 doi:10.20944/preprints202007.0415.v4

Article

Logarithmic-Time Addition for BIT-Predicate With Applications for a Simple and Linear Fast Adder And Data Structures

Juan Ramírez

jramirez@binaryprojx.com
www.binaryprojx.com
Jsalisco, México

Abstract: A construction for the systems of natural and real numbers is presented in Zermelo-Fraenkel Set Thoery, that allows for simple proofs of the properties of these systems, and practical and mathematical applications. A practical application is discussed, in the form of a Simple and Linear Fast Adder (Patent Pending). Applications to finite group theory and analysis are also presented. A method is illustrated for finding the automorphisms of any finite group G, which consists of defining a canonical block form for finite groups. Examples are given, to illustrate the procedure for finding all groups of n elements along with their automorphisms. The canonical block form of the symmetry group Δ_4 is provided along with its automorphisms. The construction of natural numbers is naturally generalized to provide a simple and sound construction of the continuum with order and addition properties, and where a real number is an infinite set of natural numbers. A basic outline of analysis is proposed with a fast derivative algorithm. Under this representation, a countable sequence of real numbers is represented by a single real number. Furthermore, an infinite $\infty \times \infty$ real-valued matrix is represented with a single real number. A real function is represented by a set of real numbers, and a countable sequence of real functions is also represented by a set of real numbers. In general, mathematical objects can be represented using the smallest possible data type and these representations are calculable. In the last section, mathematical objects of all types are well assigned to tree structures in a proposed type hierarchy.

Keywords: Structuralism, Set Theory, Arithmetic Model, Fast Adder, Arithmetic Logic Unit, Group, Data Types, Tree, Type Theory, Real Number

PACS: 02.10.v, 03.67.Lx, 03.67.a

MSC: 03-04, 03C55, 03D75, 03H15, 03C13, 05C05, 06F15

1. Introduction

The present work is part of a broader attempt in proposing an optimal universe for classical mathematics. The construction presented in [1] is the first exposition of natural and real numbers, defined as *set numbers*. The present article focuses on finite structures, and group theoretic aspects of this proposal.

In the first section appropriate definitions for operation, group, field and linear space are given to allow easy constructions in the next sections. In the second section, the system of natural numbers is described as the set of all hereditarily finite sets, **HFS**. An order $<$ and operation $\oplus$ are defined, on **HFS**, isomorphic to the natural numbers $\mathbb{N}(<,+)$. The present aim is to make the argument that this is an optimal representation of natural numbers in Z-F set theory. In particular, Von-Neumann and Zermelo-Fraenkel ordinals are embedded sub structures of the proposed construction. A Simple and Linear Fast-Adder is described (Patent Pending) based on the results of this section. The description of the adder is self contained, in a separate document, available as Supporting Material. The fast-adder is

 doi:10.20944/preprints202007.0415.v4

implemented as a sequential circuit that allows potentially faster performance and more energy efficient than other fast adders.

In the third section, a method for representing a finite function as a natural number is detailed. If A, B are two finite collections and $f : A \to B$ a function, a unique natural number N_f is assigned to the function. This is possible for abstract and concrete functions, and a linear order on all finite functions is obtained. A linear order is induced on the subset of all finite permutations, that is well defined with respect to cardinality. Specifically, if η_m, η_n are permutations of $m < n$ many objects, respectively, then $\eta_m < \mathbf{1}_n \leq \eta_n \leq \mathbf{id}_n$ where $\mathbf{1}_n$ is the one-cycle permutation of n objects and $\mathbf{id}_n$ is the identity permutation of n objects. This representation gives a good definition for equivalent functions. Two finite functions are equivalent if they are represented by the same natural number. Given a fixed finite function, equivalent objects can also be identified; an equivalence relation is defined on the objects of the finite function.

In the fourth section, a formal definition of finite groups is given in terms of natural numbers, where a single natural number is used to represent the group. Every finite group G, is well represented with a natural number N_G; if $N_G = N_H$ then H, G are in the same isomorphism class. There is a linear order on all finite groups, that is well behaved with respect to cardinality. In fact, if H, G are two finite groups such that $|H| = m < n = |G|$, then $N_H < \mathbb{Z}_n \leq N_G$. The linear order on groups is

$$\mathbb{Z}_1 < \mathbb{Z}_2 < \mathbb{Z}_3 < \mathbb{Z}_4 < \mathbb{Z}_2^2 < \mathbb{Z}_5 < \mathbb{Z}_6 < \mathbb{Z}_2 \oplus \mathbb{Z}_3 < \mathbb{Z}_7 < \mathbb{Z}_8 < Q_8 < D_8 < \mathbb{Z}_2 \oplus \mathbb{Z}_4 < \mathbb{Z}_2^3 < \mathbb{Z}_9 < \mathbb{Z}_3^2 < \cdots, \quad (1)$$

where D_n is the Dihedral group and Q_8 is the quaternion group. In general, $\mathbb{Z}_n \leq G$ if $|G| = n$ and the order is well behaved with respect to cardinality. The linear order induced on commutative groups, of n objects, also behaves well with respect to factorization of n. Intuitively, if $n = p^k$, then $\mathbb{Z}_{p^k} < \mathbb{Z}_p \oplus \mathbb{Z}_{p^{k-1}} < \mathbb{Z}_p^2 \oplus \mathbb{Z}_{p^{k-2}} < \cdots < \mathbb{Z}_p^k$. For example, $\mathbb{Z}_8 < \mathbb{Z}_2 \oplus \mathbb{Z}_4 < \mathbb{Z}_2^3$, and $\mathbb{Z}_9 < \mathbb{Z}_3^2$. If $n = p_1^{n_1} p_2^{n_2} p_3^{n_3} \cdots p_k^{n_k}$ is the prime factorization of n, then the commutative group $\mathbb{Z}_{p_1}^{n_1} \oplus \mathbb{Z}_{p_2}^{n_2} \oplus \mathbb{Z}_{p_3}^{n_3} \oplus \cdots \oplus \mathbb{Z}_{p_k}^{n_k}$ is the largest commutative group of n objects. For this purpose, a definition of canonical form for a group is given. The canonical form of a finite group is the Cayley table for the group, in a special block form. It reduces the problem of proving two finite groups are isomorphic to finding the canonical table of these groups. In the process of finding the canonical block form, the automorphisms and the minimal set of independent equations that define the group are obtained. The Supporting Material also includes an appendix where groups of less than ten objects are taken to their canonical block form. The canonical form and automorphisms of Δ_4 are also included in the appendix.

The study of real numbers has been reduced to the study of natural numbers. However, the gap (conceptual and practical) between these two kinds of objects is enormous, in most treatments. The proposed set representation of natural numbers allows for the continuum of real numbers to be constructed as a natural extension of the set of natural numbers, without having to build intermediate structures such as $\mathbb{Z}$ or $\mathbb{Q}$. A natural number is a finite subset of **HFS**, while a real number is an infinite subset of **HFS**. Just as a finite group is reduced to a natural number, similar results are true in the infinite case. For example, a real function is a set of real numbers. More surprisingly, a countable sequence of real functions is also a set of real numbers. The general idea is that the complexity of objects is reduced to the minimum possible. In the last section, mathematical objects are well assigned to tree structures. Natural numbers are finite trees (objects of type 0), real numbers are infinite trees (objects of type 1). Sets of real numbers are objects of type 2, and a set of sets of real numbers is an object of type 4. A general description of types is briefly discussed.

2. Groups, Fields and Linear Spaces

In most axiomatic constructions of numerical systems, the set of integers is defined in terms of a quotient space of $\mathbb{N} \times \mathbb{N}$. Then, the rational numbers are defined in terms of a quotient space of $\mathbb{Z} \times \mathbb{Z}$. An alternate approach is taken here, by defining the operation of a group as a function $X \to (X \to X)$. A description of fields and linear spaces is also given

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 August 2022 doi:10.20944/preprints202007.0415.v4

in this section. The definitions and propositions, of this section, allow trivial proofs in the theory of set numbers of Section 3.

Definition 1. *Let G a non empty set, and* ***Aut*** *G be the set of bijective functions of the form $G \to G$. A function $G \to$ **Aut**(G) is called an operation on the set G. A set of functions $B \subseteq$ **Aut** G is said to be balanced if $\mathbf{id}_G \in B$, and if $x \in B$ implies $x^{-1} \in B$. Let $* : G \to B$ a bijective function, for some balanced set $B \subset$ **Aut** G. If*

$$* (x) \circ *(y) = *(*(x)(y)), \tag{2}$$

for every $x, y \in G$, then $*$ is a group structure.

The functions $*(x)$ are called *operation functions of* $*$. The expression $*(x)(y) \in G$ is the image of y under the action of $*(x)$. Thus, $*(*(x)(y)) \in$ **Aut** G is the image of $*(x)(y) \in G$ under the action of $*$.

Theorem 1. *The definitions of group and group structure are equivalent.*

Proof. Let $*$ a group structure and define an operation on the elements, $x * y = *(x)(y)$. Then,

- Identity Element. There exists an object $e \in G$ such that $*(e) = \mathbf{id}_G$. Therefore, $*(e)(x) = x$ for all $x \in G$. This means $e * x = x$ for all $x \in G$. Now it must be shown $x * e = x$. It is true that $*(*(x)(e)) = *(x) \circ *(e) = *(x)$. Since $*$ is injective, it is also true that $*(x)(e) = x$.
- Inverse Element. Let $a \in G$, then there exists a unique $a^{-1} \in G$ such that $*(a^{-1}) = (*(a))^{-1}$ is the inverse function of $*(a)$. This is a direct consequence of the definition of balanced set. It will be proven that $a * a^{-1} = a^{-1} * a = e$. It is enough to prove $a^{-1} * a = e$. It can be verified that $a^{-1} * a = *(a^{-1})(a) = (*(a))^{-1}(a)$. Additionally, $*(a)(e) = a$. Therefore, the inverse function of $*(a)$ applies $(*(a))^{-1}(a) = e$.
- Associativity.

$$\begin{aligned} x * (y * z) &= *(x)(y * z) \\ &= *(x)(*(y)(z)) \\ &= (*(x) \circ *(y))(z) \\ &= *(*(x)(y))(z) \\ &= (*(x)(y)) * z \\ &= (x * y) * z. \end{aligned}$$

For the second part of this proof, it is enough to prove that a group G defines a group structure. The operation functions of the group structure are defined in terms of the cosets xG; define $*(x)$ by $g \mapsto_{*(x)} x * g$. It is easy to verify $*$ is an injective function and it is onto a balanced set. The associative property implies (2). □

The equivalence of groups and group structures is used to find their basic properties.

Theorem 2. *Let $G(*)$ a group with operation $*$. Then,*

1. *Right cancellation; $*(a)(c) = *(b)(c)$ implies $a = b$.*
2. *Left cancellation; $*(c)(a) = *(c)(b)$ implies $a = b$.*
3. *Uniqueness of identity and inverse elements.*
4. *Inverse of inverse; $(x^{-1})^{-1} = x$.*
5. *Existence of unique solutions; given $a, b \in G$ there exists a unique $x \in G$ such that $*(a)(x) = b$, and a unique $y \in G$ such that $*(y)(a) = b$.*

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 August 2022 doi:10.20944/preprints202007.0415.v4

Proof. The first part requires to apply the function $*$, so that $*(*(a)(c)) = *(*(b)(c))$ which implies $*(a) \circ *(c) = *(b) \circ *(c)$. Right cancellation of functions gives $*(a) = *(b)$. It is concluded $a = b$ because $*$ is bijective. The second part can be proven similarly if left cancellation of functions is used.

Let e_1, e_2 be identity elements. Considering e_1 as identity, then $*(e_1)(e_2) = e_2$. If e_2 is the identity, then $*(e_1)(e_2) = e_1$. Therefore $e_1 = e_2$. The uniqueness of the inverse is trivial. If a_1, a_2 are inverse elements of a, then $*a(a_1) = e = *a(a_2)$ implies $a_1 = a_2$ because of left cancellation.

Let $y = x^{-1}$, so that $*(x)$ and $*(y)$ are inverse functions; $(*(x))^{-1} = *(y)$ and $(*(y))^{-1} = *(x)$. The inverse element of $y = x^{-1}$ is the object z such that $*(z)$ is the inverse function of $*(y)$. Therefore, x is the inverse of y and it is concluded $(x^{-1})^{-1} = x$.

For the last part, consider a, b fixed. Since $*(a)$ is a bijective function $G \to G$, there exists a unique $x \in G$ such that $*(a)(x) = b$. On the other hand, a function $*(y)$ that sends a to b needs to be defined. It is easy to see that $b * (a^{-1} * a) = b$, which can be rewritten as $(*(b) \circ *(a^{-1}))(a) = b$. The function $*(b * a^{-1}) = *(*(b)(a^{-1})) = *(b) \circ *(a^{-1})$ sends a to b so that $y = b * a^{-1}$ is the solution. Suppose there exists a second object, w, that satisfies the property of y. Then $*(y)(a) = *(w)(a)$ which implies $y = w$ if right cancellation is used. □

Proposition 1. *A group structure, $*$, defines a new function $\bar{*} : G \to \boldsymbol{Aut}(G)$ such that $\bar{*}(a)(b) = *(b)(a) = b * a$. The function $\bar{*}$ is also a group structure. The two group structures $*, \bar{*}$ are equivalent in the sense that they generate isomorphic groups.*

Proof. First prove $\bar{*}$ is a group structure. It must be shown $\bar{*}$ is a function $\bar{*} : G \to B$, where the image $Im\ \bar{*} = B$ is a balanced subset of $\mathbf{Aut}(G)$. Every object $a \in G$ is assigned a unique function $\bar{*}(a)$, and $\bar{*}(e) = \mathbf{id}_G$ for exactly one object $e \in G$. Next it will be proven $\bar{*}(a)$ is bijective. First of all, it is injective. Take $\bar{*}(a)(x) = \bar{*}(a)(y)$ which is equivalent to the expression $x * a = y * a$, then $x = y$ because of right cancellation. This proves $\bar{*}(a)$ is injective. To prove $\bar{*}(a)$ is onto G, let $b \in G$, then there exists a solution x to the equation $x * a = b$ which is equivalent to $\bar{*}(a)(x) = b$. This proves $\bar{*}(a)$ is a bijection. Now it will be proven the inverse function of $\bar{*}(a)$ is equal to $(\bar{*}(a))^{-1} = \bar{*}(a^{-1}) \in Im(\bar{*})$. By definition, $\bar{*}(a^{-1})(x) = x * a^{-1}$. Also, $\bar{*}(a)$ acts by $\bar{*}(a)(x * a^{-1}) = (x * a^{-1}) * a = x$, which implies the inverse function $(\bar{*}(a))^{-1}$ acts by $(\bar{*}(a))^{-1}(x) = x * a^{-1}$. This proves $\bar{*}(a^{-1}) = (\bar{*}(a))^{-1}$. So far, it has been proven the image of $\bar{*}$ is a balanced set. To prove $\bar{*}$ is injective, take two objects $x, y \in G$ such that $\bar{*}(x) = \bar{*}(y)$. Then, $x = \bar{*}(x)(e) = \bar{*}(y)(e) = y$. Now show $\bar{*}$ satisfies the associative property. For all $a, b \in G$

$$\begin{aligned} \bar{*}(\bar{*}(a)(b))(x) &= \bar{*}(b * a)(x) \\ &= x * (b * a) \\ &= (x * b) * a \\ &= \bar{*}(a)(x * b) \\ &= \bar{*}(a)(\bar{*}(b)(x)) \\ &= (\bar{*}(a) \circ \bar{*}(b))(x), \end{aligned}$$

for all $x \in G$. This proves $\bar{*}$ is a group structure.

Let $G(*)$ be the group generated by $*$ and $G(\bar{*})$ the group generated by $\bar{*}$, then x^{-1} is the same inverse element under both operations. The inverse of $a * b$, under $*$, is equal to $b^{-1} * a^{-1}$. The inverse of $a * b = b \bar{*} a$, under $\bar{*}$, is equal to $a^{-1} \bar{*} b^{-1} = b^{-1} * a^{-1}$. These two groups are isomorphic by $x \mapsto x^{-1}$. To prove, take $\phi(a * b) = (a * b)^{-1} = b^{-1} * a^{-1} = \phi(b) * \phi(a) = \phi(a) \bar{*} \phi(b)$. □

Definition 2. *In general, the functions $*(x)$ and $\bar{*}(x)$ are not equal. When they are equal, the object x is said to commute. A group is abelian if its two generating functions are equal, $* = \bar{*}$.*

 doi:10.20944/preprints202007.0415.v4

Proposition 2. *Let $G(*)$ an operation on the set G. The following are equivalent statements.*

1. *The operation $*$ is associative.*
2. *$*(*(x)(y)) = *(x) \circ *(y)$ for all $x, y \in G$.*
3. *$*(x) \circ \bar{*}(y) = \bar{*}(y) \circ *(x)$ for all $x, y \in G$.*

Proof. The equivalence of 1. and 2. was proven in Theorem 1. Now we prove the equivalence of 1. and 3. Let $z \in G$, then

$$\begin{aligned}(*(x) \circ \bar{*}(y))(z) &= *(x)(\bar{*}(y)(z)) \\ &= *(x)(z * y) \\ &= x * (z * y) \\ &= (x * z) * y \\ &= \bar{*}(y)(x * z) \\ &= \bar{*}(y)(*(x)(z)) \\ &= (\bar{*}(y) \circ *(x))(z)\end{aligned}$$

Suppose 3. holds, then associativity can be proven,

$$\begin{aligned}x * (z * y) &= *(x)(z * y) \\ &= *(x)(\bar{*}(y)(z)) \\ &= (*(x) \circ \bar{*}(y))(z) \\ &= (\bar{*}(y) \circ *(x))(z) \\ &= \bar{*}(y)(*(x)(z)) \\ &= \bar{*}(y)(x * z) \\ &= (x * z) * y\end{aligned}$$

□

The following result is useful for consequent sections. It gives a practical means of proving associativity. If the elements of G commute and the operation functions also commute, then the operation is associative.

Proposition 3. *If $*$ is a commutative operation on the set G, and $*(x) \circ *(y) = *(y) \circ *(x)$, for all $x, y \in G$, then $*$ is associative.*

Proof. Given the hypothesis, the equalities $*(x) \circ \bar{*}(y) = *(x) \circ *(y) = *(y) \circ *(x) = \bar{*}(y) \circ *(x)$ hold true. The result follows from 3. and 1. of the last proposition. □

Definition 3. *Let $G(*)$ a group and let $H \subseteq G$ be a subset of the set G. Define $*_H$ as the function $*$ restricted to H. If $*_H$ is a group structure then it is a subgroup of $G(*)$.*

For $H \subset G$ to be a subgroup of G it is necessary that the image of H, under the action of $*_H(h)$, be equal to H, for all $h \in H$. In short, $*_H(h)[H] = H$, for all $h \in H$. This means H *is closed under the operation* $*$.

Definition 4. *Given two groups $G_1(*_1)$ and $G_2(*_2)$, a homomorphism is a function $\phi : G_1(*_1) \to G_2(*_2)$ such that*

$$\phi(*_1(a)(b)) = *_2(\phi(a))(\phi(b)),$$

for every $a, b \in G_1$. The set of all homomorphisms from $G_1(*_1)$ to $G_2(*_2)$ is represented by the notation $\mathbf{Hom}(G_1, G_2)$, when no confusion arises with respect to the operations of each group.

If the homomorphism is injective as function then it is called a monomorphism, and if it is surjective as function it is called an epimorphism. If the function is bijective it is an isomorphism, or automorphism when $\phi : G \to G$. The set of all automorphisms of $G(*)$ is represented with the notation ***Aut*** $G(*)$.

The notation **Aut**(G) and **Aut** $G(*)$ is used to differentiate between bijective functions and automorphisms.

Theorem 3. *Let X a set, then the composition operation $\circ$ is a group structure for the set of all bijective functions* ***Aut*** X. *A subset $B \subseteq$ **Aut** X that is balanced and closed under composition is a subgroup $B(\circ) \subset$ **Aut** X.*

A group structure $* : G \to B$, induces an isomorphism $* : G(*) \to B(\circ)$.

The composition operation is a group structure for the set of automorphisms ***Aut*** $G(*)$. *A balanced and closed subset, $\mathcal{B} \subseteq$ **Aut** $G(*)$, is a subgroup $\mathcal{B}(\circ) \subset$ **Aut** $G(*)$.*

Proof. For the first part, consider the function $\circ :$ **Aut** $X \to$ **Aut**(**Aut** X). If $f \in$ **Aut** X, then $\circ(f) :$ **Aut** $X \to$ **Aut** X is the function that acts by $\circ(f)(g) = f \circ g$. It will be proven $\circ$ is a bijective function onto a balanced set $Im\ \circ$. Every object in **Aut** X is assigned a function $\circ(f) \in$ **Aut**(**Aut** X). To see $\circ$ is injective, take two objects $f, g \in$ **Aut** X and suppose $\circ(f) = \circ(g)$. This implies $f = f \circ \mathbf{id}_X = g \circ \mathbf{id}_X = g$. Now, prove the image of $\circ$ is balanced. The identity of G is mapped to $\circ(\mathbf{id}_G) \in$ **Aut**(**Aut** X) which is the identity of **Aut**(**Aut** X). Also, for every $\circ(f) \in$ **Aut**(**Aut**X), the inverse function is $(\circ(f))^{-1} = \circ(f^{-1}) \in$ **Aut**(**Aut** X). The associative property is the usual associativity of composition of functions. This proves the first assertion of the first part. The second assertion of the first part is trivial. Take $B(\circ)$ balanced and closed under composition. This makes $B(\circ)$ a group.

For the second part, it must be shown $*$ is an isomorphism. From the first part of this theorem, $B(\circ)$ is a group. It is also known $*$ is a bijection. Definition 4 and associativity, in G, are used to verify $*(*(x)(y)) = *(x) \circ *(y) = \circ(*(x))(*(y))$, for all $x, y \in G$. This proves that the group structure $*$ produces an isomorphism $G(*) \to B(\circ)$, where $B(\circ)$ is the image of $*$ with the operation $\circ$.

The third part of this theorem is proven similarly to the first part of this theorem. □

The *distributive property* is defined. Rings and fields are also defined.

Definition 5. *Let $K(+)$ a group with identity 0; the set $K - \{0\}$ is represented by K_0. Let $\cdot : K_0 \to \mathcal{C} \subset$ **Hom**(K, K), an operation. The operation $\cdot$ distributes over $K(+)$, because*

$$\cdot(x)(+(a)(b)) = +(\cdot(x)(a))(\cdot(x)(b)),$$

for every $a, b, x \in K$.

*Let $R(+)$ an abelian group, and let $\cdot$ a second operation that distributes over $R(+)$. Suppose $\cdot$ is associative and suppose $\cdot 1 =$ **id**$_R$ for a unique non trivial element $1 \in R_0$. A ring $R(+, \cdot)$ has two operations, and if $\cdot$ is commutative the ring is abelian.*

*Let $K(+, \cdot)$ a ring and suppose $Im(\cdot) = \mathcal{C} \subset$ **Aut** $K(+)$ is a balanced set of automoprhisms. Then $K(+, \cdot)$ is a skew field. If the ring $K(+, \cdot)$ is abelian, $K(+, \cdot)$ is a field.*

A new notation $*x$ is used for the operation function $*(x)$. The distributive property holds when a group $K(\cdot)$ whose operation functions $\cdot x$, are homomorphisms on the original group $K(+)$. The conditions give the relations $\cdot x(0) = 0$, for all $x \in K$. Define $\cdot 0(x) = 0$. The operation function $\cdot 0$ is the trivial function $\mathbf{0} : K \to \{0\}$.

Corollary 1. *A field is an abelian group $K(+)$ together with a second abelian group $K(\cdot)$ that distributes over $K(+)$.*

Theorems 4 and 5, below, characterize linear spaces and modules. A linear space is an abelian group $V(\oplus)$, together with a field of automorphisms of $V(\oplus)$. Although these two theorems are not explicitly used in the following sections, it is useful for the last section on real numbers. Given an abelian group $V(\oplus)$ a second operation on $\mathbf{Hom}(V,V)$ is given, apart from composition. The operation $\oplus$ of V naturally induces a closed operation on $\mathbf{Hom}(V,V)$. This allows the definition of modules and linear spaces. Define addition of homomorphisms by $(f\oplus g)(x)=f(x)\oplus g(x)$. If $\mathcal{B}\subset \mathbf{Aut}\ V(\oplus)$, then the symbol $\mathcal{B}(\oplus)$ is used to emphasize that the set is being considered with addition, not composition. The trivial function $\mathbf{e}:V\to\{e\}$ acts as an identity object under addition of homomorphisms, $f=f\oplus\mathbf{e}=\mathbf{e}\oplus f$. Let $f\in\mathbf{Aut}\ V(\oplus)$, and $-f\in\mathbf{Aut}\ V(\oplus)$ the automorphism defined by $-f(x)=-(f(x))$ where $-(f(x))$ is the additive inverse of $f(x)$; the notation $-x$ is used for the inverse of x under $\oplus$. It is easily verified that $f\oplus(-f)=\mathbf{e}$. A set of automorphisms $\mathcal{B}(\oplus)$ is balanced if $\mathbf{e}\in\mathcal{B}(\oplus)$, and if $f\in\mathcal{B}(\oplus)$ implies $-f\in\mathcal{B}(\oplus)$.

Lemma 1. *Let $V(\oplus)$ an abelian group with identity e, and $\mathcal{B}(\oplus)\subset$ **Aut** $V(\oplus)$ a balanced set. If $\mathcal{B}(\oplus)$ is closed under addition of automorphisms, then $\mathcal{B}(\oplus)$ is an abelian group with identity **e**.*

Proof. This result provides an easy way of knowing if $\mathcal{B}(\oplus)$ is a group with addition of functions. It is required that $\mathcal{B}(\oplus)$ be balanced. Under addition of automorphisms, the inverse of f is the function $-f$ that acts by $x\mapsto -(f(x))$. The inverse of $\mathbf{id}_V$ is $-\mathbf{id}_V$ that makes $x\mapsto -x$. Associativity in $V(\oplus)$ implies associativity in $\mathcal{B}(\oplus)$. The commutative property in $\mathcal{B}(\oplus)$ also follows from the commutative property in $V(\oplus)$. □

Theorem 4. *Let $V(\oplus)$ an abelian group and suppose $\mathcal{B}(\circ)\subset$ **Aut** $V(\oplus)$ is a balanced, closed and commutative set of automorphisms with composition. Suppose $\mathcal{B}(\oplus)$ is balanced and closed with addition. Then $\mathcal{B}(\oplus,\circ)$ is a field, and $V(\oplus)$ is a linear space over the field of automorphisms $\mathcal{B}$. The elements of $V(\oplus)$ are called vectors.*

Proof. With respect to composition, it is sufficient to verify $\mathcal{B}(\circ)$ is balanced, closed and abelian. From the third part of Theorem 3, it is concluded $\mathcal{B}(\circ)$ is an abelian subgroup of $\mathbf{Aut}\ V(\oplus)$. If the conditions of the Lemma hold, then $\mathcal{B}(\oplus)$ is a group. Now it will be shown the distributive property holds. This is the simple statement that $\circ f$ is a homomorphism on $\mathcal{B}(\oplus)$, which is expressed by $f\circ(g\oplus h)=(f\circ g)\oplus(f\circ h)$ for every $f,g,h\in\mathcal{B}(\oplus,\circ)$. Let $x\in V$, then

$$\begin{aligned}(f\circ(g\oplus h))(x) &= f(g(x)\oplus h(x))\\ &= f(g(x))\oplus f(h(x))\\ &= (f\circ g)(x)\oplus(f\circ h)(x)\\ &= ((f\circ g)\oplus(f\circ h))(x).\end{aligned}$$

This proves $\mathcal{B}(\oplus,\circ)$ is a field. Now it will be proven the structure of a linear space, in the classic sense, has been defined. The scalar product is simply the application of an automorphism to a vector. Let $f\in\mathcal{B}$, then the scalar product of f, with a vector $v\in V$, is defined as $f\cdot v=f(v)$. First, $(f\circ g)(v)=f(g(v))=f\cdot(g\cdot v)$ because $\circ$ is the product of the field. Also, $f\cdot(u\oplus v)=(f\cdot u)\oplus(f\cdot v)$ because $f\in\mathbf{Aut}\ V(\oplus)$. By definition of addition of functions, $(f\oplus g)\cdot v=(f\cdot v)\oplus(g\cdot v)$. A linear space is defined by an abelian group V and a set of automorphisms (of V) that form a field. □

Similarly define a module M over a ring.

Theorem 5. *Let $M(\oplus)$ an abelian group and suppose $\mathcal{B}(\circ)\subset\mathbf{Hom}(M,M)$ is a closed set of homomorphisms with composition, and $\mathbf{id}_M\in\mathcal{B}(\circ)$. Suppose $\mathcal{B}(\oplus)$ is balanced and closed. Then $\mathcal{B}(\oplus,\circ)$ is a ring. The group $M(\oplus)$ is a module over the ring of homomorphisms $\mathcal{B}$. In general, the group $\mathcal{B}(\circ)$ is not abelian.*

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 August 2022 doi:10.20944/preprints202007.0415.v4

3. Finite Sets and Natural Numbers

Finding a mathematical collections of objects that behave under rules that can be interpreted as the order and operation of addition for natural and real numbers is not an easy task. This problem was taken up by many mathematicians at the beginning of the last century because formalizing arithmetic and analysis requires an understanding of the nature of numbers. The solution was found that the statements of arithmetic, and later analysis, can be formulated using an elementary concept, *set*. Attempts were then made to find set representations of numbers and to model the structure of natural numbers, using sets. Being an elementary concept, a set cannot be described in terms of other mathematical objects. Rather, mathematical objects are described using the language of sets. A set is defined linguistically, as a *collection of objects*. The symbol $\in$ is used for the binary relation of contention. The statement that an object x is *element* of a set X is represented with the symbol $x \in X$. The two most widely used models of mathematics describe natural numbers as *Hereditarily Finite Sets*. The set of all hereditarily finite sets, denoted **HFS**, consists of the sets obtained in the following procedure. The set with no objects, $\emptyset$, is in **HFS**. Also, if $x_1, x_2, \ldots, x_n$ are objects in **HFS** then $\{x_1, x_2, \ldots, x_n\} \in$ **HFS**. Construct sets using these parameters to obtain all hereditarily finite sets. The collection $\{\emptyset\}$ is an object in **HFS**. Since $\emptyset$ and $\{\emptyset\}$ are in **HFS** the collection of these two objects, $\{\emptyset, \{\emptyset\}\}$, is also in **HFS**. Then, take $\emptyset$ and $\{\emptyset, \{\emptyset\}\}$ to find $\{\emptyset, \{\emptyset, \{\emptyset\}\}\} \in$ **HFS**. The sets $\{\emptyset\}$ and $\{\emptyset, \{\emptyset\}\}$ help construct the set $\{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \in$ **HFS**, etc. The first difficulty is ordering these sets so that they model the order of natural numbers.

The solution Zermelo and Fraenkel found is to order a sub collection of **HFS**. Notice it is trivial to order the sets $\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}\}$, $\{\{\{\emptyset\}\}\}, \ldots$, all of which are elements of **HFS**. These sets are ordered by contention, where $x \in$ **HFS** implies $\{x\} \in$ **HFS**. If these are the only sets considered then the order of natural numbers is $\mathbb{N}_< = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \ldots\}$. Addition of these sets has to be defined in such a way that it serves as a model of addition of natural numbers. This simply means, a commutative and associative operation on these sets with identity element $\emptyset$ has to be defined. Proving these statements is usually tedious and laborious. But, the real difficulty arises in understanding the constructions and objects used to describe more complicated structures such as the integer numbers, rational numbers, and real numbers. Integers are described in terms of natural numbers. Rational numbers are described in terms of integers, and real numbers are defined in terms of rational numbers. The last step, in building real numbers, involves objects that are difficult to describe and work with. This leads to a gap in most undergraduate students' learning since most programs do not include these constructions. Even modern day efforts to describe the real number system do not provide an easy way to understand the nature of the object called *real number*.

The second approach taken in describing the order of natural numbers is due to Von Neumann. He orders the sets $\emptyset < \{\emptyset\} < \{\emptyset, \{\emptyset\}\} < \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} < \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\} \ldots$. As before, the set x is smaller than y if $x \in y$. But, in this case, a new natural number is formed from all the previous numbers. There is a clear parallelism between these two constructions, and the weak and strong induction principles. This approach has some advantages in simplifying some proofs for the order and addition of natural numbers. However, when building the later numerical structures, there is a similar situation as in the Zermelo-Fraenkel theory. The greater difficulty arises in building the real numbers. These constructions and their technical aspects can be consulted in [2].

The fact that there is at least two different constructions, gave way to another question, formally referred to as Benaceraff's Identification Problem. It has a great deal to do more with the Philosophy of Mathematics, than the mathematical models in use, but it still has wide implications. The main statement is set forth in a publication titled *"What Numbers Could Not Be"*, [3]. The argument has been made that numbers are actually not sets because there is no absolute way of describing them in terms of sets. For example, it cannot be known what object the number 3 is. Zermelo-Fraenkel say $3 = \{\{\{\emptyset\}\}\}$, but Von Neumann

 doi:10.20944/preprints202007.0415.v4

says $3 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$. Who is to be believed? In fact, there are infinitely many consistent set constructions of natural numbers. Are all these constructions on the same standing? Or are some more convenient than others? Both Z-F and VN provide injective functions $\mathbb{N} \to$ **HFS**. Ackermann was able to find a bijection $\mathbb{N} \to$ **HFS**. This is known as BIT-Predicate or Ackermann Coding, and it is an important part of the practical aspects this work has since mathematical systems can be modeled directly in terms of classic computational processes. It is important to note that Ackermann coding itself does not give means for adding numbers in any special manner. Although Ackermann coding represents natural numbers as sets, it still treats numbers as sequences for purposes of addition and uses the traditional means of operating. Namely, carry over algorithms with its intrinsic time delays. This section is a proposal for a representation of natural numbers with BIT-Predicate, but addition is defined as a finite state machine that reaches stable state in logarithmic time. This is done for natural numbers and real numbers, and it leads to a theory of types that is briefly discussed in the conclusions for later work.

3.1. Motivation

When adding two numbers, natural or real, there is one major difficulty involved. Addition is a special prefix problem which means that each sum bit is dependent on all equal or lower input bits, as noted in [4]. The carrying algorithm can also be consulted in [5]. When adding numbers in base 10 (or base $b > 2$), sequences of digits must be used to represent natural numbers. To write a natural number in base b, each digit in the sequence will specify how many times the corresponding power of b is considered; digits will take a value in $\{0, 1, 2, \ldots, b-1\}$. The order of the sequence is important to know how many times each power of b is added. But, with binary representation ($b = 2$), a more elementary language suffices. It is no longer needed to specify how many times a power is added. It is sufficient to specify if a power is considered or not because digits of the sequence take values in $\{0, 1\}$. Essentially, this allows for a natural number to be determined by a *set* of smaller natural numbers; those that appear as power in binary form. For example, the number $7 = 2^0 + 2^1 + 2^2$ is determined by the set $\{0, 1, 2\}$.

In this proposal, addition is treated in terms of sets, and not sequences. The sum $7 + 13 = (2^0 + 2^1 + 2^2) + (2^0 + 2^2 + 2^3)$, is the sum of sets $\{0, 1, 2\} \oplus \{0, 2, 3\}$. Two new sets are formed - symmetric difference and intersection. The powers that are not repeated $\{1, 3\}$, and the powers that repeat $\{0, 2\}$. To add a power of 2 with itself (i.e., numbers in the intersection), add "1" to that power, $2^n + 2^n = 2^{n+1}$. The sum is rewritten as $7 + 13 = (2^1 + 2^3) + (2^{0+1} + 2^{2+1})$. The first term $2^1 + 2^3$ represents symmetric difference $A \triangle B$, while the second term $2^{0+1} + 2^{2+1} = (2^0 + 2^2) + (2^0 + 2^2)$ represents the intersection. The sum has been reduced to $7 + 13 = (2^1 + 2^3) + (2^1 + 2^3)$. This step is iterated, and adding "1" to the repeated powers gives $7 + 13 = 2^{1+1} + 2^{3+1} = 2^2 + 2^4 = 20$.

If A, B are two finite sets of natural numbers, they can be added using this method and addition is isomorphic to addition of natural numbers. Form two new sets $A' = A \triangle B$ and $B' = s(A \cap B)$, where s is a function that adds 1 to the elements of $A \cap B$. Then $A \oplus B = A' \oplus B'$. It is guaranteed that in a finite number of iterations the intersection $A^{(k)} \cap B^{(k)} = \emptyset$ becomes the empty set. This yields the final answer $A^{(k+1)}$, because $A \oplus B = A^{(k+1)} \oplus B^{(k+1)} = A^{(k+1)} \oplus s(\emptyset) = A^{(k+1)}$.

Apply this reasoning with another example, $15 + 23 = 38$, from Figure 1. This is the addition $A \oplus B = \{0, 1, 2, 3\} \oplus \{0, 1, 2, 4\}$ because $15 = 2^0 + 2^1 + 2^2 + 2^3$ and $23 = 2^0 + 2^1 + 2^2 + 2^4$. First find $A' = A \triangle B = \{3, 4\}$ and $A \cap B = \{0, 1, 2\}$, so that $B' = \{0 + 1, 1 + 1, 2 + 1\} = \{1, 2, 3\}$. Iterate the process with $A'' = A' \triangle B' = \{1, 2, 4\}$ and $B'' = s(A' \cap B') = \{3 + 1\} = \{4\}$. Continuing in this manner, a stable state is reached because $A''' \cap B''' = \{1, 2\} \cap \{5\} = \emptyset$.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 August 2022 doi:10.20944/preprints202007.0415.v4

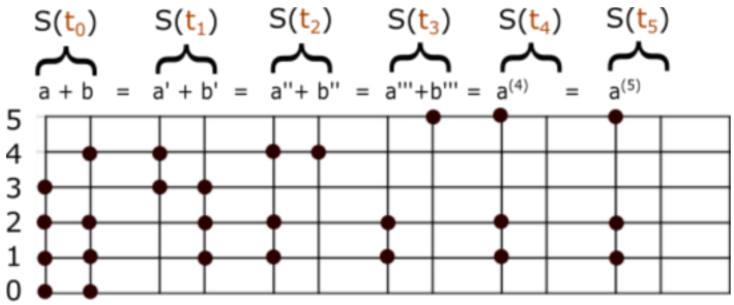

Figure 1. Graphic Representation of $15 + 23 = 38$. The sum of two sets is a process that ends in finite steps. The addition is iterated a finite number of times before the system stabilizes. In this example, the system stabilizes after three iterations. Observe that two disjoint set numbers form a stable system. This means $A \oplus B = A \cup B$ if $A \cap B = \emptyset$; the sum of disjoint sets coincides with the union.

The process described herein is a finite state machine. Each state is composed of two columns. Each column is a finite configuration of energy-levels representing one natural number, as is illustrated in Figure 1. A particle in the basic level "0" is worth 1 unit, and a particle in level "1" is worth 2 units. A particle in level "2" is worth 4 units, and in general a particle in level "n" is worth $2n$ units. A finite configuration of particles in a column represents a set number, so that each state is a pair of natural numbers. As shown in Figure 1, the initial state $S(t_0)$ is given by the inputs A, B. The next state, $S(t_1)$ is given by two new columns. The configuration of the left column is given by the energy levels that were not repeated in state $S(t_0)$. The right column in $S(t_1)$ is given by the objects that repeat but displaced one level up. The configuration of state $S(t_2)$ is defined similarly in terms of state $S(t_1)$. The left column of state $S(t_2)$ is given by the energy levels not repeated in state $S(t_1)$. The configuration in the right column of state $S(t_2)$ is given by the energy levels repeated in state $S(t_1)$ but displaced one level up. In general, the left column of state $S(t_{k+1})$ is given by the energy levels not repeated in state $S(t_k)$. The right column of state $S(t_{k+1})$ is given by a displacement, one level up, of the energy levels repeated in state $S(t_k)$. In a finite number of steps, a stable state is reached, where no particle occupies the right column. The result of the sum is given in the left column.

It should not be difficult for the reader to prove the number of steps to reach stability is bounded above by $\max(A \cup B) + 2$. The addition $A \oplus B = \{0, 1, 2\} \oplus \{0\}$ is one case that reaches the stable state in four steps (worst case scenario). Adding a unit to the string, $\{0, 1, 2, \ldots, k\} \oplus \{0\}$, gives the trivial result $\{k + 1\}$ in $k + 2$ steps. This is the set number expression for the equivalent arithmetical expression $1 + (1 + 2 + 4 + \ldots + 2^k) = 2^{k+1}$. In general, $\{n, n + 1, n + 2, \ldots, n + k\} \oplus \{n\} = \{n + k + 1\}$ is equivalent to the arithmetical expression $2^n + (2^n + 2^{n+1} + 2^{n+2} + \ldots + 2^{n+k}) = 2^{n+k+1}$. In fact, this type of string allows us to calculate the iterations for stability given a sum of two numbers. The longest string will give us the total number of iterations before stability. Going back to the bound on the number of iterations, it can be easily seen that it actually does not depend on the maximum value of the set. A more precise bound can be obtained. For example, the number of iterations for calculating $\{0, 1\} \oplus \{0\}$ is equal to the number of iterations for calculating $\{5, 6\} \oplus \{5\}$. However, the bounds are two very different numbers $\max\{0, 1\}$ and $\max\{5, 6\}$. To come up with a better bound on the number of iterations, observe that the number of iterations does not need to depend on how large the numbers are. To understand this, build a worst case scenario. Let A, B two sets such that $\#(A) + \#(B) = 3$. If the intersection $A \cap B = \emptyset$ is empty, the system is stable from the initial state. To maximize the number of iterations, build a string as above, $A = \{n, n + 1\}$ and place the third element in the bottom $B = \{n\}$. This system requires a total of two iterations to stabilize. Any other configuration of three elements will require at most one iteration to stabilize. Now, suppose a total of $k + 2$ objects; $\#(A) + \#(B) = k + 2$. A string provides a worst case scenario; a string plus the smallest number of the string. The sum $A \oplus B = \{n, n + 1, \ldots, n + k\} \oplus \{n\}$ takes a total of $k + 2$ iterations to reach stability because of the string. Now, it is easy to

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 August 2022 doi:10.20944/preprints202007.0415.v4

see that there is more than one worst case scenario. Change one of the elements from A, to the set B, and the number of iterations will be the same. Doing this with $n+1$, the result is $\{n, n+2, n+3, \ldots, n+k\} \oplus \{n, n+1\}$ which will take $k+2$ iterations to stabilize. This can be done with any of the elements of A, and with more than one. In general, if $A \triangle B = \{n+1, \ldots, n+k\}$ and $A \cap B = \{n\}$ we will have exactly $k+2$ iterations to stabilize. More generally, two sets with non empty intersection will have at least one string of this form. The longest of such strings will determine the smallest number of iterations needed for stability.

Suppose $A, B \subseteq \{0, 1, 2, \ldots, N-1\}$ are two random set numbers and let $x \leq N-1$. The probability that $x \in A \triangle B$ is equal to $\frac{2}{4} = \frac{1}{2}$. Then, the probability that there exists $n < N-1$ such that $\{n+1, \ldots, n+k\} \subseteq A \triangle B$, is equal to the probability of k consecutive heads in N fair coin tosses. Therefore, the probability of a N-bit addition taking $k \leq N$ iterations to complete, is equal to the probability of k consecutive heads in N fair coin tosses. On average, it takes $\log_2 N$ iterations to calculate a N-bit addition. The probability of taking more iterations than $\log_2 N$ decreases fast. These coin toss problems have been well studied and are standard. Solutions abound and are available in the literature as classic problems. A Simple and Linear Fast Adder is described in section 9 (Patent Pending).

In the next subsection addition is formalized for finite sets, and it is isomorphic to $\mathbb{N}_+$. An operation is a function whose image is a space of functions itself. It is an injective function $* : A \to (AfA)$ into the set AfA of all injective functions of the form $A \to A$. The operation $\oplus$ of sets is defined in terms of its operation functions $\oplus n$. Each function makes $\oplus n(x) = n \oplus x$. The function $\oplus 1$ generates the hereditarily finite sets, and it also generates the set of operation functions $\oplus n$. The functions $\oplus n$ are the powers of composition, $\oplus 2 = \oplus 1 \circ \oplus 1$, $\oplus 3 = \oplus 1 \circ \oplus 1 \circ \oplus 1$, etc. Define two base cases $0 = \emptyset$ and $1 = \{\emptyset\}$, along with a function $\oplus 1 : \mathbf{HFS} \to \mathbf{HFS}$. To add 1 to a set A, apply the function $\oplus 1$ to the set A,

$$\oplus 1(A) = (A \triangle 1) \oplus s(A \cap 1), \tag{3}$$

where $s : \mathbf{HFS} \to \mathbf{HFS}$ sends every set $X = \{x\}_{x \in X}$ to the set $s(X) = \{\oplus 1(x)\}_{x \in X}$. Applying the function s to the set X simply means $\oplus 1$ is applied to every object of X. In the following calculations, use the fact that $s(\emptyset) = \emptyset$. Furthermore, define $A \oplus \emptyset = \emptyset \oplus A = A$ which simply defines $\emptyset$ as the identity element. First, use the definition of the operation to find $\oplus 1(0) = (0 \triangle 1) \oplus s(0 \cap 1) = 1 \oplus s(\emptyset) = 1 \oplus \emptyset = 1$. The function $\oplus 1$ generates every element of **HFS** when applied successively.

$$\begin{aligned}
2 &= \oplus 1(1) = (1 \triangle 1) \oplus s(1 \cap 1) = \emptyset \oplus s(1) = s(1) = \{\oplus 1(0)\} = \{1\} \\
3 &= \oplus 1(2) = (2 \triangle 1) \oplus s(2 \cap 1) = (\{1\} \triangle \{0\}) \oplus s(\{1\} \cap \{0\}) = \{0, 1\} \oplus s(\emptyset) \\
&= \{0, 1\} \oplus \emptyset = \{0, 1\} \\
4 &= \oplus 1(3) = (3 \triangle 1) \oplus s(3 \cap 1) = \{1\} \oplus s(\{0\}) = 2 \oplus \{\oplus 1(0)\} = 2 \oplus \{1\} \\
&= 2 \oplus 2
\end{aligned}$$

Here, a new object arises. A suitable definition for $2 \oplus 2$ must be formulated, and in general a suitable definition for $A \oplus B$ is needed. Extend the definition in the obvious way,

$$A \oplus B = (A \triangle B) \oplus s(A \cap B).$$

This gives

$$2 \oplus 2 = (2 \triangle 2) \oplus s(2 \cap 2) = \emptyset \oplus s(2) = \{\oplus 1(1)\} = \{2\}.$$

Therefore,

$$4 = \{2\}.$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 August 2022 doi:10.20944/preprints202007.0415.v4

This simply means the set $\{2\} = \{\{1\}\} = \{\{\{\emptyset\}\}\}$ is the object known as *the number 4*. Continue to generate sets, by applying the function $\oplus 1$ to the result.

$$\begin{aligned}
5 &= \oplus 1(4) = (4 \triangle 1) \oplus s(4 \cap 1) = \{0,2\} \oplus s(\emptyset) = \{0,2\} \\
6 &= \oplus 1(5) = (5 \triangle 1) \oplus s(5 \cap 1) = \{2\} \oplus s\{0\} = \{2\} \oplus \{1\} = (\{2\} \triangle \{1\}) \oplus s(\{2\} \cap \{1\}) \\
&= \{1,2\} \oplus s(\emptyset) = \{1,2\} \\
7 &= \oplus 1(6) = (6 \triangle 1) \oplus s(6 \cap 1) = \{0,1,2\} \oplus s(\emptyset) = \{0,1,2\} \\
8 &= \oplus 1(7) = (7 \triangle 1) \oplus s(7 \cap 1) = \{1,2\} \oplus s\{0\} = \{1,2\} \oplus \{1\} \\
&= (\{1,2\} \triangle \{1\}) \oplus s(\{1,2\} \cap \{1\}) = \{2\} \oplus s\{1\} = \{2\} \oplus \{2\} \\
&= (\{2\} \triangle \{2\}) \oplus s(\{2\} \cap \{2\}) = \emptyset \oplus s\{2\} = s\{2\} = \{3\} \\
9 &= \oplus 1(8) = (8 \triangle 1) \oplus s(8 \cap 1) = \{0,3\} \oplus s(\emptyset) = \{0,3\} \\
10 &= \oplus 1(9) = (9 \triangle 1) \oplus s(9 \cap 1) = \{3\} \oplus s\{0\} = \{3\} \oplus \{1\} = (\{3\} \triangle \{1\}) \oplus s(\{3\} \cap \{1\}) \\
&= \{1,3\} \oplus s(\emptyset) = \{1,3\}.
\end{aligned}$$

To find the set that is the number 5, use the definition of $0 = \emptyset$ and $1 = \{\emptyset\}$. This gives $5 = \{\emptyset, \{\{\emptyset\}\}\}$. Similarly, $6 = \{\{\emptyset\}, \{\{\emptyset\}\}\}$. Notice, that the sum of two disjoint sets is the union. When referring to hereditarily finite sets, in this manner, they are called *set numbers*. Let N be a natural number with binary representation $\sum_{i=1}^{n} 2^{a_i}$, then N is the set number $\{a_1, a_2, \ldots, a_n\}$. For example, $5 = \{0,2\}$ because $5 = 2^0 + 2^2$, while $6 = \{1,2\}$ because $6 = 2^1 + 2^2$. The number $11 = \{0,1,3\}$ can easily be found.

$$11 = 5 \oplus 6 = \{0,2\} \oplus \{1,2\} = \{0,1\} \oplus s(\{2\}) = \{0,1\} \oplus \{3\} = \{0,1,3\}.$$

Another way of finding 11 is with the addition

$$11 = 7 \oplus 4 = \{0,1,2\} \oplus \{2\} = \{0,1\} \oplus s(\{2\}) = \{0,1,3\}.$$

3.2. Formalization

The constructions here described are carried out in Zermelo-Fraenkel Set Theory. A computable function is defined for describing addition of natural numbers in BIT-Predicate. The addition of two n bit numbers is a finite state machine that reaches a stable state in $\log_2 n$ iterations, on average. Given two hereditarily finite sets, an addition of sets is defined.

Definition 6. *Define the set operation* $\oplus 1$ *with* $m \oplus 1 = (m \triangle 1) \oplus s(m \cap 1)$. *The operation function* $\oplus n$ *is defined by* $m \oplus n = \oplus 1^n(m)$.

The identity function is the operation function assigned to the emptyset, $\oplus 0(m) = m$. In the last sub section it has been illustrated how to find $\oplus 1(1), \oplus 1(2), \ldots$. When carrying out the calculations for $3 \oplus 1$, it was recognized that it is necessary to know the value of $\oplus 2(2)$. Continuing to apply $\oplus 1$, more calculations of the form $\oplus n(m)$ are encountered. But, the operation function for $\oplus n$ is explicitly dependent of $\oplus 1$. The functions $\oplus n$ are defined as powers of $\oplus 1$, but to find $\oplus 1$ it is also needed to start finding $\oplus n$. The operation functions build each other simultaneously, as has been seen in the calculations above. The commutative property of $\oplus$ is trivial, using the fact that $f^n \circ f^m = f^m \circ f^n$ for a function f. The equalities are $m \oplus n = \oplus 1^n(\oplus 1^m(0)) = \oplus 1^m(\oplus 1^n(0)) = n \oplus m$.

The easiest way to prove associative property of set sum is to prove the functions $\oplus m$ and $\bar{\oplus} n$ commute, for every set numbers m, n. Given that commutativity holds, it is true that $\oplus n = \bar{\oplus} n$. Because of Proposition 3, it is sufficient to prove the commutative property holds for operation functions, $\oplus m \circ \oplus n = \oplus n \circ \oplus m$.

 doi:10.20944/preprints202007.0415.v4

Proposition 4. *The associative property holds for* $\oplus$.

Proof. By definition, the function $\oplus n$ is the function $\oplus 1$ applied a total of n times, $\oplus n(a) = \oplus 1^n(a)$. The operation functions $\oplus m$, $\oplus n$ commute,

$$\begin{aligned}(\oplus n \circ \oplus m)(a) &= \oplus n(\oplus m(a)) \\ &= \oplus 1^n(\oplus 1^m(a)) \\ &= \oplus 1^m(\oplus 1^n(a)) \\ &= \oplus m(\oplus n(a)) \\ &= (\oplus m \circ \oplus n)(a).\end{aligned}$$

□

A linear order is given, $0 \to_{\oplus 1} 1 \to_{\oplus 1} 2 \to_{\oplus 1} 3 \to_{\oplus 1} 4 \to_{\oplus 1} \cdots$. The order is defined in terms of addition. Let A, B two set numbers, then $A < B$ is true if and only if there exists a set number $m \neq \emptyset$ such that $B = A \oplus m$. Applying $\oplus n$ to B gives

$$\oplus n(B) = \oplus n(A \oplus m) = \oplus n(\oplus m(A)) = \oplus m(\oplus n(A)) = \oplus m(A \oplus n).$$

This implies $A \oplus n < B \oplus n$. That is to say, the operation preserves the order; $A < B$ implies $A \oplus n < B \oplus n$. The order is obviously transitive. Let $B = A \oplus m$ and $C = B \oplus n$. Then $C = (A \oplus m) \oplus n = A \oplus (m \oplus n)$. Since $m \oplus n$ is not the emptyset, it is true that $A < C$.

The following result provides a practical way of determining the natural order of **HFS**. Let A, B two distinct natural numbers and consider their symmetric difference $A \triangle B$ which is not empty and is bounded. That is to say, $\max(A \triangle B)$ exists. Furthermore, this maximum is in exactly one of the two sets, not in both. Compare the two sets in terms of this object, $\max(A \triangle B)$. The set that contains this object is the largest of the two. For example, $15 = \{0,1,2,3\} < \{4\} = 16$ because $A \triangle B = \{0,1,2,3,4\}$ and $\max(A \triangle B) \in \{4\} = 16$. The set number $A = \{1,5,6\} = 98$ is smaller than the set number $B = \{0,7\} = 129$ because $\max(A \triangle B) = 7 \in B$. In the following proof it will be seen that the order is anti symmetric. It will also be seen that every pair of set numbers A, B is comparable; the order is total. In particular, every set in **HFS** is obtained by applying a function of the form $\oplus 1^n = \oplus 1 \circ \oplus 1 \circ \cdots \circ \oplus 1$ to the object 0.

Theorem 6. *If A, B are two set numbers, then $A < B$ if and only if* $\max(A \triangle B) \in B$.

Proof. Let $A = \{a_1, a_2, \ldots, a_n\}$ be a set number, and suppose B is a set number such that $A < B$. From (A5), the set number B is obtained by successively adding 1 to the set number A. This means $B = \oplus 1^n(A)$ for some $n \in \mathbb{N}$. It will be proven $\max(A \triangle B) \in B$ for every $B > A$. In this proof, the fact that $A \oplus B = A \cup B$, if $A \cap B = \emptyset$, will be used. Start with $A \oplus 1 = \{a_1, a_2, \ldots, a_n\} \oplus \{0\}$. There are two cases; $0 \notin A$ or $0 \in A$. In the first case, $A \oplus 1 = \{0, a_1, a_2, \ldots a_n\}$ which implies $\max(A \triangle (A \oplus 1)) = \max\{0\} = 0 \in A \oplus 1$. Now consider the second case; suppose $a_1 = 0$. Then, $A \oplus 1 = \{0, a_2, \ldots, a_n\} \oplus \{0\} = \{a_2, a_3, \ldots, a_n\} \oplus \{1\}$. There are two sub cases; $1 \notin A$ or $1 \in A$. In the first case, $A \oplus 1 = \{1, a_2, a_3, \ldots, a_n\}$ and the result follows, $\max(A \triangle (A \oplus 1)) = \max\{0,1\} = 1 \in A \oplus 1$. In the second case, $a_2 = 1$ and this implies $A \oplus 1 = \{a_3, a_4, \ldots, a_n\} \oplus \{2\}$.

More generally, suppose k is the smallest number not in A. Then, $A = \{0, 1, \ldots, k-1, a_{k+1}, a_{k+2}, \ldots, a_n\}$, where $k < a_{k+1} < a_{k+2} < \ldots < a_n$. Applying $\oplus 1$ yields

$$A \oplus 1 = \{k, a_{k+1}, a_{k+2}, \ldots, a_n\}.$$

Then $\max(A \triangle (A \oplus 1)) = k$, which proves $\max(A \triangle (A \oplus 1)) \in A \oplus 1$. Applying $\oplus 1$ again, the result is

$$A \oplus 2 = \{k, a_{k+1}, a_{k+2}, \ldots, a_n\} \oplus \{0\} = \{0, k, a_{k+1}, a_{k+2}, \ldots, a_n\}.$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 August 2022 doi:10.20944/preprints202007.0415.v4

This means $A\triangle(A\oplus 2)=\{1,2,\ldots,k\}$ and the maximum is $k\in A\oplus 2$. Adding a unit again gives $A\oplus 3=\{1,k,a_{k+1},a_{k+2},\ldots,a_n\}$, which then implies the symmetric difference is $A\triangle(A\oplus 3)=\{0,2,3,\ldots,k\}$ with maximum in $A\oplus 3$. Then, $A\oplus 4=\{0,1,k,a_{k+1},a_{k+2},\ldots,a_n\}$ and symmetric difference $A\triangle(A\oplus 4)=\{2,3,4,\ldots,k\}$. Continue in this manner, applying $\oplus 1$, until it has been applied a total of 2^k-1 times. In each step, the set A will be smaller than. Thus far, it has been proven $\max(A\triangle B)\in B$ if $A<B<A\oplus 2^k$. Applying $\oplus 1$ once more, is simply adding the singleton $2^k=\{k\}$ to the set A. The result is $A\oplus 2^k=\{0,1,\ldots,k,a_{k+1},a_{k+2},\ldots,a_n\}$ because k is the smallest object not in A. This implies $\max(A\triangle(A\oplus 2^k))=\max\{k\}=k\in A\oplus 2^k$. It is concluded $\max(A\triangle B)\in B$ if $A<B\leq A\oplus 2^k$. The careful reader will notice the elevator argument, or an induction hypothesis is needed to justify this argument. The rest is a repetition of what has been done up to this point. To apply $\oplus 1$ to $A\oplus 2^k$, simply substitute all the elements $0,1,\ldots,k$ with $k+1$; use $2^{k+1}=1+(1+2+4+8+\cdots+2^k)$. There are two cases; $k+1\notin A$ or $k+1\in A$. In the first case, $\max(A\triangle(A\oplus 2^k\oplus 1))=k+1\in A\oplus 2^k\oplus 1$ because $A\oplus 2^k\oplus 1=\{k+1,a_{k+1},a_{k+2},\ldots,a_n\}$. In the second case, $a_{k+1}=k+1$ so that

$$A\oplus 2^k\oplus 1=\{k+1,a_{k+2},\ldots,a_n\}\oplus\{k+1\}.$$

Proceed as before, finding the second smallest number not in A. Let $p\in A$ the smallest number in $A-\{k\}$. The numbers k,p are the two smallest numbers not in A, so that $A=\{0,1,\ldots,k-1,k+1,k+2,\ldots,p-1,a_p,\ldots,a_n\}$. This implies $(A\oplus 2^k)\oplus 1=\{p,a_p,\ldots,a_n\}$. The symmetric difference with A is $\{0,1,\ldots,k-1,k+1,\ldots,p\}$. The maximum of the symmetric difference is $p\in(A\oplus 2^k)\oplus 1$. This proves $\max(A\triangle B)\in B$ if $A<B\leq(A\oplus 2^k)\oplus 1$. The symmetric difference of $(A\oplus 2^k\oplus 1)\oplus 1=\{0,p,a_p,\ldots,a_n\}$ with A, is $\{1,2,\ldots,k-1,k+1,k+2,\ldots,p-1,p\}$. The maximum of the symmetric difference is $p\in(A\oplus 2^k)\oplus 2$. This proves $\max(A\triangle B)\in B$, if $A<B\leq(A\oplus 2^k)\oplus 2$. Then, $(A\oplus 2^k)\oplus 3=\{1,p,a_p,\ldots,a_n\}$, which again gives $p=\max(A\triangle(A\oplus 2^k\oplus 3))\in A\oplus 2^k\oplus 3$. Continue in this manner. Apply $\oplus 1$ to $A\oplus 2^k$ a total of 2^k-1 times before reaching

$$(A\oplus 2^k)\oplus 2^k=\{0,1,\ldots,k-1,p,a_p,\ldots,a_n\}.$$

Here, symmetric difference is $A\triangle((A\oplus 2^k)\oplus 2^k)=\{k+1,k+2,\ldots,p\}$, and the maximum is $p\in(A\oplus 2^k)\oplus 2^k$. This proves $\max(A\triangle B)\in B$, if $A<B\leq A\oplus 2^k\oplus 2^k$. Adding 1 again, gives $A\oplus 2^k\oplus 2^k\oplus 1=\{k,p,a_p,\ldots,a_n\}$. The symmetric difference with A is the set $\{k,p\}$. The maximum is $\max\{k,p\}=p\in A\oplus 2^k\oplus 2^k\oplus 1$. Keep adding 1 until $(A\oplus 2^k)\oplus 2^p=\{0,1,\ldots,q-1,a_{n-q+1},a_{n-q+2},\ldots,a_n\}$ has been reached, where $A=\{0,1,\ldots,k-1,k+1,\ldots,p-1,p+1,\ldots,q-1,a_{q-2},a_{q-1},\ldots,a_n\}$ and $q>p$ is the third smallest number not in A. This continues, for all $k,p,q,\ldots,r$ not in A. This proves $\max(A\triangle B)\in B$ if $A<B<A\oplus 2^k\oplus 2^p\cdots\oplus 2^r$. Upon adding 1 to $A\oplus 2^k\oplus 2^p\oplus\cdots\oplus 2^r=\{0,1,\ldots,a_n\}$, the result is the singleton $\{a_n+1\}$. It is trivial to prove that the maximum of the symmetric difference is in $A\oplus 2^k\oplus 2^p\cdots\oplus 2^r\oplus 1$, since $\max(A)=\{a_n\}<\{a_n+1\}=\max(A\oplus 2^k\oplus 2^p\cdots\oplus 2^r\oplus 1)$. Observe that either $\max(X\oplus 1)=\max(X)$ or $\max(X\oplus 1)=\max(X)+1$. Therefore, the result also holds for any $B>A\oplus 2^k\oplus 2^p\cdots\oplus 2^r\oplus 1$ because

$$\max(B)\geq\max(A\oplus 2^k\oplus 2^p\cdots\oplus 2^r\oplus 1)>\max(A)$$

.

To prove the second implication, use the following observation. Let $A=\{a_1,a_2,\ldots,a_n\}$ any set number, and let $b\notin A$ the maximum $b=\max(A\triangle B)$. Add 1 to the set number A repeatedly until you get to the set number $R=\{b,a_i1,a_i2,\ldots,a_n\}$, where $\{a_i1,a_i2,\ldots,a_n\}$ are the elements of A that are greater than b. This means a set N exists such that $R=A\oplus N$. Now, add P to R, where $P=\{b_1,b_2,\ldots,b_j\}$ is the set of objects in B that are smaller than b. The result is $B=P\oplus R=P\oplus(A\oplus N)=A\oplus(N\oplus P)$ which implies $A<B$. □

Let $A=\{2,5,6,8,9\}$ and $B=\{0,1,7,8,9\}$. The largest of the two is the set that contains $\max\{0,1,2,5,6,7\}=7$, so that $A<B$. In the next sections we will have to find

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 August 2022 doi:10.20944/preprints202007.0415.v4

the order of set numbers given in a different form. For example, a set number may be written in the form $A = \{\{3,5\},\{1,2\},\{4,6\}\} = 2^{2^{2^3+2^5}+2^{2^1+2^2}+2^{2^4+2^6}}$. Compare it with $B = \{\{3,4\},\{1,2\},\{5,6\}\} = 2^{2^{2^3+2^4}+2^{2^1+2^2}+2^{2^5+2^6}}$. Obviously $A < B$ since $\max(A) = \{4,6\} < \{5,6\} = \max(B)$.

The operation function $\oplus 1$, of Definition 6, generates all **HFS** when applied successively to 0. The order in which sets are generated is an order of **HFS**, equivalent to the order of natural numbers $\mathbb{N}_{\leq}$. The operation function $\oplus n = \oplus 1^n$ is given by $\oplus 1^n(m) = m \oplus n = (m \triangle n) \oplus s(m \cap n)$.

3.3. Product of Set Numbers

The product is easy to define. Multiplication by 2 has already been defined. In binary representation $2^n + 2^n = 2^{n+1}$, and set numbers have a corresponding rule. To multiply by 2 is to apply the function $\odot 2 = s$ that adds 1 to the elements of the argument. Multiplication by 4 is $s \circ s$ which adds 2 to the elements of the argument. In general, multiplication of B by 2^k is equal to $s^k(B)$. If $B = \{b_1, \ldots, b_n\}$ then $2^k \odot B$ is equal to the set $\{b \oplus k\}_{b \in B} = \{b_1 \oplus k, \ldots, b_n \oplus k\}$. The product of a set number B with 2^k, in our graphic representation, consists of displacing the objects of the set, k units up. The set number $2^k \odot B$ is the *k-displacement of* B. The general product $A \odot B$ is defined in terms of displacements of the *base* B, and the *pivot* A.

$$A \odot B = \bigoplus_{a \in A} \{b \oplus a\}_{b \in B}. \tag{4}$$

Displacements of B are adder, one for each object of the pivot A. If $a \in A$ then the a-displacement of B is one of the displacements in our sum. Notice that multiplication by 0 results in the empty set, $0 \odot X = X \odot 0 = 0$. It is also trivial to find $1 \odot X = X \odot 1 = X$. To show that $2 = \{1\}$ is commutative under multiplication,

$$\begin{aligned} \{1\} \odot X &= \{x \oplus 1\}_{x \in X} \\ &= \bigcup_{x \in X} \{x \oplus 1\} \\ &= \bigoplus_{x \in X} \{1 \oplus x\} \\ &= X \odot \{1\}. \end{aligned}$$

This means $2 \odot X = X \odot 2 = X \oplus X$. To find the product $7 \cdot 5 = (2^0 + 2^1 + 2^2)(2^0 + 2^2)$ use distribution to obtain $2^0(2^0 + 2^2) + 2^1(2^0 + 2^2) + 2^2(2^0 + 2^2)$. Then, $(2^{0+0} + 2^{2+0}) + (2^{0+1} + 2^{2+1}) + (2^{0+2} + 2^{2+2}) = (2^0 + 2^2) + (2^1 + 2^3) + (2^2 + 2^4)$. Carrying out the addition gives $7 \cdot 5 = 2^0 + 2^1 + 2^2 + 2^5 = 35$. Before proving general properties, calculate $5 \odot 15 = \{0,2\} \odot \{0,1,2,3\}$ in two different ways to verify these numbers commute. First make $A = 5$ and $B = 15$. Two displacements of $B = \{0,1,2,3\}$ will be added. The first displacement is $\{0 \oplus 0, 1 \oplus 0, 2 \oplus 0, 3 \oplus 0\} = \{0,1,2,3\}$, and the second displacement is $\{0 \oplus 2, 1 \oplus 2, 2 \oplus 2, 3 \oplus 2\} = \{2,3,4,5\}$. Adding the two, gives $\{0,1,2,3\} \oplus \{2,3,4,5\} = \{0,1,3,6\} = 75$. Now, make $A = 15$ and $B = 5$. Four displacements of $5 = \{0,2\}$, each corresponding to an element of $15 = \{0,1,2,3\}$. The displacements of 5 are $\{0,2\}, \{1,3\}, \{2,4\}, \{3,5\}$. Adding these four displacements results in $(\{0,2\} \oplus \{1,3\}) \oplus (\{2,4\} \oplus \{3,5\}) = \{0,1,2,3\} \oplus \{2,3,4,5\} = 75$, using associativity of addition.

Figure 2 shows the graphic representation of $7 \odot 9$. To formalize this, first verify $\odot 2$ is a morphism for addition of set numbers; verify $s(A \oplus B) = s(A) \oplus s(B)$. Use $X \oplus X = s(X)$ to prove $s(A \oplus B) = (A \oplus B) \oplus (A \oplus B) = (A \oplus A) \oplus (B \oplus B) = s(A) \oplus s(B)$. This implies

$$s^k(A \oplus B) = s^k(A) \oplus s^k(B), \tag{5}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 August 2022 doi:10.20944/preprints202007.0415.v4

for every $k \in \mathbb{N}$. To prove the distributive property use (5) and the commutative and associative properties of addition of sets.

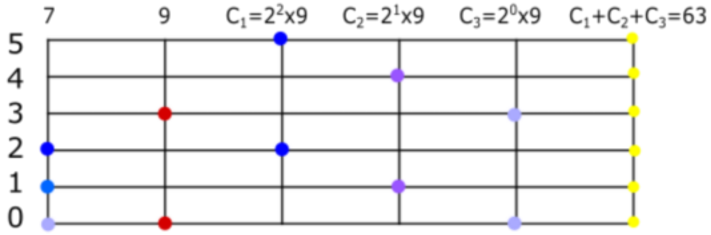

Figure 2. We find the product $7 \odot 9$. The first and second columns are the pivot and base, respectively. The next three columns correspond to the displacements of the base. The last column is the sum of the displacements. The result is equal to $63 = \{0, 1, 2, 3, 4, 5\}$.

$$\begin{aligned} A \odot (B \oplus C) &= \bigoplus_{a \in A} \{x \oplus a\}_{x \in B \oplus C} \\ &= \bigoplus_{a \in A} s^a(B \oplus C) \\ &= \bigoplus_{a \in A} (s^a(B) \oplus s^a(C)) \\ &= \bigoplus_{a \in A} s^a(B) \oplus \bigoplus_{a \in A} s^a(C) \\ &= (A \odot B) \oplus (A \odot C) \end{aligned}$$

To prove multiplication is commutative, let $a \in A$ fixed. The set $\{b \oplus a\}_{b \in B} = \{b_1 \oplus a, b_2 \oplus a, \ldots, b_n \oplus a\} = \{b_1 \oplus a\} \oplus \{b_2 \oplus a\} \oplus \ldots \oplus \{b_n \oplus a\}$ can be expressed as a sum of disjoint singletons, $\bigoplus_{b \in B} \{b \oplus a\}$. Therefore,

$$\begin{aligned} A \odot B &= \bigoplus_{a \in A} \{b \oplus a\}_{b \in B} \\ &= \bigoplus_{a \in A} \bigoplus_{b \in B} \{b \oplus a\} \\ &= \bigoplus_{b \in B} \bigoplus_{a \in A} \{a \oplus b\} \\ &= \bigoplus_{b \in B} \{a \oplus b\}_{a \in A} \\ &= B \odot A. \end{aligned}$$

The commutative property of addition and multiplication of sets has been proven. Together with the distributive property, these imply

$$(A \oplus B) \odot C = (A \odot C) \oplus (B \odot C). \tag{6}$$

To prove that multiplication is associative, the following proposition is needed.

Proposition 5. *The operation function $\odot N$ acts on sets by $\odot N(X) = \oplus X^N(0)$.*

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 August 2022 doi:10.20944/preprints202007.0415.v4

Proof. This is proven by mathematical induction. It is true for 1, since $1 \odot X = X$. Suppose it is true for N, then using the distributive property of (6),

$$\begin{aligned} \odot(N \oplus 1)(X) &= \odot N(X) \oplus \odot 1(X) \\ &= \oplus X^N(0) \oplus X \\ &= \oplus X(\oplus X^N(0)) \\ &= \oplus X^{N+1}(0). \end{aligned}$$

□

Now it can be proven that the associative property holds for the product of set numbers. Because of Proposition 3, it is sufficient to verify the operation functions of $\odot$ commute. The notation $\bigoplus_{i=1}^{N} X$ to represent the number $\oplus X^N(0)$ will be used. The expression $\oplus_{j=1}^{A}\left(\oplus_{i=1}^{B} X\right)$ indicates to add X a total of B times, to obtain the set number $B \odot X$. Then, add $B \odot X$ a total of A times to obtain $A \odot (B \odot X)$. This equates to adding X a total number of $A \odot B = B \odot A$ times. Consider a rectangular matrix of size $A \times B$ and every entry is equal to X, then add all the entries. Considering the matrix of size $B \times A$ and proceeding to adding the entries, is equivalent to rearranging the order of the sum. To prove associativity of product, apply Proposition 5 twice, then use commutativity and associativity of addition to find the third equality. Then, apply Proposition 5 again.

$$\begin{aligned} (\odot A \circ \odot B)(X) &= \odot A\left(\bigoplus_{i=1}^{B} X\right) \\ &= \bigoplus_{j=1}^{A}\left(\bigoplus_{i=1}^{B} X\right) \\ &= \bigoplus_{i=1}^{B}\left(\bigoplus_{j=1}^{A} X\right) \\ &= \odot B\left(\bigoplus_{i=1}^{A} X\right) \\ &= (\odot B \circ \odot A)(X) \end{aligned}$$

To prove the properties of multiplication of sets, they have been expressed in terms of addition, where the commutative, associative properties have been proven. In [1], there is a description of subtraction, division and powers of set numbers.

3.4. Integers

The structure of integers is not necessary to construct the structure of real numbers. However, a construction of $\mathbb{Z}$ is provided because it introduces methods and concepts of previous and later sections. Operation functions and their inverse functions are used to describe integers. A positive integer $\mathbf{n} \in \mathbb{Z}$ is an operation function $\oplus n$, while its negative integer **-n** $\in \mathbb{Z}$ is the inverse function $(\oplus n)^{-1}$. Notice one important fact. Negative integers can easily be distinguished from positive integers. A negative integer is a function of the form **-n** : $\{n, n \oplus 1, n \oplus 2, \ldots\} \to \mathbb{N}$, while a positive integer is a function of the form $\mathbf{n} : \mathbb{N} \to \{n, n \oplus 1, n \oplus 2, \ldots\}$. This will have to be considered when defining addition of integers; it does not represent any difficulty but the reader must be careful. The integer **0** is the identity function of $\mathbb{N}$. The set of negative integers will be represented with the symbol $-\mathbb{N}$. It will be said that $X \subset \mathbb{Z}$ is a *non negative subset of* $\mathbb{Z}$ if $-\mathbb{N} \cap X = \emptyset$, and the like.

The sum of integers is defined in the obvious way, using composition. Let $\mathbf{m} = \oplus m$ and $\mathbf{n} = \oplus n$ positive integers. The composition of these is a positive integer. Define the addition of two positive integers by the relation **m+n** $= \oplus m \circ \oplus n$. The sum, **-m-n**, of

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 August 2022 doi:10.20944/preprints202007.0415.v4

negative integers **-m** $= (\oplus m)^{-1}$ and **-n** $= (\oplus n)^{-1}$, is defined as the composition of inverse functions $(\oplus m)^{-1} \circ (\oplus n)^{-1} = (\oplus n \circ \oplus m)^{-1}$. Given commutativity $\oplus n \circ \oplus m = \oplus m \circ \oplus n$, it follows that **-m-n** is equal to the negative integer $(\oplus m \circ \oplus n)^{-1}$ = **-(m+n)**. The sum of one negative integer **-m** and one positive integer **n** is defined as follows. There are two possible cases. If the corresponding natural numbers satisfy $m < n$, there is a natural number x such that $n = m + x$. Define **-m+n** = **x**, where **x** $= \oplus x : \mathbb{N} \to \{n - m, n - m + 1, n - m + 2, \ldots\}$. In the contrary case that the natural numbers satisfy $n < m$, then $m = n + x$ for some natural number x. Define addition of these integers by **-m+n** = **-x**, where **-x** $= (\oplus x)^{-1} : \{m - n, m - n + 1, m - n + 2, \ldots\} \to \mathbb{N}$. The order relation between m, n determines if **-m+n** is a positive integer or a negative integer. In both cases, the relation **-m+n** $= (\oplus m)^{-1} \circ \oplus n$ holds. But, how is **n-m** defined? Consider the composition $\oplus n \circ (\oplus m)^{-1}$. In both cases, $m < n$ or $n < m$, the composition is $\oplus n \circ (\oplus m)^{-1} : \{m, m + 1, \ldots\} \to \{n, n + 1, \ldots\}$. Although $\oplus n \circ (\oplus m)^{-1}$ is a well defined composition, it is not an integer. The functions $\oplus n \circ (\oplus m)^{-1}$ and $(\oplus m)^{-1} \circ \oplus n$ are not the same function. However, in the intersection of the domains, these compositions are equal functions. Thus, defining the sum of integers as commutative, **n-m** = **-m+n**, is justified. To prove addition of integers is associative, let **x**, **y**, **z** integers. Eight different cases have to be proven. The different combinations for x, y, z being positive or negative. Suppose first, y is positive. Then, **x+y** $= \oplus x \circ \oplus y$. Consider two sub cases. If z is positive, the associative property holds for **(x+y)+z** = **x+(y+z)** because the associative property holds for composition of functions. Suppose z is negative. Then **(x+y)+z** = **z+(x+y)** = **z+(y+x)** = **(z+y)+x** = **x+(z+y)** = **x+(y+z)**. Going back to the assumption of y, now suppose y is negative and x is positive. The equality **(x+y)+z** = **(y+x)+z** = **y+(x+z)** = **y+(z+x)** = **(y+z)+x** = **x+(y+z)** holds. If **x** and **y** are negative, then **x+y** $= \oplus x \circ \oplus y$. This implies **(x+y)+z** $= (\oplus x \circ \oplus y) \circ \oplus z = \oplus x \circ (\oplus y \circ \oplus z)$ = **x+(y+z)**. This proves addition of integers is associative. The addition of integers **5-3** is equal to the function $\oplus 2$, while the result of **3-5** is $(\oplus 2)^{-1}$.

Ordering integers is natural, in this context. Two integers **x**, **y** satisfy the inequality **x** $<$ **y** if and only if **x**$(n) <$ **y**(n), for any $n \in \mathbb{N}$. For example, **-5** $<$ **2** because **-5**$(5) = 0 <$ $7 =$ **2**(5). Of course, the order is well defined so that there is no natural number n such that **2**$(n) <$ **-5**(n). To prove **-6** $<$ **-3** a set number in the domain of **-3** and **-6** is chosen. Say, the number 6. Then, **-6**$(6) = 0 < 3 =$ **-3**(6).

4. Finite Functions and Permutations

In this section, the set of finite functions on $\mathbb{N}$ is described and an injective function from this set of functions, into the set of natural numbers is defined. The important quality of this representation is that functions are equivalent if and only if they are represented by the same number. Natural numbers are assigned to abstract functions, also. There will be a distinct difference when working with an abstract function or a concrete function. When working with abstract functions, two functions are defined to have the same *structure* if they are assigned the same natural number. Concrete functions, on the other hand, can have the same structure but different numeric representation. For example, consider the functions f, g defined by

$$\begin{array}{ll} f(a) = b & g(a) = a \\ f(b) = a & g(b) = c \\ f(c) = c & g(c) = b. \end{array}$$

These two abstract functions have the same structure, and have the same numeric representation. They will be considered to be the same function. However, if the objects are not abstract, so that $a, b, c \in \mathbb{N}$ take specific values, then f, g are distinct concrete functions and they will be represented by distinct numbers. In the previous example, let $a = 3$, $b = 5$ and $c = 0$. The functions f, g defined by $f(3) = 5$, $f(5) = 3$, $f(0) = 0$, and $g(3) = 3$, $g(5) = 0$, $g(0) = 5$ are different and they will be represented by different numbers.

In this section, the set of finite functions of natural numbers is given a linear order. Then, an equivalence definition for abstract finite functions is given, which also orders the

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 August 2022 doi:10.20944/preprints202007.0415.v4

equivalence classes. In the process, a canonical order for the elements of a given abstract finite function is defined, and it can be determined which objects of the function can be considered to be equivalent. In the example, the objects a, b, are equivalent in the function f, while c is not equivalent to another object. The objects b, c are equivalent in the function g, and a is not equivalent to another object.

4.1. Ordered Pairs

To represent finite functions as natural numbers, first it is necessary to find a way of representing ordered pairs as natural numbers. An ordered pair of natural numbers should be an object (m, n) from which two natural numbers are represented in a predetermined order. This means that (m, n) and (n, m) should not be the same object. The first ordered pair, (m, n), represents two natural numbers in order; first m, then n. The second ordered pair (n, m) means first n, then m. A set of two natural numbers $\{X, Y\}$ is not an ordered pair because it is a set of two objects without a predetermined order; a collection of objects X, Y and they are not ordered.

To solve this, a method of coding an ordered pair of natural numbers using odd/even numbers to represent the first/second component, respectively, is outlined. An odd set number is a set number A with $0 \in A$. An even set number is a set number B such that $0 \notin B$. Any set number X, has associated to it the odd number $s(X) \oplus 1$, and the even number $s(X \oplus 1)$. For example, 0 is associated to the odd number $s(0) \oplus 1 = 1$ and the even number $s(0 \oplus 1) = 2$. The number 1 is associated the odd number $s(1) \oplus 1 = 2 \oplus 1 = 3$ and the even number $s(1 \oplus 1) = s(2) = 4$. In general, the natural number k has odd and even representations $2k + 1$ and $2(k + 1)$, respectively, as shown in Table 1.

X	ODD	EVEN
0	1	2
1	3	4
2	5	6
3	7	8
4	9	10
5	11	12
⋮	⋮	⋮

Table 1. Every natural number is uniquely associated an odd and even number.

To find the odd representation of the set number X, displace the objects of X one unit up, then add the object 0 to $s(X)$ to obtain $s(X) \oplus 1 = s(X) \cup \{0\}$. The even representation, $s(X \oplus 1)$, is obtained by displacing the elements of $x \oplus 1$ one unit up. The ordered pair (m, n) is a set number of one odd and one even number, $\{2m + 1, 2(n + 1)\}$. This allows to differentiate the two components. The odd number represents the first component, while the even number is used for the second component. The set number $P = \{2m + 1, 2(n + 1)\}$ is the ordered pair (m, n). The set number $P = \{2m + 1, 2(n + 1)\}$ represents the ordered pair (m, n). The set number representing $(0, 0)$ is $\{1, 2\} = 2^{2(0)+1} + 2^{2(0+1)} = 6$. The ordered pair $(4, 5)$ is represented by the natural number $2^{2(4)+1} + 2^{2(5+1)} = 2^9 + 2^{12}$. In summary, $P = \{A, B\} \in \mathbb{N}$, with $0 \in A$ and $0 \notin B$, represents the ordered pair (m, n), where $m, n \in \mathbb{N}$ are the unique natural numbers that satisfy $s(m) \oplus 1 = A$ and $s(n \oplus 1) = B$. Solving for m, n gives $m = \frac{A-1}{2}$ and $n = \frac{B}{2} - 1$, where $\frac{X}{2} = s^{-1}(X)$. The function s^{-1} displaces the set number X one unit down, $s^{-1}(X) = \{x - 1\}_{x \in X}$.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 August 2022 doi:10.20944/preprints202007.0415.v4

Definition 7. *Consider the family of sets*

$$\begin{aligned}(0,) &= \{6, 18, 66, 258, 1026, \ldots, 2 + 2^{2(n+1)}, \ldots\} \\ (1,) &= \{12, 24, 72, 264, 1032, \ldots, 8 + 2^{2(n+1)}, \ldots\} \\ (2,) &= \{36, 48, 96, 288, 1056, \ldots, 32 + 2^{2(n+1)}, \ldots\} \\ \vdots &\quad \vdots \\ (m,) &= \{2^{2m+1} + 4, 2^{2m+1} + 16, \ldots, 2^{2m+1} + 2^{2(n+1)}, \ldots\}. \end{aligned} \tag{7}$$

Any element $x \in \bigcup_i (i,)$, in the above family of sets, is an ordered pair. The ordered pair (m, n) is the $n+1$-st element of the set $(m,)$. A finite relation is a finite subset $R \subset \bigcup_i (i,)$; elements of R are called components.

There are a few important remarks to be made. Every ordered pair of natural numbers is identified with a unique natural number. Two ordered pairs are the same if and only they are represented by the same set number. And, every natural number representing an ordered pair is a multiple of 6 (the converse is obviously not true). This is a good definition for ordered pairs (m, n). An ordered pair $(0, n)$ is any element of the set $(0,)$. The ordered pair $(0, 0)$ is represented by $6 = 2^{2(0)+1} + 2^{2(0+1)}$, and $(0, 1)$ is $18 = 2^{2(0)+1} + 2^{2(1+1)}$. The third number of the set $(0,)$ represents the ordered pair $(0, 2)$, etc. An element of $(1,)$ is an ordered pair of the form $(1, n)$. The ordered pair $(1, 0)$ is represented by $12 = 2^{2(1)+1} + 2^{2(0+1)}$, the first object of $(1,)$. The ordered pair $(1, 1)$ is $24 = 2^{2(1)+1} + 2^{2(1+1)}$, the second object of $(1,)$. The third object of $(1,)$ represents the ordered pair $(1, 2)$, etc. Now, an important jump can be made, which is a continuation to the last definition. A *relation* will be defined. A finite collection of ordered pairs is a natural number,

$$\{\{A_1, B_1\}, \ldots, \{A_n, B_n\}\} = 2^{2^{A_1}+2^{B_1}} + \cdots + 2^{2^{A_n}+2^{B_n}} \tag{8}$$

where A_i are odd and the B_i are even. Under this definition, a set of ordered pairs is a relation, as is usual. The information of a finite relation is stored in a single natural number, and the structure is obtainable from that number. The relation $\{(0, 0), (0, 1), (0, 2), (2, 1)\}$ is represented by the set number $2^{2^{2(0)+1}+2^{2(0+1)}} + 2^{2^{2(0)+1}+2^{2(1+1)}} + 2^{2^{2(0)+1}+2^{2(2+1)}} + 2^{2^{2(2)+1}+2^{2(1+1)}}$. Two finite relations are the same if and only if they are represented by the same natural number. For another example, take the relation $\{(2, 1), (2, 2), (4, 2), (4, 4)\}$ given by the set number $2^{2^{2(2)+1}+2^{2(1+1)}} + 2^{2^{2(2)+1}+2^{2(2+1)}} + 2^{2^{2(4)+1}+2^{2(2+1)}} + 2^{2^{2(4)+1}+2^{2(4+1)}}$. This allows for finite functions to be described as natural numbers.

4.2. Concrete Functions

In this section, the definition of finite relations is used to represent a finite function of natural numbers. Going back to the definition of relation, it is additionally required that no odd number be repeated. A finite function is represented by a set number of the form (8), where all the A_i are distinct.

Definition 8. *A function $f : A \to B$ is a set number $f = \{\{A_1, B_1\}, \{A_2, B_2\}, \ldots, \{A_n, B_n\}\} = 2^{2^{A_1}+2^{B_1}} + 2^{2^{A_2}+2^{B_2}} + \cdots + 2^{2^{A_n}+2^{B_n}}$, where all the A_i are distinct odd numbers and B_i are even numbers. A function is called bijective if, additionally, all the B_i are distinct. Every element of f is an arrow component. The function f maps $m \mapsto n$ if and only if $2^{2m+1} + 2^{2(n+1)} \in f$.*

A permutation $\{0, 1, 2, \ldots, n\} \to \{0, 1, 2, \ldots, n\}$ is particularly easy to identify. It is a set of $n+1$ ordered pairs, in which every element of $\{1, 2, 3, 4, \ldots, 2n, 2n+1, 2(n+1)\}$ appears in exactly one ordered pair. Examples of permutations are

 doi:10.20944/preprints202007.0415.v4

$$
\begin{aligned}
\{\{1,2\},\{3,4\}\} &= 2^{2^1+2^2}+2^{2^3+2^4}\\
\{\{1,4\},\{3,2\}\} &= 2^{2^1+2^4}+2^{2^3+2^2}\\
\{\{1,2\},\{3,4\},\{5,6\},\{7,8\}\} &= 2^{2^1+2^2}+2^{2^3+2^4}+2^{2^5+2^6}+2^{2^7+2^8}\\
\{\{1,6\},\{3,8\},\{5,2\},\{7,4\}\} &= 2^{2^1+2^6}+2^{2^3+2^8}+2^{2^5+2^2}+2^{2^7+2^4}\\
\{\{1,4\},\{3,10\},\{5,6\},\{7,8\},\{9,2\}\} &= 2^{2^1+2^4}+2^{2^3+2^{10}}+2^{2^5+2^6}+2^{2^7+2^8}+2^{2^9+2^2}\\
\{\{1,6\},\{3,8\},\{5,2\},\{7,10\},\{9,4\}\} &= 2^{2^1+2^6}+2^{2^3+2^8}+2^{2^5+2^2}+2^{2^7+2^{10}}+2^{2^9+2^4}
\end{aligned}
$$

The first permutation is the identity permutation $(0)(1)$. The second set number is representing the one-cycle permutation $(0,1)$. The third and fourth numbers represent $(0)(1)(2)(3)$ and $(0,2)(1,3)$, respectively. The fifth and sixth permutations are $(0,1,4,)(2)(3)$, and $(0,2)(1,3,4)$. A linear order is provided for the set of finite functions, and in particular permutations. The order is well behaved in several ways. If $f:\{0,1,2,\ldots,m\}\to\{0,1,2,\ldots,m\}$, and $g:\{0,1,2,\ldots,n\}\to\{0,1,2,\ldots,n\}$ are permutations and $m<n$, then the representation of f is smaller than the representation of g. This representation is not very good for measuring how much movement a permutation causes. This manner of assigning natural numbers to functions makes a distinction between functions with the same structure. For example, the functions f,g defined by $f(0)=0$ and $g(1)=1$ have the same structure but are assigned different numbers. In the following section, this issue is addressed. A natural number is assigned to any abstract finite function, in such a way that two functions are represented by the same number if and only if they have the same structure. This will be taken as definition of equivalent structure for two functions, because it gives a *modulo-structure* representation of concrete functions.

4.3. Abstract Functions

Consider the permutations $(1,2)(3,4)$ and $(1,3)(2,4)$. These will be represented by the numbers $2^{2^3+2^6}+2^{2^5+2^4}+2^{2^7+2^{10}}+2^{2^9+2^8}$ and $2^{2^3+2^8}+2^{2^7+2^4}+2^{2^5+2^{10}}+2^{2^9+2^6}$, respectively. These numbers are different. It would be useful to have a good definition, modulo the structure, for the two functions above, so that they are assigned the same natural number. In other words, it would be advantageous to number finite functions in such a manner that functions with the same structure will be represented by the same natural number. Let $f:A\to B$ a concrete function, where $A,B\in\mathbb{N}$. The first step is to forget the numeric value assigned to the elements of the components. This means that the sets A,B are no longer thought of as set numbers. Think of the elements of A and B as abstract objects with a function defined on them. Every object in $A\cup B$ is given a non-numeric symbol. For example, the function f defined by

$$
\begin{aligned}
f(2) &= 2\\
f(5) &= 6\\
f(6) &= 5\\
f(8) &= 6\\
f(10) &= 15
\end{aligned}
$$

depends on the distinct objects $2,5,6,8,10,15$ and it will be considered an abstract function f^* defined by

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 August 2022 doi:10.20944/preprints202007.0415.v4

$$\begin{aligned} f^*(a) &= a \\ f^*(b) &= c \\ f^*(c) &= b \\ f^*(d) &= c \\ f^*(p) &= q \end{aligned} \tag{9}$$

Now, a way of assigning a natural number N_{f^*} to the abstract function f^*, in a sufficiently reasonable manner, is to be described. To do this, go back to the realm of numeric values. Take a fixed bijection $\eta : \{a, b, c, d, p, q\} \to \{0, 1, 2, 3, 4, 5\}$, and call it a *naming function of* f^*. Using the procedure of the last section, there is an associated representation $N_{f^*}(\eta) \in \mathbb{N}$ that depends on the naming function η and the abstract function f^*. Now consider the set of all representations $\{N_{f^*}(\eta)\}_\eta$; let η variable over all possible naming functions. In the example, there are 6! possibilities.

To find the modulo-structure representation of a concrete function f, first find the abstract function f^* corresponding to f, then proceeded to find all the possible naming functions of f^*. There is a total of $\#(A \cup B)!$ naming functions. Each naming function η provides a representation $N_{f^*}(\eta)$, so that there is a set of representations $\{N_{f^*}(\eta)\}_\eta$.

Definition 9. *Let f be a concrete function and f^* its corresponding abstract function. Their exists at least one naming, ρ, such that $N_{f^*}(\rho)$ is equal to the maximum element of the set $\{N_{f^*}(\eta)\}_\eta$. This maximum is the modulo-structure representation of f, and the symbol $N_{f^*} = N_{f^*}(\rho)$ is used.*

Let f^* and g^* abstract functions such that $N_{f^*}(\eta) = N_{g^*}(\mu)$, for some naming functions η of f^* and μ of g^*. Then $f^* = g^*$, and η, μ *are equivalent naming functions for* f^*. The sets of representations for f^*, g^* are disjoint if f^*, g^* are different functions; $f^* \neq g^*$ implies $\{N_{f^*}(\eta)\}_\eta \cap \{N_{g^*}(\mu)\}_\mu = \emptyset$. This is a good representation of abstract functions as natural numbers, because $\{N_{f^*}(\eta)\}_\eta$ is a natural number and two functions are assigned different numbers if and only if they have different structure. A linear order for finite functions has been defined. The representation of a function is a large natural number because $\#\{N_{f^*}(\eta)\}_\eta = (\#(Dom(f^*) \cup Im(f^*)))!$. If f^* is a permutation of k objects, the representation of f^* is the sum of $k!$ many powers of 2. The representation of a permutation of 10 objects would be a natural number somewhere close to $10^{10^{230}}$. This representation can be made smaller, and the order of the functions will be invariant. In Definition 9, the fact that sets of representations are disjoint for different functions, is used. The function f is assigned the maximum of the representations, for the following reason. Let $A \cap B = \emptyset$, then the order relation of the maximum elements, $\max(A) < \max(B)$, determines the order relation $A < B$. Assigning to f the set of representations $\{N_{f^*}(\eta)\}_\eta$, or the maximum element, $N_{f^*} = \max\{N_{f^*}(\eta)\}_\eta$, defines the same order on the set of finite functions.

A definition for equivalent objects of a finite function $f : A \to B$ can also be defined. Let $\rho_1, \rho_2 : (A \cup B) \to \{0, 1, 2, \ldots, n-1\}$ two canonical naming functions so that $N_{f^*} = N_{f^*}(\rho_1) = N_{f^*}(\rho_2)$, where $n = \#(A \cup B)$. Suppose the naming functions are not equal, so that $\rho_1(x) \neq \rho_2(x)$, for some $x \in A \cup B$. Naming functions are bijections, so there exists $y \neq x$ such that $\rho_1(y) = \rho_2(x)$. Then x, y are equivalent objects because there are two distinct canonical naming functions ρ_1, ρ_2 that assign the same numerical value to x, y.

Definition 10. *Let $f : A \to B$ a finite function. Two distinct objects $x, y \in A \cup B$ are equivalent if there exist canonical naming functions ρ_1, ρ_2 such that $\rho_1(x) = \rho_2(y)$. This gives an equivalence relation on the set of objects $A \cup B$.*

This method provides two things. The set of all finite functions can be ordered (modulo structure), and a canonical naming function on the objects $Dom(f) \cup Im(f)$ is also obtained. The set of abstract finite permutations can be ordered. Also the elements of any abstract

finite permutation can be ordered, and it is easy to know which objects of f are equivalent. Most importantly these orders are well behaved in several ways. Here the focus is on ordering finite permutations, and a general exposition of finite functions is left for future work. Nonetheless, some examples of general functions are given below. The representation of the first finite functions will be found, to get an intuitive grasp of this order.

The first example is of course the trivial function f_0 that sends $a \to a$. This function depends of a single object so we use the set $\{0\}$ to name the set of objects $\{a\}$. Recalling the definition of ordered pair, the ordered pair $0 \to 0$ is represented by the number $6 = 2^1 + 2^2$. The odd number is used to represent the preimage and the even number to represent the image; a 0 in the preimage means 1 is an element of the ordered pair and a 0 in the image means 2 is an element of the ordered pair. The function consists of one component. Its only element is 6, so $N_{f_0} = 2^{2^1+2^2}$. To construct all finite functions in order of their representation, the next logical choice is a function f_1 defined by one component, $f_1(a) = b$. In this case, there are two objects so a naming of this function is a bijection $\{a, b\} \to \{0, 1\}$. Choosing the naming $a = 0$ and $b = 1$ gives the representation $2^{2^1+2^4}$. With the naming $a = 1$ and $b = 0$ the representation is $2^{2^3+2^2}$. It is concluded that the canonical representation of f_1 is the number $N_{f_1} = 2^{2^1+2^4}$ corresponding to the first naming function $a = 0$ and $b = 1$ because that is the maximum of the representations. These are the only two possible abstract functions of one component; namely $f_0(0) = 0$ (trivial function) and $f_1(0) = 1$ (one object sent to a different object).

Finite functions are ordered isomorphic to $\mathbb{N}$; every finite function is assigned a unique natural number. There is a set of natural numbers $\{N_f\}_{f:finite\ function}$ representing finite functions; every finite function f is represented by a unique set number N_f. Being a set of natural numbers, the set $\{N_f\}_{f:finite\ function}$ can be ordered $N_0 < N_1 < N_2 < \ldots$. Every finite function f is assigned an index; $N_f = N_k$ for some index k. The first few functions $N_0 < N_1 < N_2 < \ldots$ will be found as examples. The first two functions are the one component functions $N_0 = 2^{2^1+2^2}$ and $N_1 = 2^{2^1+2^4}$ from above. Now consider functions of two components. To find the next function, $f_2 = N_2$, add a component. But, intuitively, our order will also assign a larger representation to a function with more objects, holding fixed the number of components. Consider finite functions of two components, and two objects. There is a total of 3 functions that satisfy this conditions and they are the functions N_2, N_3, N_4. The function N_2 is given by two components that switch the objects in the domain, $f_2(a) = b$ and $f_2(b) = a$. This means the two objects of the function f_2 are equivalent. The naming $a = 0, b = 1$ or the naming $a = 1, b = 0$ give the same representation $N_2 = 2^{2^1+2^4} + 2^{2^3+2^2}$. The next function in order is the identity function on two objects, $f_3(a) = a$ and $f_3(b) = b$. Again, both objects are equivalent and they give the representation $N_3 = 2^{2^1+2^2} + 2^{2^3+2^4}$. The function N_4 is the *trivial function* that sends two objects to one of the two; the components are $f_4(a) = a$ and $f_4(b) = a$. The canonical representation $N_4 = 2^{2^1+2^4} + 2^{2^3+2^4}$ is given by the naming $a = 1$ and $b = 0$. The first summand represents $f_4(0) = 1$ and the second summand represents $f_4(1) = 1$. It is easy to verify the alternative naming function gives a smaller representation. The naming $a = 0$ and $b = 1$, gives the representation $2^{2^1+2^2} + 2^{2^3+2^2}$, of f_4. Notice that the function that seems to cause more movement, f_2, is represented by the smallest number of the three. The function that sends everything to a is the largest of the three, and the identity is the middle number. This observation will be important in the special case of ordering permutations.

Now consider functions of two components and three objects, the next functions in the order, N_5, N_6, N_7. Each function is given with its canonical naming, and some of the other non canonical representations.

$f_5(a) = b$, $f_5(b) = c$ has canonical naming $a = 1$, $b = 2$, $c = 0$ giving ordered pairs $(1, 2), (2, 0)$ with representation

$$N_5 = 2^{2^3+2^6} + 2^{2^5+2^2}.$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 August 2022 doi:10.20944/preprints202007.0415.v4

Other, non canonical, representations are $a = 0$, $b = 1$, $c = 2$ with representation $2^{2^1+2^4} + 2^{2^3+2^6}$; $a = 0$, $b = 2$, $c = 1$ with representation $2^{2^1+2^6} + 2^{2^5+2^4}$; $a = 1$, $b = 0$, $c = 2$ with representation $2^{2^3+2^2} + 2^{2^1+2^6}$; $a = 2$, $b = 1$, $c = 0$ with representation $2^{2^5+2^4} + 2^{2^3+2^2}$; $a = 2$, $b = 0$, $c = 1$ with representation $2^{2^5+2^2} + 2^{2^1+2^4}$, etc.

$f_6(a) = c$, $f_6(b) = c$ has canonical naming $a = 0$, $b = 1$, $c = 2$ giving the ordered pairs $(0, 2), (1, 2)$ with representation

$$N_6 = 2^{2^1+2^6} + 2^{2^3+2^6}.$$

The naming $a = 1$, $b = 0$, $c = 2$ is also canonical; a, b are equivalent objects of f. Other, non canonical, representations are $a = 2$, $b = 1$, $c = 0$ with representation $2^{2^5+2^2} + 2^{2^3+2^2}$; $a = 2$, $b = 0$, $c = 1$ with representation $2^{2^5+2^4} + 2^{2^1+2^4}$. etc.

$f_7(a) = a$, $f_7(b) = c$ has canonical naming $a = 2$, $b = 0$, $c = 1$ giving the ordered pairs $(2, 2), (0, 1)$ with representation

$$N_7 = 2^{2^1+2^4} + 2^{2^5+2^6}.$$

Other, non canonical, naming functions are $a = 2$, $b = 1$, $c = 0$ with representation $2^{2^3+2^2} + 2^{2^5+2^6}$; $a = 0$, $b = 1$, $c = 2$ with representation $2^{2^1+2^2} + 2^{2^3+2^6}$; $a = 0$, $b = 2$, $c = 1$ with representation $2^{2^1+2^2} + 2^{2^5+2^4}$; etc.

So far, the first eight numbers $N_0, N_1, \ldots, N_7$ have been found. To find the next numbers representing functions, in order, one must be careful. There is one function of two components and four objects. However, it is not next in order, because the functions of three components and three objects have smaller representation. We see that the order of functions is determined first in terms of objects. Let $f : A \to B$ and $g : C \to D$ finite functions and suppose $\#(A \cup B) < \#(C \cup D)$, then $f < g$. If $\#(A \cup B) = \#(C \cup D)$, we check the number of components. The function with more components has larger representation; $\#(f) < \#(g)$ implies $f < g$. Let A_n^m a finite function of n objects and m components. The following inequalities hold.

$$A_1^1 < A_2^1 < A_2^2 < A_3^2 < A_3^3 < A_4^2 < A_4^3 < A_4^4 < A_5^3 < A_5^4 < A_5^5 < A_6^3 < A_6^4 < A_6^5 < A_6^6 < \cdots$$

This simply means that apart from being well defined, the order given to finite functions is well behaved in the sense just described. The table below states the number of functions with n objects and m components. There is one function of one object and one component ($a \to a$). There is one function of two objects and one component ($a \to b$). There are three functions of two objects and two components. Three functions of three objects and two components, have also been found. This is shown in Table 2.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 August 2022 doi:10.20944/preprints202007.0415.v4

Functions	Objects	Components
1	1	1
1	2	1
3	2	2
3	3	2
7	3	3
1	4	2
9	4	3
	4	4
3	5	3
	5	4
	5	5
1	6	3
	6	4
	6	5
	6	6
⋮	⋮	⋮

Table 2. The first column indicates how many distinct functions of n objects and m components. There is no general way of calculating the number of functions, except to find all possible functions and to determine which ones are equivalent.

Suppose f, g have the same number of objects $\#O(f) = \#O(g)$, and the same number of components $\#C(f) = \#C(g)$. Then, we have to find their canonical representations $N_f, N_g \in \mathbb{N}$, and the order relation $N_f < N_g$ of the representations determines the order relation $f < g$ of the functions. Therefore, to compare two finite functions, it is sufficient to compute their canonical representations and compare these numbers. To find the index k such that $N_k = N_f$, is slightly more complicated. It is so far known how to find the canonical representation N_f of f. But, to know its position in the order more information than just its canonical representation is needed. The total number of functions there are of less objects, and the total number of functions that have the same number of objects but less components. Then, the canonical representation of all functions with the same number of objects and same number of components has to be found. In the table above, there are seven functions of three components and three objects. These seven functions are listed below. For simplicity of exposition, arrows are used to represent the components of a function. For example, the function defined by the three components $f(a) = f(b) = f(c) = a$ is the set of arrows of the last column. These are shown in Table 3.

N_8	N_9	N_{10}	N_{11}	N_{12}	N_{13}	N_{14}
$a \to b$	$a \to b$	$a \to a$	$a \to a$	$a \to a$	$a \to a$	$a \to a$
$b \to c$	$b \to a$	$b \to c$	$b \to b$	$b \to a$	$b \to a$	$b \to a$
$c \to a$	$c \to a$	$c \to b$	$c \to c$	$c \to c$	$c \to b$	$c \to a$

Table 3. There is a total of seven functions of three objects and three components.

Any function of three components and three objects is equivalent to one of these seven. These functions are next in the canonical ordering of finite functions; they are represented by the numbers $N_8, N_9, \ldots, N_{14}$. To know which of these seven functions is N_8, find the canonical representation of all seven and the one with smallest canonical representation is the function N_8, then the function N_9 is the function with second smallest representation, etc. Of these seven functions, the one with largest representation is the function N_{14}. Here they are given in order from smallest to largest (left to right). It is left as an exercise for the reader, to verify the canonical representations of these functions.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 August 2022 doi:10.20944/preprints202007.0415.v4

$$\begin{aligned}
N_8 &= 2^{2^1+2^4} + 2^{2^3+2^6} + 2^{2^5+2^2} \\
N_9 &= 2^{2^1+2^6} + 2^{2^3+2^6} + 2^{2^5+2^4} \\
N_{10} &= 2^{2^1+2^4} + 2^{2^3+2^2} + 2^{2^5+2^6} \\
N_{11} &= 2^{2^1+2^2} + 2^{2^3+2^4} + 2^{2^5+2^6} \\
N_{12} &= 2^{2^1+2^2} + 2^{2^3+2^6} + 2^{2^5+2^6} \\
N_{13} &= 2^{2^1+2^4} + 2^{2^3+2^6} + 2^{2^5+2^6} \\
N_{14} &= 2^{2^1+2^6} + 2^{2^3+2^6} + 2^{2^5+2^6}
\end{aligned}$$

To find the canonical representation of N_8, observe the objects are all equivalent. Let $a = 2$, then make $b = 0$ and $c = 1$, to maximize the term where a is image. The naming functions $b = 2, a = 1, c = 0$ and $c = 2, b = 1, a = 0$ also give the canonical representation. The canonical naming function of N_9 is also easy to find. Start by naming $a = 2$, since a is the most frequent object. Then make $b = 1$ because b is the object that has more relations with a. In N_{10} make $a = 2$ because a is a fixed point; this ensures the term $2^5 + 2^6$ is in the function and maximizes the value. The objects b, c are equivalent in the function N_{10} because there are two canonical naming functions $a = 2, b = 1, c = 0$ and $a = 2, b = 0, c = 1$. The rest of the canonical naming functions are easily found.

Now consider the function of two components and four objects defined by $f_{15}(a) = c$ and $f_{15}(b) = d$. The objects in the image have priority for being assigned larger numbers, so start with naming $c = 3$ because c is in the image. Now, things change between choosing a, b, d for the value 2. Instead of assigning 2 to d, which is also in the image, use the object that is related to $c = 3$. That would be the object $a = 2$. Then, assign the values $d = 1$ and $b = 0$. The components of the function are the ordered pairs $(2, 3)$ and $(0, 1)$ stating $f_{15}(2) = 3$ and $f_{15}(0) = 1$. The set of these ordered pairs is the canonical representation $N_{15} = 2^{2^1+2^4} + 2^{2^5+2^8}$; the summand $2^{2^1+2^4}$ represents the pair $(0, 1)$ and the second summand $2^{2^5+2^8}$ represents the pair $(2, 3)$. The naming function $d = 3, b = 2, c = 1$, $a = 0$ gives components $(0, 1)$ and $(2, 3)$ so that this is also a canonical naming function. Equivalent objects are those that can be assigned the same numerical value under different canonical naming functions. Therefore, a, b are equivalent and c, d are equivalent.

Next in order are the functions of three components and four objects. Each of these nine functions is represented by one of the numbers $N_{16}, N_{17}, \ldots, N_{24}$. Any function of three components and four objects is equivalent to one of these nine. Table 4 provides these functions.

N_{16}	N_{17}	N_{18}	N_{19}	N_{20}	N_{21}	N_{22}	N_{23}	N_{24}
$a \to c$	$a \to b$	$a \to b$	$a \to c$	$a \to d$	$a \to a$	$a \to a$	$a \to a$	$a \to a$
$b \to a$	$b \to a$	$b \to d$	$b \to c$	$b \to d$	$b \to c$	$b \to d$	$b \to b$	$b \to a$
$c \to d$	$c \to d$	$c \to d$	$c \to d$	$c \to d$	$c \to d$	$c \to d$	$c \to d$	$c \to d$

Table 4. There is a total of nine functions of four objects and three components.

The smallest of these nine functions is $N_{16} = 2^{2^1+2^6} + 2^{2^5+2^8} + 2^{2^7+2^4}$ given by the canonical naming function $a = 2$, $b = 0$, $c = 3$, $d = 1$. The next function is $N_{17} = 2^{2^1+2^4} + 2^{2^5+2^8} + 2^{2^7+2^6}$ with canonical naming function $a = 3$, $b = 2$, $c = 0$, $d = 1$ and a, b are equivalent objects. The third is $N_{18} = 2^{2^1+2^6} + 2^{2^3+2^8} + 2^{2^5+2^8}$ under the naming $a = 0$, $b = 2$, $c = 1$, $d = 3$. Next is the function $N_{19} = 2^{2^3+2^8} + 2^{2^5+2^8} + 2^{2^7+2^2}$ with the naming function $a = 2$, $b = 1$, $c = 3$, $d = 0$ and a, b are equivalent objects. The function $N_{20} = 2^{2^1+2^8} + 2^{2^3+2^8} + 2^{2^5+2^8}$ has naming $a = 2$, $b = 1$, $c = 0$, $d = 3$ and a, b, c are equivalent objects. The function $N_{21} = 2^{2^3+2^6} + 2^{2^5+2^2} + 2^{2^7+2^8}$ is given by $a = 3$, $b = 1$, $c = 2$, $d = 0$. The next function is $N_{22} = 2^{2^1+2^6} + 2^{2^3+2^6} + 2^{2^7+2^8}$ with $a = 3$, $b = 1$, $c = 0$, $d = 2$ and b, c equivalent. The second largest is $N_{23} = 2^{2^1+2^4} + 2^{2^5+2^6} + 2^{2^7+2^8}$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 August 2022 doi:10.20944/preprints202007.0415.v4

with naming $a = 3$, $b = 2$, $c = 0$, $d = 1$ and a, b equivalent. The largest function is $N_{24} = 2^{2^1+2^4} + 2^{2^5+2^8} + 2^{2^7+2^8}$ with $a = 3$, $b = 2$, $c = 0$, $d = 1$. By now it can be appreciated that it is not trivial to find the canonical naming function of a finite function, in the general case. Careful observations have to be made to calculate the canonical naming functions, without having to find all possible representations. There are two main problems to solve in the general case, and these computational strategies will be treated in future work. 1) Finding the canonical naming function of any finite function, and 2) Finding the total number of distinct abstract functions of n objects and m components. In the next section, the suborder of finite permutations will be discussed and it proves much easier to work with.

The next functions in the order of all finite functions are functions of four objects and four components. The general analysis is left for future work, because finding all the possible distinct functions of four objects and four components is laborious. The functions that come after those are functions of three components and five objects. There is a total of three such functions.

$$\begin{array}{lll} a \to a & a \to b & a \to d \\ b \to d & b \to d & b \to d \\ c \to p & c \to p & c \to p \end{array}$$

There is one function of three components and six objects.

$$\begin{array}{l} a \to d \\ b \to p \\ c \to q \end{array}$$

The representation of f^*, in example (9) is a set of five natural numbers. The canonical naming will have to assign $\eta(a) = 5$. This guarantees $2^{11} + 2^{12} \in N_{f^*}(\rho)$ represents $f^*(a) = a$. No object has a relation with a. Any of the remaining objects b, c, d, p, q can still be assigned the value 4. If $\rho(c) = 4$ the representation is maximized because two components have power 2^{10}; namely, the components $f^*(b) = c$ and $f^*(d) = c$. Choose $\rho(b) = 3$ instead of $\rho(d) = 3$ because b is related to c by two components. This leaves us with $\rho(d) = 2$. Now we have to assign $q = 1$ and $p = 0$. The canonical representation of

$$\begin{array}{rcl} f^*(a) & = & a \\ f^*(b) & = & c \\ f^*(c) & = & b \\ f^*(d) & = & c \\ f^*(p) & = & q \end{array}$$

under the canonical naming ρ is

$$N_{f^*} = 2^{2^1+2^4}2^{2^5+2^{10}} + 2^{2^7+2^{10}} + 2^{2^9+2^8} + 2^{2^{11}+2^{12}}$$

and there are no equivalent objects.

4.4. Finite Permutations

The suborder of permutations is easier to find, in part because it is well behaved with respect to cardinality. Let f a permutation on m objects and g a permutation on $n > m$ objects, then $N_f < N_g$. Furthermore, permutations are ordered by complexity.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 August 2022 doi:10.20944/preprints202007.0415.v4

Given permutations f, g of the same size, they can be ordered and the interpretation is that a larger number is a simpler permutation. The identity permutation of size n has larger representation than all other permutations of size n. The one cycle permutation of n objects has the smallest representation. The number of distinct abstract permutations of size n, is equal to the number of additive partitions of n. The order of the first few permutations is given. The unique permutation P_0, of size 1, is the function f_0 of one component, represented by $N_0 = 2^6$. There are two permutations of size 2, the functions N_2 and N_3. There is a total of three permutations of size 3. These are the functions N_8, N_{10}, N_{11}. The smallest of these three numbers, N_8, represents the one cycle permutation. The middle permutation, N_{10}, leaves one object fixed. The largest, N_{11}, represents the identity permutation. Call these first six permutations $P_0 = N_0, P_1 = N_2, P_2 = N_3, P_3 = N_8, P_4 = N_{10}, P_5 = N_{11}$. Let us order the five distinct permutations of size 4. These are given in order in Table 5.

P_6	P_7	P_8	P_9	P_{10}
$a \to b$	$a \to b$	$a \to a$	$a \to a$	$a \to a$
$b \to c$	$b \to a$	$b \to c$	$b \to b$	$b \to b$
$c \to d$	$c \to d$	$c \to d$	$c \to d$	$c \to c$
$d \to a$	$d \to c$	$d \to b$	$d \to c$	$d \to d$

Table 5. There is a total of five permutations of four objects.

If two objects are in the same cycle, then they are equivalent. The converse is not true. For example, all objects of P_{10} are equivalent but they are all in different cycles. Function P_6 has canonical naming function $a = 3, b = 1, c = 0, d = 2$. To find its naming function, observe that all the objects are equivalent. Choose $a = 3$, without loss of generality. Next, the term where a is in the image has to be maximized. To this end, define $d = 2$. Then, to maximize the term where a is in the preimage, make $b = 1$. This implies $c = 0$. In permutation P_7 there are two pairs of equivalent objects a, b and c, d, and a canonical naming function is $a = 3, b = 2, c = 1, d = 0$. Permutation P_8 has canonical naming $a = 3$, $b = 2, c = 0, d = 1$, and objects b, c, d are equivalent. The canonical naming function of permutation P_9 is $a = 3, b = 2, c = 1\ d = 0$, and objects c, d are equivalent. Any naming function of P_{10} is canonical; all objects are equivalent. The fact that all objects are equivalent does not imply every naming is the canonical naming. An example of this is P_6.

$$\begin{aligned}
P_6 &= 2^{2^1+2^6} + 2^{2^3+2^2} + 2^{2^5+2^8} + 2^{2^7+2^4} \\
P_7 &= 2^{2^1+2^4} + 2^{2^3+2^2} + 2^{2^5+2^8} + 2^{2^7+2^6} \\
P_8 &= 2^{2^1+2^4} + 2^{2^3+2^6} + 2^{2^5+2^2} + 2^{2^7+2^8} \\
P_9 &= 2^{2^1+2^4} + 2^{2^3+2^2} + 2^{2^5+2^6} + 2^{2^7+2^8} \\
P_{10} &= 2^{2^1+2^2} + 2^{2^3+2^4} + 2^{2^5+2^6} + 2^{2^7+2^8}
\end{aligned}$$

To find the maximum $N_f(\rho) = \max_\eta\{N_f(\eta)\}$, over all possible naming η, find the canonical naming providing the largest representation. To maximize the set number $\{\{a, f(a)\}, \{b, f(b)\}, \{c, f(c)\}, \{d, f(d)\}\}$. The number $2^7 + 2^8 \in f$ is a component of the canonical naming function if $f(x) = x$ for some $x \in \{a, b, c, d\}$. If there are no fixed points, look for cycles of two objects, and continue looking for the smallest possible cycle, to assign the largest values of the naming. The largest possible set number that can represent an abstract permutation of four elements is $2^{2^1+2^2} + 2^{2^3+2^4} + 2^{2^5+2^6} + 2^{2^7+2^8}$, representing the identity permutation $(0)(1)(2)(3)$. The number N_f measures and compares the *movement* a permutation causes. If f has more movements than g, then $N_f < N_g$. The more complicated a permutation becomes the smaller its representation becomes (holding fixed the permutated set). Intuitively, assigning larger values to objects in smaller cycles helps to maximize the representation.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 August 2022 doi:10.20944/preprints202007.0415.v4

One more example of permutations will be given before applying the same method to define groups. Let f be the permutation $(a)(b,c)(p,q,r)$ on the set of abstract objects $\{a,b,c,d,p,q,r\}$. A canonical ordering of its elements, and the canonical representation N_f will be found. It should result in $a=5$, $b=4$, $c=3$, $p=2$, $q=0$, $r=1$, or one of its equivalent numbering functions, and

$$N_f = 2^{2^1+2^4} + 2^{2^3+2^6} + 2^{2^5+2^2} + 2^{2^7+2^{10}} + 2^{2^9+2^8} + 2^{2^{11}+2^{12}}.$$

In all the equivalent numbering functions, $a=0$. An alternative is $b=3$ and $c=4$. The values of the objects in the 3-cycle can also be changed. Make $q=2$, then $p=1$ and $r=0$. If $r=2$, then $q=1$ and $p=0$.

5. Finite Groups

Using the results from the previous sections, finite groups can be represented with natural numbers. A finite group $G(*)$ is a bijection that assigns permutations, of the set G, to objects of G. Operation functions are the elements in the image of $*: G \to Aut(G)$. Consider a naming η of the set G. Then the objects of G, and the operation functions of G are set numbers. Thus, $*$ is a function of the form $M \to N$, where $\max(M) < \min(N)$. If the group has k elements, the domain $M = Dom(*)$ is the set number $\{0,1,2,\ldots,k-1\}$. The image $N = Im(*)$ is the set number $\{N_{*0}(\eta), N_{*1}(\eta), N_{*2}(\eta), \ldots, N_{*(k-1)}(\eta)\}$, where the operation functions N_{*x} are concrete functions. The operation function $*x$ is represented by a natural number $N_{*x}(\eta)$, given a naming function η. The definition of group satisfies the definition of function. Every finite group is a set number whose elements are ordered pairs. The ordered pairs are sets of two objects; one odd and one even. The first components are odd numbers $2i+1$, for every $i \in \{0,1,2,\ldots,k-1\}$. The second components are even numbers representing permutations, $2(N_{*x}(\eta)+1)$. Every naming function η defines a natural number $N_G(\eta)$, that depends on the group and the naming function of that group. There is a finite number of these representations. The maximum representation is the canonical representation $N_G = \max\{N_G(\eta)\}_\eta$ of the group G. This canonical representation gives us a canonical ordering of the elements of G, as well. It behaves much like the representations of permutations. The largest value is assigned to the identity element, $e = k-1$, in any canonical naming function. A group is a set number of the form

$$\begin{aligned}
&2^{2^{2(k-1)+1}+2^{2\left(2^{(2^1+2^2)}+2^{(2^3+2^4)}+\cdots+2^{(2^{2k-1}+2^{2(k-1+1)})}+1\right)}} + 2^{2^{2(k-2)+1}+2^{2\left(2^{(2^1+2^a)}+2^{(2^3+2^b)}+\cdots+2^{(2^{2k-1}+2^{2(k-2+1)})}+1\right)}} + \\
&+2^{2^{2(k-3)+1}+2^{2\left(2^{(2^1+2^c)}+2^{(2^3+2^d)}+\cdots+2^{(2^{2k-1}+2^{2(k-3+1)})}+1\right)}} + 2^{2^{2(k-4)+1}+2^{2\left(2^{(2^1+2^x)}+2^{(2^3+2^y)}+\cdots+2^{(2^{2k-1}+2^{2(k-4+1)})}+1\right)}} + \ldots \\
&\ldots + 2^{2^{2(0)+1}+2^{2\left(2^{(2^1+2^z)}+2^{(2^3+2^w)}+\cdots+2^{(2^{2k-1}+2^{2(0+1)})}+1\right)}},
\end{aligned}$$

where the $k-1$ numbers $a,b,\ldots$ are distinct elements of $\{2,4,6,\ldots,2k-6,2k-4,2k\}$, the numbers $c,d,\ldots$ are distinct elements of $\{2,4,6,\ldots,2k-6,2k-2,2k\}$, the numbers $x,y,\ldots$ are distinct elements of $\{2,4,6,\ldots,2k-8,2k-4,2k-2,2k\}$, etc. The numbers $z,w,\ldots$ are distinct elements of $\{4,6,\ldots,2k\}$. Also, the numbers $a,c,x,z,\ldots$ are all pairwise different and distinct from 2. All the $b,d,y,w,\ldots$ are pairwise different and distinct from 4, etc. The first term, $2^{2^{2(k-1)+1}+2^{2\left(2^{(2^1+2^2)}+2^{(2^3+2^4)}+\cdots+2^{(2^{2k-1}+2^{2(k-1+1)})}+1\right)}}$, indicates that $e=k-1$ is assigned to the identity function. Not all natural numbers of this form are groups. It is additionally required that the associative property holds. Later in this section it will be seen that verifying the associative property is a straightforward process. The composition of functions will be analyzed numerically. With abstract permutations there was a canonical representation, given by a canonical naming of the objects. Here a similar situation arises, now in the context of groups.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 August 2022 doi:10.20944/preprints202007.0415.v4

Theorem 7. *Let G a finite group of order k, then a naming function $\rho : G \to \{0, 1, 2, \ldots, k-1\}$ exists and a corresponding canonical representation $N_G = \max_{\eta} N_G(\eta) = N_G(\rho)$. The bijection ρ is the canonical ordering of G, and $\rho(e) = k - 1$. Two distinct group objects x, y are equivalent if their exists two distinct canonical orderings ρ_1, ρ_2 such that $\rho_1(x) = \rho_2(y)$.*

This gives a well defined linear order on the set of finite groups. Two groups have the same canonical representation if and only if they are isomorphic. This linear order is well behaved with respect to cardinality; $|G| < |H|$ if and only if $N_G < N_H$.

The order of a group element, $|g|$, is the smallest power n such that $g^n = e$. The identity element is assigned to $k - 1$, to maximize the representation. Then identify the objects of smallest order, in G; this number is the smallest prime number that divides $|G|$. The number $k - 2$ will be assigned to one of these objects. The first groups are constructed to illustrate the procedure of finding canonical representation of a group. Start with the trivial group of one object. The group G_0 is determined by the relation $*a(a) = a$. We have the trivial naming $a = 0$ and the operation function N_{*0} is the one component function $P_0 = N_0 = 2^{2^1+2^2}$. Although the numeric value of a group G is used with the notation N_G, groups will also be represented with G_k for a natural index k. This is true because all finite groups will be generated in a linear order. The first group is the trivial group. The canonical representation is

$$G_0 = 2^{2^{2(0)+1}+2^{2\left(2^{2^1+2^2}+1\right)}} = 2^{2^1+2^{2\left(2^6+1\right)}}.$$

Before continuing on to more groups, the use of a table notation is explained, to represent a set of permutations. The same notation of a column of arrows to represent a single permutation is used, but the arrows are not written. For example, the permutation $(a, b)(c)(d)$ can be written as

$$\begin{matrix} a & b \\ b & a \\ c & c \\ d & d \end{matrix}$$

To represent several permutations of the same size, a single rectangular grid is required. For this, one column is used as a pivot for the rest. For example, the set of permutations $\{(a, b)(c)(d),\ (a, b, c, d),\ (a)(c, b)(d), (a)(b)(c)(d)\}$ can be written as a single rectangular grid of $4 + 1$ columns. The first (left-most) column serves as pivot by which all other columns are defined. The second column represents the permutation $(a, b)(c)(d)$, the third column represents the permutation (a, b, c, d), the fourth column is $(a)(b, c)(d)$ and the fifth column is $(a)(b)(c)(d)$.

$$\begin{matrix} a & b & b & a & a \\ b & a & c & c & b \\ c & c & d & b & c \\ d & d & a & d & d \end{matrix}$$

In the particular case of groups, the table is square and rows and columns do not repeat objects. Additionally, one column must be equal to the identity permutation, so the following convention is adopted. The left-most column will represent the identity permutation and we only need to write it once. The identity object will be in the upper left hand corner. The second column is representing the operation function of the second object in the first column. The third column represents the operation function of the third object in the first column. In general, if a is the $k - th$ object in the first column, then the operation function $*a$ is represented in the $k - th$ column of the table. Therefore, an operation is written in the usual table form, so that the following table has products such as $e * e = e$, $a * e = a$, $b * b = e$, $a * c = e$, and the like.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 August 2022 doi:10.20944/preprints202007.0415.v4

$$\begin{matrix} e & a & b & c \\ a & b & c & e \\ b & c & e & a \\ c & e & a & b \end{matrix}$$

This simply means that given any fixed position, the object in that position is expressed in terms of the operation between the first objects of that column and row. This form of writing the operation functions coincides with the multiplication table of the group. If x is the $k-th$ object in the first row, then the $k-th$ column gives the corresponding operation function $*x$. In the process of finding the canonical representation of a group, the associative property will have to be verified frequently. This is given by a simple rule on the group table. Let x be any object in a group G of order n. The element x appears in the table exactly n times; once in each column/row. Each one of the positions where x appears, is an expression for x in terms of two objects; a factorization $x = y * z$. Given a table representing a set of operation functions, the operation satisfies the associative property table if and only if $*x = *y \circ *z$, for every factorization $x = y * z$ of every $x \in G$. In the table example above, $b = a * a$ so that $*b = *a \circ *a$ has to be verified. To verify this is true, prove $*b(g) = (*a \circ *a)(g)$ for every $g \in G$. To find $b * c = *b(c)$, the arrows $c \to_{*a} e \to_{*a} a$ are used. Also, $b * a = *b(a)$ given by the arrows $a \to_{*a} b \to_{*a} c$. For another example, take the product $e = c * a$. It must be proven $*c \circ *a$ is the identity function. First find $(*c \circ *a)(b)$, which is given by the arrows $b \to_{*a} c \to_{*c} b$. Also, $(*c \circ *a)(a)$ is given by $a \to_{*a} b \to_{*c} a$, etc. The construction of groups can be continued having in mind the above rules. A group of two objects will have a table of the form

$$\begin{matrix} e & g_1 \\ g_1 & \end{matrix}$$

Of course g_1 has an inverse $\neq e$, so that $g_1 * g_1 = e$

$$\begin{matrix} e & g_1 \\ g_1 & e \end{matrix}$$

The canonical naming function is trivial to find. To maximize the representation, make $e = 1$, $g_1 = 0$. The group has numeric table

$$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$$

The group is an operation. This operation is a concrete function of two components. The first component is $*(e) = \mathbf{id}$, that sends $e = 1$ to the identity function $(0)(1)$. The second component of the operation is $*(g_1) = (0,1)$, that sends the object $0 = g_1$ to the permutation $(0,1)$. The canonical representation has two terms. The first term representing the first component is $2^{2^{2(1)+1}+2^{2\left(2^{2^3+2^4}+2^{2^1+2^2}+1\right)}}$. The second component is given by the expression $2^{2^{2(0)+1}+2^{2\left(2^{2^3+2^2}+2^{2^1+2^4}+1\right)}}$. The canonical representation of the group $\mathbb{Z}_2$ is

$$N_{\mathbb{Z}_2} = G_1 = 2^{2^{2(1)+1}+2^{2\left(2^{2^3+2^4}+2^{2^1+2^2}+1\right)}} + 2^{2^{2(0)+1}+2^{2\left(2^{2^3+2^2}+2^{2^1+2^4}+1\right)}} = 2^{2^3+2^{2\left(2^6+2^{24}+1\right)}} + 2^{2+2^{2\left(2^{18}+2^{12}+1\right)}}.$$

Why is (10) the canonical representation? The canonical representation is the maximum of the representations. In this case there are two possible representations, one for each naming function. The naming defined by $e = 0$ and $g_1 = 1$, has the representation

$$2^{2^{2(0)+1}+2^{2\left(2^{2^3+2^4}+2^{2^1+2^2}+1\right)}} + 2^{2^{2(1)+1}+2^{2\left(2^{2^3+2^2}+2^{2^1+2^4}+1\right)}}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 August 2022 doi:10.20944/preprints202007.0415.v4

because now 0 is assigned to the identity function, while 1 is assigned the permutation $(0,1)$. This representation is smaller than the canonical representation above. The reader should understand why this is true, before moving on to the next examples. Remember, the largest number of the naming will be assigned to the identity object because this maximizes the representation. Naming the rest of the objects, to obtain the canonical representation, will be described below.

To make a distinction, a *term* is a number representing a component $x \to_* *x$ of the operation. The upper terms are the numbers representing components of the operation functions; we call them *sub terms*. For example, $2^{2^3+2^2}$ is a sub term of the term $2^{2^{2(0)+1}+2^{2\left(2^{2^3+2^2}+2^{2^1+2^4}+1\right)}}$. Terms are ordered pairs; they are elements of the set $\bigcup_i (i,)$, defined at the beginning of section 4.1. Notice in the second equality, that sub terms are also ordered pairs. For example, the sub term $2^{2^3+2^2}$ and the term $2^3 + 2^{2\left(2^6+2^{24}+1\right)}$ are both numbers of the form $2^{2m+1} + 2^{2(n+1)}$. They are both concrete arrows.

5.1. $|G| = 3$

Next are groups of three objects. Start with the table

$$\begin{array}{ccc} e & g_1 & g_2 \\ g_1 & & \\ g_2 & & \end{array}$$

All objects of G have to satisfy $|g| \,|\, 3$, so that $|g| = 3$ for all $g \in G$. This means $g_1^2 \neq e$. Since g_1 is not the identity element, $g_1^2 \neq g_1$. Therefore g_1^2 is a new object g_2, and $g_1 * g_2 = g_1^3 = e$.

$$\begin{array}{ccc} e & g_1 & g_2 \\ g_1 & g_2 & \\ g_2 & e & \end{array}$$

Use the associative rule to find the column of g_2. We know $g_2 = g_1^2$ so that $*g_2$ is the function $*g_1 \circ *g_1$. To find $*g_2(g_1)$ we follow the arrows $g_1 \to_{*g_1} g_2 \to_{*g_1} e$ so that $g_2 * g_1 = e$. In the same way, we find $g_2 \to_{*g_1} e \to_{*g_1} g_1$ so that $g_2^2 = g_1$.

$$\begin{array}{ccc} e & g_1 & g_2 \\ g_1 & g_2 & e \\ g_2 & e & g_1 \end{array}$$

This is the group $\mathbb{Z}_3$. We wish to find the canonical naming function. We start with $e = 2$. One of the non trivial objects g_1, g_2 will have to be assigned the value 1 and the other will be assigned the value 0. We wish to know which of the two objects will be assigned the value 1 and which will be assigned the value 0. There are two different canonical naming functions. These are $\rho_1 : e = 2, g_1 = 1, g_2 = 0$, and $\rho_2 : e = 2, g_1 = 0, g_2 = 1$. Either of these naming functions will give the numerical table

$$\begin{array}{ccc} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{array} \tag{10}$$

The canonical representation of the group is a concrete function of three components. We can use either of the two canonical naming functions to find it. The first component is the ordered pair that assigns 2 to the identity permutation $(0)(1)(2)$ because $e = 2$ is the identity object. This ordered pair is represented by the number

$$2^{2^{2(2)+1}+2^{2\left(2^{2^5+2^6}+2^{2^3+2^4}+2^{2^1+2^2}+1\right)}}.$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 August 2022 doi:10.20944/preprints202007.0415.v4

The second component assigns $g_1 = 1$ to the concrete permutation $(2,1,0)$ because this is the permutation represented by the column of 1, in (10). This component is given by

$$2^{2^{2(1)+1}+2^{2\left(2^{2^5+2^4}+2^{2^3+2^2}+2^{2^1+2^6}+1\right)}}.$$

The object $g_2 = 0$ is assigned the permutation $(2,0,1)$ because this is the permutation given by the column of 0, in table (10). The third term is the number

$$2^{2^{2(0)+1}+2^{2\left(2^{2^5+2^2}+2^{2^3+2^6}+2^{2^1+2^4}+1\right)}}.$$

The canonical representation is the number

$$\begin{aligned} N_{\mathbb{Z}_3} = G_2 &= 2^{2^{2(2)+1}+2^{2\left(2^{2^5+2^6}+2^{2^3+2^4}+2^{2^1+2^2}+1\right)}} + 2^{2^{2(1)+1}+2^{2\left(2^{2^5+2^4}+2^{2^3+2^2}+2^{2^1+2^6}+1\right)}} + 2^{2^{2(0)+1}+2^{2\left(2^{2^5+2^2}+2^{2^3+2^6}+2^{2^1+2^4}+1\right)}} \\ &= 2^{2^5+2^{2\left(2^6+2^{24}+2^{96}+1\right)}} + 2^{2^3+2^{2\left(2^{66}+2^{12}+2^{48}+1\right)}} + 2^{2^1+2^{2\left(2^{18}+2^{72}+2^{36}+1\right)}} \end{aligned}$$

5.2. $|G| = 4$

Before moving on to finding groups of four objects, one more thing is brought to attention. Given a finite group G, the list of all the operations $a * b$ is a system of equations that defines the group. The procedure used here for finding groups, will provide a *minimal set of independent equations* that determine each group. The group $\mathbb{Z}_2$ is determined by the expression $a^2 = e$ (trivial expressions, $e^2 = e$ and $e * x = x * e = x$ do not have to be written down). So, $\mathbb{Z}_2$ is a group determined by one equation. The group $\mathbb{Z}_3$ is given by the expressions $a^2 = b$ and $a^3 = e$. From these two equations, the complete list of operations of the group can be derived.

Klein 4-Group. Start with a set $\{e, g_1, g_2, g_3\}$. There is at least one object with order equal to the smallest prime divisor of 4; suppose $g_1^2 = e$, without loss of generality.

$$\begin{array}{cccc} e & g_1 & g_2 & g_3 \\ g_1 & e & g_3 & g_2 \\ g_2 & g_3 & & \\ g_3 & g_2 & & \end{array} \tag{11}$$

There are two possibilities $g_2^2 = e$ or $g_2^2 = g_1$. Suppose the first case is true. Then the Klein four-group, $K(4)$, is determined. Any group of of four elements e, g_1, g_2, g_3 such that $e = g_1^2 = g_2^2$, is isomorphic to $K(4)$.

$$\begin{array}{cccc} e & g_1 & g_2 & g_3 \\ g_1 & e & g_3 & g_2 \\ g_2 & g_3 & e & g_1 \\ g_3 & g_2 & g_1 & e \end{array}$$

To find the canonical naming functions start with $e = 3$. there are three remaining objects g_1, g_2, g_3. To find their values start with the list of objects in table form. All the non trivial objects are second order objects whoever $2, 1, 0$ may be.

$$\begin{array}{cccc} 3 & 2 & 1 & 0 \\ 2 & 3 & & \\ 1 & & 3 & \\ 0 & & & 3 \end{array}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 August 2022 doi:10.20944/preprints202007.0415.v4

Already, this determines the group.

$$\begin{array}{cccc} 3 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \\ 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 3 \end{array}$$

This means that any naming function with $e = 3$ is a canonical naming function. The objects g_1, g_2, g_3 are all equivalent, so that $K(4)$ has a total of six canonical naming functions. The object 2 can be chosen from three possible options. The object 1 can be chosen from the remaining two objects and 0 is determined as the remaining object. A naming function will be represented by a sequence. For example, the naming function $e = 3$, $g_1 = 2$, $g_2 = 1$, $g_3 = 0$ is written as $\eta(e, g_1, g_2, g_3)$. The six naming functions are

$$\begin{array}{ccc} \eta(e,g_1,g_2,g_3) & \eta(e,g_2,g_1,g_3) & \eta(e,g_3,g_1,g_2) \\ \eta(e,g_1,g_3,g_2) & \eta(e,g_2,g_3,g_1) & \eta(e,g_3,g_2,g_1) \end{array}$$

Any naming function with $e = 3$, gives the numeric table

$$\begin{array}{cccc} 3 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \\ 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 3 \end{array}$$

and canonical representation

$$\begin{aligned} N_{K(4)} \;=\; & 2^{2^7+2^{\left(2^{(2^7+2^8)}+2^{(2^5+2^6)}+2^{(2^3+2^4)}+2^{(2^1+2^2)}+1\right)}} + 2^{2^5+2^{\left(2^{(2^7+2^6)}+2^{(2^5+2^8)}+2^{(2^3+2^2)}+2^{(2^1+2^4)}+1\right)}} \\ & + 2^{2^3+2^{\left(2^{(2^7+2^4)}+2^{(2^5+2^2)}+2^{(2^3+2^8)}+2^{(2^1+2^6)}+1\right)}} + 2^{2^1+2^{\left(2^{(2^7+2^2)}+2^{(2^5+2^4)}+2^{(2^3+2^6)}+2^{(2^1+2^8)}+1\right)}}. \end{aligned}$$

The first term is the component that sends 3 to the identity function $(0)(1)(2)(3)$, while the second term is the component that sends 2 to the permutation $(0,1)(2,3)$, etc. The group $K(4)$ has a total of six automorphisms, and there are a total of six distinct canonical naming functions. This in not coincidental. Each of these six naming functions determines an automorphism of $K(4)$. Hold one of these fixed as *pivot*. For example, take the pivot $A = \eta(e, g_3, g_1, g_2)$ which will be held fixed. Choose a second canonical naming function $B = \eta(e, g_1, g_3, g_2)$. The function that sends the first component of A to the first component of B, and the second component of A to the second component of B, etc. is called a *component function*. The component function $\phi : A \to B$ is an automorphism, for every canonical naming function B. That is to say, ϕ that acts by $e \mapsto e$, $g_1 \mapsto g_3$, $g_2 \mapsto g_2$, $g_3 \mapsto g_1$ is an automorphism of $K(4)$. Choosing $B = A$ then the identity automorphism is being described. Let each of the of the canonical naming functions defines an automorphism. The initial choice of A is inconsequential; any choice for A gives the same set of functions.

Cyclic group $\mathbb{Z}_4$. Going back to table (11), consider the second case, $g_2^2 = g_1$. This determines the table

$$\begin{array}{cccc} e & g_1 & g_2 & g_3 \\ g_1 & e & g_3 & g_2 \\ g_2 & g_3 & g_1 & e \\ g_3 & g_2 & e & g_1 \end{array}$$

This is the cyclic group $\mathbb{Z}_4$, determined by the equations $g_1^2 = e$ and $g_2^2 = g_1$. To find the canonical representation, be careful to assign values. It is not known which object of $\mathbb{Z}_4$ takes each value of $0, 1, 2$. If the value 2 is assigned to g_2, g_3, the numeric table

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 August 2022 doi:10.20944/preprints202007.0415.v4

$$\begin{array}{cccc} 3 & 2 & 1 & 0 \\ 2 & 1 & & \\ 1 & & & \\ 0 & & & \end{array}$$

On the other hand, if 2 is assigned to the second order object, g_1, the table is

$$\begin{array}{cccc} 3 & 2 & 1 & 0 \\ 2 & 3 & & \\ 1 & & & \\ 0 & & & \end{array}$$

The latter maximizes the representation. Intuitively, try to assign the larger numbers by giving priority to the left-most columns. Within a column give priority to the objects of upper rows. Place larger numbers further to the left and then further to the top of the table. The rest of the table is determined.

$$\begin{array}{cccc} 3 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \\ 1 & 0 & 2 & 3 \\ 0 & 1 & 3 & 2 \end{array}$$

Any naming function with $e = 3, g_1 = 2$ is a canonical naming function. One canonical naming function is $e = 3$, $g_1 = 2$, $g_2 = 1$, $g_3 = 0$ which is written as $\eta(e, g_1, g_2, g_3)$. The other canonical naming function is $\eta(e, g_1, g_3, g_2)$. This implies g_2, g_3 are equivalent objects. There are two automorphisms. Fix $A = \eta(e, g_1, g_3, g_2)$ then $B = \eta(e, g_1, g_2, g_3)$. This determines the automorphism with components $e \mapsto e$, $g_1 \mapsto g_1$, $g_2 \mapsto g_3$, $g_3 \mapsto g_2$. If $B = A$ the identity automorphism is determined. The canonical representation is

$$\begin{aligned} N_{\mathbb{Z}_4} = {} & 2^{2^7+2^{2\left(2^{(2^7+2^8)}+2^{(2^5+2^6)}+2^{(2^3+2^4)}+2^{(2^1+2^2)}+1\right)}} + 2^{2^5+2^{2\left(2^{(2^7+2^6)}+2^{(2^5+2^8)}+2^{(2^3+2^2)}+2^{(2^1+2^4)}+1\right)}} \\ & + 2^{2^3+2^{2\left(2^{(2^7+2^4)}+2^{(2^5+2^2)}+2^{(2^3+2^6)}+2^{(2^1+2^8)}+1\right)}} + 2^{2^1+2^{2\left(2^{(2^7+2^2)}+2^{(2^5+2^4)}+2^{(2^3+2^8)}+2^{(2^1+2^6)}+1\right)}}. \end{aligned}$$

The groups $K(4)$ and $\mathbb{Z}_4$ are compared in terms of the order of natural numbers. The odd number of the terms do not determine the order because the even numbers, representing the operation functions, are larger than the odd numbers. The group with the largest operation function, that is not in both groups, is the larger of the two. The group $K(4)$ has larger numeric representation than $\mathbb{Z}_4$ because $K(4)$ has the largest operation function that is not in both groups. The cyclic group has the smallest representation, $N_{\mathbb{Z}_4} < N_{K(4)}$, just as the one cycle permutation (a, b, c, d) has smaller representation than $(a, b)(c, d)$. Try to find all groups with less than ten objects. The minimum independent set of equations, the canonical naming functions of its elements, the set of automorphisms, the canonical table and canonical numeric representation are given in the second appendix.

6. Infinite Sets and Real Numbers

In this section, the structure of real numbers is built using the same principles of the construction of natural numbers. The methods are simply extended to the case of infinite sets. First of all, notice that any real number in the unit interval $(0, 1]$ can be given in negative powers of 2. For example, the number $\frac{1}{2} = 2^{-1}$ and $\frac{3}{4} = 2^{-1} + 2^{-2}$.

A second observation is made. Consider the energy level graph of a sum, as in Figure 1. Notice that a vertical displacement of the configuration of points, gives another true statement. What happens if a displacement is made into negative integers? The

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 August 2022 doi:10.20944/preprints202007.0415.v4

statement still holds true. See Figure 3. This is true because negative powers of two are still operated with the same rule. The expression $2^n + 2^n = 2^{n+1}$ holds for any integer n, not only positive integers. For example, to add the numbers $\frac{1}{2} + \frac{3}{4}$, the equality is $2^{-1} + 2^{-1} + 2^{-2} = 2^0 + 2^{-2} = 1 + \frac{1}{4}$. This is used in formalizing real numbers.

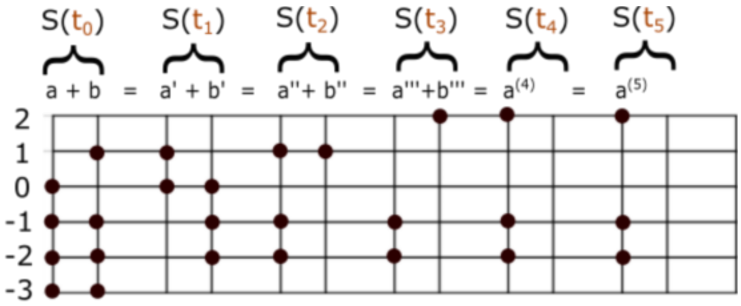

Figure 3. The energy level interpretation can be taken to negative levels. Particles occupying these levels represent negative powers of 2. In Figure 1 this represented $15 + 23 = 38$. Here, the statement is $1.875 + 2.875 = 4.75$.

Natural numbers are represented as hereditarily finite sets, and $\mathbb{N} =$ **HFS**. Axiom (A3) implies that any sub collection of $\mathbb{N}$, is a set. In particular, infinite sub collections of $\mathbb{N}$ are sets. In this section it will be proven that these sets are the real numbers. The section is divided in three main parts.

1. **Continuum** $[0,1]$**.** Any real number in the unit interval $(0,1]$ is the sum of infinitely many negative powers of 2. Moreover, every infinite set of natural numbers defines a unique real number in the unit interval.
2. **Real Numbers.** The constructions of $\mathbb{N}$ and $[0,1]$ are generalized to represent positive real numbers as infinite subsets of $\mathbb{Z}$. Then, the set of infinite subsets of $\mathbb{N}$ is given the structure of $\mathbb{R}$.
3. **Limits and Continuity.** The concept of limit and continuity has a simple description in terms of these constructions. An initial description of Analysis, in terms of $\mathbb{N}_<$, is provided.

6.1. Continuum $[0,1]$

A real number $x \in (0,1]$ can be expressed as a sum of negative powers of 2, so that $x = \sum_{i \in X} 2^{-i}$ for some set $X \subseteq \mathbb{N}$. The set X is the set number corresponding to x. The set number X can be a finite set (for some rational numbers). However, notice that a number 2^{-k} can be seen as an infinite sum $\left(\sum_{i=k+1}^{\infty} 2^{-i} \right)$. Thus, every $x \in (0,1]$ is represented by a unique infinite set of natural numbers. The symbol $\mathbb{N}_1 = \{1,2,3,\ldots\}$ is used for the set of natural numbers greater than 0. A bijection $\mathbb{N}_{inf} \to (0,1]$, where $\mathbb{N}_{inf}$ is the set of all infinite subsets of $\mathbb{N}_1$, will be given in this section. Infinite subsets of $\mathbb{N}$ are called *infinite set numbers* and they are ordered similarly to finite set numbers, but with one difference. The smaller powers of 2 represent larger numbers. For example, $2^{-5} < 2^{-1}$. Instead of using the maximum of the set difference, now it is the minimum. Therefore, $A < B$ if and only if $\min(A \triangle B) \in B$. Notice that $1 \in \mathbb{R}$ is the set number $\mathbb{N}_1$. Obviously, any two objects in $\mathbb{N}_{inf}$ are comparable in terms of this order relation, $<$, because the symmetric difference is non empty for set numbers $A \neq B$. Then, $\min(A \triangle B)$ exists because of the well order principle.

The order for the continuum has been defined in terms of the minimum of symmetric difference, and not in terms of addition as was the case with the order of natural numbers. This is because addition is not yet defined for infinite set numbers. It is trivial to verify the

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 August 2022 doi:10.20944/preprints202007.0415.v4

order is anti symmetric. The order is also transitive. Suppose $A < B$ and $B < C$. There is an object $b_1 \in B/A$ such that $b_1 < a$ for every $a \in A/B$. Also, there exists an object $c_0 \in C/B$ such that $c_0 < b$ for every $b \in B/C$. Suppose there exists $a_2 \in A/C$ such that $a_2 < c$ for every $c \in C/A$. Treat two cases and in each arrive at a contradiction, proving $A < C$.

1. Suppose $a_2 \in B$. But it is also true that $a_2 \notin C$. Then $c_0 < a_2$. Therefore, $c_0 \in A$. It is also true $c_0 \notin B$. This implies $b_1 < c_0$, which in turn means $b_1 \in C$. And, given that $b_1 \notin A$, it follows that $a_2 < b_1$. Use transitivity in $\mathbb{N}$ to find contradiction.
2. Suppose $a_2 \notin B$. This implies $b_1 < a_2$. It is true that $b_1 \notin A$, so that $b_1 \in C$ implies $a_2 < b_1$ which is a contradiction. It must be the case that $b_1 \notin C$. Then, $c_0 < b_1$. For $c_0 < a_2$ to be true, it must be true that $c_0 \in A$. But, this would imply $b_1 < c_0$, again a contradiction.

This proves the order on $\mathbb{N}_{inf}$ is transitive. The collection $\mathbb{N}_{inf}$ has been ordered isomorphic to $(0, 1]$. The real number 1 is the set $\mathbb{N}$. To include the real number 0, in the order, consider $\mathbb{N}^*_{inf} = \mathbb{N}_{inf} \cup \{\emptyset\}$. This is the collection whose objects are the infinite subsets of $\mathbb{N}$, plus the empty set. The order of $[0, 1]$ is given in terms of sets, where $0 = \emptyset$ and $1 = \mathbb{N}_1$, and every $x \in (0, 1]$ is an infinite set of natural numbers. The most important aspect in the order of a continuum is the supremum property. The supremum exists for this order. Let $\mathbf{X} \subseteq \mathbb{N}_{inf}$; every element of $\mathbf{X}$ is an infinite set of natural numbers. Define $x_1 = \min(\bigcup \mathbf{X})$ and $Y_1 = \{A \in \mathbf{X} | x_1 \in A\}$. Let

$$x_{n+1} = \min\Big(\bigcup Y_n - \{x_i\}_{i=1}^n\Big),$$

where $Y_n = \{A \in Y_{n-1} | x_n \in A\}$. The set number $\{x_i\}_i \in \mathbb{N}_{inf}$ is the supremum of X, by construction. This is shown in Figure 4.

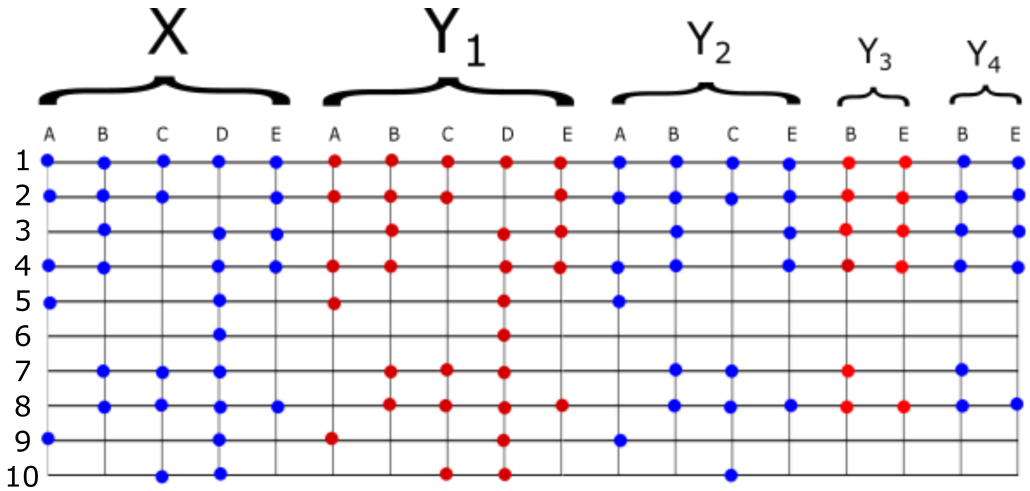

Figure 4. The iterations for finding the supremeum of the family $X = \{A, B, C, D, E\}$ is represented graphically. The elements of X are set numbers in the unit interval. For example, $A = 2^{-1} + 2^{-2} + 2^{-4} + 2^{-5} + 2^{-9} = 0.845703125$.

The next step, after defining the order for infinite set numbers, is to define their addition operation. Let $r = s^{-1}$; the inverse function of s. Recall, this function subtracts 1 one unit to the elements of the argument. Given two infinite set numbers $A = \{a_1, a_2, \ldots\}$ and $B = \{b_1, b_2, \ldots\}$, let $A_n = \{a_k\}_{k=1}^n$ and $B_n = \{b_k\}_{k=1}^n$ be the sets of the first n objects. Define

$$A_n \oplus B_n = (A_n \triangle B_n) \oplus r(A_n \cap B_n).$$

The addition $A \oplus B$ is the supremum of the finite sums,

$$A \oplus B = \sup_n (A_n \oplus B_n).$$

 doi:10.20944/preprints202007.0415.v4

6.2. Real Numbers

Based on the observation of Figure 3, the previous constructions of $\mathbb{N}$ and $[0,1]$ can be generalized, into the structure of positive real numbers, $\mathbb{R}_0^+$. But first, it must be proven that integers are sets. Take the integer $1 \in \mathbb{Z}$; it is the function $\oplus 1$. A finite function is a finite set of set numbers. Then, the function $\oplus 1$ is the object $\{\{1,4\},\{3,6\},\{5,8\},\{7,10\},\ldots\}$. Thus, the integer $1 \in \mathbb{Z}$ is an infinite set number and, in particular, a set. The integer $2 \in \mathbb{Z}$ is the infinite set $\{\{1,6\},\{3,8\},\{5,10\},\{7,12\},\ldots\} \in \mathbb{N}_{inf}^*$, etc. The negative integer $-1 \in \mathbb{Z}$ is the object $\{\{\{3,2\},\{5,4\},\{7,6\},\{9,8\},\ldots\}$. The negative integer $-2 \in \mathbb{Z}$ is the object $\{\{\{5,2\},\{7,4\},\{9,6\},\{11,8\},\ldots\}$, etc. That is to say, integers are a sub collection of the collection $\mathbb{N}_{inf}^*$. To prove the collection of integers is a set, it is sufficient to prove $\mathbb{N}_{inf}^*$ is a set, because of the axiom of subsets. Although the elements of $\mathbb{N}_{inf}^*$ are sets, it is not possible to go any further with the current collection of axioms. It cannot be proven $\mathbb{N}_{inf}^*$ is a set. The power set axiom is needed. **(A6)** If x is a set, then the subsets of x form a set $P(x)$. It trivially follows that $\mathbb{Z}$ is a set.

Let $\bar{\mathbb{Z}} \subset P(\mathbb{N}_{inf}^*)$ the set whose objects are subsets of $\mathbb{Z}$, that are bounded above, in terms of the order defined for integers (which is different than the order on infinite set numbers). Put differently, $A \in \bar{\mathbb{Z}}$ if and only if $A \subset \mathbb{Z}$ and $\max(A)$ exists, where $\max(A)$ is the maximum function in terms of the order of integers. The set A is a positive real number well represented by the sum of its integer powers of 2. The non negative integers represent the whole part of the real number, while the negative integers are the fractional part of the real number. This simply means $A \cap \mathbb{N}$ is the whole part of A, and $A \cap -\mathbb{N}$ is the fractional part. The set of all non negative real numbers is $\mathbb{R}_0^+ = \bar{\mathbb{Z}} \cup \{\emptyset\}$. Two positive real numbers are order related $A < B$ if and only if $\max(A \triangle B) \in B$. The addition is defined as before by $A \oplus B = (A \triangle B) \oplus s(A \cap B)$. The supremum can also be found in this structure of sets. At this point, negative real numbers can be built using the same technique used to build the negative integers. Every $A \in \mathbb{R}_0^+$ is identified with a function $\oplus A : \mathbb{R}_0^+ \to [A, \infty)$. The new set of positive real numbers is the set of bijections $\mathbb{R}_0^+ \to [A, \infty)$, for all $A \in \mathbb{R}_0^+$. Negative real numbers are the inverse functions of these. This construction will not be discussed further. A different approach will be taken.

An alternative method of building the set of real numbers, $\mathbb{R}$, which depends only on natural numbers is now discussed. A construction of the unit interval $(0,1]$ was given, where every number real number in the unit interval was represented as an object in $\mathbb{N}_{inf}^*$. Now, the set $\mathbb{N}_{inf}^*$ will be used to represent all of the real numbers.

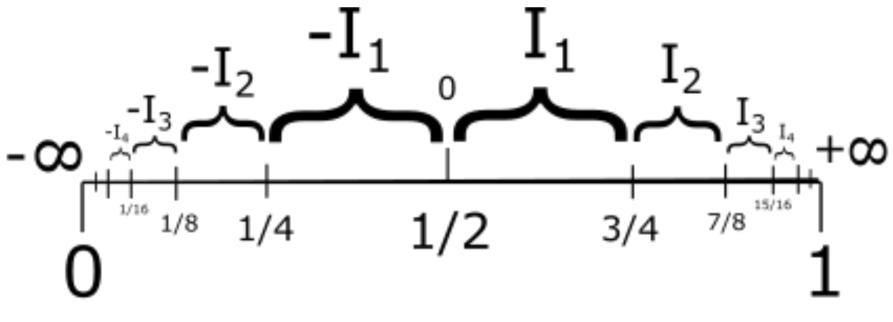

Figure 5. Use the fact that $(0,1]$ is bijective to any interval $(\frac{n}{2^k}, \frac{n+1}{2^k}]$. Under this representation, the real number $0 \in \mathbb{R}$ is the set $\{2,3,4,5,\ldots\}$. Infinities are the empty set $-\infty = \emptyset$, and the set of positive integers $+\infty = \mathbb{N}_1$

Each of the positive intervals $I_1 = (0,1]$, $I_2 = (1,2]$,..., and negative intervals $-I_1 = (-1,0]$, $-I_2 = (-2,-1]$,..., is isomorphic to the interval $(\frac{i}{2^k}, \frac{i+1}{2^k}]$, for any $i,k \in \mathbb{N}$. Real number intervals of unit length, I_i, will be compressed into smaller and smaller intervals so as to fit them all in the unit interval as in Figure 5. The interval $I_1 \subset \mathbb{R}$ is identified with the

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 August 2022 doi:10.20944/preprints202007.0415.v4

interval $(\frac{1}{2}, \frac{3}{4}] \subset \mathbb{N}_{inf}$. The interval $I_2 \subset \mathbb{R}$ is the interval $(\frac{3}{4}, \frac{7}{8}] \subset \mathbb{N}_{inf}$, etc. The negative interval $-I_1 \subset \mathbb{R}$ is the interval $(\frac{1}{4}, \frac{1}{2}] \subset \mathbb{N}_{inf}$, etc. Let $X \in (\frac{1}{2}, \frac{3}{4}]$, then it is an infinite set number such that $1 \in X$ and $2 \notin X$. A set number $X \in (\frac{3}{4}, \frac{7}{8}]$ is an infinite set number such that $1 \in X$ and $2 \in X$, but $3 \notin X$, etc. A set number $X \in (\frac{1}{4}, \frac{1}{2}]$ is an infinite set number such that $1 \notin X$ and $2 \in X$. A set number $X \in (\frac{1}{8}, \frac{1}{4}]$ is an infinite set number such that $1, 2 \notin X$ and $3 \in X$, etc. This is easily interpreted in defining the set of all real numbers. Let $X = \{1, 2, 3, \ldots, n, k_1, k_2, k_3, \ldots\} \in \mathbb{N}_{inf}$, where $3 \leq n+2 \leq k_1 < k_2 < k_3 < \cdots$, then X is positive real number. A negative real number is $X = \{n, k_1, k_2, k_3, \ldots\}$ with $3 \leq n+1 \leq k_1 < k_2 < k_3 < \cdots$. This simply means a positive real number with integer part equal to $n-1$ is an infinite set number X with $1, 2, \ldots, n \in X$ and $n+1 \notin X$. A negative real number with integer part equal to $-(n-2)$ is an infinite set number X with $\min(X) = n$. The fractional part will be given by the remaining objects $k_1, k_2, k_3 \ldots$. We can immediately differentiate a positive set number from a negative set number. For example, The set number $\{1, 2, 3, 4, 10, 11, 12, 13, \ldots\}$ is positive with integer part equal to 3. The set number $\{4, 5, 10, 11, 12, 13, \ldots\}$ is negative with integer part equal to -2. The set number $\{1, 2, 6, 7, 8, \ldots\}$ has integer part equal to 1, while $\{6, 8, 9, 10, \ldots\}$ has integer part -4.

How is the fractional part of a set number found, in this context? The first natural numbers are place holders for identifying the integer part. Let $X \in \mathbb{N}^*_{inf}$ an infinite set number. The objects $k_1, k_2, k_3, \ldots$ are representing the fractional part of X. The fractional part of X is $r^{n+2}(\{k_1, k_2, k_3, \ldots\})$. To store the information of a real number $X \in \mathbb{R}$ as an infinite set number use the first n natural numbers to determine the integer. But then there are still infinitely many natural numbers left to determine the fractional part. So all that has to be done is displace the fractional part $n+2$ places, so that the fractional part and integer part do not interfere. In the case of positive real numbers, one natural number is left out as a queue that the integer part ends there. Displacement up, $n+2$ times, is equivalent to applying s^{n+2}. Now, to recover the fractional part, displace the k_i's back $n+2$ times by applying r^{n+2}. For every $x \in \mathbb{N}^*_{inf}$, the numbers $\min(X)$ and $\min(X^c)$ are well defined, because of the well ordering principle. Exactly one of these two is equal to 1 and the other is larger than 1. A *positive real number* is an infinite set number with $min(X) = 1$. A *negative real number* is an infinite set number with $min(X^c) = 1$. More specifically, if $0 < X \leq 1$ then $\min(X^c) = 2$, and if $-1 < X \leq 0$ then $\min(X) = 2$. The equality $\min(X^c) = 3$ is equivalent to $1 < X \leq 2$, and $\min(x) = 3$ is equivalent to $-2 < X \leq -1$. If $2 < X \leq 3$ then we have $\min(X^c) = 4$, and if $-3 < X \leq -2$ then we have $\min(X) = 4$. In general, $X \in (n-1, n]$ if and only if $\min(X^c) = n+1$, and $X \in (-(n-1), -(n-2)]$ if and only if $\min(X) = n$.

For example, the fractional part of π is equal to $.141596\ldots = 2^{-3} + 2^{-6} + 2^{-11} + 2^{-12} + 2^{-13} + 2^{-14} + \cdots$ given by the set $\{3, 6, 11, 12, 13, 14\ldots\}$ Therefore, the numbers π and $-\pi$ are represented by

$$\begin{aligned} \pi &= \{1, 2, 3, 4, 3+5, 6+5, 11+5, 12+5, 13+5, 14+5, \ldots\} \\ &= \{1, 2, 3, 4, 8, 11, 16, 17, 18, 19\ldots\} \\ -\pi &= \{5, 3+5, 6+5, 11+5, 12+5, 13+5, 14+5\ldots\} \\ &= \{5, 8, 11, 16, 17, 18, 19\ldots\} \end{aligned}$$

The set of infinite set numbers $\mathbb{N}^*_{inf}$ is defined as $\mathbb{R}$. Real numbers and natural numbers are different types of sets. Natural numbers are finite sets, of finite sets. Given that **HFS** is the set of finite sets, it can be said *"A natural number is a finite subset of **HFS**"*. On the other hand, *"A real number is an infinite subset of **HFS**"*. Adequate definitions can be made for addition of real numbers combining addition of natural numbers and addition of infinite set numbers defined for the continuum $(0, 1]$.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 August 2022 doi:10.20944/preprints202007.0415.v4

6.3. Limits and Continuity

Suitable and practical expressions of the concepts of analysis are proposed. The first concept formalized is *limit point*. Let $P, X \in \mathbb{N}^*_{inf}$ two infinite set numbers. Intuitively, these two objects are close, if their first terms coincide. Take as an example the set numbers $P = \{2,4,5,8,9,10,11,12,13,\ldots\} = 2^{-2}+2^{-4}+2^{-5}+2^{-8}+2^{-9}+2^{-10}+\ldots$ and $X = \{2,4,5,8,9,14,15,16,17\ldots\} = 2^{-2}+2^{-4}+2^{-5}+2^{-8}+2^{-9}+2^{-14}+\ldots$. They are relatively close because the first terms (the largest terms) coincide. The first elements coinciding is equivalent to the two set numbers being "close". Another way of saying this is that $\min(P\triangle X)$ is a large number. The larger $\min(P\triangle X)$, the larger the elements of $P\triangle X$ become, making the smaller powers (larger terms) coincide. In the part, larger natural numbers represent the smaller terms of the real number.

Let $P \in \mathbb{R}$ an infinite set number, and let $\mathbf{X}$ a set of infinite set numbers. Then, P *is a limit point of* $\mathbf{X}$ if there exists $X_N \in \mathbf{X}$ such that $\min(P\triangle X_N) > N$, for every $N \in \mathbb{N}$. There is one exception to this definition. The fractional part of some real numbers can be expressed as sum of finite many negative powers of 2. The definition of limit point for these numbers is defined separately. Suppose $P \in \mathbb{N}^*_{inf}$ is an infinite set number that has finite representation $P = \{p_1, p_2, \ldots, p_k\}$, where $p_1 < p_2 < \cdots < p_k$. That is to say, $P = \{p_1, p_2, \ldots, p_{k-1}, p_k+1, p_k+2, p_k+3, \ldots\}$. The last term 2^{-p_k} replaces the terms $2^{-(p_k+1)}+2^{-(p_k+2)}+2^{-(p_k+3)}+\ldots$. A set of infinite set numbers is provided such that theses numbers become arbitrarily close to P, from above. Let $X_1 = \{p_1, p_2, \ldots, p_k, p_k+1\}$, and $X_2 = \{p_1, p_2, \ldots, p_k, p_k+2\}$. In general define $X_i = \{p_1, p_2, \ldots, p_k, p_k+i\}$. These set numbers X_i are getting closer to P but the previous definition of limit point is not satisfied. The minimum element of the symmetric difference is not getting larger. In fact, it is constant the $\min(P\triangle X_i) = p_k$. Therefore, a different definition is required for this case. Let P an infinite set number with finite representation. The number P is a limit point of $\mathbf{X}$ if for every $N \in \mathbb{N}$ there exists $X_N \in \mathbf{X}$ such that $X_N = \{p_1, p_2, \ldots, p_k, p_k+N, p_{n_1}, p_{n_2}, \ldots\}$, where $p_k + N < p_{n_1} < p_{n_2} < \cdots$. This in turn makes the difference bounded, $|P - X_N| \leq \frac{1}{2^{p_k+N-1}} = \{p_k + N - 1\}$. Let P, X two infinite set numbers with their integer parts equal and suppose their fractional parts coincide in the first elements,

$$\begin{aligned} P &= 2^{m_1}+2^{m_2}+\ldots+2^{m_k}+2^{n_1}+2^{n_2}+2^{n_3}+\ldots+2^{n}+2^{\alpha_1}+2^{\alpha_2}+2^{\alpha_3}+\ldots \\ X &= 2^{m_1}+2^{m_2}+\ldots+2^{m_k}+2^{n_1}+2^{n_2}+2^{n_3}+\ldots+2^{n}+2^{\beta_1}+2^{\beta_2}+2^{\beta_3}+\ldots \end{aligned}$$

where $m_1 < m_2 < \ldots < m_k < n_1 < n_2 < \ldots < n < \alpha_1 < \alpha_2 < \cdots$ and $n < \beta_1 < \beta_2 < \cdots$. The numbers m_i determine the integer part and n_i, n are the elements that coincide in the fractional part (the first negative powers of 2 that coincide). Then, the difference $|P - X| < \frac{1}{2^n}$ is bounded, by $\{n\}$.

Infinite set numbers with finite representations can be handled in another, informal, manner. For simplicity, consider set numbers of $(0,1] \subseteq \mathbb{N}_{inf}$. For example, $1/2 \in [0,1]$ has the representations $\{1\} = \{2,3,4,\ldots\}$. The set number $P = 1/2$ should be a limit point of the set $\mathbf{X} = \{A_1, A_2, A_3, A_4, \ldots\}$ where the A_i are

$$\begin{aligned} A_1 &= 1 &&= \{1,2,3,4,5,6,\ldots\} \\ A_2 &= 3/4 &&= \{1,3,4,5,6,7,\ldots\} \\ A_3 &= 5/8 &&= \{1,4,5,6,7,8,\ldots\} \\ A_4 &= 9/16 &&= \{1,5,6,7,8,9,\ldots\} \\ &\vdots &&\vdots \end{aligned}$$

If $P = \{2,3,4,5,\ldots\}$ then $\min(P\triangle A_i) = 1$, for every A_i. Using the finite representation $P = \{1\}$, gives the symmetric differences: $P\triangle A_1 = \{2,3,4,\ldots\}$, $P\triangle A_2 = \{3,4,5,\ldots\}$, $P\triangle A_3 = \{4,5,6,\ldots\}$, $P\triangle A_4 = \{5,6,7,\ldots\}$,... In effect, satisfying the condition that for every $N \in \mathbb{N}$ there exists $X_N \in \mathbf{X}$ such that $\min(P\triangle X_N) > N$.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 August 2022 doi:10.20944/preprints202007.0415.v4

If P has finite representation and the set numbers A_i tend to P, from below, the same problem cannot occur. For example, take $P = 1/2 = \{2,3,4,5,\ldots\}$ and the set $\mathbf{X} = \{A_1, A_2, \ldots\}$ defined by

$$\begin{aligned} A_1 &= 3/8 &&= \{2,4,5,6,7,8\ldots\} \\ A_2 &= 7/16 &&= \{2,3,5,6,7,8\ldots\} \\ A_3 &= 15/32 &&= \{2,3,4,6,7,8\ldots\} \\ A_4 &= 31/64 &&= \{2,3,4,5,7,8\ldots\} \\ A_5 &= 63/128 &&= \{2,3,4,5,6,8\ldots\} \\ \vdots &\quad \vdots &&\quad\ \vdots \end{aligned}$$

It is easily verified that for every $N \in \mathbb{N}$ there exists X_N such that $\min(P \triangle X_N) > N$. The symmetric differences are $P \triangle A_1 = \{3\}$, $P \triangle A_2 = \{4\}$, $P \triangle A_3 = \{5\}$, $P \triangle A_4 = \{6\}$, $P \triangle A_5 = \{7\}, \ldots$.

Continuity is described in terms of the order of natural orders. In the next section a formal definition for real function is provided. It is used provisionally, for the sake of illustration.

Definition 11. *Let $f : \boldsymbol{A} \subseteq \mathbb{R} \to \boldsymbol{B} \subseteq \mathbb{R}$ a real function, and let p a limit point of the domain $\boldsymbol{A}$. The function f has limit point p, and the limit is equal to q, if and only if for every $N \in \mathbb{N}$ there exists $M \in \mathbb{N}$ such that $\min(p \triangle x) > M$ implies $\min(f(p) \triangle q) > N$.*

The function is continuous in p if and only if for every $N \in \mathbb{N}$ there exists $M \in \mathbb{N}$ such that $\min(p \triangle x) > M$ implies $\min(f(p) \triangle f(x)) > N$.

The theory of convergence and topological aspects of $\mathbb{R}$ are expressed directly in terms of the order of natural numbers. Using these general indications and the subtraction algorithm, given in [I], it is possible to define the derivative. The derivative can be treated in two ways. The subtraction algorithm allows for the traditional definition of derivative, for finding the numerical value $f'(p)$. However, to prove the *existence* of the derivative, there is an alternative definition of a *discrete derivative*. The quotient of two powers of 2 is obtained by subtracting the powers, $\frac{2^n}{2^m} = 2^{n-m}$. For a fixed $x \in A$ in the domain of f, consider the difference $D_x = \min(f(p) \triangle f(x)) - \min(p \triangle x)$. If the derivative, at p, is $f'(p) = 1$ it will have to be true that D_x tends to 0 as x tends to p. If the function has derivative $f'(p) = 0$ it must be true D_x is unbounded as x tends to p. If, instead, D_x tends to a positive integer, as x tends to p, then the derivative satisfies the inequality $0 < f'(p) < 1$. The last case, when D_x tends to a negative number, as x tends to p, means the derivative is $f'(p) > 1$, greater than 1. These last calculations are approximations to the absolute value of the derivative.

The discrete derivative is a criteria for the existence and absolute value of the magnitude of the derivative. In exchange, for not knowing the exact numerical value of the derivative, we can say that finding the discrete derivative is computationally much faster. We are substituting the quotient $\frac{fp-fx}{p-x}$ of floating point numbers, with finding the difference of natural numbers, $|\min(fp \triangle fx) - \min(p \triangle x)|$. The end result is that instead of having to calculate two subtractions and one division of real numbers, we find the minimum element for two sets of natural numbers and the difference of these natural numbers. The proposed set theory has clear advantages in its simplicity of description of objects and efficient algorithms. That is to say, it stands above other set theories in theoretical and practical terms. Alternative constructions of real numbers are found in modern references [6],[7],[8],[9]. These involve complex objects and cumbersome methods for proving the existence of the real number system.

7. Type Theory and Trees

In this section an account of representing general objects of modern mathematics is given. This will be a superficial description but enough is illustrated to be clear on the

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 August 2022 doi:10.20944/preprints202007.0415.v4

extent of constructions possible. This universe of sets can be well represented in terms of trees. A brief description of the theory of types induced by this axiomatic base is also outlined. A consistent hierarchy of types and universe is proposed.

7.1. Basic Objects In Mathematics

Ordered pairs, and finite sets of ordered pairs, are natural numbers. To define an ordered n-tuple of natural numbers, recall that even and odd natural numbers were used to tell apart the first component from the second. One might initially want to solve in the following manner. To well represent ordered 3-tuples use $\{1, 4, 7, 10, \ldots, 3k-2, \ldots\}$ to represent the first component, then use $\{2, 5, 8, 11, \ldots, 3k-1, \ldots\}$ to represent the second component, and multiples of three, $\{3, 6, 9, 12, \ldots, 3k, \ldots\}$ to represent the third component. This will give a table similar to (7), of ordered pairs. Table 6 allows to describe ordered 3-tuples.

X	$3k-2$	$3k-1$	$3k$
0	1	2	3
1	4	5	6
2	7	8	9
3	10	11	12
4	13	14	15
5	16	17	18
6	19	20	21
⋮	⋮	⋮	⋮

Table 6. The elements of this table allow us to represent an ordered 3-tuple as a natural number.

The ordered 3-tuple $(0, 0, 0)$ is the set number $2^1 + 2^2 + 2^3 = \{1, 2, 3\}$. Also, $(1, 2, 3)$ is equal to $2^4 + 2^8 + 2^{12} = \{4, 8, 12\}$. To represent 4-tuples, a new Table, 7, is needed.

X	$4k-3$	$4k-2$	$4k-1$	$4k$
0	1	2	3	4
1	5	6	7	8
2	9	10	11	12
3	13	14	15	16
4	17	18	19	20
5	21	22	23	24
6	25	26	27	28
⋮	⋮	⋮	⋮	⋮

Table 7. The elements of this table allows to represent an ordered 4-tuple as a natural number.

This manner of defining finite sequences has two big disadvantages that will become clear, when defining a second method for representing ordered n-tuples. The first is quite obvious: it is not possible to define an infinite sequence of natural numbers. The easiest way to solve this is by going back to the definition of ordered pairs. The sets given in (7) are of great importance in the constructions of this section. It is shown again, for reference in Table 8. Here it is used differently. Only the first two rows are needed to define ordered pairs; it will only be necessary to use the first two sets $(0,)$ and $(1,)$. The pair (i, j) will be a set of two numbers; its elements will be the $i+1$-th object of the first row and the $j+1$-th object of the second row.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 August 2022 doi:10.20944/preprints202007.0415.v4

6	18	66	258	1026	...	$2+2^{2(n+1)}$	...
12	24	72	264	1032	...	$8+2^{2(n+1)}$	...
36	48	96	288	1056	...	$32+2^{2(n+1)}$	...
128	144	192	382	1152	...	$128+2^{2(n+1)}$	...
516	528	576	768	1536	...	$512+2^{2(n+1)}$	...
⋮	⋮	⋮	⋮	⋮		⋮	
$2^{2m+1}+4$	$2^{2m+1}+16$	$2^{2m+1}+64$	$2^{2m+1}+256$	$2^{2m+1}+1024$	...	$2^{2m+1}+2^{2(n+1)}$	...
⋮	⋮	⋮	⋮	⋮		⋮	

Table 8. The elements of this table allow to represent an ordered pair as a natural number. The elements of the first row are used to represent the first component, while the elements of the second row are used to represent the second component. This table is also used for finding a good representation of sequences of real numbers.

A definition of ordered pair is given, that supersedes the one given before. An ordered pair is a set number (i,j) where $i \in (0,)$ and $j \in (1,)$. Specifically, the ordered pair (i,j), is the set number $\{2^1+2^{2(i+1)}, 2^3+2^{2(j+1)}\}$. The $i+1$-st element of $(0,)$ is included to show that i is in the first component. Include the $j+1$-st element of $(1,)$ to indicate j is in the second component. For example, the ordered pair $(0,0)$ is the set number $2^6+2^{12}=\{6,12\}$. The ordered pair $(0,1)$ is $2^6+2^{24}=\{6,24\}$.

Data types are being defined for different mathematical objects. Different kinds of mathematical objects and relations can be represented as natural and real numbers. An extension of this new representation allows to define *infinite sequence of natural numbers*. Select one element, n_k, from the set $(k,)$, for every $k \in \mathbb{N}$. Then, $2^1+2^{2(n_1+1)} \in S$ means n_1 is the first natural number of the sequence. The second number is given by $2^3+2^{2(n_2+1)} \in S$, and so on. The set number $\{2^1+2^{2(n_1+1)}, 2^3+2^{2(n_2+1)}, 2^5+2^{2(n_3+1)}, \ldots\}$ represents the sequence $(n_1, n_2, n_3, \ldots)$. For example, the sequence $(1,3,2,5,4,\ldots)$ is given by

$$\{2+2^{2(1+1)}, 8+2^{2(3+1)}, 32+2^{2(2+1)}, 128+2^{2(5+1)}, 512+2^{2(4+1)}, \ldots\} = \{18, 264, 96, 4224, 1536, \ldots\}. \quad (12)$$

Of course, to define a finite sequence, a k-tuple, use the first k sets, $(1,), (2,), \ldots (k,)$. A finite sequence of natural numbers is a set number of the form

$$\{2^1+2^{2(n_1+1)}, 2^3+2^{2(n_2+1)}, 2^5+2^{2(n_3+1)}, \ldots, 2^{2k+1}+2^{2(n_k+1)}\}.$$

A *natural function*, $\mathbb{N} \to \mathbb{N}$, is an infinite set of ordered pairs. A function of this form is a set number

$$\{\{6, B_1\}, \{18, B_2\}, \{66, B_3\}, \{258, B_4\}, \ldots\}$$

where B_i are elements of $(1,)$. If the B_i are all distinct, the function is an injection. If every element of $(1,)$ is a B_i the function is onto $\mathbb{N}$. This represents natural functions as real numbers. There exists a bijective function from the set of all natural functions, onto a proper subset of real numbers.

How can a *sequence of real numbers* be represented? The same question stated differently, How can a (finite or infinite) sequence of infinite set numbers be well defined? It would be advantageous to find a way of storing and rescuing the information that determines a sequence $\xi = (r_1, r_2, r_3, \ldots)$ where each $r_i = \{n_1^i, n_2^i, n_3^i, \ldots\}$ is a real number. Use the set $(0,)$ to represent the elements of r_1. Use the set $(1,)$ to represent the elements of r_2, etc. Then, $2^{2(i)+1}+2^{2(n_j^i+1)} \in \xi$ if and only if $n_j^i \in r_i$. The infinite sequence of real numbers, $(r_1, r_2, \ldots)$, is represented by the real number $\bigcup_i r_i$. The union of all the r_i's is a real number that represents the infinite sequence $(r_1, r_2, \ldots)$; it is an infinite set number with infinitely many objects from each set $(i,)$. Actually, any set number with infinitely many elements of each $(i,)$ is representing a unique sequence of real numbers.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 August 2022 doi:10.20944/preprints202007.0415.v4

If we have an infinite set $X \subset (0,)$ this determines a real number. The set $X = \{2 + 2^{2(x_1+1)}, 2 + 2^{2(x_2+1)}, 2 + 2^{2(x_3+1)}, \ldots\}$ determines the real number $X^* = \{x_1, x_2, x_3, \ldots\}$. In the same manner, $Y = \{8 + 2^{2(y_1+1)}, 8 + 2^{2(y_2+1)}, 8 + 2^{2(y_3+1)}, \ldots\} \subset (1,)$ determines the real number $Y^* = \{y_1, y_2, y_3, \ldots\}$. The infinite set number $X \cup Y$ is a real number, whose objects are in $(0,) \cup (1,)$, and the objects of $(0,)$ are distinguishable from the objects of $(1,)$. The objects in $(0,)$ give the first component, and the second component is given by the elements of $(1,)$. This provides a good representation of the ordered pair of real numbers, (X^*, Y^*), as a single real number $X \cup Y$. To represent ordered 3-tuples of real numbers, use the set $(2,)$, also. Let $Z^* = \{z_1, z_2, z_3, \ldots\} \subset \mathbb{N}$ a real number, then $Z = \{32 + 2^{2(z_1+1)}, 32 + 2^{2(z_2+1)}, 32 + 2^{2(z_3+1)}, \ldots\} \subset (2,)$. And, the ordered 3-tuple (X^*, Y^*, Z^*) is the real number $X \cup Y \cup Z$. An infinite sequence of real numbers $x_1, x_2, \ldots$ is represented by a single real number. A bijective function from the set of all real sequences onto a proper subset of real numbers has been described. A sequence of real numbers is well represented by a single real number. And, it has also been shown that a function $\mathbb{N} \to \mathbb{N}$ is well represented by a real number. Consequently, a *sequence* $(f_1, f_2, \ldots)$, *of functions* $f_i : \mathbb{N} \to \mathbb{N}$, can be represented as a single real number. In summary, a second definition for ordered pairs is given, that is a more powerful definition than the first because it allows to represent an infinite sequence of natural numbers, as a real number. Moreover, if ξ is a countable sequence of real numbers, it is also represented as a real number. Therefore, a good representation of functions $\mathbb{N} \to \mathbb{N}$, and sequences of these functions, is obtained.

A representation of sequences of sequences can also be described. Consider first the simplest kind, a *sequence* $T = (S_1, S_2, \ldots)$ *of sequences,* S_i*, of natural numbers*. Use subsets of $(i,)$, to find these representations. A subset of $(0,)$ is used to represent the first sequence $S_1 = (n_1^1, n_2^1, n_3^1, \ldots)$,

$$2 + 2^{2^{\left(\left(2+2^{2\left(n_1^1+1\right)}\right)+1\right)}}, 2 + 2^{2^{\left(\left(2+2^{2\left(n_2^1+1\right)}\right)+1\right)}}, \ldots \in T$$

for every $i = 1, 2, 3, \ldots$. Use a subset of $(1,)$ to represent the second sequence. If $S_2 = (n_1^2, n_2^2, n_3^2, \ldots)$ is the second sequence, then

$$8 + 2^{2^{\left(\left(2+2^{2\left(n_1^2+1\right)}\right)+1\right)}}, 8 + 2^{2^{\left(\left(2+2^{2\left(n_2^2+1\right)}\right)+1\right)}}, \ldots \in T$$

for every $i = 1, 2, 3 \ldots$. The third term is

$$32 + 2^{2^{\left(\left(2+2^{2\left(n_1^3+1\right)}\right)+1\right)}}, 32 + 2^{2^{\left(\left(2+2^{2\left(n_2^3+1\right)}\right)+1\right)}}, \ldots \in T,$$

etc. In general, the set $(k,)$ is used to represent the sequence S_k, for every $k \in \mathbb{N}$. The sequence of sequences can be reconstructed from the real number

$$\begin{aligned} T = {} & 2^{2+2^{2^{\left(\left(2+2^{2\left(n_1^1+1\right)}\right)+1\right)}}} + 2^{2+2^{2^{\left(\left(2+2^{2\left(n_2^1+1\right)}\right)+1\right)}}} + \ldots + 2^{8+2^{2^{\left(\left(8+2^{2\left(n_1^2+1\right)}\right)+1\right)}}} + 2^{8+2^{2^{\left(\left(2+2^{2\left(n_2^2+1\right)}\right)+1\right)}}} + \ldots + \\ & + 2^{32+2^{2^{\left(\left(2+2^{2\left(n_1^3+1\right)}\right)+1\right)}}} + 2^{32+2^{2^{\left(\left(2+2^{2\left(n_2^3+1\right)}\right)+1\right)}}} + \ldots + \ldots + \ldots\ldots \end{aligned}$$

Now, let $\xi_i = (r_1^i, r_2^i, r_3^i, \ldots)$ a sequence of real numbers, for every $i \in \mathbb{N}$, and let $\Xi = (\xi_1, \xi_2, \xi_3, \ldots)$ the sequence of those. It is easy to construct a real number representing this object. This is true because every sequence ξ_i, of real numbers, is represented by a real number. The sequence of real numbers, Ξ, can in turn be reduced to a single real number. A real matrix of infinitely (countable) many columns and rows can be represented by a single real number.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 August 2022 doi:10.20944/preprints202007.0415.v4

There are similarities between natural numbers and real numbers. A finite natural function is a finite set of natural numbers. An infinite natural function is a set of infinitely many natural numbers. Similarly, a real function will be represented by a set of real numbers. An ordered pair of real numbers has been defined, now a *real function* can be defined. Let us proceed with this construction. A function is a collection of components $f_x = (x, fx)$, and every ordered pair of real numbers $f_x \in \mathbb{R}$ is a real number. Therefore, the function $f : \mathbb{R} \to \mathbb{R}$ can be represented by a set of real numbers $\{f_x\}_{x\in\mathbb{R}}$. Every real function $\mathbb{R} \to \mathbb{R}$ is an uncountable set of real numbers

$$f = \{\{a_1^x, a_2^x, \ldots, b_1^x, b_2^x, \ldots\}\}_{x\in\mathbb{R}},$$

where $x = \{a_i^x\}_i \subset (0,)$ and $f(x) = \{b_i^x\}_i \subset (1,)$. This means $f_x = x \cup f(x) = \{a_1^x, a_2^x, \ldots, b_1^x, b_2^x, \ldots\}$. The function is injective if $f(x) \triangle f(y) \neq \emptyset$ for $x \neq y$. The function f is onto $\mathbb{R}$ if for every infinite subset $A \subset (1,)$, there exists an object $x \in \mathbb{R}$ such that $A = f_x \cap (1,)$. A real function is bijective if for every infinite subset $A \subset (1,)$ there exists exactly one $x \in \mathbb{R}$ such that $A = f_x \cap (1,)$.

We can extend our results to represent any sequence of real functions, $(f_1, f_2, \ldots)$, as a set of real numbers. Just as $(0,)$ and $(1,)$ are used to define a function $f_1 : \mathbb{R} \to \mathbb{R}$, the set $(2,)$ and $(3,)$ can be used to define a function $f_2 : \mathbb{R} \to \mathbb{R}$. In the same way $(4,)$ and $(5,)$ are used to define a function $f_3 : \mathbb{R} \to \mathbb{R}$, etc.

There is another consequence of representing a real function as a set of real numbers. Given that any function $\mathbb{R} \to \mathbb{R}$ is a subset of $\mathbb{R}$, any function $\mathbb{R} \to (\mathbb{R} \to \mathbb{R})$ can be represented as a set of real numbers. For any finite amount of iterations, an object $\mathbb{R} \to (\mathbb{R} \to (\mathbb{R} \to (\mathbb{R} \to \cdots (\mathbb{R} \to \mathbb{R}) \cdots)))$ is a set of real numbers. In the next subsection, a type theory is described using trees.

7.2. Trees

It has been shown that natural numbers are the finite sets that can be built recursively with the function $\oplus 1$. These sets can be well represented by finite tree structures. Trees are used to represent natural numbers first, then all types of objects. The concept of set is equivalent to the concept of tree. A finite set number is a set of finitely many smaller set numbers. The definition of trees is equivalent. A tree is a *trunk* (the principle node); the set X. Every branch of the trunk is an element of the set X. For example, a single trunk with no branches is the set number 0. Suppose the tree of X is known. How is the tree corresponding to $X \oplus 1$ found? Add a branch that is a 0-tree (add 1 unit). The set number 1 is a trunk with one 0-branch; $1 = \{0\}$. This in illustrated in Figure 6.

A tree is a graph of nodes and edges such that (i) A *trunk* can be identified: a principle edge with a finite number of *branches* attached to one of its nodes. All branches are attached to the same node of the trunk. (ii) Each branch on the tree is a tree. (iii) A single edge is a tree; the 0-tree. The successor of a tree is obtained by adding a single edge to the trunk; attach a 0-tree to the trunk. Adding an edge to the 0-tree gives its successor, the 1-tree, which is two edges joined together at one node. Adding an edge to the 1-tree, yields its successor, the 2-tree, etc.

An extra rule for defining an equivalence class on finite trees is needed. If a tree has two identical branches, substitute these two identical branches with a single branch, the successor. This process is called *reduction*. If a tree can be reduced to obtain another tree, they are in the same equivalence class. An irreducible tree is said to be in canonical form. Reducing the 2-tree, gives the canonical form. To reduce the 2-tree, substitute the two identical 0-trees with a single 1-tree. Adding a single edge to the result of that, results in the canonical form of the 3-tree because it contains no identical branches. If an edge is added to the 3-tree, reduction of branches will have to be applied two times before reaching the canonical form of the 4-tree. First take away the two 0-trees and add a 1-tree. But, there is already another 1-tree; there are two identical 1-trees. Replace those trees with a single 2-tree. Every natural number is associated an equivalence class of finite trees, and a single canonical tree. Every branch on the canonical tree of a set number X corresponds to a

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 August 2022 doi:10.20944/preprints202007.0415.v4

natural number $k \in X$. Every tree is made up of smaller trees, and a well defined method of building trees is provided. The canonical tree associated to the set number X, has $\#(X)$ many branches. Each branch is defined in the same way. A natural number is defined by its cardinality; and the cardinality of its elements; and the cardinality of the elements of its' elements; etc. Trees are used to represent real numbers, also. Consider trees with infinitely many branches. Each branch must be a finite tree and they are not allowed to be repeated. If two branches are identical, reduce the tree. Consider a third kind of tree, with infinitely many branches. But, each of these branches is a tree of infinite branches. The object just described is a collection of real numbers. The next sub section is a formalization of the concept of types.

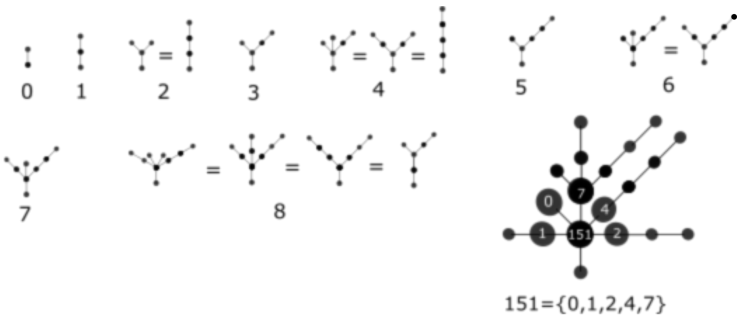

Figure 6. Canonical trees can be built easily, given a set number. The tree representation of $6 = \{1, 2\}$ is a tree with two branches; a 1-tree and a 2-tree. The canonical tree for $7 = \{0, 1, 2\}$ has three branches. One branch is the 0-tree, the second branch is the 1-tree and the third branch is the 2-tree. The canonical tree of $8 = \{3\}$ is a trunk with one branch, which is the 3-tree. The canonical tree of $151 = \{0, 1, 2, 4, 7\}$ has five branches: $0, 1, 2, 4, 7$-trees.

7.3. Type Theory

Finite trees are *objects of Type-0*. Trees of infinite branches with each branch being an object of type-0 are called *objects of Type-1*. For example, a natural number is an object of Type-0 and a real number is an object of Type-1. A tree whose branches are all objects of Type-1 is an *object of Type-2*. An example of an object of Type-2 is a set of real numbers. Use the Replacement Axiom to build a tree with branches of Type-0 and Type-1; an *object of Type-3*. A set consisting of natural and real numbers is an object of Type-3. The power set of a Type-2 object gives an object of Type-4 which is a tree whose branches are Type-2 objects. A set of sets of real numbers is an *object of Type-4*. For example, the construction of $\mathbb{Z}$ made it an object of Type-2, while $\bar{\mathbb{Z}}$ is of Type-4. Define Type-n objects in a manner analogous to the definition of natural numbers. The power set axiom is required for the existence of $P(\mathbb{N}), P(P(\mathbb{N})), P(P(P(\mathbb{N}))), P(P(P(P(\mathbb{N})))), \ldots, A, P(A), \ldots$ which are sets of type $2, 4, 16, 2^{16}, \ldots, n, 2^n, \ldots$, respectively. A Type-8 object is a tree whose branches are Type-3 objects. The power set of a Type-4 object is a Type-16 object, etc. Other types are found using the replacement axiom to combine subsets of these power sets. For example, a Type-7 object consists of objects of three different types, $0, 1, 2$. In Section 7 sequences of real numbers are represented using Type-1 objects, and real functions are represented using Type-2 objects.

The next step in classifying types is to consider trees with infinite many types of branches; trees with branches of Type-n_1, Type-n_2, Type-$n_3 \ldots$ for infinite many types. This is called an *object of infinite Type-1,0*. An *object of infinite Type-1,1* is a tree that only has branches of infinite Type-1,0. An *object of Type-1,2* is a tree that has only branches of Type-1,1. A tree with branches of both Type-1,0 and infinite Type-1,1 is an *object of infinite Type-1,3*. If all the branches of a tree are objects of Type-1,2, it is an *object of infinite Type-1,4*. All infinite Types-1, k are constructed in a manner analogous to natural numbers.

 doi:10.20944/preprints202007.0415.v4

Consider a tree whose branches are all objects of infinite type-1, k, and suppose there are infinite many types of objects of infinite type; the branches are objects of Type-1, n_1, Type-1, n_2, Type-1, $n_3, \ldots$ for infinitely many infinite types. A tree built with objects of infinite many infinite-types, is an object of infinite Type-2, 0. Trees whose objects are only objects of Type-2, 0 are called *objects of Type*-2, 1. A tree whose objects are all of Type-2, 1 is an *object of Type*-2, 2. A tree with objects of Type-2, 0 and Type-2, 1 is an *object of Type*-2, 3, etc.

An object of infinite Type-3, 0 is a tree that has branches of finite type and infinite Type-1, k. A tree with objects of Type-3,0 is an object of Type-3,1, and so on. An object of infinite Type-4,0 is a tree with infinite many types of objects of Type-2, k. An object of Type-4, 0 has objects of Type-2, n_1, Type-2, n_2, Type-2, $n_3 \ldots$ for infinite many Type-2, k objects. An object of Type-4, 1 is a tree with branches of Type-4, 0, etc. An infinite Type-5,0 object consists of branches of finite type and infinite types 2, k. A Type-6, 0 object consists of objects of types 1, k and types 2, k, etc. Continue in this manner until all objects of Type-m, n, for every $m, n \in \mathbb{N}$, have been described. Higher hierarchy types are left for future analysis.

8. Conclusions

The importance of the axiomatic base is usually undermined because it does not bring any new results or methods into most practical areas of mathematics. Instead, the axiomatic base of mathematics is seen as a *stone in the path*; an obstacle to be dealt with and forgotten. The natural number system provided here differs from others in the fact that natural constructions for classic structures of mathematics are obtained, allowing for a natural description of finite structures. Finite groups are described using natural numbers. Finding all finite groups of n objects is still not trivial but a better notion of attacking this problem is acquired. In the process of finding all groups, with $|G| = n$, a minimum set of independent equations that defines each group is given. To prove isomorphism of two groups the canonical representation of both groups has to be the same natural number. This happens if and only if the canonical numeric table of the groups is identical. We are provided a linear order for the set of all finite groups, that is isomorphic to $\mathbb{N}$. This linear order on finite groups is well behaved with respect to cardinality. In particular, the commutative group $\mathbb{Z}_n$ is the smallest group of n objects; $\mathbb{Z}_n < G$ for every group G such that $|G| = n$. If $n = p_1^{n_1} p_2^{n_2} p_3^{n_3} \cdots p_k^{n_k}$ is the prime factorization of n, then the commutative group $\mathbb{Z}_{p_1}^{n_1} \oplus \mathbb{Z}_{p_2}^{n_2} \oplus \mathbb{Z}_{p_3}^{n_3} \oplus \cdots \oplus \mathbb{Z}_{p_k}^{n_k}$ is the largest commutative group of n objects. This last behavior was not treated with detail, and is left for future work. Finite groups are also ordered internally. The elements of any finite group are ordered through the canonical naming functions. A criteria for defining equivalent objects of a fixed finite group is obtained, that provides the automorphisms of the group. The set theory for natural numbers was extended to describe infinite mathematical objects such as real numbers, real functions, real valued matrices, sets of real numbers, and structures derived from those, etc. More results can be pursued in future work. This can include a thorough description of groups, rings, fields and linear spaces, in the finite and infinite cases. This work has served as introduction of set numbers in the area of finite groups and structures. Another line of work will include a more comprehensive description of the calculus of real numbers. Revisions on the classic theory of types and the Continuum Hypothesis are also in order.

There are a variety of ways for codifying the information of mathematical structures and the natural data types for some structures has been provided, although this library of types must be completed. Trees are used to represent any type of mathematical object. The general procedure for expressing mathematical objects using the smallest type possible is described. For example, real functions have the same data type as sets of real numbers. These computational aspects can also be treated with detail, focusing on physical models to represent the arithmetic of Energy Levels. In [10], the author mentions the possibility that *"...we will be able to compare between different Set Theories according to what type of mathematical hinterland they provide for theoretical Physics."* Aside from classic computational schemes that can be improved, such as the one proposed for a simple and linear fast adder, modern computational schemes can also be explored. Encoding and storing mathematical objects

 doi:10.20944/preprints202007.0415.v4

(structures of information), is an option to be considered for future work. On the other hand, the linear sum of two waves, in phase, with equal wavelength and frequency, is equal a wave with double the amplitude. The linear superposition of constructive interference from two coherent sources satisfies the numeric principle for addition, $2^n + 2^n = 2^{n+1}$. Thus, measurements on the amplitude of waves can be used as a computational arithmetic model. This could provide a valid approach, for a linear optical computing scheme. Most recently, in [11], it has been noted that *"...the wave nature of light and related inherent operations such as interference and diffraction, can play a major role in enhancing computational throughput..."* And that *"In this view, photons are an ideal match for computing node-distributed networks."* An implementation of the finite-state machine of addition can be a system of coherent wave sources.

Funding: This research received no external funding.

Acknowledgments: Special Thanks to my Professors at undergraduate school. I am specifically thankful to Ms. Sofia Ortega Castillo who has helped me to prepare and organize this material, for talks given at the National Congress of Mathematics (2019, Monterrey, México), and encouraged me to pursue publication of the material. I am indebted to my professor in group theory, Alonso Castillo Ramírez who has assisted me with all kinds of questions that came up during the time I wrote some of the details. And, to my professor in analysis and probability theory, Victor Pérez Abreu Carreón who has always been a great teacher and friend, whose conversations and classes have inspired a great deal of the work I have tried to carry out. Any corrections or changes to be made are sole responsibility of the author.

Conflicts of Interest: The author declares no conflict of interest. This article has been submitted to peer review AFTER a Patent Filing Date has been assigned.

References

1. Ramírez, J.P. A New Set Theory for Analysis; *Axioms* ***2019***, *8, 31.*
2. Bernays, Paul. Axiomatic Set Theory; *Dover: New York, NY, USA,* ***1991***.
3. Benacerraf, Paul. What Numbers Could Not Be; *Philos. Rev.* ***1965***, *74.*
4. R. E. Ladner and M. J. Fischer. Parallel Prefix Computation; *Journal of the ACM, 27(4), pp. 831-838, October* ***1980***
5. Metropolis, N.; Rota, G.C.; Tanny, S. Significance Arithmetic: The Carrying Algorithm; *Journal of Combinatorial Theory, Series A,* ***1973***, *14, 386–421*.
6. A'Campo, N. A Natural Construction for the Real Numbers. *arXiv,* ***2003***; arXiv:math.GN/0301015 v1.
7. Arthan, R.D. The Eudoxus Real Numbers. *arXiv,* ***2004***; arXiv:math/0405454.
8. De Bruijn, N.G. Definig Reals Without the Use of Rationals; *Koninkl. Nederl. Akademie Van Wetenschappen: Amsterdam, The Netherlands*, ***1976***.
9. Knopfmacher, A.; Knopfmacher, J. Two Concrete New Constructions of the Real Numbers. *Rocky Mt. J. Math.* ***1988***, *18, 813–824.*
10. Magidor, Menachem. Some Set Theories are More Equal. Preliminary Draft.
11. Miscuglio, Mario. *Appl. Phys. Rev.* 7, 031404 (2020); https://doi.org/10.1063/5.0001942