_5\},n^{t}_{A_2^C\cup A_2^N}(x_5)=\{x_1,x_3,x_4,x_5\},n^{t}_{A_2^C\cup A_2^N}(x_6)=\{x_2,x_6\}.$$

$$n^{t+1}_{A_1^C\cup A_1^N}(x_1)=\{x_1,x_5\},n^{t+1}_{A_1^C\cup A_1^N}(x_2)=\{x_2,x_4\},n^{t+1}_{A_1^C\cup A_1^N}(x_3)=\{x_3,x_6\},$$
$$n^{t+1}_{A_1^C\cup A_1^N}(x_4)=\{x_2,x_4\},n^{t+1}_{A_1^C\cup A_1^N}(x_5)=\{x_1,x_5\},n^{t+1}_{A_1^C\cup A_1^N}(x_6)=\{x_3,x_6\}.$$

$$n^{t+1}_{A_2^C\cup A_2^N}(x_1)=\{x_1,x_5\},n^{t+1}_{A_2^C\cup A_2^N}(x_2)=\{x_2\},n^{t+1}_{A_2^C\cup A_2^N}(x_3)=\{x_3\},$$
$$n^{t+1}_{A_2^C\cup A_2^N}(x_4)=\{x_4\},n^{t+1}_{A_2^C\cup A_2^N}(x_5)=\{x_1,x_5\},n^{t+1}_{A_2^C\cup A_2^N}(x_6)=\{x_6\}.$$

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By Definition 3 and Definition 4 we have

$$\underline{\sum_{k=1}^{m}\Delta_{A_k}^{t}}^{O}(X)=\{x_2,x_3,x_4,x_6\},\underline{\sum_{k=1}^{m}\Delta_{A_k}^{t}}^{P}(X)=\{x_2,x_6\},\overline{\sum_{k=1}^{m}\Delta_{A_k}^{t}}^{O}(X)=\{x_2,x_3,x_4,x_6\},\overline{\sum_{k=1}^{m}\Delta_{A_k}^{t}}^{P}(X)=U.$$

$$\overline{\sum_{k=1}^{m}\Delta_{A_k}^{t}}^{P}(X)-\underline{\sum_{k=1}^{m}\Delta_{A_k}^{t}}^{P}(X)=U-\{x_2,x_6\}=\{x_1,x_3,x_4,x_5\}.$$

By Theorem 10 we have

$$\underline{\Delta\sum_{k=1}^{m}\Delta_{A_k}}^{O}(X)=\{x_3,x_4\},\underline{\Delta\sum_{k=1}^{m}\Delta_{A_k}}^{P}(X)=\{x_3,x_4\},\overline{\Delta\sum_{k=1}^{m}\Delta_{A_k}}^{O}(X)=\{x_1,x_5\},\overline{\Delta\sum_{k=1}^{m}\Delta_{A_k}}^{P}(X)=\{x_1,x_5\}.$$

and

$$\underline{\sum_{k=1}^{m}\delta_{A_k}^{t+1}}^{O}(X)=\underline{\sum_{k=1}^{m}\Delta_{A_k}^{t}}^{O}(X)\cup\underline{\Delta\sum_{k=1}^{m}\Delta_{A_k}}^{O}(X)=\{x_2,x_3,x_4,x_6\}\cup\{x_3,x_4\}=\{x_2,x_3,x_4,x_6\},$$

$$\overline{\sum_{k=1}^{m}\delta_{A_k}^{t+1}}^{O}(X)=\overline{\sum_{k=1}^{m}\Delta_{A_k}^{t}}^{O}(X)-\overline{\Delta\sum_{k=1}^{m}\Delta_{A_k}}^{O}(X)=\{x_2,x_3,x_4,x_6\}-\{x_1,x_5\}=\{x_2,x_3,x_4,x_6\},$$

$$\underline{\sum_{k=1}^{m}\delta_{A_k}^{t+1}}^{P}(X)=\underline{\sum_{k=1}^{m}\Delta_{A_k}^{t}}^{P}(X)\cup\underline{\Delta\sum_{k=1}^{m}\Delta_{A_k}}^{P}(X)=\{x_2,x_6\}\cup\{x_3,x_4\}=\{x_2,x_3,x_4,x_6\},$$

$$\overline{\sum_{k=1}^{m}\delta_{A_k}^{t+1}}^{P}(X)=\overline{\sum_{k=1}^{m}\Delta_{A_k}^{t}}^{P}(X)-\overline{\Delta\sum_{k=1}^{m}\Delta_{A_k}}^{P}(X)=U-\{x_1,x_5\}=\{x_2,x_3,x_4,x_6\}.$$

**Definition 8.** Let $NIS^t=(U,AT^t,\Delta)$ be a neighborhood information system at time $t$, $NIS^{t+1}=(U,AT^{t+1},\delta)$ be a neighborhood information system at time $t+1$. For any $X\subseteq U$, the dynamic lower approximation character sets of X in NMGRS while neighborhood classes decreasing can be defined as:

$$\Delta_{A_k}^{L}(X)=\left\{x|x\in\overline{\sum_{k=1}^{m}\Delta_{A_k}^{t}}^{P}(X)-\underline{\sum_{k=1}^{m}\Delta_{A_k}^{t}}^{P}(X)\ \wedge n_{A_k^C\cup A_k^N}^{t+1}(x)\subseteq X\right\},\forall k\in\{1,2,\cdots,m\}.$$

**Definition 9.** Let $NIS^t=(U,AT^t,\Delta)$ be a neighborhood information system at time $t$, $NIS^{t+1}=(U,AT^{t+1},\delta)$ be a neighborhood information system at time $t+1$. For any $X\subseteq U$, the dynamic upper approximation character sets of X in NMGRS while neighborhood classes decreasing can be defined as:

$$\Delta_{A_k}^{U}(X)=\left\{x|x\in\overline{\sum_{k=1}^{m}\Delta_{A_k}^{t}}^{P}(X)-\underline{\sum_{k=1}^{m}\Delta_{A_k}^{t}}^{P}(X)\ \wedge n_{A_k^C\cup A_k^N}^{t+1}(x)\cap X=\emptyset\right\},\forall k\in\{1,2,\cdots,m\}.$$

**Theorem 11.** *Let $NIS^t=(U,AT^t,\Delta)$ be a neighborhood information system at time t, $NIS^{t+1}=(U,AT^{t+1},\delta)$ be a neighborhood information system at time $t+1$. For any $X\subseteq U$, the following results hold.*

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(i) $\underline{\sum_{k=1}^{m}\Delta_{A_k}^{t+1}}^{O}(X)=\underline{\sum_{k=1}^{m}\Delta_{A_k}^{t}}^{O}(X)\cup(\cup_{k=1}^{m}\Delta_{A_k}^{L}(X))$,

(ii) $\overline{\sum_{k=1}^{m}\Delta_{A_k}^{t+1}}^{O}(X)=\overline{\sum_{k=1}^{m}\Delta_{A_k}^{t}}^{O}(X)-(\cup_{k=1}^{m}\Delta_{A_k}^{U}(X))$,

(iii) $\underline{\sum_{k=1}^{m}\Delta_{A_k}^{t+1}}^{P}(X)=\underline{\sum_{k=1}^{m}\Delta_{A_k}^{t}}^{P}(X)\cup(\cap_{k=1}^{m}\Delta_{A_k}^{L}(X))$,

(iv) $\overline{\sum_{k=1}^{m}\Delta_{A_k}^{t+1}}^{P}(X)=\overline{\sum_{k=1}^{m}\Delta_{A_k}^{t}}^{P}(X)-(\cap_{k=1}^{m}\Delta_{A_k}^{L}(X))$.

**Proof.**

(i) *By Theorem 10 and Definition 8 we have* $\underline{\Delta\sum_{k=1}^{m}\Delta_{A_k}}^{O}(X)=\cup_{k=1}^{m}\Delta_{A_k}^{L}(X)$ *and* $\underline{\sum_{k=1}^{m}\delta_{A_k}^{t+1}}^{O}(X)=\underline{\sum_{k=1}^{m}\Delta_{A_k}^{t}}^{O}(X)\cup(\cup_{k=1}^{m}\Delta_{A_k}^{L}(X))$.

(ii) *By Theorem 10 and Definition 9 we have* $\overline{\Delta\sum_{k=1}^{m}\Delta_{A_k}}^{O}(X)=\cup_{k=1}^{m}\Delta_{A_k}^{U}(X)$ *and* $\overline{\sum_{k=1}^{m}\delta_{A_k}^{t+1}}^{O}(X)=\overline{\sum_{k=1}^{m}\Delta_{A_k}^{t}}^{O}(X)-(\cup_{k=1}^{m}\Delta_{A_k}^{L}(X))$.

(iii) *It is similar to i.*

(iv) *It is similar to ii.*

□

**Example 5.** (Continuation of Example 4)By Definition 8 we have $\Delta_{A_1}^{L}(X)=\{x_3,x_4\},\Delta_{A_2}^{L}(X)=\{x_3,x_4\}$. By Definition 9 we have $\Delta_{A_1}^{U}(X)=\{x_1,x_5\},\Delta_{A_2}^{U}(X)=\{x_1,x_5\}$. By Theorem 11 we have

$$\underline{\sum_{k=1}^{2}\delta_{A_k}^{t+1}}^{O}(X)=\underline{\sum_{k=1}^{2}\Delta_{A_k}^{t}}^{O}(X)\cup(\cup_{k=1}^{2}\Delta_{A_k}^{L}(X))=\{x_2,x_3,x_4,x_6\}\cup\{x_3,x_4\}=\{x_2,x_3,x_4,x_6\},$$

$$\underline{\sum_{k=1}^{2}\delta_{A_k}^{t+1}}^{P}(X)=\underline{\sum_{k=1}^{2}\Delta_{A_k}^{t}}^{P}(X)\cup(\cap_{k=1}^{2}\Delta_{A_k}^{L}(X))=\{x_2,x_6\}\cup\{x_3,x_4\}=\{x_2,x_3,x_4,x_6\},$$

$$\overline{\sum_{k=1}^{2}\delta_{A_k}^{t+1}}^{O}(X)=\underline{\sum_{k=1}^{2}\Delta_{A_k}^{t}}^{O}(X)-(\cup_{k=1}^{2}\Delta_{A_k}^{U}(X))=\{x_2,x_3,x_4,x_6\}-\{x_1,x_5\}=\{x_2,x_3,x_4,x_6\},$$

$$\overline{\sum_{k=1}^{2}\delta_{A_k}^{t+1}}^{P}(X)=\underline{\sum_{k=1}^{2}\Delta_{A_k}^{t}}^{O}(X)-(\cap_{k=1}^{2}\Delta_{A_k}^{U}(X))=U-\{x_1,x_5\}=\{x_2,x_3,x_4,x_6\}.$$

**Corollary 12.** *Let* $IS^t=(U,AT^t,V_{AT^t},f^t)$ *be an information system at time t,* $IS^{t+1}=\left(U,AT^{t+1},V_{AT^{t+1}},f^{t+1}\right)$ *be an information system at time* $t+1$. *For any* $X\subseteq U$, *the following results hold.*

(i) $V(\underline{\sum_{k=1}^{m}\delta_{A_k}^{t+1}}^{O}(X))=V(\underline{\sum_{k=1}^{m}\Delta_{A_k}^{t}}^{O}(X))\vee(\vee_{k=1}^{m}V(\Delta_{A_k}^{L}(X))$,

(ii) $V(\overline{\sum_{k=1}^{m}\delta_{A_k}^{t+1}}^{O}(X))=V(\overline{\sum_{k=1}^{m}\Delta_{A_k}^{t}}^{O}(X))\wedge\sim(\vee_{k=1}^{m}V(\Delta_{A_k}^{U}(X))$,

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*(iii)* $V(\underline{\sum_{k=1}^{m}\delta_{A_k}^{t+1}}^{P}(X)) = V(\underline{\sum_{k=1}^{m}\Delta_{A_k}^{t}}^{P}(X)) \vee (\wedge_{k=1}^{m} V(\Delta_{A_k}^{L}(X)),$

*(iv)* $V(\overline{\sum_{k=1}^{m}\delta_{A_k}^{t+1}}^{P}(X)) = V(\overline{\sum_{k=1}^{m}\Delta_{A_k}^{t}}^{P}(X)) \wedge \sim (\wedge_{k=1}^{m} V(\Delta_{A_k}^{L}(X))).$

**Proof.** *This corollary is the matrix representation of Theorem 11.* □

**Example 6.** (Continuation of Example 5)By Corollary 12 we have

$$V(\underline{\sum_{k=1}^{2}\delta_{A_k}^{t+1}}^{O}(X)) = V(\underline{\sum_{k=1}^{2}\Delta_{A_k}^{t}}^{O}(X)) \vee (\vee_{k=1}^{2} V(\Delta_{A_k}^{L}(X))) = [0,1,1,1,0,1] \vee [0,0,1,1,0,0] = [0,1,1,1,0,1],$$

$$V(\underline{\sum_{k=1}^{2}\delta_{A_k}^{t+1}}^{P}(X)) = V(\underline{\sum_{k=1}^{2}\Delta_{A_k}^{t}}^{P}(X)) \vee (\wedge_{k=1}^{2} V(\Delta_{A_k}^{L}(X))) = [0,0,1,1,0,0] \vee [0,1,1,1,0,1] = [0,0,1,1,0,0],$$

$$V(\overline{\sum_{k=1}^{2}\delta_{A_k}^{t+1}}^{O}(X)) = V(\overline{\sum_{k=1}^{2}\Delta_{A_k}^{t}}^{O}(X)) \wedge \sim (\vee_{k=1}^{2} V(\Delta_{A_k}^{U}(X))) = [0,1,1,1,0,1] \wedge [0,1,1,1,0,1] = [0,1,1,1,0,1],$$

$$V(\overline{\sum_{k=1}^{2}\delta_{A_k}^{t+1}}^{P}(X)) = V(\overline{\sum_{k=1}^{2}\Delta_{A_k}^{t}}^{P}(X)) \wedge \sim (\wedge_{k=1}^{2} V(\Delta_{A_k}^{U}(X))) = [1,1,1,1,1,1] \wedge [0,1,1,1,0,1] = [0,1,1,1,0,1].$$

### *3.2. Matrix-based approaches for updating approximations while neighborhood classes are increasing*

In this subsection, we present matrix-based dynamic approaches for updating approximations in NMGRS while neighborhood classes are increasing. We denote $NIS^t = (U, AT^t, \delta)$ a neighborhood information system at time $t$, $NIS^{t+1} = (U, AT^{t+1}, \Delta)$ be a neighborhood information system at time $t+1$ where $\delta$ and $\Delta$ are two nonnegative numbers and $\delta \leqslant \Delta$. For all $A_k^t \in AT^t (k \leqslant m)$, there exists $A_k^{t+1} \in AT^{t+1}$, such that $(A_k^C)^{t+1} \subseteq (A_k^C)^t$ and $(A_k^N)^{t+1} \subseteq (A_k^N)^t$. Also, for all $x \in U$, we denote neighborhood class of $x$ at time $t$ by $n^t_{A_k^C \cup A_k^N}(x)$. Denote neighborhood class of $x$ at time $t+1$ by $n^{t+1}_{A_k^C \cup A_k^N}(x)$. Denote approximations in PNMGRS by $\underline{\sum_{k=1}^{m}\delta_{A_k}^{t}}^{P}(X)$ and $\overline{\sum_{k=1}^{m}\delta_{A_k}^{t}}^{P}(X)$ at time $t$. Denote approximations in PNMGRS by $\underline{\sum_{k=1}^{m}\Delta_{A_k}^{t+1}}^{P}(X)$ and $\overline{\sum_{k=1}^{m}\Delta_{A_k}^{t+1}}^{P}(X)$ at time $t+1$. Denote approximations in ONMGRS by $\underline{\sum_{k=1}^{m}\delta_{A_k}^{t}}^{O}(X)$ and $\overline{\sum_{k=1}^{m}\delta_{A_k}^{t}}^{O}(X)$ at time $t$. Denote approximation in ONMGRS by $\underline{\sum_{k=1}^{m}\Delta_{A_k}^{t+1}}^{O}(X)$ and $\overline{\sum_{k=1}^{m}\Delta_{A_k}^{t+1}}^{O}(X)$ at time $t+1$.

**Theorem 13.** *Let $NIS^t = (U, AT^t, \delta)$ be a neighborhood information system at time t, $NIS^{t+1} = (U, AT^{t+1}, \Delta)$ be a neighborhood information system at time $t+1$. For any $X \subseteq U$, the following result holds.*

$$n^{t+1}_{A_k^C \cup A_k^N}(x) \supseteq n^t_{A_k^C \cup A_k^N}(x)$$

**Proof.**

- If $(A_k^C)^{t+1} \subseteq (A_k^C)^t$, we have $n^t_{A_k^C}(x) \supseteq n^{t+1}_{A_k^C}(x) \Rightarrow n^{t+1}_{A_k^C \cup A_k^N}(x) \supseteq n^t_{A_k^C \cup A_k^N}(x)$;
- If $(A_k^N)^{t+1} \subseteq (A_k^N)^t$, we have $n^t_{A_k^N}(x) \supseteq n^{t+1}_{A_k^N}(x) \Rightarrow n^{t+1}_{A_k^C \cup A_k^N}(x) \supseteq n^t_{A_k^C \cup A_k^N}(x)$;

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*(iv)* *if* $\overline{\Delta\sum_{k=1}^{m}\delta_{A_k}}^{P}(X)=\left\{x|\exists k\in\{1,2,\cdots,m\},n^{t+1}_{A_k^C\cup A_k^N}(x)\cap X\neq\emptyset,x\in\underline{\sum_{k=1}^{m}\delta^t_{A_k}}^{O}(X)\cup(U-\overline{\sum_{k=1}^{m}\delta^t_{A_k}}^{O}(X)\right\}$, *then* $\overline{\sum_{k=1}^{m}\Delta^{t+1}_{A_k}}^{P}(X)=\overline{\sum_{k=1}^{m}\delta^t_{A_k}}^{P}(X)\cup\overline{\Delta\sum_{k=1}^{m}\delta_{A_k}}^{P}(X)$.

**Proof.**

*(i)* $''\Rightarrow'':$ *Since we have* $\underline{\sum_{k=1}^{m}\Delta^{t+1}_{A_k}}^{O}(X)\subseteq\underline{\sum_{k=1}^{m}\delta^t_{A_k}}^{O}(X)$, *and* $\underline{\Delta\sum_{k=1}^{m}\delta_{A_k}}^{O}(X)\cap\underline{\sum_{k=1}^{m}\Delta^{t+1}_{A_k}}^{O}(X)=\emptyset$, *we have* $\underline{\sum_{k=1}^{m}\Delta^{t+1}_{A_k}}^{O}(X)\subseteq\underline{\sum_{k=1}^{m}\delta^t_{A_k}}^{O}(X)-\underline{\Delta\sum_{k=1}^{m}\delta_{A_k}}^{O}(X)$.
$''\Leftarrow'':\forall x\notin\underline{\sum_{k=1}^{m}\delta^t_{A_k}}^{O}(X)$, $x\notin\underline{\sum_{k=1}^{m}\delta^t_{A_k}}^{O}(X)\Rightarrow n^t_{A_k^C\cup A_k^N}(x)\not\subseteq X\Rightarrow n^{t+1}_{A_k^C\cup A_k^N}(x)\not\subseteq X\Rightarrow x\notin\underline{\sum_{k=1}^{m}\Delta^{t+1}_{A_k}}^{O}(X)\wedge x\in\underline{\Delta\sum_{k=1}^{m}\delta_{A_k}}^{O}(X)$, *we have that* $\underline{\sum_{k=1}^{m}\Delta^{t+1}_{A_k}}^{O}(X)-\underline{\Delta\sum_{k=1}^{m}\delta_{A_k}}^{O}(X)\subseteq\underline{\sum_{k=1}^{m}\delta^t_{A_k}}^{O}(X)$. *This completes the proof.*

*(ii)* $''\Rightarrow'':\forall x\in\overline{\sum_{k=1}^{m}\Delta^{t+1}_{A_k}}^{O}(X)$, $x\in\overline{\sum_{k=1}^{m}\Delta^{t+1}_{A_k}}^{O}(X)\Rightarrow\forall k\in\{1,2,\cdots,m\},n^{t+1}_{A_k^C\cup A_k^N}(x)\cap X\neq\emptyset\Rightarrow$
$\forall k\in\{1,2,\cdots,m\},n^t_{A_k^C\cup A_k^N}(x)\cap X\neq\emptyset\Rightarrow x\in\overline{\sum_{k=1}^{m}\delta^t_{A_k}}^{O}(X)\vee x\in\overline{\Delta\sum_{k=1}^{m}\delta_{A_k}}^{O}(X)$, *we have* $\overline{\sum_{k=1}^{m}\delta^{t+1}_{A_k}}^{O}(X)\subseteq\overline{\sum_{k=1}^{m}\delta^t_{A_k}}^{O}(X)\cup\overline{\Delta\sum_{k=1}^{m}\delta_{A_k}}^{O}(X)$.
$''\Leftarrow'':\forall x\notin\overline{\sum_{k=1}^{m}\Delta^{t+1}_{A_k}}^{O}(X),x\notin\overline{\sum_{k=1}^{m}\Delta^{t+1}_{A_k}}^{O}(X)\Rightarrow\exists k\in\{1,2,\cdots,m\},n^{t+1}_{A_k^C\cup A_k^N}(x)\cap X=\emptyset\Rightarrow$
$\exists k\in\{1,2,\cdots,m\},n^t_{A_k^C\cup A_k^N}(x)\cap X=\emptyset\Rightarrow x\notin\overline{\sum_{k=1}^{m}\delta^t_{A_k}}^{O}(X)\wedge x\notin\overline{\Delta\sum_{k=1}^{m}\delta_{A_k}}^{O}(X)$, *thus we have* $\overline{\sum_{k=1}^{m}\delta^t_{A_k}}^{O}(X)\cup\overline{\Delta\sum_{k=1}^{m}\delta_{A_k}}^{O}(X)\subseteq\overline{\sum_{k=1}^{m}\Delta^{t+1}_{A_k}}^{O}(X)$. *This completes the proof.*

*(iii)* *It is similar to i.*

*(iv)* *It is similar to ii.*

□

**Example 7.** (Continuation of Example 1)Supposed $AT^t=\{A^t_1,A^t_2\},A^t_1=(A^C_1)^t\cup(A^N_1)^t$, $(A^C_1)^t=\{a_1,a_2\}$ and $(A^N_1)^t=\emptyset,A^t_2=(A^C_2)^t\cup(A^N_2)^t$, $(A^C_2)^t=\emptyset$ and $(A^N_2)^t=\{a_3,a_4\}$, $AT^{t+1}=\{A^{t+1}_1,A^{t+1}_2\},A^{t+1}_1=(A^C_1)^{t+1}\cup(A^N_1)^{t+1}$, $(A^C_1)^{t+1}=\{a_2\}$ and $(A^N_1)^{t+1}=\emptyset$, $A^{t+1}_2=(A^C_2)^{t+1}\cup(A^N_2)^{t+1}$, $(A^C_2)^{t+1}=\emptyset$ and $(A^N_2)^{t+1}=\{a_4\}$. Let $X=\{x_2,x_3,x_4,x_6\}$ and $\delta=0.5,\Delta=0.5$.

By Definition 1 we have

$$n^t_{A^C_1\cup A^N_1}(x_1)=\{x_1\},n^t_{A^C_1\cup A^N_1}(x_2)=\{x_2\},n^t_{A^C_1\cup A^N_1}(x_3)=\{x_3\},$$
$$n^t_{A^C_1\cup A^N_1}(x_4)=\{x_4\},n^t_{A^C_1\cup A^N_1}(x_5)=\{x_5\},n^t_{A^C_1\cup A^N_1}(x_6)=\{x_6\}.$$

$$n^t_{A^C_2\cup A^N_2}(x_1)=\{x_1,x_4,x_5\},n^t_{A^C_2\cup A^N_2}(x_2)=\{x_2\},n^t_{A^C_2\cup A^N_2}(x_3)=\{x_3\},$$
$$n^t_{A^C_2\cup A^N_2}(x_4)=\{x_1,x_4\},n^t_{A^C_2\cup A^N_2}(x_5)=\{x_1,x_5\},n^t_{A^C_2\cup A^N_2}(x_6)=\{x_6\}.$$

$$n^{t+1}_{A^C_1\cup A^N_1}(x_1)=\{x_1,x_4\},n^{t+1}_{A^C_1\cup A^N_1}(x_2)=\{x_2,x_6\},n^{t+1}_{A^C_1\cup A^N_1}(x_3)=\{x_3,x_5\},$$
$$n^{t+1}_{A^C_1\cup A^N_1}(x_4)=\{x_1,x_4\},n^{t+1}_{A^C_1\cup A^N_1}(x_5)=\{x_3,x_5\},n^{t+1}_{A^C_1\cup A^N_1}(x_6)=\{x_2,x_6\}.$$

$$n^{t+1}_{A^C_2\cup A^N_2}(x_1)=\{x_1,x_2,x_4,x_5\},n^{t+1}_{A^C_2\cup A^N_2}(x_2)=\{x_1,x_2,x_4,x_5\},n^{t+1}_{A^C_2\cup A^N_2}(x_3)=\{x_3,x_6\},$$
$$n^{t+1}_{A^C_2\cup A^N_2}(x_4)=\{x_1,x_2,x_4,x_5\},n^{t+1}_{A^C_2\cup A^N_2}(x_5)=\{x_1,x_2,x_4,x_5\},n^{t+1}_{A^C_2\cup A^N_2}(x_6)=\{x_3,x_6\}.$$

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By Definition 4 and Definition 3 we have

$$\underline{\sum_{k=1}^{2}\delta_{A_k}^{t}}^{O}(X)=\{x_2,x_3,x_4,x_6\},\underline{\sum_{k=1}^{2}\delta_{A_k}^{t}}^{P}(X)=\{x_2,x_3,x_6\},\overline{\sum_{k=1}^{2}\delta_{A_k}^{t}}^{O}(X)=\{x_2,x_3,x_4,x_6\},\overline{\sum_{k=1}^{2}\delta_{A_k}^{t}}^{P}(X)=U.$$

$$\overline{\sum_{k=1}^{2}\delta_{A_k}^{t}}^{O}(X)\cup(U-\underline{\sum_{k=1}^{2}\delta_{A_k}^{t}}^{O}(X))=\{x_2,x_3,x_4,x_6\}\cup(U-\{x_2,x_3,x_4,x_6\}=U.$$

By Theorem 10 we have

$$\underline{\Delta\sum_{k=1}^{2}\delta_{A_k}}^{O}(X)=\{x_1,x_4,x_5\},\underline{\Delta\sum_{k=1}^{2}\delta_{A_k}}^{P}(X)=\{x_1,x_2,x_3,x_4,x_5\},\overline{\Delta\sum_{k=1}^{2}\delta_{A_k}}^{O}(X)=U,\overline{\Delta\sum_{k=1}^{2}\delta_{A_k}}^{P}(X)=U.$$

and

$$\underline{\sum_{k=1}^{2}\Delta_{A_k}^{t+1}}^{O}(X)=\underline{\sum_{k=1}^{2}\delta_{A_k}^{t}}^{O}(X)-\underline{\Delta\sum_{k=1}^{2}\delta_{A_k}}^{O}(X)=\{x_2,x_3,x_4,x_6\}-\{x_1,x_4,x_5\}=\{x_2,x_3,x_6\},$$

$$\overline{\sum_{k=1}^{m}\Delta_{A_k}^{t+1}}^{O}(X)=\overline{\sum_{k=1}^{2}\delta_{A_k}^{t}}^{O}(X)\cup\overline{\Delta\sum_{k=1}^{2}\delta_{A_k}}^{O}(X)=\{x_2,x_3,x_4,x_6\}\cup U=U,$$

$$\underline{\sum_{k=1}^{2}\Delta_{A_k}^{t+1}}^{P}(X)=\underline{\sum_{k=1}^{2}\delta_{A_k}^{t}}^{P}(X)-\underline{\Delta\sum_{k=1}^{2}\delta_{A_k}}^{P}(X)=\{x_2,x_3,x_6\}-\{x_1,x_2,x_3,x_4,x_5\}=\{x_6\},$$

$$\overline{\sum_{k=1}^{2}\Delta_{A_k}^{t+1}}^{P}(X)=\overline{\sum_{k=1}^{2}\delta_{A_k}^{t}}^{P}(X)\cup\overline{\Delta\sum_{k=1}^{2}\delta_{A_k}}^{P}(X)=U\cup U=U.$$

**Definition 10.** Let $NIS^t=(U,AT^t,\delta)$ be a neighborhood information system at time $t$, $NIS^{t+1}=(U,AT^{t+1},\Delta)$ be a neighborhood information system at time $t+1$. For any $X\subseteq U$, the dynamic lower approximation character set of X in NMGRS while neighborhood classes increasing can be defined as:

$$\delta_{A_k}^{L}(X)=\left\{x|x\in\overline{\sum_{k=1}^{m}\delta_{A_k}^{t}}^{O}(X)\cup(U-\overline{\sum_{k=1}^{m}\delta_{A_k}^{t}}^{O}(X))\wedge n_{A_k^C\cup A_k^N}^{t+1}(x)\not\subseteq X\right\},\forall k\in\{1,2,\cdots,m\}.$$

**Definition 11.** Let $NIS^t=(U,AT^t,\delta)$ be a neighborhood information system at time $t$, $NIS^{t+1}=(U,AT^{t+1},\Delta)$ be a neighborhood information system at time $t+1$. For any $X\subseteq U$, the dynamic upper approximation character set of X in NMGRS while neighborhood classes increasing can be defined as:

$$\delta_{A_k}^{U}(X)=\left\{x|x\in\overline{\sum_{k=1}^{m}\delta_{A_k}^{t}}^{O}(X)\cup(U-\overline{\sum_{k=1}^{m}\delta_{A_k}^{t}}^{O}(X))\wedge n_{A_k^C\cup A_k^N}^{t+1}(x)\cap X\neq\emptyset\right\},\forall k\in\{1,2,\cdots,m\}.$$

**Theorem 17.** *Let $NIS^t=(U,AT^t,\delta)$ be a neighborhood information system at time t, $NIS^{t+1}=(U,AT^{t+1},\Delta)$ be a neighborhood information system at time $t+1$. For any $X\subseteq U$, the following results hold.*

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(i) $\underline{\sum_{k=1}^{m}\Delta_{A_k}^{t+1}}^{O}(X)=\underline{\sum_{k=1}^{m}\delta_{A_k}^{t}}^{O}(X)-(\cap_{k=1}^{m}\delta_{A_k}^{L}(X))$,

(ii) $\overline{\sum_{k=1}^{m}\Delta_{A_k}^{t+1}}^{O}(X)=\overline{\sum_{k=1}^{m}\delta_{A_k}^{t}}^{O}(X)\cup(\cap_{k=1}^{m}\delta_{A_k}^{U}(X))$,

(iii) $\underline{\sum_{k=1}^{m}\Delta_{A_k}^{t+1}}^{P}(X)=\underline{\sum_{k=1}^{m}\delta_{A_k}^{t}}^{P}(X)-(\cup_{k=1}^{m}\delta_{A_k}^{L}(X))$,

(iv) $\overline{\sum_{k=1}^{m}\Delta_{A_k}^{t+1}}^{P}(X)=\overline{\sum_{k=1}^{m}\delta_{A_k}^{t}}^{P}(X)\cup(\cup_{k=1}^{m}\delta_{A_k}^{L}(X))$.

**Proof.**

1. *By Theorem 16 and Definition 10 we have that* $\underline{\Delta\sum_{k=1}^{m}\delta_{A_k}}^{O}(X)=\cap_{k=1}^{m}\delta_{A_k}^{L}(X)$ *and* $\underline{\sum_{k=1}^{m}\Delta_{A_k}^{t+1}}^{O}(X)=\underline{\sum_{k=1}^{m}\delta_{A_k}^{t}}^{O}(X)-(\cap_{k=1}^{m}\delta_{A_k}^{L}(X))$.
2. *By Theorem 16 and Definition 11 we have that* $\overline{\Delta\sum_{k=1}^{m}\delta_{A_k}}^{O}(X)=\cap_{k=1}^{m}\delta_{A_k}^{U}(X)$ *and* $\overline{\sum_{k=1}^{m}\Delta_{A_k}^{t+1}}^{O}(X)=\overline{\sum_{k=1}^{m}\delta_{A_k}^{t}}^{O}(X)\cup(\cap_{k=1}^{m}\delta_{A_k}^{U}(X))$.
3. *It is similar to i.*
4. *It is similar to ii.*

□

**Example 8.** (Continuation of Example 7)By Definition 10 we have

$$\delta_{A_1}^{L}(X)=\{x_1,x_3,x_4,x_5\},\delta_{A_2}^{L}(X)=\{x_1,x_2,x_4,x_5\}.$$

By Definition 11 we have

$$\delta_{A_1}^{U}(X)=U,\delta_{A_2}^{U}(X)=U.$$

By Theorem 17 we have

$$\underline{\sum_{k=1}^{2}\Delta_{A_k}^{t+1}}^{O}(X)=\underline{\sum_{k=1}^{2}\delta_{A_k}^{t}}^{O}(X)-(\cap_{k=1}^{2}\delta_{A_k}^{L}(X))=\{x_2,x_3,x_4,x_6\}-\{x_1,x_4,x_5\}=\{x_2,x_3,x_6\},$$

$$\underline{\sum_{k=1}^{2}\Delta_{A_k}^{t+1}}^{P}(X)=\underline{\sum_{k=1}^{2}\delta_{A_k}^{t}}^{P}(X)-(\cup_{k=1}^{2}\delta_{A_k}^{L}(X))=\{x_2,x_3,x_6\}-\{x_1,x_2,x_3,x_4,x_5\}=\{x_6\},$$

$$\overline{\sum_{k=1}^{2}\Delta_{A_k}^{t+1}}^{O}(X)=\overline{\sum_{k=1}^{2}\delta_{A_k}^{t}}^{O}(X)\cup(\cap_{k=1}^{2}\delta_{A_k}^{U}(X))=\{x_2,x_3,x_4,x_6\}\cup U=U,$$

$$\overline{\sum_{k=1}^{2}\Delta_{A_k}^{t+1}}^{P}(X)=\overline{\sum_{k=1}^{2}\delta_{A_k}^{t}}^{O}(X)\cup(\cup_{k=1}^{2}\delta_{A_k}^{U}(X))=U\cup U=U.$$

**Corollary 18.** *Let* $NIS^t=(U,AT^t,\delta)$ *be a neighborhood information system at time t,* $NIS^{t+1}=(U,AT^{t+1},\Delta)$ *be a neighborhood information system at time* $t+1$. *For any* $X\subseteq U$, *the following results hold.*

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1. $V(\underline{\sum_{k=1}^{m}\Delta_{A_k}^{t+1}}^{O}(X)) = V(\underline{\sum_{k=1}^{m}\delta_{A_k}^{t}}^{O}(X))\wedge\sim(\wedge_{k=1}^{m}V(\delta_{A_k}^{L}(X)))$,
2. $V(\overline{\sum_{k=1}^{m}\Delta_{A_k}^{t+1}}^{O}(X)) = V(\overline{\sum_{k=1}^{m}\delta_{A_k}^{t}}^{O}(X))\vee(\wedge_{k=1}^{m}V(\delta_{A_k}^{U}(X)))$,
3. $V(\underline{\sum_{k=1}^{m}\Delta_{A_k}^{t+1}}^{P}(X)) = V(\underline{\sum_{k=1}^{m}\delta_{A_k}^{t}}^{P}(X))\wedge\sim(\vee_{k=1}^{m}V(\delta_{A_k}^{L}(X)))$,
4. $V(\overline{\sum_{k=1}^{m}\Delta_{A_k}^{t+1}}^{P}(X)) = V(\overline{\sum_{k=1}^{m}\delta_{A_k}^{t}}^{P}(X))\vee(\vee_{k=1}^{m}V(\delta_{A_k}^{L}(X)))$.

**Proof.** *This theorem is the matrix representation of Theorem 17.* □

**Example 9.** (Continuation of Example 8)By Corollary 18 we have

$$V(\underline{\sum_{k=1}^{2}\Delta_{A_k}^{t+1}}^{O}(X)) = V(\underline{\sum_{k=1}^{2}\delta_{A_k}^{t}}^{O}(X))\wedge\sim(\wedge_{k=1}^{2}V(\delta_{A_k}^{L}(X))) = [0,1,1,1,0,1]\wedge[0,1,1,0,0,1] = [0,1,1,0,0,1],$$
$$V(\underline{\sum_{k=1}^{2}\Delta_{A_k}^{t+1}}^{P}(X)) = V(\underline{\sum_{k=1}^{2}\delta_{A_k}^{t}}^{P}(X))\wedge\sim(\vee_{k=1}^{2}V(\delta_{A_k}^{L}(X))) = [0,1,1,0,0,1]\wedge[0,0,0,0,0,1] = [0,0,0,0,0,1],$$
$$V(\overline{\sum_{k=1}^{2}\Delta_{A_k}^{t+1}}^{O}(X)) = V(\underline{\sum_{k=1}^{2}\delta_{A_k}^{t}}^{O}(X))\vee(\wedge_{k=1}^{2}V(\delta_{A_k}^{U}(X))) = [0,1,1,1,0,1]\vee[1,1,1,1,1,1] = [1,1,1,1,1,1],$$
$$V(\overline{\sum_{k=1}^{2}\Delta_{A_k}^{t+1}}^{P}(X)) = V(\underline{\sum_{k=1}^{2}\delta_{A_k}^{t}}^{O}(X))\vee(\vee_{k=1}^{2}V(\delta_{A_k}^{U}(X))) = [1,1,1,1,1,1]\vee[1,1,1,1,1,1] = [1,1,1,1,1,1].$$

## 4. Matrix-based dynamic algorithms for updating approximations while neighborhood classes increasing or decreasing

Based on Corollary 12, we proposed a matrix-based Algorithm 2 for updating approximations in NMGRS while neighborhood classes are decreasing. The total time complexity of Algorithm 2 is $O(|AT^t||\overline{\sum_{k=1}^{m}\Delta_{A_k}^{t}}^{P}(X) - \underline{\sum_{k=1}^{m}\Delta_{A_k}^{t}}^{P}(X)||U|)$. Steps 3-12 are to calculate $\Delta_{A_k}^{L}$ and $\Delta_{A_k}^{U}$ ($k\in\{1,2,\cdots,m\}$) with time complexity $O(|AT^t||\overline{\sum_{k=1}^{m}\Delta_{A_k}^{t}}^{P}(X) - \underline{\sum_{k=1}^{m}\Delta_{A_k}^{t}}^{P}(X)||U|)$. Steps 17-22 are to compute $\Delta\underline{\sum_{k=1}^{m}\Delta_{A_k}^{t}}^{O}(X)$, $\Delta\overline{\sum_{k=1}^{m}\Delta_{A_k}^{t}}^{O}(X)$, $\Delta\underline{\sum_{k=1}^{m}\Delta_{A_k}^{t}}^{P}(X)$ and $\Delta\overline{\sum_{k=1}^{m}\Delta_{A_k}^{t}}^{P}(X)$ with time complexity $O(|AT^t||U|)$. Steps 24-27 are to update the approximations in NMGRS while neighborhood classes are decreasing with time complexity $O(|U|)$.

Since the time complexity of Algorithm 1 is $O(|AT^t||X||U|)$, and in general, $O(|AT^t||\overline{\sum_{k=1}^{m}\Delta_{A_k}^{t}}^{P}(X) - \underline{\sum_{k=1}^{m}\Delta_{A_k}^{t}}^{P}(X)||U|) \leqslant O(|AT^t||X||U|)$ does not hold, so Algorithm 3 is proposed to make sure the total time complexity is no more than $O(|AT^t||X||U|)$. In other words, when $|\overline{\sum_{k=1}^{m}\Delta_{A_k}^{t}}^{P}(X) - \underline{\sum_{k=1}^{m}\Delta_{A_k}^{t}}^{P}(X)| > |X|$, we call Algorithm 1, otherwise, we call Algorithm 2.

Based on Corollary 18, we proposed a matrix-based Algorithm 4 for updating approximations in NMGRS while neighborhood classes are increasing. The total time complexity of Algorithm 4 is $O(|AT^t||\underline{\sum_{k=1}^{m}\delta_{A_k}^{t+1}}^{O}(X)\cup$

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**Algorithm 2** Matrix-based algorithm for updating approximations in NMGRS while neighborhood classes decreasing

**Require:** (1)$NIS^t = (U, AT^t, \Delta)$. (2)$NIS^{t+1} = (U, AT^{t+1}, \delta)$. (3)A target concept $X \subseteq U$. (4)$\underline{\sum_{k=1}^{m} \Delta_{A_k}^t}^O(X)$, $\overline{\sum_{k=1}^{m} \Delta_{A_k}^t}^O(X)$, $\underline{\sum_{k=1}^{m} \Delta_{A_k}^t}^P(X)$ and $\overline{\sum_{k=1}^{m} \Delta_{A_k}^t}^P(X)$. (5)neighborhood classes $n^{t+1}_{A_k^C \cup A_k^N}(x), \forall x \in U, \forall k \in \{1,2,\cdots,m\}$.

**Ensure:** $\underline{\sum_{k=1}^{m} \delta_{A_k}^{t+1}}^O(X)$, $\overline{\sum_{k=1}^{m} \delta_{A_k}^{t+1}}^O(X)$, $\underline{\sum_{k=1}^{m} \delta_{A_k}^{t+1}}^P(X)$ and $\overline{\sum_{k=1}^{m} \delta_{A_k}^{t+1}}^P(X)$.

$m \longleftarrow |AT^t|$
$n \longleftarrow |U|$
**for** $k = 1 \to m$ **do**
  **for** $i = 1 \to n$ **do**
    **if** $v_i(\overline{\sum_{k=1}^{m} \Delta_{A_k}^t}^P(X) - \underline{\sum_{k=1}^{m} \Delta_{A_k}^t}^P(X)) = 1 \wedge V(n^{t+1}_{A_k^C \cup A_k^N}(x_i)) \cdot V^T(\sim(X)) = 0$ **then**
$v_i(\Delta^L_{A_k}(X)) = 1$
    **end if**
    **if** $v_i(\overline{\sum_{k=1}^{m} \Delta_{A_k}^t}^P(X) - \underline{\sum_{k=1}^{m} \Delta_{A_k}^t}^P(X)) = 1 \wedge V(n^{t+1}_{A_k^C \cup A_k^N}(x_i)) \cdot V^T(X) = 0$ **then**
$v_i(\Delta^U_{A_k}(X)) = 1$
    **end if**
  **end for**
**end for**
$V(\underline{\Delta \sum_{k=1}^{m} \Delta_{A_k}}^O(X)) \longleftarrow V(\Delta^L_{A_1}(X))$
$V(\overline{\Delta \sum_{k=1}^{m} \Delta_{A_k}}^O(X)) \longleftarrow V(\Delta^U_{A_1}(X))$
$V(\underline{\Delta \sum_{k=1}^{m} \Delta_{A_k}}^P(X)) \longleftarrow V(\Delta^L_{A_1}(X))$
$V(\overline{\Delta \sum_{k=1}^{m} \Delta_{A_k}}^P(X)) \longleftarrow V(\Delta^U_{A_1}(X))$
**for** $k = 2 \to m$ **do**
  $V(\underline{\Delta \sum_{k=1}^{m} \Delta_{A_k}}^O(X)) \longleftarrow V(\underline{\Delta \sum_{k=1}^{m} \Delta_{A_k}}^O(X)) \vee V(\Delta^L_{A_k}(X))$
  $V(\overline{\Delta \sum_{k=1}^{m} \Delta_{A_k}}^O(X)) \longleftarrow V(\overline{\Delta \sum_{k=1}^{m} \Delta_{A_k}}^O(X)) \vee V(\Delta^U_{A_k}(X))$
  $V(\underline{\Delta \sum_{k=1}^{m} \Delta_{A_k}}^P(X)) \longleftarrow V(\underline{\Delta \sum_{k=1}^{m} \Delta_{A_k}}^P(X)) \wedge V(\Delta^L_{A_k}(X))$
  $V(\overline{\Delta \sum_{k=1}^{m} \Delta_{A_k}}^P(X)) \longleftarrow V(\overline{\Delta \sum_{k=1}^{m} \Delta_{A_k}}^P(X)) \wedge V(\Delta^U_{A_k}(X))$
**end for**

$V(\underline{\sum_{k=1}^{m} \delta_{A_k}^{t+1}}^O(X)) \longleftarrow V(\underline{\sum_{k=1}^{m} \Delta_{A_k}^t}^O(X)) \vee V(\underline{\Delta \sum_{k=1}^{m} \Delta_{A_k}}^O(X))$
$V(\overline{\sum_{k=1}^{m} \delta_{A_k}^{t+1}}^O(X)) \longleftarrow V(\overline{\sum_{k=1}^{m} \Delta_{A_k}^t}^O(X)) \wedge \sim V(\overline{\Delta \sum_{k=1}^{m} \Delta_{A_k}}^O(X))$
$V(\underline{\sum_{k=1}^{m} \delta_{A_k}^{t+1}}^P(X)) \longleftarrow V(\underline{\sum_{k=1}^{m} \Delta_{A_k}^t}^P(X)) \vee V(\underline{\Delta \sum_{k=1}^{m} \Delta_{A_k}}^P(X))$
$V(\overline{\sum_{k=1}^{m} \delta_{A_k}^{t+1}}^P(X)) \longleftarrow V(\overline{\sum_{k=1}^{m} \Delta_{A_k}^t}^P(X)) \wedge \sim V(\overline{\Delta \sum_{k=1}^{m} \Delta_{A_k}}^P(X))$
Return $\underline{\sum_{k=1}^{m} \delta_{A_k}^{t+1}}^O(X)$, $\overline{\sum_{k=1}^{m} \delta_{A_k}^{t+1}}^O(X)$, $\underline{\sum_{k=1}^{m} \delta_{A_k}^{t+1}}^P(X)$ and $\overline{\sum_{k=1}^{m} \delta_{A_k}^{t+1}}^P(X)$.

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**Algorithm 3** Ensure total time complexity of updating approximations in NMGRS while decreasing neighborhood classes no more than $O(|AT^t||X||U|)$

1: Require (1)$NIS^t = (U, AT^t, \Delta)$. (2)$NIS^{t+1} = (U, AT^{t+1}, \delta)$. (3)A target concept $X \subseteq U$. (4)$\underline{\sum_{k=1}^{m} \Delta_{A_k}^t}^O(X)$, $\overline{\sum_{k=1}^{m} \Delta_{A_k}^t}^O(X)$. $\underline{\sum_{k=1}^{m} \Delta_{A_k}^t}^P(X)$ and $\overline{\sum_{k=1}^{m} \Delta_{A_k}^t}^P(X)$.(4)neighborhood classes $n^{t+1}_{A_k^C \cup A_k^N}(x), \forall x \in U, \forall k \in \{1,2,\cdots,m\}$.

**Ensure:** $\underline{\sum_{k=1}^{m} \Delta_{A_k}^{t+1}}^O(X)$, $\overline{\sum_{k=1}^{m} \Delta_{A_k}^{t+1}}^O(X)$, $\underline{\sum_{k=1}^{m} \Delta_{A_k}^{t+1}}^P(X)$ and $\overline{\sum_{k=1}^{m} \Delta_{A_k}^{t+1}}^P(X)$.

2: **if** $|\overline{\sum_{k=1}^{m} \Delta_{A_k}^t}^P(X) - \underline{\sum_{k=1}^{m} \Delta_{A_k}^t}^P(X)| \leqslant |X|$ **then** Call Algorithm 2

3: **end if**

4: **if** $|\overline{\sum_{k=1}^{m} \Delta_{A_k}^t}^P(X) - \underline{\sum_{k=1}^{m} \Delta_{A_k}^t}^P(X)| > |X|$ **then** Call Algorithm 1

5: **end if**

6: Return $\underline{\sum_{k=1}^{m} \delta_{A_k}^{t+1}}^O(X)$, $\overline{\sum_{k=1}^{m} \delta_{A_k}^{t+1}}^O(X)$, $\underline{\sum_{k=1}^{m} \delta_{A_k}^{t+1}}^P(X)$ and $\overline{\sum_{k=1}^{m} \delta_{A_k}^{t+1}}^P(X)$.

$(U - \overline{\sum_{k=1}^{m} \delta_{A_k}^{t+1}}^O(X))||U|)$. Steps 3-12 are to calculate $\delta_{A_k}^L$ and $\delta_{A_k}^U$ $(k \in \{1,2,\cdots,m\})$ with time complexity $O(|AT^t||\underline{\sum_{k=1}^{m} \delta_{A_k}^{t+1}}^O(X) \cup (U - \overline{\sum_{k=1}^{m} \delta_{A_k}^{t+1}}^O(X))||U|)$. Steps 17-22 are to compute $\underline{\Delta \sum_{k=1}^{m} \delta_{A_k}}^O(X)$, $\overline{\Delta \sum_{k=1}^{m} \delta_{A_k}}^O(X)$, $\underline{\Delta \sum_{k=1}^{m} \delta_{A_k}}^P(X)$ and $\overline{\Delta \sum_{k=1}^{m} \delta_{A_k}}^P(X)$ with time complexity $O(|AT^t||U|)$. Steps 24-27 are to update the approximations in NMGRS while neighborhood classes are increasing with time complexity $O(|U|)$.

Since the time complexity of Algorithm 1 is $O(|AT^t||X||U|)$, and in general, $O(|AT^t||\underline{\sum_{k=1}^{m} \delta_{A_k}^{t+1}}^O(X) \cup (U - \overline{\sum_{k=1}^{m} \delta_{A_k}^{t+1}}^O(X))||U|) \leqslant O(|AT^t||X||U|)$ does not hold, Algorithm 5 was proposed to make sure the total time complexity is no more than $O(|AT^t||X||U|)$, in other words, when $|\underline{\sum_{k=1}^{m} \delta_{A_k}^{t+1}}^O(X) \cup (U - \overline{\sum_{k=1}^{m} \delta_{A_k}^{t+1}}^O(X))| > |X|$, we call Algorithm 1, otherwise, we call Algorithm 4.

## 5. Experimental evaluations

In this section, several experiments were conducted to evaluate the effectiveness and the efficiency of the algorithms we proposed, namely Algorithm 1(MB), Algorithm 3(DMB) and Algorithm 5(DMB). We chose four algorithms to compare, namely, set operation based static algorithm(SOB)[37], set operation based dynamic algorithm(DSOB)[37], relation matrix based static algorithm(RMB)[6] and relation matrix based dynamic algorithm(DRMB)[6]. Six data sets were chosen from UCI machine learning repository. The details of these data sets are listed in Table 2. All the experiments were carried out on a personal computer with 64-bit windows 10, Inter(R) Core(TM) i7 6700HQ CPU @2.60GHz, and 16GB memory. The program language was Matlab r2015b.

### *5.1. Comparison of computational time using date sets with different size*

The computational time are compared among MB,DMB,SOB,DSOB,RMB and DRMB when the size of data sets increases. With neighborhood classes decreasing or increasing. First of all, we randomly chose the categorical and numerical attributes $\widehat{a_C}$ and $\widehat{a_N}$ in the data set and randomly divided the rest of the categorical and numerical attributes into three parts randomly to contribute to three granular structures respectively. While neighborhood classes decrease, we added the categorical and numerical attributes $\widehat{a_C}$ and $\widehat{a_N}$ into the three

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**Algorithm 5** Ensure total time complexity of updating approximations in NMGRS while increasing neighborhood classes no more than $O(|AT^t||X||U|)$

1: Require (1)$NIS^t = (U, AT^t, \delta)$(2)$NIS^{t+1} = (U, AT^{t+1}, \Delta)$(3)A target concept $X \subseteq U$ (4)$\underline{\sum_{k=1}^{m} \delta_{A_k}^t}^O(X)$, $\overline{\sum_{k=1}^{m} \delta_{A_k}^t}^O(X)$, $\underline{\sum_{k=1}^{m} \delta_{A_k}^t}^P(X)$ and $\overline{\sum_{k=1}^{m} \delta_{A_k}^t}^P(X)$.(4)neighborhood classes $n^{t+1}_{A_k^C \cup A_k^N}(x), \forall x \in U, \forall k \in \{1, 2, \cdots, m\}$.

**Ensure:** $\underline{\sum_{k=1}^{m} \Delta_{A_k}^{t+1}}^O(X)$, $\overline{\sum_{k=1}^{m} \Delta_{A_k}^{t+1}}^O(X)$, $\underline{\sum_{k=1}^{m} \Delta_{A_k}^{t+1}}^P(X)$ and $\overline{\sum_{k=1}^{m} \Delta_{A_k}^{t+1}}^P(X)$.

2: **if** $|\underline{\sum_{k=1}^{m} \delta_{A_k}^t}^O(X) \cup (U - \overline{\sum_{k=1}^{m} \delta_{A_k}^t}^O(X))| > |X|$ **then** Call Algorithm 2

3: **end if**

4: **if** $|\underline{\sum_{k=1}^{m} \delta_{A_k}^t}^O(X) \cup (U - \overline{\sum_{k=1}^{m} \delta_{A_k}^t}^O(X))| \leqslant |X|$ **then** Call Algorithm 1

5: **end if**

6: Return $\underline{\sum_{k=1}^{m} \Delta_{A_k}^{t+1}}^O(X)$, $\overline{\sum_{k=1}^{m} \Delta_{A_k}^{t+1}}^O(X)$, $\underline{\sum_{k=1}^{m} \Delta_{A_k}^{t+1}}^P(X)$ and $\overline{\sum_{k=1}^{m} \Delta_{A_k}^{t+1}}^P(X)$.

**Table 2.** Details of data sets

| No. | Data sets | Samples | Attributes |
|---|---|---|---|
| 1 | Cylinder Bands Data Set | 541 | 40 |
| 2 | Dermatology | 366 | 20 |
| 3 | Flags | 194 | 30 |
| 4 | LasVegas Trip Advisor Reviews Data set | 504 | 20 |
| 5 | Soybean Large | 307 | 36 |
| 6 | Student | 396 | 33 |

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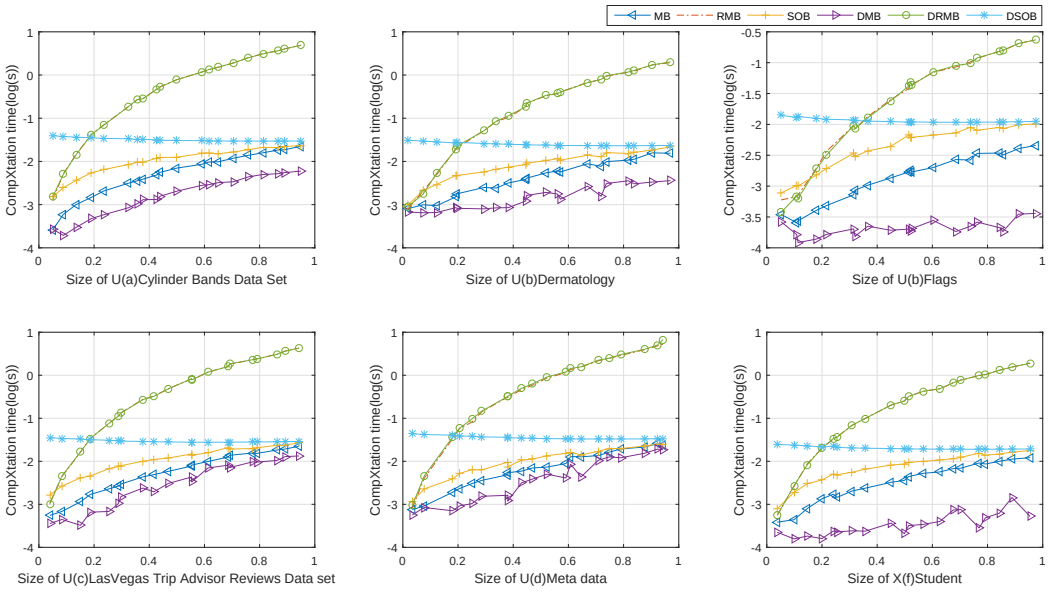


**Figure 1.** Comparison of Algorithm 3 when the size of $U$ increase gradually

granular structures at the same time. While neighborhood classes increase, we combined the categorical and numerical attributes $\widehat{a_C}$ and $\widehat{a_N}$ with each granular structure and deleted the categorical and numerical attributes $\widehat{a_C}$ and $\widehat{a_N}$ from the three granular structures at the same time. We randomly divided every data set $U$ into ten subsets which is denoted by $\{U_1, U_2, \cdots, U_{10}\}$. Then $U_1$ was chosen as the first temporary data set. After that, some samples of the temporary data set were randomly selected to contribute to the target concept $X$. The size of the target concept $X$ was about 0.95 times the size of each temporary data set. We calculated the four approximations in NMGRS using MB,DMB,SOB,DSOB,RMB and DRMB ten times and compared the averages. Then we made $U_1 \cup U_2$ the second temporary data set and repeat the whole process.

When the size of data sets increases gradually, results of the six algorithms while neighborhood classes increasing or decreasing in NMGRS are shown in Figure 1 and Figure 2. We can see that DMB is the most efficient algorithm when the size of data set increases gradually. From Figure 1 we can see that DMB is effective and it reduces the computational time.

### *5.2. Comparison of computational time using target concepts with different size*

Similarly, instead of construct temporary data sets in Section 5.1, we construct temporary target concepts. The process of constructing and varying the three granular structures is similar to Section 5.1. We randomly divided each data set into ten subsets $\{X_1, X_2, \cdots, X_{10}\}$. And then $X_1$ was chosen as the first temporary target concept. Finally, we calculated the four approximations in MGRS by the the four algorithms ten times and compared the averages. Then we made $X_1 \cup X_2$ the second temporary target concept and the whole process was repeated.

When the size of target concept increases gradually, results of MB, DMB, SOB, DSOB, RMB and DRMB are shown in Figure 3 and Figure 4. When the size of target concept increases gradually, RMB is always the most time consuming algorithm among the six algorithms. DMB and MB are more efficient than SOB,DSOB,RMB and DRMB. The computational time of DMB is always less than or equal to MB, so DMB is more efficient than any other algorithms among all the 6 algorithms considered. Sometimes the computational time of DMB is a

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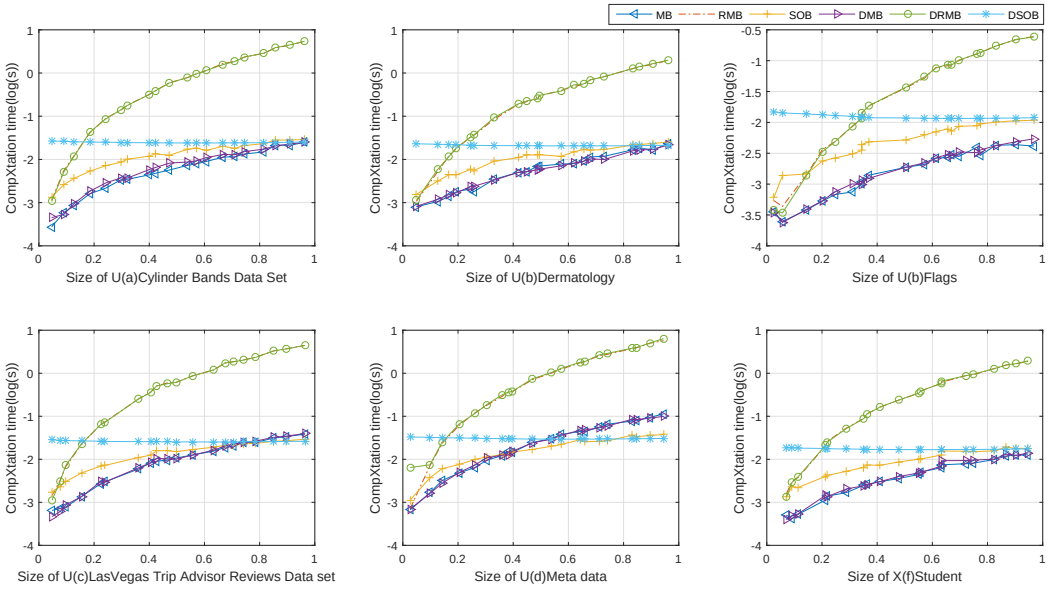


**Figure 2.** Comparison of Algorithm 5 when the size of $U$ increase gradually

little more than the MB while deleting attributes, which is due to additional computation in DMB and it is within the expected range.

## 6. Conclusion

Data sets in real life applications sometimes are complex and huge. The granular structures often increase and decrease in some data sets. Sometimes it is difficult to handle. Therefore it is important to design algorithms to update approximations in NMGRS while neighborhood classes decreasing or increasing. In this paper, four algorithms have been proposed to ensure that the time complexity of the incremental algorithm is less than or equal to the static algorithm. Experimental results showed that the computational time of the incremental algorithms is no more than the static algorithm in most of the situations considered here.

We have examined the advantage and disadvantage of our proposed methods by experimental evaluation. The theoretical time complexity of DMB is more efficient than the other algorithms. However, the disadvantage of DMB and MB is also obvious: when the average size of granules is too small, the time complexity of matrix production is $|U|$ and the time complexity of set operation is far less than $|U|$, which makes the algorithms based on the two approaches hard to compare. In the worst situation, DMB is more efficient than SOB and DSOB.

Approximation computation is a basic process of attribute reduction. In the future, we will further investigate attribute reduction algorithms using the approaches we proposed.

## 7. Acknowledge

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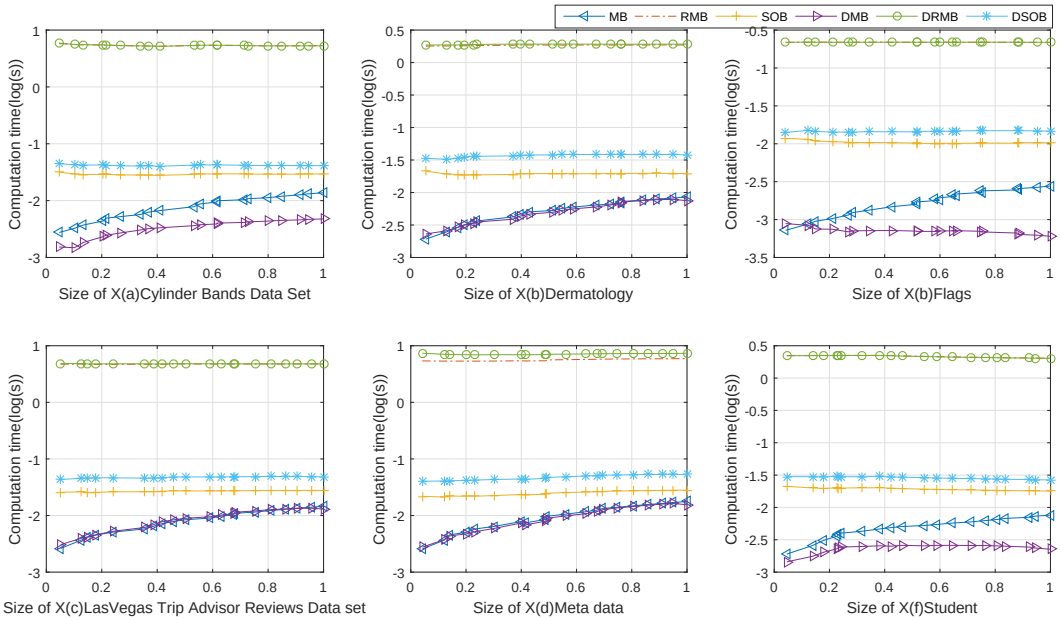


**Figure 3.** Comparison of Algorithm 3 when the size of $X$ increase gradually

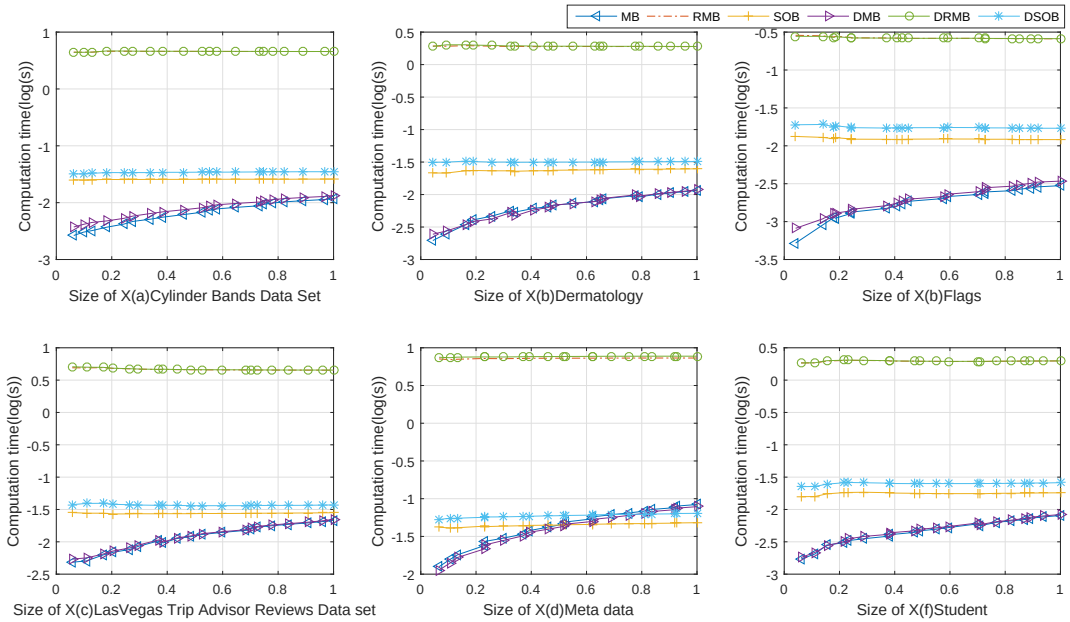


**Figure 4.** Comparison of Algorithm 5 when the size of $X$ increase gradually

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