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COMPLETENESS IN QUASI-PSEUDOMETRIC SPACES

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Abstract. The aim of this paper is to discuss the relations between various notions of sequential completeness and the corresponding notions of completeness by nets or by filters in the setting of quasi-metric spaces. We propose a new definition of right K-Cauchy net in a quasi-metric space for which the corresponding completeness is equivalent to the sequential completeness. In this way we complete some results of R. A. Stoltenberg, Proc. London Math. Soc. **17** (1967), 226–240, and V. Gregori and J. Ferrer, Proc. Lond. Math. Soc., III Ser., **49** (1984), 36. A discussion on nets defined over ordered or pre-ordered directed sets is also included.

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1. Introduction

For a mapping $d : X \times X \to \mathbb{R}$ on a set X consider the following conditions:

$$\begin{aligned}
&(\mathrm{M1}) \qquad d(x,y) \geqslant 0 \quad \text{and} \quad d(x,x) = 0;\\
&(\mathrm{M2}) \qquad d(x,y) = d(y,x);\\
&(\mathrm{M3}) \qquad d(x,z) \leqslant d(x,y) + d(y,z);\\
&(\mathrm{M4}) \qquad d(x,y) = 0 \Rightarrow x = y;\\
&(\mathrm{M4'}) \qquad d(x,y) = d(y,x) = 0 \Rightarrow x = y,
\end{aligned}$$

for all $x, y, z \in X$.

The mapping d is called a *pseudometric* if it satisfies (M1), (M2) and (M3) and a *metric* if it further satisfies (M4).

The *open* and *closed balls* in a pseudometric space (X, d) are defined by

$$B_d(x,r) = \{y \in X : d(x,y) < r\} \quad \text{and} \quad B_d[x,r] = \{y \in X : d(x,y) \leqslant r\},$$

respectively.

A *filter* on a set X is a nonempty family $\mathcal{F}$ of nonempty subsets of X satisfying the conditions

$$\begin{aligned}
&(\mathrm{F1}) \qquad F \subseteq G \ \text{ and } \ F \in \mathcal{F} \ \Rightarrow \ G \in \mathcal{F};\\
&(\mathrm{F2}) \qquad F \cap G \in \mathcal{F} \quad \text{for all} \quad F, G \in \mathcal{F}.
\end{aligned}$$

It is obvious that (F2) implies

$$(\mathrm{F2'}) \qquad F_1, \dots, F_n \in \mathcal{F} \ \Rightarrow \ F_1 \cap \ldots \cap F_n \in \mathcal{F}.$$

for all $n \in \mathbb{N}$ and $F_1, \dots, F_n \in \mathcal{F}$.

A *base* of a filter $\mathcal{F}$ is a subset $\mathcal{B}$ of $\mathcal{F}$ such that every $F \in \mathcal{F}$ contains a $B \in \mathcal{B}$.

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A nonempty family $\mathcal{B}$ of nonempty subsets of X such that

$$\text{(BF1)} \qquad \forall B_1, B_2 \in \mathcal{B},\ \exists B \in \mathcal{B},\ B \subseteq B_1 \cap B_2\,.$$

generates a filter $\mathcal{F}(\mathcal{B})$ given by

$$\mathcal{F}(\mathcal{B}) = \{U \subseteq X : \exists\, B \in \mathcal{B},\ B \subseteq U\}\,.$$

A family $\mathcal{B}$ satisfying (BF1) is called a *filter base*.

A *uniformity* on a set X is a filter $\mathcal{U}$ on $X \times X$ such that

$$\begin{aligned} &\text{(U1)} \qquad \Delta(X) \subseteq U,\ \forall U \in \mathcal{U};\\ &\text{(U2)} \qquad \forall U \in \mathcal{U},\ \exists V \in \mathcal{U},\ \text{ such that } V \circ V \subseteq U\,,\\ &\text{(U3)} \qquad \forall U \in \mathcal{U},\ U^{-1} \in \mathcal{U}. \end{aligned}$$

where

$$\begin{aligned} \Delta(X) &= \{(x, x) : x \in X\} \text{ denotes the diagonal of } X,\\ M \circ N &= \{(x, z) \in X \times X : \exists y \in X,\ (x, y) \in M \text{ and } (y, z) \in N\}, \text{ and}\\ M^{-1} &= \{(y, x) : (x, y) \in M\}\,, \end{aligned}$$

for any $M, N \subseteq X \times X$.

The sets in $\mathcal{U}$ are called *entourages*. A *base* for a uniformity $\mathcal{U}$ is a base of the filter $\mathcal{U}$. The composition $V \circ V$ is denoted sometimes simply by V^2. Since every entourage contains the diagonal $\Delta(X)$, the inclusion $V^2 \subseteq U$ implies $V \subseteq U$.

For $U \in \mathcal{U}$, $x \in X$ and $Z \subseteq X$ put

$$U(x) = \{y \in X : (x, y) \in U\} \quad\text{and}\quad U[Z] = \bigcup\{U(z) : z \in Z\}\,.$$

A uniformity $\mathcal{U}$ generates a topology $\tau(\mathcal{U})$ on X for which the family of sets

$$\{U(x) : U \in \mathcal{U}\}$$

is a base of neighborhoods of any point $x \in X$.

A *base* for a uniformity $\mathcal{U}$ is any base of the filter $\mathcal{U}$. The following characterization of bases can be found in Kelley [6].

Proposition 1.1. *A nonempty family $\mathcal{B}$ of subsets of a set $X \times X$ is a base of a uniformity $\mathcal{U}$ if and only if*

(B1) $\Delta(X) \subseteq B$ *for any* $B \in \mathcal{B}$;
(B2) $\forall B \in \mathcal{B},\ \exists C \in \mathcal{B},\ C^2 \subseteq B$;
(B3) $\forall B_1, B_2 \in \mathcal{B},\ \exists C \in \mathcal{B},\ C \subseteq B_1 \cap B_2$;
(B4) $\forall B \in \mathcal{B},\ \exists C \in \mathcal{B},\ C \subseteq B^{-1}$.

The corresponding uniformity is given by

$$\mathcal{U} = \{U \subseteq X \times X : \exists B \in \mathcal{B},\ B \subseteq U\}\,.$$

A *subbase* of a uniformity $\mathcal{U}$ is a family $\mathcal{B} \subseteq U$ such that any $U \in \mathcal{U}$ contains the intersection of a finite family of sets in $\mathcal{B}$.

Remark 1.2. *It can be shown (see, e.g., Kelley* [6]*) that a nonempty family $\mathcal{B}$ of subsets of a set $X \times X$ is a subbase of a uniformity $\mathcal{U}$ if and only if it satisfies the conditions (B1), (B2) and (B4) from Proposition 1.1.*

In this case the corresponding uniformity is given by

$$\mathcal{U} = \{U \subseteq X \times X : \exists n \in \mathbb{N},\ \exists B_1, \ldots, B_n \in \mathcal{B},\ B_1 \cap \cdots \cap B_n \subseteq U\}\,.$$

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Let (X, d) be a pseudometric space. Then the pseudometric d generates a topology τ_d for which

$$B_d(x, r),\ r > 0,$$

is a base of neighborhoods for every $x \in X$.

The pseudometric d generates also a uniform structure $\mathcal{U}_d$ on X having as basis of entourages the sets

$$U_\varepsilon = \{(x, y) \in X \times X : d(x, y) < \varepsilon\},\ \ \varepsilon > 0\,.$$

Since

$$U_\varepsilon(x) = B_d(x, \varepsilon),\ x \in X,\ \varepsilon > 0,$$

it follows that the topology $\tau(\mathcal{U}_d)$ agrees with the topology τ_d generated by the pseudometric d.

A sequence (x_n) in X is called *Cauchy* (or *fundamental*) if for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that

$$d(x_n, x_m) < \varepsilon \ \text{ for all }\ m, n \in \mathbb{N}\ \text{ with }\ m, n \geqslant n_\varepsilon\,,$$

a condition written also as

$$\lim_{m,n\to\infty} d(x_m, x_n) = 0\,.$$

A sequence (x_n) in a uniform space $(X, \mathcal{U})$ is called $\mathcal{U}$-*Cauchy* (or simply *Cauchy*) if for every $U \in \mathcal{U}$ there exists $n_0 \in \mathbb{N}$ such that

$$(x_m, x_n) \in U \ \text{ for all }\ m, n \in \mathbb{N}\ \text{ with }\ m, n \geqslant n_\varepsilon\,.$$

It is obvious that in the case of a pseudometric space (X, d) a sequence is Cauchy with respect to the pseudometric d if and only if it is Cauchy with respect to the uniformity $\mathcal{U}_d$.

The Cauchyness of nets in pseudometric or in uniform spaces is defined by analogy with that of sequences.

A filter $\mathcal{F}$ in a uniform space $(X, \mathcal{U})$ is called $\mathcal{U}$-*Cauchy* (or simply *Cauchy*) if for every $U \in \mathcal{U}$ there exists $F \in \mathcal{F}$ such that

$$F \times F \subseteq U\,.$$

Definition 1.3. A pseudometric space (X, d) is called *complete* if every Cauchy sequence in X converges. A uniform space $(X, \mathcal{U})$ is called *sequentially complete* if every $\mathcal{U}$-Cauchy sequence in X converges and *complete* if every $\mathcal{U}$-Cauchy net in X converges (or, equivalently, if every $\mathcal{U}$-Cauchy filter in X converges).

Remark 1.4. *We can define the completeness of a subset Y of X by the condition that every Cauchy sequence in Y converges to some element of Y. A closed subset of a pseudometric space is complete and a complete subset of a metric space is closed. A complete subset of a pseudometric space need not be closed.*

The following result holds in the metric case.

Theorem 1.5. *For a pseudometric space (X, d) the following conditions are equivalent.*

1. *The metric space X is complete.*
2. *Every Cauchy net in X is convergent.*
3. *Every Cauchy filter in $(X, \mathcal{U}_d)$ is convergent.*

An important result in metric spaces is Cantor characterization of completeness.

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Theorem 1.6 (Cantor theorem). *A pseudometric space (X,d) is complete if and only if every descending sequence of nonempty closed subsets of X with diameters tending to zero has nonempty intersection. This means that for any family F_n, $n \in \mathbb{N}$, of nonempty closed subsets of X*

$$F_1 \supseteq F_2 \supseteq \dots \quad \text{and} \quad \lim_{n\to\infty} \operatorname{diam}(F_n) = 0 \;\Rightarrow\; \bigcap_{n=1}^{\infty} F_n \neq \emptyset\,.$$

If d is a metric then this intersection contains exactly one point.

The *diameter* of a subset Y of a pseudometric space (X,d) is defined by

$$\operatorname{diam}(Y) = \sup\{d(x,y) : x,y \in Y\}\,. \tag{1.1}$$

2. Quasi-pseudometric and quasi-uniform spaces

2.1. **Quasi-pseudometric spaces.** Dropping the symmetry condition (M2) in the definition of a metric one obtains the notion of quasi-pseudometric, that is, a *quasi-pseudometric* on an arbitrary set X is a mapping $d : X \times X \to \mathbb{R}$ satisfying the conditions (M1) and (M3). If d satisfies further (M4′) then it called a *quasi-metric*. The pair (X,d) is called a *quasi-pseudometric space*, respectively a *quasi-metric space*[1].

The conjugate of the quasi-pseudometric d is the quasi-pseudometric $\bar{d}(x,y) = d(y,x)$, $x,y \in X$. The mapping $d^s(x,y) = \max\{d(x,y), \bar{d}(x,y)\}$, $x,y \in X$, is a pseudometric on X which is a metric if and only if d is a quasi-metric.

If (X,d) is a quasi-pseudometric space, then for $x \in X$ and $r > 0$ we define the balls in X by the formulae

$$B_d(x,r) = \{y \in X : d(x,y) < r\} \text{ - the open ball, and}$$
$$B_d[x,r] = \{y \in X : d(x,y) \leqslant r\} \text{ - the closed ball.}$$

The topology τ_d (or $\tau(d)$) of a quasi-pseudometric space (X,d) can be defined through the family $\mathcal{V}_d(x)$ of neighborhoods of an arbitrary point $x \in X$:

$$\begin{aligned} V \in \mathcal{V}_d(x) &\iff \exists r > 0 \text{ such that } B_d(x,r) \subseteq V \\ &\iff \exists r' > 0 \text{ such that } B_d[x,r'] \subseteq V. \end{aligned}$$

The topological notions corresponding to d will be prefixed by d- (e.g. d-closure, d-open, etc).

The convergence of a sequence (x_n) to x with respect to τ_d, called d-convergence and denoted by $x_n \xrightarrow{d} x$, can be characterized in the following way

$$x_n \xrightarrow{d} x \iff d(x,x_n) \to 0. \tag{2.1}$$

Also

$$x_n \xrightarrow{\bar{d}} x \iff \bar{d}(x,x_n) \to 0 \iff d(x_n,x) \to 0. \tag{2.2}$$

As a space equipped with two topologies, τ_d and $\tau_{\bar{d}}$, a quasi-pseudometric space can be viewed as a bitopological space in the sense of Kelly [7].

Asymmetric normed spaces

Let X be a real vector space. A mapping $p : X \to \mathbb{R}$ is called an *asymmetric seminorm* on X if

[1] In [4] the term "quasi-semimetric" is used instead of "quasi-pseudometric"

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$$
\begin{aligned}
&\text{(AN1)} \qquad p(x) \geqslant 0;\\
&\text{(AN2)} \qquad p(\alpha x) = \alpha p(x);\\
&\text{(AN3)} \qquad p(x+y) \leqslant p(x) + p(y)\,,
\end{aligned}
$$

for all $x, y \in X$ and $\alpha \geqslant 0$.

If, further,

$$
\text{(AN4)} \qquad p(x) = p(-x) = 0 \;\Rightarrow\; x = 0\,,
$$

for all $x \in X$, then p is called an *asymmetric norm.*

To an asymmetric seminorm p one associates a quasi-pseudometric d_p given by

$$
d_p(x, y) = p(y - x), \quad x, y \in X,
$$

which is a quasi-metric if p is an asymmetric norm. All the topological and metric notions in an asymmetric normed space are understood as those corresponding to this quasi-pseudometric d_p (see [4]).

The following topological properties are true for quasi-pseudometric spaces.

Proposition 2.1 (see [4])**.** *If (X, d) is a quasi-pseudometric space, then the following hold.*

1. *The ball $B_d(x, r)$ is d-open and the ball $B_d[x, r]$ is $\bar{d}$-closed. The ball $B_d[x, r]$ need not be d-closed.*
2. *The topology d is T_0 if and only if d is a quasi-metric.*
   The topology d is T_1 if and only if $d(x, y) > 0$ for all $x \neq y$ in X.
3. *For every fixed $x \in X$, the mapping $d(x, \cdot) : X \to (\mathbb{R}, |\cdot|)$ is d-usc and $\bar{d}$-lsc.*
   For every fixed $y \in X$, the mapping $d(\cdot, y) : X \to (\mathbb{R}, |\cdot|)$ is d-lsc and $\bar{d}$-usc.

Remark 2.2. *It is known that the topology τ_d of a pseudometric space (X, d) is Hausdorff (or T_2) if and only if d is a metric if and only if any sequence in X has at most one limit.*

The characterization of Hausdorff property of quasi-pseudometric spaces can also be given in terms of uniqueness of the limits, as in the metric case: the topology of a quasi-pseudometric space (X, d) is Hausdorff if and only if every sequence in X has at most one d-limit if and only if every sequence in X has at most one $\bar{d}$-limit (see [17]*).*

In the case of an asymmetric seminormed space there exists a characterization in terms of the asymmetric seminorm (see [4]*, Proposition 1.1.40).*

Recall that a topological space (X, τ) is called:

- T_0 if, for every pair of distinct points in X, at least one of them has a neighborhood not containing the other;
- T_1 if, for every pair of distinct points in X, each of them has a neighborhood not containing the other;
- T_2 (or *Hausdorff*) if every two distinct points in X admit disjoint neighborhoods;
- *regular* if, for every point $x \in X$ and closed set A not containing x, there exist the disjoint open sets U, V such that $x \in U$ and $A \subseteq V$.

2.2. **Quasi-uniform spaces.** Again, the notion of quasi-uniform space is obtained by dropping the symmetry condition (U3) from the definition of a uniform space, that is, a *quasi-uniformity* on a set X is a filter $\mathcal{U}$ in $X \times X$ satisfying the conditions (U1) and (U2). The sets in $\mathcal{U}$ are called *entourages* and the pair $(X, \mathcal{U})$ is called a *quasi-uniform space*, as in the case of uniform spaces.

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As uniformities, a quasi-uniformity $\mathcal{U}$ generates a topology $\tau(\mathcal{U})$ on X in a similar way: the sets

$$\{U(x) : U \in \mathcal{U}\}$$

form a base of neighborhoods of any point $x \in X$.

The topology $\tau(\mathcal{U})$ is T_0 if and only if $\bigcap \mathcal{U}$ is a partial order on X, and T_1 if and only if $\bigcap \mathcal{U} = \Delta(X)$.

The family of sets

$$\mathcal{U}^{-1} = \{U^{-1} : U \in \mathcal{U}\} \tag{2.3}$$

is another quasi-uniformity on X called the *quasi-uniformity conjugate* to $\mathcal{U}$. Also $\mathcal{U} \cup \mathcal{U}^{-1}$ is a subbase of a uniformity $\mathcal{U}^s$ on X, called the associated uniformity to the quasi-uniformity $\mathcal{U}$. It is the coarsest uniformity on X finer than both $\mathcal{U}$ and $\mathcal{U}^{-1}$, $\mathcal{U}^s = \mathcal{U} \vee \mathcal{U}^{-1}$. A basis for $\mathcal{U}^s$ is formed by the sets $\{U \cap U^{-1} : U \in \mathcal{U}\}$.

If (X, d) is a quasi-pseudometric space, then

$$U_\varepsilon = \{(x, y) \in X \times X : d(x, y) < \varepsilon\},\ \varepsilon > 0\ ,$$

is a basis for a quasi-uniformity $\mathcal{U}_d$ on X. The family

$$U_\varepsilon^- = \{(x, y) \in X \times X : d(x, y) \leqslant \varepsilon\},\ \varepsilon > 0\ ,$$

generates the same quasi-uniformity. Since $U_\varepsilon(x) = B_d(x, \varepsilon)$ and $U_\varepsilon^-(x) = B_d[x, \varepsilon]$, it follows that the topologies generated by the quasi-pseudometric d and by the quasi-uniformity $\mathcal{U}_d$ agree, i.e., $\tau_d = \tau(\mathcal{U}_d)$.

In this case

$$\mathcal{U}_d^{-1} = \mathcal{U}_{\bar{d}} \quad \text{and} \quad \mathcal{U}_d^s = \mathcal{U}_{d^s}\ .$$

3. Cauchy sequences and sequential completeness in quasi-pseudometric and quasi-uniform spaces

In contrast to the case of metric or uniform spaces, completeness, total boundedness and compactness look very different in quasi-metric and quasi-uniform spaces, due to the lack of symmetry of the distance. The present paper is concerned only with completeness. There are several notions of completeness in quasi-metric and quasi-uniform spaces, all agreeing with the usual notion of completeness in the case of metric or uniform spaces, each of them having its advantages and weaknesses.

We introduce now some of these notions following [10] (see also [4]).

Definition 3.1. A sequence (x_n) in (X, d) is called

- *left (right) d-Cauchy* if for every $\varepsilon > 0$ there exist $x \in X$ and $n_0 \in \mathbb{N}$ such that
$$d(x, x_n) < \varepsilon \ \ (\text{respectively}\ \ d(x_n, x) < \varepsilon)$$
for all $n \geqslant n_0$;
- *d^s-Cauchy* if it is a Cauchy sequence is the pseudometric space (X, d^s), that is for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that
$$d^s(x_n, x_k) < \varepsilon \ \text{ for all }\ n, k \geqslant n_0\ ,$$
or, equivalently,
$$d(x_n, x_k) < \varepsilon \ \text{ for all }\ n, k \geqslant n_0\ ;$$
- *left (right) K-Cauchy* if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that
$$d(x_k, x_n) < \varepsilon \ \ (\text{respectively}\ \ d(x_n, x_k) < \varepsilon)$$
for all $n, k \in \mathbb{N}$ with $n_0 \leqslant k \leqslant n$;

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- *weakly left (right) K-Cauchy* if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$d(x_{n_0}, x_n) < \varepsilon \quad (\text{respectively} \quad d(x_n, x_{n_0}) < \varepsilon)\,,$$

  for all $n \geqslant n_0$.

Sometimes, to emphasize the quasi-pseudometric d, we shall say that a sequence is left d-K-Cauchy, etc.

It seems that K in the definition of a left K-Cauchy sequence comes from Kelly [7] who considered first this notion.

Some remarks are in order.

Remark 3.2 ([10])**.** *Let (X, d) be a quasi-pseudometric space.*

1. *These notions are related in the following way:*

   d^s*-Cauchy* $\Rightarrow$ *left K-Cauchy* $\Rightarrow$ *weakly left K-Cauchy* $\Rightarrow$ *left d-Cauchy.*

   The same implications hold for the corresponding right notions. No one of the above implications is reversible.
2. *A sequence is left Cauchy (in some sense) with respect to d if and only if it is right Cauchy (in the same sense) with respect to $\bar{d}$.*
3. *A sequence is d^s-Cauchy if and only if it is both left and right d-K-Cauchy.*
4. *A d-convergent sequence is left d-Cauchy and a $\bar{d}$-convergent sequence is right d-Cauchy. For the other notions, a convergent sequence need not be Cauchy.*
5. *If each convergent sequence in a regular quasi-metric space (X, d) admits a left K-Cauchy subsequence, then X is metrizable* ([9])*.*

We also mention the following simple properties of Cauchy sequences.

Proposition 3.3 ([2, 11])**.** *Let (x_n) be a left or right K-Cauchy sequence in a quasi-pseudometric space (X, d).*

1. *If (x_n) has a subsequence which is d-convergent to x, then (x_n) is d-convergent to x.*
2. *If (x_n) has a subsequence which is $\bar{d}$-convergent to x, then (x_n) is $\bar{d}$-convergent to x.*
3. *If (x_n) has a subsequence which is d^s-convergent to x, then (x_n) is d^s-convergent to x.*

To each of these notions of Cauchy sequence corresponds two notions of sequential completeness, by asking that the corresponding Cauchy sequence be d-convergent or d^s-convergent. Due to the equivalence d-left Cauchy $\iff$ $\bar{d}$-right Cauchy one obtains nothing new by asking that a d-left Cauchy sequence is $\bar{d}$-convergent. For instance, the $\bar{d}$-convergence of any left d-K-Cauchy sequence is equivalent to the right K-completeness of the space $(X, \bar{d})$.

Definition 3.4 ([10])**.** A quasi-pseudometric space (X, d) is called:

- *sequentially d-complete* if every d^s-Cauchy sequence is d-convergent;
- *sequentially left d-complete* if every left d-Cauchy sequence is d-convergent;
- *sequentially weakly left (right) K-complete* if every weakly left (right) K-Cauchy sequence is d-convergent;
- *sequentially left (right) K-complete* if every left (right) K-Cauchy sequence is d-convergent;
- *sequentially left (right) Smyth complete* if every left (right) K-Cauchy sequence is d^s-convergent;
- *bicomplete* if the associated pseudometric space (X, d^s) is complete, i.e., every d^s-Cauchy sequence is d^s-convergent. A bicomplete asymmetric normed space (X, p) is called a *biBanach space.*

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As we noticed (see Remark 3.2.4), each d-convergent sequence is left d-Cauchy, but for each of the other notions there are examples of d-convergent sequences that are not Cauchy, which is a major inconvenience. Another one is that a complete (in some sense) subspace of a quasi-metric space need not be closed.

The implications between these completeness notions are obtained by reversing the implications between the corresponding notions of Cauchy sequence from Remark 3.2.1.

Remark 3.5. *(a) These notions of completeness are related in the following way:*

sequentially d-complete $\Rightarrow$ sequentially weakly left K-complete $\Rightarrow$ sequentially left K-complete $\Rightarrow$ sequentially left d-complete.

The same implications hold for the corresponding notions of right completeness.

(b) sequentially left or right Smyth completeness implies bicompleteness.

No one of the above implication is reversible (see [10]), excepting that between weakly left and left K-sequential completeness, as it was surprisingly shown by Romaguera [12].

Proposition 3.6 ([12], Proposition 1)**.** *A quasi-pseudometric space is sequentially weakly left K-complete if and only if it is sequentially left K-complete.*

A series $\sum_n x_n$ in an asymmetric seminormed space (X, p) is called *convergent* if there exists $x \in X$ such that $x = \lim_{n\to\infty} \sum_{k=1}^n x_k$ (i.e., $\lim_{n\to\infty} p\,(\sum_{k=1}^n x_k - x) = 0$). The series $\sum_n x_n$ is called *absolutely convergent* if $\sum_{n=1}^\infty p(x_n) < \infty$. It is well-known that a normed space is complete if and only if every absolutely convergent series is convergent. A similar result holds in the asymmetric case too.

Proposition 3.7. *Let (X, d) be a quasi-pseudometric space.*

1. *If a sequence (x_n) in X satisfies $\sum_{n=1}^\infty d(x_n, x_{n+1}) < \infty$ $(\sum_{n=1}^\infty d(x_{n+1}, x_n) < \infty)$, then it is left (right) d-K-Cauchy.*
2. *The quasi-pseudometric space (X, d) is sequentially left (right) d-K-complete if and only if every sequence (x_n) in X satisfying $\sum_{n=1}^\infty d(x_n, x_{n+1}) < \infty$ (resp. $\sum_{n=1}^\infty d(x_{n+1}, x_n) < \infty$) is d-convergent.*
3. *An asymmetric seminormed space (X, p) is sequentially left K-complete if and only if every absolutely convergent series is convergent.*

Cantor type results

Concerning Cantor-type characterizations of completeness in terms of descending sequences of closed sets (the analog of Theorem 1.6) we mention the following result. The *diameter* of a subset A of a quasi-pseudometric space (X, d) is defined by

$$\operatorname{diam}(A) = \sup\{d(x, y) : x, y \in A\}\,. \tag{3.1}$$

It is clear that, as defined, the diameter is, in fact, the diameter with respect to the associated pseudometric d^s. Recall that a quasi-pseudometric space is called sequentially d-complete if every d^s-Cauchy sequence is d-convergent (see Definition 3.4).

Theorem 3.8 ([10], Theorem 10)**.** *A quasi-pseudometric space (X, d) is sequentially d-complete if and only if each decreasing sequence $F_1 \supseteq F_2 \dots$ of nonempty closed sets with $\operatorname{diam}(F_n) \to 0$ as $n \to \infty$ has nonempty intersection, which is a singleton if d is a quasi-metric.*

The following characterization of right K-completeness was obtained in [3], using a different terminology.

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Proposition 3.9. *A quasi-pseudometric space (X,d) is sequentially right K-complete if and only if any decreasing sequence of closed $\bar{d}$-balls*

$$B_{\bar{d}}[x_1,r_1] \supseteq B_{\bar{d}}[x_2,r_2] \supseteq \ldots \quad \text{with} \quad \lim_{n\to\infty} r_n = 0\,,$$

has nonempty intersection.

If the topology d is Hausdorff, then $\bigcap_{n=1}^{\infty} B_{\bar{d}}[x_n,r_n]$ contains exactly one element.

4. Completeness by nets and filters

The Cauchy properties of a net $(x_i : i \in I)$ in a quasi-pseudometric space (X,d) are defined by analogy with that of sequences, by replacing in Definition 3.1 the natural numbers with the elements of the directed set I.

The situation is good for left Smyth completeness (see Definition 3.4).

Proposition 4.1 ([13], Prop. 1)**.** *For a quasi-metric space (X,d) the following are equivalent.*

1. *Every left d-K-Cauchy sequence is d^s-convergent.*
2. *Every left d-K-Cauchy net is d^s-convergent.*

A quasi-uniform space $(X,\mathcal{U})$ is called *bicomplete* if $(X,\mathcal{U}^s)$ is a complete uniform space. This notion is useful and easy to handle, because one can appeal to well known results from the theory of uniform spaces, but it is not appropriate for the study of the specific properties of quasi-uniform spaces, so one introduces adequate definitions, by analogy with quasi-pseudometric spaces.

Definition 4.2. Let $(X,\mathcal{U})$ be a quasi-uniform space.

A filter $\mathcal{F}$ on $(X,\mathcal{U})$ is called:

- *left* (*right*) *$\mathcal{U}$-Cauchy* if for every $U\in\mathcal{U}$ there exists $x \in X$ such that $U(x)\in\mathcal{F}$ (respectively $U^{-1}(x)\in\mathcal{F}$);
- *left* (*right*) *$\mathcal{U}$-K-Cauchy* if for every $U\in\mathcal{U}$ there exists $F\in\mathcal{F}$ such that $U(x)\in\mathcal{F}$ (resp. $U^{-1}(x)\in\mathcal{F}$) for all $x\in F$.

A net $(x_i : i\in I)$ in $(X,\mathcal{U})$ is called:

- *left $\mathcal{U}$-Cauchy* (*right $\mathcal{U}$-Cauchy*) if for every $U\in\mathcal{U}$ there exists $x\in X$ and $i_0\in I$ such that $(x,x_i)\in U$ (respectively $(x_i,x)\in U$) for all $i\geqslant i_0$;
- *left $\mathcal{U}$-K-Cauchy* (*right $\mathcal{U}$-K-Cauchy*) if

$$\forall U\in\mathcal{U},\ \exists i_0\in I,\ \forall i,j\in I,\ i_0\leqslant i\leqslant j \ \Rightarrow\ (x_i,x_j)\in U \quad (\text{resp.}\ \ (x_j,x_i)\in U\,. \tag{4.1}$$

The notions of left and right $\mathcal{U}$-K-Cauchy filter were defined by Romaguera in [12].

Observe that

$$(x_j,x_i)\in U \iff (x_i,x_j)\in U^{-1}\,,$$

so that a filter is right $\mathcal{U}$-K-Cauchy if and only if it is left $\mathcal{U}^{-1}$-K-Cauchy. A similar remark applies to $\mathcal{U}$-nets.

Definition 4.3. A quasi-uniform space $(X,\mathcal{U})$ is called:

- *left $\mathcal{U}$-complete by filters* (*left K-complete by filters*) if every left $\mathcal{U}$-Cauchy (respectively, left $\mathcal{U}$-K-Cauchy) filter in X is $\tau(\mathcal{U})$-convergent;
- *left $\mathcal{U}$-complete by nets* (*left $\mathcal{U}$-K-complete by nets*) if every left $\mathcal{U}$-Cauchy (respectively, left $\mathcal{U}$-K-Cauchy) net in X is $\tau(\mathcal{U})$-convergent;
- *Smyth left $\mathcal{U}$-K-complete by nets* if every left K-Cauchy net in X is $\mathcal{U}^s$-convergent.

The notions of right completeness are defined similarly, by asking the $\tau(\mathcal{U})$-convergence of the corresponding right Cauchy filter (or net) with respect to the topology $\tau(\mathcal{U})$ (or with respect to $\tau(\mathcal{U}^s)$ in the case of Smyth completeness).

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As we have mentioned in Introduction, in pseudometric spaces the sequential completeness is equivalent to the completeness defined in terms of filters, or of nets. Romaguera [12] proved a similar result for the left K-completeness in quasi-pseudometric spaces.

Remark 4.4. *In the case of a quasi-pseudometric space the considered notions take the following form.*

A filter $\mathcal{F}$ in a quasi-pseudometric space (X,d) is called left K-Cauchy *if it left $\mathcal{U}_d$-K-Cauchy. This is equivalent to the fact that for every $\varepsilon>0$ there exists $F_\varepsilon\in\mathcal{F}$ such that*

$$\forall x\in F_\varepsilon,\quad B_d(x,\varepsilon)\in\mathcal{F}\,. \tag{4.2}$$

Also a net $(x_i:i\in I)$ is called left K-Cauchy *if it is left $\mathcal{U}_d$-K-Cauchy or, equivalently, for every $\varepsilon>0$ there exists $i_0\in I$ such that*

$$\forall i,j\in I,\ \ i_0\leqslant i\leqslant j\ \Rightarrow\ d(x_i,x_j)<\varepsilon\,. \tag{4.3}$$

Proposition 4.5 ([12])**.** *For a quasi-pseudometric space (X,d) the following are equivalent.*

1. *The space (X,d) is sequentially left K-complete.*
2. *Every left K-Cauchy filter in X is d-convergent.*
3. *Every left K-Cauchy net in X is d-convergent.*

In the case of left $\mathcal{U}_d$-completeness this equivalence does not hold in general.

Proposition 4.6 (Künzi [8])**.** *A Hausdorff quasi-metric space (X,d) is sequentially left d-complete if and only if the associated quasi-uniform space $(X,\mathcal{U}_d)$ is left $\mathcal{U}_d$-complete by filters.*

4.1. **Right K-completeness in quasi-pseudometric spaces.** It is strange that for the right completeness the things look worse than for the left completeness.

As remarked Stoltenberg [15, Example 2.4] a result similar to Proposition 4.5 does not hold for right K-completeness: there exists a sequentially right K-complete T_1 quasi-metric space which is not right K-complete by nets. Actually, Stoltenberg [15] proved that the equivalence holds for a more general definition of a right K-Cauchy net, see Proposition 4.15.

An analog of Proposition 4.5 for right K-completeness can be obtained only under some supplementary hypotheses on the quasi-pseudometric space X.

A quasi-pseudometric space (X,d) is called R_1 if for all $x,y\in X$, d-cl$\{x\}\neq d$-cl$\{y\}$ implies the existence of two disjoint d-open sets U,V such that $x\in U$ and $y\in V$.

Proposition 4.7 ([1])**.** *Let (X,d) be a quasi-pseudometric space. The following are true.*

1. *If X is right K-complete by filters, then every right K-Cauchy net in X is convergent. In particular, every right K-complete by filters quasi-pseudometric space is sequentially right K-complete.*
2. *If the quasi-pseudometric space (X,d) is R_1 then X is right K-complete by filters if and only if it is sequentially right K-complete.*

Stoltenberg's example

As we have mentioned, Stoltenberg [15, Example 2.4] gave an example of a sequentially right K-complete T_1 quasi-metric space which is not right K-complete by nets, which we shall present now.

Denote by $\mathcal{A}$ the family of all countable subsets of the interval $[0,\frac{1}{3}]$. For $A\in\mathcal{A}$ let

$$A_1^A=A,\ \ X_{k+1}^A=A\cup\left\{\frac{1}{2},\frac{3}{4},\dots,\frac{2^k-1}{2^k}\right\},\ k\in\mathbb{N},\ \text{and}$$

$$X_\infty^A=A\cup\left\{\frac{2^k-1}{2^k}:k\in\mathbb{N}\right\}=\bigcup\{X_k^A:k\in\mathbb{N}\}\,.$$

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Put $\mathcal{S} = \left\{X_k^A : A \in \mathcal{A},\, k \in \mathbb{N} \cup \{\infty\}\right\}$ and define $d : \mathcal{S} \times \mathcal{S} \to [0, \infty)$ by

$$d(X_k^A, X_j^B) = \begin{cases} 0 & \text{if } A = B \text{ and } k = j,\ A, B \in \mathcal{A},\ k, j \in \mathbb{N} \cup \{\infty\} \\ 2^{-j} & \text{if } X_j^B \subsetneqq X_k^A,\ A, B \in \mathcal{A},\ k \in \mathbb{N} \cup \{\infty\},\ j \in \mathbb{N}, \\ 1 & \text{otherwise}\,. \end{cases}$$

Proposition 4.8. *$(\mathcal{S}, d)$ is a sequentially right K-complete T_1 quasi-metric space which is not right K-complete by nets.*

Proof. The proof that d is a T_1 quasi-metric on $\mathcal{S}$ is straightforward.

I. *$(\mathcal{S}, d)$ is sequentially right K-complete.*

Let $(X_n)_{n\in\mathbb{N}}$ be a right K-Cauchy sequence in $\mathcal{S}$. Then there exists $n_0 \in \mathbb{N}$ such that

$$d(X_m, X_n) < 1 \quad \text{for all} \quad m, n \in \mathbb{N} \text{ with } n_0 \leqslant n \leqslant m\,.$$

For $i \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ let

$$X_{n_0+i} = X_{k_i}^{A_i} \text{ where } A_i \in \mathcal{A} \text{ and } k_i \in \mathbb{N} \cup \{\infty\}\,.$$

Since

$$d\left(X_{k_{i+1}}^{A_{i+1}}, X_{k_i}^{A_i}\right) < 1\,,$$

it follows $k_i \in \mathbb{N}$ for all $i \in \mathbb{N}_0$. For $0 < \varepsilon < 1$ there exists $i_0 \in \mathbb{N}$ such that

$$d(X_{n_0+i+1}, X_{n_0+i}) < \varepsilon \quad \text{for all } i \geqslant i_0\,,$$

which means that

$$2^{-k_i} = d\left(X_{k_{i+1}}^{A_{i+1}}, X_{k_i}^{A_i}\right) < \varepsilon \quad \text{for all } i \geqslant i_0\,.$$

This shows that $\lim_{i\to\infty} k_i = \infty$.

Let $A = \bigcup\{A_i : i \in \mathbb{N}_0\}$ and $X = X_\infty^A$. Then $X_{k_i}^{A_i} \subsetneqq X_\infty^A$, so that

$$d(X, X_{n_0+i}) = d\left(X_\infty^A, X_{k_i}^{A_i}\right) = 2^{-k_i} \to 0 \ \text{ as } \ i \to \infty\,.$$

which shows that the sequence (X_n) is d-convergent to X.

II. *The quasi-metric space $(\mathcal{S}, d)$ is not right K-complete by nets.*

Let $\mathcal{S}_0 = \{X_k^A : A \in \mathcal{A},\, k \in \mathbb{N}\}$ ordered by

$$X \leqslant Y \iff X \subseteq Y, \text{ for } X, Y \in \mathcal{S}_0\,.$$

We have

$$X_i^A \leqslant X_j^B \iff \begin{cases} A \subseteq B & \text{and} \\ i \leqslant j \end{cases}$$

for $X_i^A, X_j^B \in \mathcal{S}_0$, $(\mathcal{S}_0, \leqslant)$ is directed and the mapping $\phi : \mathcal{S}_0 \to \mathcal{S}$ defined by $\phi(X) = X$, $X \in \mathcal{S}_0$, is a net in $\mathcal{S}$.

Let us show first that the net ϕ is right K-Cauchy. For $\varepsilon > 0$ choose $k \in \mathbb{N}$ such that $2^{-k} < \varepsilon$. For some $C \in \mathcal{A}$, X_k^C belongs to $\mathcal{S}_0$ and

$$d(X_j^A, X_i^B) = 2^{-i} \leqslant 2^{-k} < \varepsilon$$

for all

$$X_j^A, X_i^B \in \mathcal{S}_0 \ \text{ with } \ X_k^C \leqslant X_i^B \leqslant X_j^A,\ X_j^A \neq X_i^B\,,$$

showing that the net ϕ is right K-Cauchy.

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Let $X = X_k^C$ be an arbitrary element in $\mathcal{S}$. We show that for every $X_i^A \in \mathcal{S}_0$ there exists $X_j^B \in \mathcal{S}_0$ with $X_i^A \leqslant X_j^B$ such that $d(X, X_j^B) = 1$, which will imply that the net ϕ is not d-convergent to X.

Since C is a countable set, there exists $x_0 \in [0, \frac{1}{3}] \setminus C$. For an arbitrary $X_i^A \in \mathcal{S}_0$ let $B = A \cup \{x_0\}$. Then $X_i^B \in \mathcal{S}_0$, $X_i^A \leqslant X_i^B$ and $X_k^C \nsubseteq X_j^B$, so that, by the definition of the metric d, $d(X_k^C, X_i^B) = 1$. □

Stoltenberg-Cauchy nets

Stoltenberg [15] also considered a more general definition of a right K-Cauchy net as a net $(x_i : i \in I)$ satisfying the condition: for every $\varepsilon > 0$ there exists $i_\varepsilon \in I$ such that

$$d(x_i, x_j) < \varepsilon \quad \text{for all} \quad i, j \geqslant i_\varepsilon \quad \text{with} \quad i \nleqslant j\,. \tag{4.4}$$

Let us call such a net *Stoltenberg-Cauchy* and *Stoltenberg completeness* the completeness with respect to Stoltenberg-Cauchy nets.

It follows that, for this definition,

$$d(x_j, x_i) < \varepsilon \quad \text{and} \quad d(x_j, x_i) < \varepsilon \quad \text{for all} \;\; i, j \geqslant i_\varepsilon \;\; \text{with} \quad i \nsim j,$$

where $i \nsim j$ means that i, j are incomparable (that is, no one of the relations $i \leqslant j$ or $j \leqslant i$ holds).

Gregori-Ferrer-Cauchy nets

Later, Gregori and Ferrer [5] found a gap in the proof of Theorem 2.5 from [15] and provided a counterexample to it, based on Example 2.4 of Stoltenberg (see Proposition 4.8).

Example 4.9 ([5])**.** Let $\mathcal{A}$, $(\mathcal{S}, d)$ be as in the preamble to Proposition 4.8 and $I = \mathbb{N} \cup \{a, b\}$, where the set $\mathbb{N}$ is considered with the usual order and a, b are two distinct elements not belonging to $\mathbb{N}$ with

$$k \leqslant a, \quad k \leqslant b, \quad \text{for all} \;\; k \in \mathbb{N},$$
$$a \leqslant a, \quad b \leqslant b \quad a \leqslant b, \quad b \leqslant a.$$

Consider two sets $A, B \in \mathcal{A}$ with $A \subsetneqq B$ and let $\phi : I \to \mathcal{S}$ be given by

$$\phi(k) = X_k^A,\ k \in \mathbb{N}, \quad \phi(a) = X_\infty^A, \quad \phi(b) = X_\infty^B\,.$$

Then the net ϕ is right Cauchy in the sense of (4.4) but not convergent in $(\mathcal{S}, d)$.

Indeed, for $0 < \varepsilon < 1$ let $k_0 \in \mathbb{N}$ be such that $2^{-k_0} < \varepsilon$.

Since

$$i \leqslant a,\ i \leqslant b,\ i \geqslant k_0,\ \forall i \in \mathbb{N},\ k_0 \leqslant a \leqslant b,\ k_0 \leqslant b \leqslant a,$$

it follows that the condition $i \nleqslant j$ can hold for some $i, j \in I$, $i, j \geqslant k_0$, in the following cases:

(a) $i, j \in \mathbb{N},\ i, j \geqslant k_0,\ j < i;$

(b) $i = a,\ j \in \mathbb{N},\ j \geqslant k_0$

(c) $i = b,\ j \in \mathbb{N},\ j \geqslant k_0$

In the case (a), $X_j^A \subsetneqq X_i^A$ and

$$d(\phi(i), \phi(j)) = d(X_i^A, X_j^A) = 2^{-j} \leqslant 2^{-k_0} < \varepsilon\,.$$

In the case (b), $X_j^A \subsetneqq X_\infty^A$ and again

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$$d(\phi(a),\phi(j)) = d(X^A_\infty, X^A_j) = 2^{-j} \leqslant 2^{-k_0} < \varepsilon.$$

The case (c) is similar to (b).

To show that ϕ is not convergent let $X \in \mathcal{S}\setminus\{X^B_\infty\}$. Then $b \geqslant i$ for any $i\in I$ and

$$d(X,\phi(b)) = d(X, X^B_\infty) = 1\,,$$

so that ϕ does not converge to X. If $X = X^B_\infty$, then $a \geqslant i$ for any $i \in I$ and

$$d(X^B_\infty,\phi(a)) = d(X^B_\infty, X^A_\infty) = 1\,.$$

Gregori and Ferrer [5] proposed a new definition of a right K-Cauchy net, for which the equivalence to sequential completeness holds.

Definition 4.10. A net $(x_i : i \in I)$ in a quasi-metric space (X, d) is called *GF-Cauchy* if one of the following conditions holds:

(a) for every maximal element $j \in I$ the net (x_i) converges to x_j;
(b) I has no maximal elements and the net (x_i) converges;
(c) I has no maximal elements and the net (x_i) satisfies the condition (4.4).

Maximal elements and net convergence

For a better understanding of this definition we shall analyze the relations between maximal elements in a preordered set and the convergence of nets. Recall that in the definition of a directed set $(I, \leqslant)$ the relation $\leqslant$ is supposed to be only a preorder, i.e. reflexive and transitive and not necessarily antireflexive (see [6]). Notice that some authors suppose that in the definition of a directed set $\leqslant$ is a partial order (see, e.g., [16]). For a discussion of this matter see [14, §7.12, p. 160].

Let $(I, \leqslant)$ be a preordered set. An element $j \in I$ is called:

- *strictly maximal* if there is no $i \in I \setminus \{j\}$ with $j \leqslant i$, or, equivalently,

$$j \leqslant i \;\Rightarrow\; i = j, \quad \text{for every } \; i \in I; \tag{4.5}$$

- *maximal* if

$$j \leqslant i \;\Rightarrow\; i \leqslant j, \quad \text{for every } \; i \in I\,. \tag{4.6}$$

Remark 4.11. *Let $(I, \leqslant)$ be a preordered set.*

1. A strictly maximal element is maximal, and if $\leqslant$ is an order, then these notions are equivalent.

Suppose now that the set I is further directed. Then the following hold.

2. Every maximal element j of I is a maximum for I, i.e. $i \leqslant j$ for all $i \in I$.

3. If j is a maximal element and $j' \in I$ satisfies $j \leqslant j'$, then j' is also a maximal element.

4. (Uniqueness of the strictly maximal element) If j is a strictly maximal element, then $j' = j$ for any maximal element j' of I.

Proof. 1. These assertions are obvious.

2. Indeed, suppose that $j \in I$ satisfies (4.6). Then, for arbitrary $i \in I$, there exists $i' \in I$ with $i' \geqslant j, i$. But $j \leqslant i'$ implies $i' \leqslant j$ and so $i \leqslant i' \leqslant j$. (We use the notation $i \geqslant j, k$ for $i \geqslant j \wedge i \geqslant k$.)

3. Let $i \in I$ be such that $j' \leqslant i$. Then $j \leqslant i$ and, by the maximality of j, $i \leqslant j \leqslant j'$.

4. If j is strictly maximal and j' is a maximal element of I, then, by 2, $j' \leq j$ so that, by (4.6) applied to j', $j \leq j'$ and so, by (4.5) applied to j, $j' = j$. □

We present now some remarks on maximal elements and net convergence.

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Remark 4.12. *Let (X,d) be a quasi-metric space, $(I,\leqslant)$ a directed sets and $(x_i : i\in I)$ a net in X.*

1. If $(I,\leqslant)$ has a strictly maximal element j, then the net (x_i) is convergent to x_j.

2.(a) *If the net (x_i) converges to $x\in X$, then $d(x,x_j)=0$ for every maximal element j of I. If the topology τ_d is T_1 then, further, $x_j=x$.*

(b) *If the net (x_i) converges to x_j and to $x_{j'}$, where j,j' are maximal elements of I, then $x_j=x_{j'}$.*

(c) *If I has maximal elements and, for some $x\in X$, $x_j=x$ for every maximal element j, then the net (x_i) converges to x.*

Proof. 1. For an arbitrary $\varepsilon>0$ take $i_\varepsilon=j$. Then $i\geqslant j$ implies $i=j$, so that

$$d(x_j,x_i)=d(x_j,x_j)=0<\varepsilon\,.$$

2.(a) For every $\varepsilon>0$ there exists $i_\varepsilon\in I$ such that $d(x,x_i)<\varepsilon$ for all $i\geqslant i_\varepsilon$. By Remark 4.11.2, $j\geqslant i_\varepsilon$ for every maximal j, so that $d(x,x_j)<\varepsilon$ for all $\varepsilon>0$, implying $d(x,x_j)=0$.

If the topology τ_d is T_1, then, by Proposition 2.1.2, $x_j=x$.

(b) By (a), $d(x_j,x_{j'})=0$ and $d(x_{j'},x_j)=0$, so that $x_j=x_{j'}$.

(c) Let $x\in X$ be such that $x_j=x$ for every maximal element j of I and let j be a fixed maximal element of I. For any $\varepsilon>0$ put $i_\varepsilon=j$. Then, by Remark 4.11.3, any $i\in I$ such that $i\geqslant j$ is also a maximal element of I, so that $x_i=x$ and $d(x,x_i)=0<\varepsilon$. □

Let us say that a quasi-metric space (X,d) is *GF-complete* if every GF-Cauchy net (i.e. satisfying the conditions (a),(b),(c) from Definition 4.10) is convergent. Remark that, with this definition, condition (b) becomes tautological and so superfluous, so it suffices to ask that every net satisfying (a) and (c) be convergent.

By Remarks 4.11.1 and 4.12.1, (a) always holds if $\leqslant$ is an order, so that, in this case, a net satisfying condition (c) is a GF-Cauchy net and so GF-completeness agrees with that given by Stoltenberg.

Strongly Stoltenberg-Cauchy nets

In order to avoid the shortcomings of the preorder relation, as, for instance, those put in evidence by Example 4.9, we propose the following definition.

Definition 4.13. A net $(x_i : i\in I)$ in a quasi-metric space (X,d) is called *strongly Stoltenberg-Cauchy* if for every $\varepsilon>0$ there exists $i_\varepsilon\in I$ such that, for all $i,j\geqslant i_\varepsilon$,

$$(j\leqslant i\vee i\nsim j)\ \Rightarrow\ d(x_i,x_j)<\varepsilon\,. \tag{4.7}$$

We present now some remarks on the relations of this notion with other notions of Cauchy net (Stoltenberg and GF), as well as the relations between the corresponding completeness notions. It is obvious that in the case of a sequence $(x_k)_{k\in\mathbb{N}}$ each of these three notions agrees with the right K-Cauchyness of (x_k).

Remark 4.14. *Let $(x_i : i\in I)$ be a net in a quasi-metric space (X,d).*

1.(a) We have

$$i\nleqslant j\ \Rightarrow\ (j\leqslant i\vee i\nsim j)\,, \tag{4.8}$$

for all $i,j\in I$. If $\leqslant$ is an order, then the reverse implication also holds for all $i,j\in I$ with $i\neq j$.

(b) If the net $(x_i : i\in I)$ satisfies (4.7) then it satisfies (4.4), i.e. every strong Stoltenberg-Cauchy net is Stoltenberg-Cauchy. If $\leqslant$ is an order, then these notions are equivalent.

Hence, net completeness with respect to (4.4) (i.e. Stoltenberg completeness) implies net completeness with respect to (4.7);

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2. Suppose that the net $(x_i : i \in I)$ *satisfies* (4.7).

(a) If j, j' *are maximal elements of* I*, then* $x_j = x_{j'}$*. Hence, if* I *has maximal elements, then there exists* $x \in X$ *such that* $x_j = x$ *for every maximal element* j *of* I*, and the net* (x_i) *converges to* x*.*

(b) Consequently, the net (x_i) *also satisfies the conditions (a) and (c) from Definition 4.10, so that, GF-completeness implies completeness with respect to* (4.7).

Proof. 1.(a) Let $i, j \in I$ with $i \not\leqslant j$. Since $j \leqslant i$ if i, j are comparable, the implication (4.8) holds. If $\leqslant$ is an order and $i \neq j$, then $j \leqslant i \;\Rightarrow\; i \not\leqslant j$ and $i \nsim j \;\Rightarrow\; i \not\leqslant j$.

(b) Since it suffices to ask that (4.4) and (4.7) hold only for distinct $i, j \geqslant i_\varepsilon$, the equivalence of these notions in the case when $\leqslant$ is an order follows.

Suppose that the net (x_i) satisfies (4.7). For $\varepsilon > 0$ choose $i_\varepsilon \in I$ according to (4.7) and let $i, j \geqslant i_\varepsilon$ with $i \not\leqslant j$. Taking into account (4.8) it follows $d(x_i, x_j) < \varepsilon$, i.e. (x_i) satisfies (4.4).

Suppose now that every net satisfying (4.4) converges and let (x_i) be a net in X satisfying (4.7). Then it satisfies (4.4) so it converges.

2.(a) Let j, j' be maximal elements of I. For $\varepsilon > 0$ choose i_ε according to (4.7). By Remark 4.11.2, $j, j' \geqslant i_\varepsilon$, $j \leqslant j'$, $j' \leqslant j$, so that $d(x_{j'}, x_j) < \varepsilon$ and $d(x_j, x_{j'}) < \varepsilon$. Since these inequalities hold for every $\varepsilon > 0$, it follows $d(x_{j'}, x_j) = 0 = d(x_j, x_{j'})$ and so $x_j = x_{j'}$. The convergence of the net (x_i) follows from Remark 4.12.2.(c).

(b) The assertions on GF-Cauchy nets follow from (a). □

We show now that completeness by nets with respect to (4.7) is equivalent to sequential right K-completeness.

Proposition 4.15 ([15], Theorem 2.5)**.** *A* T_1 *quasi-metric space* (X, d) *is sequentially right* K*-complete if and only if every net in* X *satisfying* (4.7) *is* d*-convergent.*

Proof. We have only to prove that the sequential right K-completeness implies that every net in X satisfying (4.7) is d-convergent.

Let $(x_i : i \in I)$ be a net in X satisfying (4.7). Let $i_k \geqslant i_{k-1},\ k \geqslant 2$, be such that (4.7) holds for $\varepsilon = 1/2^k,\ k \in \mathbb{N}$.

This is possible. Indeed, take i_1 such that (4.7) holds for $\varepsilon = 1/2$. If i_2' is such that (4.7) holds for $\varepsilon = 2^{-2}$, then pick $i_2 \in I$ such that $i_2 \geqslant i_1, i_2'$. Continuing by induction one obtains the desired sequence $(i_k)_{k\in\mathbb{N}}$.

We distinguish two cases.

Case I. $\exists j_0 \in I,\ \exists k_0 \in \mathbb{N},\quad \forall k \geqslant k_0,\ i_k \leqslant j_0\,.$

Let $i \geqslant j_0$. Then for every k, $i_k \leqslant j_0 \leqslant i$ implies $d(x_i, x_{j_0}) < 2^{-k}$ so that $d(x_i, x_{j_0}) = 0$. Since the quasi-metric space (X, d) is T_1, it follows $x_i = x_{j_0}$ for all $i \geqslant j_0$ (see Proposition 2.1), so that the net $(x_i : i \in I)$ is d-convergent to x_{j_0}.

Case II. $\forall j \in I,\ \forall k \in \mathbb{N},\ \exists k' \geqslant k,\quad i_{k'} \not\leqslant j\,.$

The inequalities $d(x_{i_{k+1}}, x_{i_k}) < 2^{-k},\ k \in \mathbb{N}$, imply that the sequence $(x_{i_k})_{k\in\mathbb{N}}$ is right K-Cauchy (see Proposition 3.7), so it is d-convergent to some $x \in X$.

For $\varepsilon > 0$ choose $k_0 \in \mathbb{N}$ such that $2^{-k_0} < \varepsilon$ and $d(x, x_{i_k}) < \varepsilon$ for all $k \geqslant k_0$.

Let $i \in I$, $i \geqslant i_{k_0}$. By hypothesis, there exists $k \geqslant k_0$ such that $i_k \not\leqslant i$, implying $i \leqslant i_k \vee i_k \nsim i$. Since $i_{k_0} \leqslant i_k, i$, by the choice of i_{k_0}, $d(x_{i_k}, x_i) < 2^{-k_0} < \varepsilon$ in both of these cases. But then

$$d(x, x_i) \leqslant d(x, x_{i_k}) + d(x_{i_k}, x_i) < 2\varepsilon\,,$$

proving the convergence of the net (x_i) to x. □

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The proof of Proposition 4.15 in the case of GF-completeness

As the result in [5] is given without proof, we shall supply one.

Proposition 4.16. *A T_1 quasi-metric space (X,d) is right K-sequentially complete if and only if every net satisfying the conditions (a) and (c) from Definition 4.10 is convergent.*

Proof. Obviously, a proof is needed only for the case (c).

Suppose that the directed set $(I,\leqslant)$ has no maximal elements and let $(x_i : i\in I)$ be a net in a quasi-metric space (X,d) satisfying (4.4).

The proof follows the ideas of the proof of Proposition 4.15 with some further details. Let $i_k\leqslant i_{k+1}$, $k\in\mathbb{N}$, be a sequence of indices in I such that $d(x_i,x_j)<2^{-k}$ for all $i,j\geqslant i_k$ with $i\not\leqslant j$. We show that we can further suppose that $i_{k+1}\not\leqslant i_k$.

Indeed, the fact that I has no maximal elements implies that for every $i\in I$ there exists $i'\in I$ such that

$$i\leqslant i' \text{ and } i'\not\leqslant i. \tag{4.9}$$

Let $i_1'\in I$ be such that (4.4) holds for $\varepsilon=2^{-1}$. Take i_1 such that $i_1'\leqslant i_1$ and $i_1\not\leqslant i_1'$. Let $i_2'\geqslant i_1$ be such that (4.4) holds for $\varepsilon=2^{-2}$ and let $i_2\in I$ satisfying $i_2'\leqslant i_2$ and $i_2\not\leqslant i_2'$. Then $i_1\leqslant i_2$ and $i_2\not\leqslant i_1$, because $i_2\leqslant i_1\leqslant i_2'$ would contradict the choice of i_2.

By induction one obtains a sequence (i_k) in I satisfying $i_k\leqslant i_{k+1}$ and $i_{k+1}\not\leqslant i_k$ such that (4.4) is satisfied with $\varepsilon=2^{-k}$ for every i_k.

We shall again consider two cases.

Case I. $\exists j_0\in I,\ \exists k_0\in\mathbb{N},\quad \forall k\geqslant k_0,\ i_k\leqslant j_0\,.$

Let $i\geqslant j_0$. By (4.9) there exists $i'\in I$ such that $i\leqslant i'$ and $i'\not\leqslant i$, implying $d(x_{i'},x_i)<2^{-k}$ for all $k\geqslant k_0$, that is $d(x_{i'},x_i)=0$, so that, by T_1, $x_{i'}=x_i$.

We also have $i'\not\leqslant j_0$ because $i'\leqslant j_0$ would imply $i'\leqslant i$, in contradiction to the choice of i'. But then, $d(x_{i'},x_{j_0})<2^{-k}$ for all $k\geqslant k_0$, so that, as above, $d(x_{i'},x_i)=0$ and $x_{i'}=x_{j_0}$.

Consequently, $x_i=x_{j_0}$ for every $i\geqslant j_0$, proving the convergence of the net (x_i) to x_{j_0}.

Case II. $\forall j\in I,\ \forall k\in\mathbb{N},\ \exists k'\geqslant k,\quad i_{k'}\not\leqslant j\,.$

The condition $d(x_{i_k},x_{i_{k+1}})<2^{-k}$, $k\in\mathbb{N}$, implies that the sequence $(x_{i_k})_{k\in\mathbb{N}}$ is right K-Cauchy, so that there exists $x\in X$ with $d(x,x_{i_k})\to 0$ as $k\to\infty$.

For $\varepsilon>0$ let $k_0\in\mathbb{N}$ be such that $2^{-k_0}<\varepsilon$ and $d(x,x_{i_k})<\varepsilon$ for all $k\geqslant k_0$.

Let $i\geqslant i_{k_0}$. By II, for $j=i$ and $k=k_0$, there exists $k\geqslant k_0$ such that $i_k\not\leqslant i$. The conditions $k\geqslant k_0$, $i_{k_0}\leqslant i$, $i_{k_0}\leqslant i_k$ and $i_k\not\leqslant i$ imply

$$d(x,x_{i_k})<\varepsilon \text{ and } d(x_{i_k},x_i)<\varepsilon\,,$$

so that

$$d(x,x_i)\leqslant d(x,x_{i_k})+d(x_{i_k},x_i)<2\varepsilon,$$

for all $i\geqslant i_{k_0}$. □

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