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New Type of Degenerate Poly-Frobenius-Euler Polynomials and Numbers

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Abstract. Motivated by Kim-Kim [19] introduced the new type of degenerate poly-Bernoulli polynomials by means of the degenerate polylogarithm function. In this paper, we define the degenerate poly-Frobenius-Euler polynomials, called the new type of degenerate poly-Frobenius-Euler polynomials, by means of the degenerate polylogarithm function. Then, we derive explicit expressions and some identities of those numbers and polynomials.

Keywords: polylogarithm function; Frobenius-Euler polynomials; new type of degenerate poly-Frobenius-Euler polynomials

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1. Introduction

Throughout this presentation, we use the following standard notions $\mathbb{N} = \{1, 2, \cdots\}$, $\mathbb{N}_0 = \{0, 1, 2, \cdots\} = \mathrm{N} \cup \{0\}$, $\mathbb{Z}^- = \{-1, -2, \cdots\}$. Also as usual $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers.

The classical Bernoulli $B_n(x)$, Euler $E_n(x)$ and Genocchi $G_n(x)$ polynomial are defined by means of the following generating function as follows

$$\frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \mid t \mid < 2\pi, \ \frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \mid t \mid < \pi,$$

and

$$\frac{2t}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}, \mid t \mid < \pi, \text{(see [1-25])} \tag{1.1}$$

respectively.

For $u \in \mathbb{C}$ with $u \neq 1$, the classical Frobenius-Euler polynomials $H_n^{(\alpha)}(x; u)$ of order α are defined by means of the following generating function

$$\left(\frac{1-u}{e^t-u}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} H_n^{(\alpha)}(x; u)\frac{t^n}{n!}, \text{(see [1, 5, 11, 12])}. \tag{1.2}$$

In the special case when $x = 0$, $H_n^{(\alpha)}(u) = H_n^{(\alpha)}(0; u)$ are called n^{th} Frobenius-Euler numbers of order α. For $\alpha = 1$ into (1.2), $H_n^{(1)}(x, u) = H_n(x, u)$, are called the Frobenius-Euler polynomials and $H_n^{(\alpha)}(0; u) = h_n^{(\alpha)}(u)$, are called the Frobenius-Euler numbers of order α. Substituting $u = -1$ into (1.2), $H_n(x; -1) = E_n(x)$, are called the Euler polynomials, (see [8, 22, 24, 25]).

In (2017), Kurt [10] introduced the poly-Frobenius-Euler polynomials are given by

$$\frac{(1-u)\mathrm{Li}_k(1-e^{-t})}{t(e^t-u)}e^{xt} = \sum_{n=0}^{\infty} H_n^{(k)}(x; u)\frac{t^n}{n!}. \tag{1.3}$$

In the case when $x = 0$, $H_n^{(k)}(u) = H_n^{(k)}(0; u)$ are called the poly-Frobenius-Euler numbers.

For any non-zero $\lambda \in \mathbb{R}$ (or $\mathbb{C}$), the degenerate exponential function is defined by

$$e_\lambda^x(t) = (1 + \lambda t)^{\frac{x}{\lambda}}, e_\lambda(t) = (1 + \lambda t)^{\frac{1}{\lambda}}, \text{(see [13, 18, 19])}. \tag{1.4}$$

By binomial expansion, we get

$$e_\lambda^x(t) = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}, \text{(see [14, 17])}, \tag{1.5}$$

where $(x)_{0,\lambda} = 1$, $(x)_{n,\lambda} = (x - \lambda)(x - 2\lambda) \cdots (x - (n - 1)\lambda), (n \geq 1)$.

Note that

$$\lim_{\lambda \longrightarrow 0} e_\lambda^x(t) = \sum_{n=0}^{\infty} x^n \frac{t^n}{n!} = e^{xt}.$$

In [2, 3] Carlitz introduced the degenerate Bernoulli and degenerate Euler polynomials are defined by

$$\frac{z}{e_\lambda(z) - 1} e_\lambda^x(z) = \sum_{j=0}^{\infty} B_{j,\lambda}(x) \frac{z^j}{j!}, \quad \frac{2}{e_\lambda(z) + 1} e_\lambda^x(z) = \sum_{j=0}^{\infty} E_{j,\lambda}(x) \frac{z^j}{j!}, \tag{1.6}$$

respectively.

In the case when $x = 0$, $B_{j,\lambda} = B_{j,\lambda}(0)$ are called the degenerate Bernoulli numbers and $x = 0$, $E_{j,\lambda} = E_{j,\lambda}(0)$ are called the degenerate Euler numbers.

Obviously

$$\lim_{\lambda \longrightarrow 0} \beta_n(x; \lambda) = B_n(x), \lim_{\lambda \longrightarrow 0} E_n(x; \lambda) = E_n(x).$$

Kim et al. [15] introduced the degenerate Frobenius-Euler polynomials are defined by means of the generating function as follows

$$\frac{1 - u}{(1 + \lambda t)^{\frac{1}{\lambda}} - u} (1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} h_{n,\lambda}(x|u) \frac{t^n}{n!}, \tag{1.7}$$

At the value $x = 0$, $h_{n,\lambda}(u) = h_{n,\lambda}(0|u)$ are called the degenerate Frobenius-Euler numbers.

It is readily seen that

$$\lim_{\lambda \longrightarrow 0} h_{n,\lambda}(x|u) = H_n(x|u), (n \geq 0).$$

For $s \in \mathbb{Z}$, the polylogaritm function is defined by a power series in z as

$$\mathrm{Li}_s(z) = \sum_{j=1}^{\infty} \frac{j^n}{j^s} = z + \frac{z^2}{2^s} + \frac{z^3}{3^s}, (|\ z\ | < 1), \text{(see, [4, 9])}. \tag{1.8}$$

It is notice that

$$\mathrm{Li}_1(z) = \sum_{j=1}^{\infty} \frac{z^j}{j} = -\log(1 - z). \tag{1.9}$$

For $\lambda \in \mathbb{R}$, Kim-Kim [19] defined the degenerate version of the logarithm function, denoted by $\log_\lambda(1+z)$ as follows:

$$\log_\lambda(1+z) = \sum_{j=1}^{\infty} \lambda^{j-1}(1)_{j,1/\lambda}\frac{z^j}{j!}, (\text{see, [18]}) \tag{1.10}$$

being the inverse of the degenerate version of the exponential function $e_\lambda(z)$ as has been shown below

$$e_\lambda(\log_\lambda(z)) = \log_\lambda(e_\lambda(z)) = z.$$

It is noteworthy to mention that

$$\lim_{\lambda\to 0}\log_\lambda(1+z) = \sum_{j=1}^{\infty}(-1)^{j-1}\frac{z^j}{j!} = \log(1+z).$$

The degenerate polylogarithm function [19] is defined by Kim-Kim to be

$$l_{k,\lambda}(z) = \sum_{j=1}^{\infty}\frac{(-\lambda)^{j-1}(1)_{j,1/\lambda}}{(j-1)!j^k}z^n, (k \in \mathbb{Z}, \mid z \mid < 1). \tag{1.11}$$

It is clear that

$$\lim_{\lambda\to 0} l_{k,\lambda}(z) = \sum_{j=1}^{\infty}\frac{z^j}{j^k} = \mathrm{Li}_k(z), (\text{see [4, 9]}).$$

From (1.10) and (1.11), we get

$$l_{1,\lambda}(z) = \sum_{j=1}^{\infty}\frac{(-\lambda)^{j-1}(1)_{j,1/\lambda}}{j!}z^j = -\log_\lambda(1-z). \tag{1.8}$$

Very recently, Kim-Kim [19] introduced the new type degenerate version of the Bernoulli polynomials and numbers, by using the degenerate polylogarithm function as follows

$$\frac{l_{k,\lambda}(1-e_\lambda(-z))}{1-e_\lambda(-z)}e_\lambda^u(z) = \sum_{j=0}^{\infty}\beta_{j,\lambda}^{(k)}(u)\frac{z^j}{j!}. \tag{1.12}$$

In the special case $x=0$, $\beta_{j,\lambda}^{(k)} = \beta_{j,\lambda}^{(k)}(0)$ are called the degenerate poly-Bernoulli numbers.

It is well known that the Stirling numbers of the first kind are defined by

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l, (\text{see [24, 25]}), \tag{1.13}$$

where $(x)_0 = 1$, and $(x)_n = x(x-1)\cdots(x-n+1), (n \geq 1)$. From (1.13), it is easily to see that

$$\frac{1}{k!}(\log(1+t))^k = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!},\ (k \geq 0), (\text{see [12, 13, 18]}). \tag{1.14}$$

In the inverse expression to (1.14), the Stirling numbers of the second kind are defined by

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l, (\text{see [10, 12, 22]}). \tag{1.15}$$

From (1.15), it is easily to see that

$$\frac{1}{k!}(e^t-1)^k = \sum_{n=l}^{\infty} S_2(n,l)\frac{t^n}{n!}, (\text{see [22, 24, 25]}). \tag{1.16}$$

For $n \geq 0$, the degenerate Stirling numbers of the second kind [7, 8, 17] are defined by

$$\frac{1}{n!}(e_\lambda(t)-1)^n = \sum_{l=n}^{\infty} S_{2,\lambda}(l,n)\frac{t^l}{l!}, (n \geq 0). \tag{1.17}$$

In this paper, we construct the degenerate poly-Frobenius-Euler polynomials and numbers, called the new type of poly-Frobenius-Euler polynomials and numbers by using the degenerate polylogarithm function and derive several properties on the degenerate poly-Frobenius-Euler polynomials and numbers.

2. New type of degenerate poly-Frobenius-Euler polynomials

Let $\lambda, u \in \mathbb{C}$ with $u \neq 1$ and $k \in \mathbb{Z}$, by using the degenerate polylogarithm function, we define the new type of degenerate poly-Frobenius-Euler polynomials as follows

$$\frac{(1-u)l_{k,\lambda}(1-e_\lambda(-t))}{t(e_\lambda(t)-u)}e_\lambda^x(t) = \sum_{n=0}^{\infty} H_{n,\lambda}^{(k)}(x;u)\frac{t^n}{n!}. \tag{2.1}$$

In the special case, $x = 0$, $H_{n,\lambda}^{(k)}(u) = H_{n,\lambda}^{(k)}(0;u)$ are called the new type of degenerate poly-Frobenius-Euler numbers.

For $k = 1$ in (2.1), we get

$$\frac{1-u}{e_\lambda(t)-u}e_\lambda^x(t) = \sum_{n=0}^{\infty} h_{n,\lambda}(x;u)\frac{t^n}{n!}, (\text{see } [15]) \tag{2.2}$$

where $h_{n,\lambda}(x;u)$ are called the degenerate Frobenius-Euler polynomials.

Theorem 2.1. For $n \geq 0$, we have

$$H_{n,\lambda}^{(k)}(x;u) = \sum_{m=0}^{n} \binom{n}{m} H_{n-m,\lambda}^{(k)}(u)(x)_{m,\lambda}.$$

Proof. From (2.1), we have

$$\sum_{n=0}^{\infty} H_{n,\lambda}^{(k)}(x;u)\frac{t^n}{n!} = \left(\frac{(1-u)l_{k,\lambda}(1-e_\lambda(-t))}{t(e_\lambda(t)-u)}\right)e_\lambda^x(t)$$

$$= \sum_{n=0}^{\infty} H_{n,\lambda}^{(k)}(u)\frac{t^n}{n!}\sum_{m=0}^{\infty}(x)_{m,\lambda}\frac{t^m}{m!}$$

$$L.H.S = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \binom{n}{m} H_{n-m,\lambda}^{(k)}(u)(x)_{m,\lambda}\frac{t^n}{n!}. \tag{2.3}$$

Therefore, by (2.1) and (2.3), we require at the desired result. □

Theorem 2.2. For $k \in \mathbb{Z}$ and $n \geq 0$, we have

$$H_{n,\lambda}^{(k)}(x;u) = \sum_{l=0}^{n}\binom{n}{l}\sum_{m=0}^{l}\frac{(1)_{m+1,1/\lambda}(-\lambda)^m(-1)^{l-m}}{(m+1)^{k-1}}\frac{S_{2,\lambda}(l+1,m+1)}{l+1}h_{n-l,\lambda}(x;u).$$

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Proof. It is proved by using (1.7), (1.11) and (2.1) that

$$\sum_{n=0}^{\infty} H_{n,\lambda}^{(k)}(x;u)\frac{t^n}{n!} = \left(\frac{(1-u)e_\lambda^x(t)}{t(e_\lambda(t)-u)}\right) l_{k,\lambda}(1-e_\lambda(-t))$$

$$= \left(\frac{(1-u)e_\lambda^x(t)}{t(e_\lambda(t)-u)}\right)\sum_{m=1}^{\infty}\frac{(1)_{m,1/\lambda}(-\lambda)^{m-1}}{(m-1)!m^k}(1-e_\lambda(-t))^m$$

$$= \left(\frac{(1-u)e_\lambda^x(t)}{t(e_\lambda(t)-u)}\right)\sum_{m=0}^{\infty}\frac{(1)_{m+1,1/\lambda}(-\lambda)^{m}}{(m+1)^{k-1}m!}(1-e_\lambda(-t))^{m+1}$$

$$= \left(\frac{(1-u)e_\lambda^x(t)}{t(e_\lambda(t)-u)}\right)\sum_{m=0}^{\infty}\frac{(1)_{m+1,1/\lambda}(-\lambda)^{m}}{(m+1)^{k-1}}\sum_{l=m+1}^{\infty}S_{2,\lambda}(l,m+1)(-1)^{l-m-1}\frac{t^l}{l!}$$

$$= \left(\frac{(1-u)e_\lambda^x(t)}{e_\lambda(t)-u}\right)\sum_{l=0}^{\infty}\sum_{m=0}^{l}\frac{(1)_{m+1,1/\lambda}(-\lambda)^{m}(-1)^{l-m}}{(m+1)^{k-1}}\frac{S_{2,\lambda}(l+1,m+1)}{l+1}\frac{t^l}{l!}$$

$$= \sum_{n=0}^{\infty} h_{n,\lambda}(x;u)\frac{t^n}{n!}\sum_{l=0}^{\infty}\sum_{m=0}^{l}\frac{(1)_{m+1,1/\lambda}(-\lambda)^{m}(-1)^{l-m}}{(m+1)^{k-1}}\frac{S_{2,\lambda}(l+1,m+1)}{l+1}\frac{t^l}{l!}$$

$$L.H.S = \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\sum_{m=0}^{l}\frac{(1)_{m+1,1/\lambda}(-\lambda)^{m}(-1)^{l-m}}{(m+1)^{k-1}}\frac{S_{2,\lambda}(l+1,m+1)}{l+1}h_{n-l,\lambda}(x;u)\right)\frac{t^n}{n!}. \tag{2.4}$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides, we get the result. □

Corollary 2.1. For $k \in \mathbb{Z}$ and $n \geq 0$, we have

$$H_{n,\lambda}^{(k)}(u) = \sum_{l=0}^{n}\binom{n}{l}\sum_{m=0}^{l}\frac{(1)_{m+1,1/\lambda}(-\lambda)^{m}(-1)^{l-m}}{(m+1)^{k-1}}\frac{S_{2,\lambda}(l+1,m+1)}{l+1}h_{n-l,\lambda}(u).$$

Corollary 2.2. For $n \geq 0$, we have

$$H_{n,\lambda}(x;u) = \sum_{l=0}^{n}\binom{n}{l}\sum_{m=0}^{l}\frac{(1)_{m+1,1/\lambda}(-\lambda)^{m}(-1)^{l-m}S_{2,\lambda}(l+1,m+1)}{l+1}h_{n-l,\lambda}(x;u).$$

Corollary 2.3. For $n \geq 0$, we have

$$E_{n,\lambda}(x) = \sum_{l=0}^{n}\binom{n}{l}\sum_{m=0}^{l}\frac{(1)_{m+1,1/\lambda}(-\lambda)^{m}(-1)^{l-m}S_{2,\lambda}(l+1,m+1)}{l+1}E_{n-l,\lambda}(x).$$

Moreover,

$$\sum_{l=0}^{n}\binom{n}{l}\sum_{m=0}^{l}\frac{(1)_{m+1,1/\lambda}(-\lambda)^{m}(-1)^{l-m}S_{2,\lambda}(l+1,m+1)}{l+1}E_{n-l,\lambda}(x) = 0.$$

Theorem 2.3. For $n \geq 0$, we have

$$\sum_{l=0}^{n}\binom{n}{l}\sum_{m=0}^{l}\frac{(1)_{m+1,1/\lambda}(-\lambda)^{m}(-1)^{l-m}}{(m+1)^{k-1}}\frac{S_{2,\lambda}(l+1,m+1)}{l+1}(x)_{n-l,\lambda}$$

$$= \frac{1}{1-u}\left[\sum_{m=0}^{n}\binom{n}{m}H_{n-m,\lambda}^{(k)}(x;u)(1)_{m,\lambda} - uH_{n,\lambda}^{(k)}(x;u)\right].$$

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Proof. From (2.1), we have

$$\frac{(1-u)l_{k,\lambda}(1-e_\lambda(-t))}{t}e_\lambda^x(t) = e_\lambda(t)\sum_{n=0}^{\infty} H_{n,\lambda}^{(k)}(x;u)\frac{t^n}{n!} - u\sum_{n=0}^{\infty} H_{n,\lambda}^{(k)}(x;u)\frac{t^n}{n!}$$
$$= \sum_{m=0}^{\infty}(1)_{m,\lambda}\frac{t^m}{m!}\sum_{n=0}^{\infty} H_{n,\lambda}^{(k)}(x;u)\frac{t^n}{n!} - u\sum_{n=0}^{\infty} H_{n,\lambda}^{(k)}(x;u)\frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}H_{n-m,\lambda}^{(k)}(x;u)(1)_{m,\lambda} - uH_{n,\lambda}^{(k)}(x;u)\right)\frac{t^n}{n!}. \tag{2.5}$$

On the other hand,

$$\frac{(1-u)l_{k,\lambda}(1-e_\lambda(-t))}{t}e_\lambda^x(t)$$
$$= \left(\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}\right)\frac{(1-u)}{t}\left(\sum_{m=1}^{\infty}\frac{(1)_{m,1/\lambda}(-\lambda)^{m-1}}{(m-1)!m^k}(1-e_\lambda(-t))^m\right)$$
$$= \left(\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}\right)\frac{(1-u)}{t}\left(\sum_{m=0}^{\infty}\frac{(1)_{m+1,1/\lambda}(-\lambda)^{m}}{(m+1)^{k-1}m!}(1-e_\lambda(-t))^{m+1}\right)$$
$$= (1-u)\left(\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}\right)\left(\sum_{l=0}^{\infty}\sum_{m=0}^{l}\frac{(1)_{m+1,1/\lambda}(-\lambda)^{m}(-1)^{l-m}}{(m+1)^{k-1}}\frac{S_{2,\lambda}(l+1,m+1)}{l+1}\frac{t^l}{l!}\right)$$
$$L.H.S = (1-u)\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\sum_{m=0}^{l}\frac{(1)_{m+1,1/\lambda}(-\lambda)^{m}(-1)^{l-m}}{(m+1)^{k-1}}\frac{S_{2,\lambda}(l+1,m+1)}{l+1}(x)_{n-l,\lambda}\right)\frac{t^n}{n!}. \tag{2.6}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides, we get the result. □

Theorem 2.4. For $n \geq 0$, we have

$$H_{n,\lambda}^{(k)}(u) = \frac{(1-u)}{x(e_\lambda(x)-u)}\underbrace{\int_0^x \frac{e_\lambda^{1-\lambda}(-t)}{1-e_\lambda(-t)}\int_0^t \frac{e_\lambda^{1-\lambda}(-t)}{1-e_\lambda(-t)}\cdots\int_0^t \frac{e_\lambda^{1-\lambda}(-t)}{1-e_\lambda(-t)}}_{(k-2)-\text{times}}tdtdt\cdots dt.$$

Proof. Using (1.11), we first consider the following expression

$$\frac{d}{dx}l_{k,\lambda}(1-e_\lambda(-x)) = \frac{d}{dx}\sum_{n=1}^{\infty}\frac{(1)_{n,1/\lambda}}{(n+1)!n^k}(1-e_\lambda(-x))^n$$
$$= \frac{1}{1-e_\lambda(-x)}l_{k-1,\lambda}(1-e_\lambda(-x)). \tag{2.7}$$

From (2.7), $k \geq 2$, we have

$$\sum_{n=0}^{\infty}H_{n,\lambda}^{(k)}(u)\frac{x^n}{n!} = \frac{(1-u)}{x(e_\lambda(x)-u)}l_{k,\lambda}(1-e_\lambda(-x)). \tag{2.8}$$
$$= \frac{(1-u)}{x(e_\lambda(x)-u)}\int_0^x \frac{e_\lambda^{1-\lambda}(-t)}{1-e_\lambda(-t)}\int_0^t \frac{e_\lambda^{1-\lambda}(-t)}{1-e_\lambda(-t)}\cdots\int_0^t \frac{e_\lambda^{1-\lambda}(-t)}{1-e_\lambda(-t)}tdtdt\cdots dt$$
$$= \frac{(1-u)}{x(e_\lambda(x)-u)}\underbrace{\int_0^x \frac{e_\lambda^{1-\lambda}(-t)}{1-e_\lambda(-t)}\int_0^t \frac{e_\lambda^{1-\lambda}(-t)}{1-e_\lambda(-t)}\cdots\int_0^t \frac{e_\lambda^{1-\lambda}(-t)}{1-e_\lambda(-t)}}_{(k-2)-\text{times}}tdtdt\cdots dt. \tag{2.9}$$

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By (2.9), we obtain at the desired result. Thus, we complete the proof. □

Theorem 2.5. Let $n \geq 0$. Then

$$H_{n,\lambda}^{(2)}(u) = \sum_{m=0}^{n} \binom{n}{m} (-1)^m \frac{B_{m,\lambda}(1-\lambda)}{m+1} h_{n-m,\lambda}(u) = \sum_{m=0}^{n} \binom{n}{m} (-1)^{n-m} \frac{B_{n-m,\lambda}(1-\lambda)}{n-m+1} h_{m,\lambda}(u).$$

Proof. By using (1.17) and Theorem 2.4, we get

$$\begin{aligned}
\sum_{n=0}^{\infty} H_{n,\lambda}^{(2)}(u) \frac{x^n}{n!} &= \frac{1-u}{x(e_\lambda(x)-u)} \int_0^x \frac{-t}{e_\lambda(-t)-1} e_\lambda^{1-\lambda}(-t)dt \\
&= \frac{1-u}{x(e_\lambda(x)-u)} \int_0^x \sum_{n=0}^{\infty} B_{n,\lambda}(1-\lambda) \frac{(-t)^n}{n!} dt \\
&= \frac{1-u}{x(e_\lambda(x)-u)} \sum_{m=0}^{\infty} (-1)^m \frac{B_{m,\lambda}(1-\lambda)}{m+1} \frac{x^m}{m!} \\
&= \left(\sum_{n=0}^{\infty} h_{n,\lambda}(u) \frac{x^n}{n!} \right) \left(\sum_{m=0}^{\infty} (-1)^m \frac{B_{m,\lambda}(1-\lambda)}{m+1} \frac{x^m}{m!} \right) \\
L.H.S &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} (-1)^m \frac{B_{m,\lambda}(1-\lambda)}{m+1} h_{n-m,\lambda}(u) \right) \frac{x^n}{n!}. \qquad (2.10)
\end{aligned}$$

Therefore, by (2.10), we get the following theorem.

□

Theorem 2.6. Let $k \in \mathbb{Z}$ and $n \geq 0$, we have

$$\begin{aligned}
H_{n,\lambda}^{(k)}(u) = \sum_{m=0}^{n} \binom{n}{m} \sum_{m_1+m_2+\cdots+m_{k-1}=m} \binom{m}{m_1, m_2, \cdots, m_k} h_{n-m.\lambda}(u) \\
\times \frac{\beta_{m_1,\lambda}(\lambda-1)}{m_1+1} \frac{\beta_{m_2,\lambda}(\lambda-1)}{m_2+m_2+1} \cdots \frac{\beta_{m_{k-1},\lambda}(\lambda-1)}{m_1+m_2+\cdots+n_{k-1}+1}.
\end{aligned}$$

Proof. In general, from (2.9), we note that

$$\sum_{n=0}^{\infty} H_{n,\lambda}^{(k)}(u) \frac{x^n}{n!} = \frac{1-u}{x(e_\lambda(x)-u)} \int_0^x \underbrace{\frac{e_\lambda^{1-\lambda}(-t)}{1-e_\lambda(-t)} \int_0^t \frac{e_\lambda^{1-\lambda}(-t)}{1-e_\lambda(-t)} \cdots \int_0^t \frac{e_\lambda^{1-\lambda}(-t)}{1-e_\lambda(-t)}}_{(k-2)-\text{times}} tdtdt \cdots dt$$

$$\begin{aligned}
= \frac{1-u}{e_\lambda(x)-u} \sum_{m=0}^{\infty} \sum_{m_1+m_2+\cdots+m_{k-1}=m}^{\infty} \binom{m}{m_1, m_2, \cdots, m_k} \frac{\beta_{m_1,\lambda}(\lambda-1)}{m_1+1} \frac{\beta_{m_2,\lambda}(\lambda-1)}{m_2+m_2+1} \\
\times \cdots \frac{\beta_{m_{k-1},\lambda}(\lambda-1)}{m_1+m_2+\cdots+n_{k-1}+1} \frac{x^m}{m!}
\end{aligned}$$

$$\begin{aligned}
L.H.S = \sum_{n=0}^{\infty} \Bigg(\sum_{m=0}^{n} \binom{n}{m} \sum_{m_1+m_2+\cdots+m_{k-1}=m} \binom{m}{m_1, m_2, \cdots, m_k} h_{n-m.\lambda}(u) \\
\times \frac{\beta_{m_1,\lambda}(\lambda-1)}{m_1+1} \frac{\beta_{m_2,\lambda}(\lambda-1)}{m_2+m_2+1} \cdots \frac{\beta_{m_{k-1},\lambda}(\lambda-1)}{m_1+m_2+\cdots+n_{k-1}+1} \Bigg) \frac{x^n}{n!}. \qquad (2.11)
\end{aligned}$$

Therefore, by comparing the coefficients of t^n on both sides, we obtain the result. □

Theorem 2.7. For $n \geq 0$, we have

$$H_{n,\lambda}^{(k)}(x+y;u) = \sum_{m=0}^{n} \binom{n}{m} H_{n-m,\lambda}^{(k)}(x;u)(y)_{m,\lambda}.$$

Proof. From (2.1), we have

$$\sum_{n=0}^{\infty} H_{n,\lambda}^{(k)}(x+y;u)\frac{t^n}{n!} = \left(\frac{(1-u)l_{k,\lambda}(1-e_\lambda(-t))}{t(e_\lambda(t)-u)}\right) e_\lambda^{x+y}(t)$$

$$= \left(\sum_{n=0}^{\infty} H_{n,\lambda}^{(k)}(x;u)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}(y)_{m,\lambda}\frac{t^m}{m!}\right)$$

$$L.H.S = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m} H_{n-m,\lambda}^{(k)}(x;u)(y)_{m,\lambda}\right)\frac{t^n}{n!}. \tag{2.12}$$

Comparing the coefficients on both sides of (2.12), we get the result. □

Theorem 2.8. For $n \geq 0$, we have

$$H_{n,\lambda}^{(k)}(x+1;u) = \sum_{m=0}^{n} \binom{n}{m} H_{n-m,\lambda}^{(k)}(x;u)(1)_{m,\lambda}.$$

Proof. By (2.1), we observe that

$$\sum_{n=0}^{\infty}\left[H_{n,\lambda}^{(k)}(x+1;u) - H_{n,\lambda}^{(k)}(x;u)\right]\frac{t^n}{n!} = \left(\frac{(1-u)l_{k,\lambda}(1-e_\lambda(-t))}{t(e_\lambda(t)-u)}\right) e_\lambda^{x}(t)\left[e_\lambda(t)-1\right]$$

$$L.H.S = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m} H_{n-m,\lambda}^{(k)}(x;u)(1)_{m,\lambda}\frac{t^n}{n!} - \sum_{n=0}^{\infty} H_{n,\lambda}^{(k)}(x;u)\frac{t^n}{n!}. \tag{2.13}$$

Comparing the coefficients of t^n on both sides, we obtain the result. □

Theorem 2.9. For $n \geq 0$, we have

$$H_{n,\lambda}^{(k)}(x;u) = \sum_{m=0}^{n}\sum_{q=0}^{m}\binom{n}{m}(x)_q S_\lambda^{(2)}(m,q) H_{n-m,\lambda}^{(k)}(u).$$

Proof. From (2.1), we have

$$\sum_{n=0}^{\infty} H_{n,\lambda}^{(k)}(x;u)\frac{t^n}{n!} = \left(\frac{(1-u)l_{k,\lambda}(1-e_\lambda(-x))}{t(e_\lambda(t)-u)}\right) e_\lambda^{x}(t)$$

$$= \left(\frac{(1-u)l_{k,\lambda}(1-e_\lambda(-x))}{t(e_\lambda(t)-u)}\right)\left[e_\lambda(t)-1+1\right]^x$$

$$= \left(\frac{(1-u)l_{k,\lambda}(1-e_\lambda(-x))}{t(e_\lambda(t)-u)}\right)\left(\sum_{q=0}^{\infty}(x)_q\sum_{l=q}^{\infty} S_\lambda^{(2)}(l,q)\frac{t^l}{l!}\right)$$

$$L.H.S = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{q=0}^{m}\binom{n}{m}(x)_q S_\lambda^{(2)}(m,q) H_{n-m,\lambda}^{(k)}(u)\right)\frac{t^n}{n!}. \tag{2.14}$$

By comparing the coefficients of t^n on both sides, we get the result. □

Theorem 2.10. For $n \geq 0$, we have

$$H_{n,\lambda}^{(k)}(x+\alpha|u) = \sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l} x^m m! S_{2,\lambda}(l+\alpha, m+\alpha) H_{n-l,\lambda}^{(k)}(u).$$

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Proof. Replacing x by $x+\alpha$ in (2.1), we have

$$\sum_{n=0}^{\infty} H_{n,\lambda}^{(k)}(x+\alpha;u)\frac{t^n}{n!} = \left(\frac{(1-u)l_{k,\lambda}(1-e_\lambda(-x))}{t(e_\lambda(t)-u)}\right) e_\lambda^{x+\alpha}(t)$$

$$= \left(\frac{(1-u)l_{k,\lambda}(1-e_\lambda(-x))}{t(e_\lambda(t)-u)}\right) e_\lambda^{\alpha}(t) \left(\sum_{m=0}^{\infty} x^m (e_\lambda(t)-1)^m\right)$$

$$= \left(\frac{(1-u)l_{k,\lambda}(1-e_\lambda(-x))}{t(e_\lambda(t)-u)}\right) \left(e_\lambda^{\alpha}(t) \sum_{m=0}^{\infty} x^m m! \frac{(e_\lambda(t)-1)^m}{m!}\right)$$

$$= \left(\frac{(1-u)l_{k,\lambda}(1-e_\lambda(-x))}{t(e_\lambda(t)-u)}\right) \left(e_\lambda^{\alpha}(t) \sum_{m=0}^{\infty} x^m m! \sum_{l=m}^{\infty} S_{2,\lambda}(l,m)\frac{t^l}{l!}\right)$$

$$= \sum_{n=0}^{\infty} H_{n,\lambda}^{(k)}(u)\frac{t^n}{n!} \left(\sum_{l=0}^{\infty}\sum_{m=0}^{l} x^m m! S_{2,\lambda}(l+\alpha, m+\alpha)\frac{t^l}{l!}\right)$$

$$L.H.S = \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n}\sum_{m=0}^{l} \binom{n}{l} x^m m! S_{2,\lambda}(l+\alpha, m+\alpha) H_{n-l,\lambda}^{(k)}(u)\right)\frac{t^n}{n!}. \tag{2.15}$$

Therefore, by (2.1) and (2.15), we obtain the result. □

4. Conclusions

Motivated by the definition of the degenerate poly-Bernoulli polynomials introduced by Kim-Kim [19], in the present paper, we have considered a class of new generating function for the degenerate Frobenius-Euler polynomials, called the new type of degenerate poly-Frobenius-Euler polynomials, by means of the degenerate polylogarithm function. Then, we have derived some useful relations and properties. We have showed that the new type of degenerate poly-Frobenius-Euler polynomials equal a linear combination of the degenerate Frobenius-Euler polynomials and Stirlings numbers of the first and second kind. In a special case, we have given a relation between the new type of degenerate Frobenius-Euler polynomials and Bernoulli polynomials of order n.

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