Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 10 July 2020 doi:10.20944/preprints202007.0202.v1

A note on a new type of degenerate poly-Euler numbers and polynomials

Waseem A. Khan

Department of Mathematics and Natural Sciences, Prince Mohammad Bin Fahd University, P.O Box 1664, Al Khobar 31952, Kingdom of Saudi Arabia

E-mail: wkhan1@pmu.edu.sa

Abstract. Kim-Kim [12] introduced the new type of degenerate Bernoulli numbers and polynomials arising from the degenerate logarithm function. In this paper, we introduce a new type of degenerate poly-Euler polynomials and numbers, are called degenerate poly-Euler polynomials and numbers, by using the degenerate polylogarithm function and derive several properties on the degenerate poly-Euler polynomials and numbers. In the last section, we also consider the degenerate unipoly-Euler polynomials attached to an arithmetic function, by using the degenerate polylogarithm function and investigate some identities of those polynomials. In particular, we give some new explicit expressions and identities of degenerate unipoly polynomials related to special numbers and polynomials.

2010 Mathematics Subject Classification.: 11B73, 11B83, 05A19.
Keywords: Degenerate polylogarithm functions, degenerate poly-Euler polynomials, degenerate unipoly functions, degenerate unipoly-Euler polynomials.

1. Introduction

In [1, 2], Carlitz initiated a study of degenerate versions of some special polynomials and numbers, namely the degenerate Bernoulli and Euler polynomials and numbers. Kim and Kim et al. [6-19] have studied the degenerate versions of special numbers and polynomials actively. This idea provides a powerful tool to define special numbers and polynomials of their degenerate versions. We can say that the notion of degenerate version forms a special class of polynomials because of their great applicability. The most important applications of these polynomials are in the theory of finite differences, analytic number theory, application in classical analysis, and statistics. Despite the applicability of special functions in classical analysis and statistics, they also arise in communications systems, quantum mechanics, nonlinear wave propagation, electric circuit theory, electromagnetic theory, etc.

As is well known, the classical Bernoulli polynomials $B_j(u)$ and the classical Euler polynomials $E_j(u)$ are respectively, usually defined by means of the follwing generating function as follows (see [5, 6, 8, 16, 18, 22]):

$$\frac{t}{e^z-1}e^{uz}=\sum_{j=0}^{\infty}B_j(u)\frac{z^j}{j!},\mid z\mid<2\pi,\ \frac{2}{e^z+1}e^{uz}=\sum_{j=0}^{\infty}E_j(u)\frac{z^j}{j!},\mid z\mid<\pi. \tag{1.1}$$

In case when $u=0$, $B_j=B_j(0)$ and $E_j=E_j(0)$ are respectively called the Bernoulli numbers and the Euler numbers.

From (1.1), we see

$$B_j:=B_j(0)=(-1)^jB_j,\ \ E_n:=2^nE_n\left(\frac{1}{2}\right),(n\in\mathbb{N}_0). \tag{1.2}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 10 July 2020 doi:10.20944/preprints202007.0202.v1

The notion of degenerate exponential function

$$e_{\lambda}^{u}(z)=(1+\lambda z)^{\frac{u}{\lambda}}$$

is considered without the limit case. That is, the degenerate of the exponential function e^{z} is equal to $(1+\lambda z)^{\frac{u}{\lambda}}$. It follows from here that the degenerate of a parameter z is $\frac{\log(1+\lambda z)}{z}$. This idea was firstly considered for degenerate Bernoulli and degenerate Euler polynomials, respectively given by Carlitz [1, 2] as follows:

$$\frac{z}{e_{\lambda}(z)-1}e_{\lambda}^{u}(z)=\sum_{j=0}^{\infty}B_{j,\lambda}(u)\frac{z^{j}}{j!},\ \frac{2}{e_{\lambda}(z)+1}e_{\lambda}^{u}(z)=\sum_{j=0}^{\infty}E_{j,\lambda}(u)\frac{z^{j}}{j!}. \tag{1.3}$$

In the case when $u=0$, $B_{j,\lambda}=B_{j,\lambda}(0)$ are called the degenerate Bernoulli numbers and $u=0$, $E_{j,\lambda}=E_{j,\lambda}(0)$ are called the degenerate Euler numbers.

Let $(u)_{n,\lambda}$ be the degenerate falling factorial sequence given by

$$(u)_{j,\lambda}=u(u-\lambda)\cdots(u-(j-1)\lambda),(j\geq 1),$$

with the assumption $(u)_{0,\lambda}=1$.

For $s\in\mathbb{Z}$, the polylogaritm function is defined by a power series in z as

$$\mathrm{Li}_{s}(z)=\sum_{j=1}^{\infty}\frac{j^{n}}{j^{s}}=z+\frac{z^{2}}{2^{s}}+\frac{z^{3}}{3^{s}},(\mid z\mid<1),(\text{see, [3, 4, 20]}). \tag{1.4}$$

It is notice that

$$\mathrm{Li}_{1}(z)=\sum_{j=1}^{\infty}\frac{z^{j}}{j}=-\log(1-z). \tag{1.5}$$

For $\lambda\in\mathbb{R}$, Kim-Kim [12] defined the degenerate version of the logarithm function, denoted by $\log_{\lambda}(1+z)$ as follows:

$$\log_{\lambda}(1+z)=\sum_{j=1}^{\infty}\lambda^{j-1}(1)_{j,1/\lambda}\frac{z^{j}}{j!},(\text{see, [11]}) \tag{1.6}$$

being the inverse of the degenerate version of the exponential function $e_{\lambda}(z)$ as has been shown below

$$e_{\lambda}(\log_{\lambda}(z))=\log_{\lambda}(e_{\lambda}(z))=z.$$

It is noteworthy to mention that

$$\lim_{\lambda\to 0}\log_{\lambda}(1+z)=\sum_{j=1}^{\infty}(-1)^{j-1}\frac{z^{j}}{j!}=\log(1+z).$$

The degenerate polylogarithm function [12] is defined by Kim-Kim to be

$$l_{k,\lambda}(z)=\sum_{j=1}^{\infty}\frac{(-\lambda)^{j-1}(1)_{j,1/\lambda}}{(j-1)!j^{k}}z^{n},(k\in\mathbb{Z},\mid z\mid<1). \tag{1.7}$$

It is clear that

$$\lim_{\lambda\to 0}l_{k,\lambda}(z)=\sum_{j=1}^{\infty}\frac{z^{j}}{j^{k}}=\mathrm{Li}_{k}(z),(\text{see [3, 4]}).$$

From (1.6) and (1.7), we get

$$l_{1,\lambda}(z)=\sum_{j=1}^{\infty}\frac{(-\lambda)^{j-1}(1)_{j,1/\lambda}}{j!}z^{j}=-\log_{\lambda}(1-z). \tag{1.8}$$

Very recently, Kim-Kim [12] introduced the new type degenerate version of the Bernoulli polynomials and numbers, by using the degenerate polylogarithm function as follows

$$\frac{l_{k,\lambda}(1-e_\lambda(-z))}{1-e_\lambda(-z)}e_\lambda^u(z)=\sum_{j=0}^{\infty}\beta_{j,\lambda}^{(k)}(u)\frac{z^j}{j!}. \tag{1.9}$$

On setting $x=0$, $\beta_{j,\lambda}^{(k)}=\beta_{j,\lambda}^{(k)}(0)$ are called the degenerate poly-Bernoulli numbers.

Lee-Kim-Jang [21] introduced the type 2 degenerate poly-Euler polynomials and numbers as follows

$$\frac{\mathrm{Ei}_k(\log(1+2t))}{t(e_\lambda(t)+1)}e_\lambda^x(t)=\sum_{n=0}^{\infty}E_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}. \tag{1.10}$$

When $x=0$, $E_{n,\lambda}^{(k)}=E_{n,\lambda}^{(k)}(0)$ are called the type 2 degenerate poly-Euler numbers.

The degenerate Stirling numbers of the first kind [17] are defined by

$$\frac{1}{k!}(\log_\lambda(1+z))^k=\sum_{j=k}^{\infty}S_{1,\lambda}(j,k)\frac{z^j}{j!},\ (k\geq 0), (\text{see } [19]). \tag{1.11}$$

It is clear that

$$\lim_{\lambda\to 0}S_{1,\lambda}(j,k)=S_1(j,k),$$

are calling the Stirling numbers of the first kind given by

$$\frac{1}{k!}(\log(1+z))^k=\sum_{j=k}^{\infty}S_1(j,k)\frac{z^j}{j!},\ (k\geq 0), (\text{see } [7, 11]).$$

The degenerate Stirling numbers of the second kind [9] are given by

$$\frac{1}{k!}(e_\lambda(z)-1)^k=\sum_{j=k}^{\infty}S_{2,\lambda}(j,k)\frac{z^j}{j!},\ (k\geq 0). \tag{1.12}$$

Note here that

$$\lim_{\lambda\to 0}S_{2,\lambda}(j,k)=S_2(j,k),$$

standing for the Stirling numbers of the second kind given by means of the following generating function:

$$\frac{1}{k!}(e^z-1)^k=\sum_{j=k}^{\infty}S_2(j,k)\frac{z^j}{j!},\ (k\geq 0), (\text{see, } [1\text{-}19]).$$

In this paper, we construct new type degenerate poly-Euler polynomials and numbers by using degenerate polylogarithm function and derive several properties on the degenerate poly-Euler numbers and polynomials. Furthermore, we introduce the degenerate unipoly-Euler polynomials attached to an arithmetic function, by using degenerate polylogarithm function and investigate some properties for those polynomials. Also, we give some new explicit expressions and identities of degenerate unipoly polynomials related to special numbers and polynomials.

2. A new type of degenerate poly-Euler numbers and polynomials

In this section, we define the new type degenerate Euler numbers and polynomials by using the degenerate polylogarithm function which are called the new type of degenerate poly-Euler numbers and polynomials as follows.

For $k \in \mathbb{Z}$, we define the new type degenerate Euler numbers, which are called the new type of degenerate poly-Euler numbers, as

$$\frac{1}{u} l_{k,\lambda}(u)|_{u=z(e_\lambda(z)+1)} = \frac{1}{e_\lambda(z)+1} l_{k,\lambda}(1-e_\lambda(-2z)) = \sum_{q=0}^{\infty} E_{q,\lambda}^{(k)} \frac{z^q}{q!}. \tag{2.1}$$

Note that

$$\sum_{q=0}^{\infty} E_{q,\lambda}^{(1)} \frac{z^q}{q!} = \frac{1}{z(e_\lambda(z)+1)} l_{1,\lambda}(1-e_\lambda(-2z)) = \frac{2}{e_\lambda(z)+1} = \sum_{q=0}^{\infty} E_{q,\lambda} \frac{z^q}{q!}. \tag{2.2}$$

Thus, we have

$$E_{q,\lambda}^{(1)} = E_{q,\lambda}, (q \geq 0).$$

Now, we consider the new type degenerate Euler polynomials which are called the new type of degenerate poly-Euler polynomials and fiven by

$$\frac{l_{k,\lambda}(1-e_\lambda(-2z))}{z(e_\lambda(z)+1)} e_\lambda^u(z) = \sum_{q=0}^{\infty} E_{q,\lambda}^{(k)}(u) \frac{z^q}{q!}. \tag{2.3}$$

Note here that $E_{q,\lambda}^{(k)} = E_{q,\lambda}^{(k)}(0)$. From (2.1) and (2.3), we see that

$$\begin{aligned} \sum_{p=0}^{\infty} E_{p,\lambda}^{(k)}(u) \frac{z^p}{p!} &= \frac{l_{k,\lambda}(1-e_\lambda(-2z))}{z(e_\lambda(z)+1)} e_\lambda^u(z) \\ &= \sum_{m=0}^{\infty} E_{m,\lambda}^{(k)} \frac{t^m}{m!} \sum_{n=0}^{\infty} \frac{(u)_{n,\lambda}}{n!} t^n \\ &= \sum_{p=0}^{\infty} \left(\sum_{i=0}^{p} \binom{p}{i} E_{i,\lambda}^{(k)}(u)_{p-i,\lambda} \right) \frac{z^p}{p!}. \end{aligned} \tag{2.4}$$

Therefore, by equation (2.4), we obtain the following theorem.

Theorem 2.1. Let $p \geq 0$. Then

$$E_{p,\lambda}^{(k)}(u) = \sum_{i=0}^{p} \binom{p}{i} E_{i,\lambda}^{(k)}(u)_{p-i,\lambda}.$$

Now, we observe that

$$\begin{aligned} \frac{d}{dx} e_\lambda(-2x) &= \frac{d}{dx} \sum_{m=0}^{\infty} \frac{(1)_{m,\lambda}}{m!} (-2x)^m \\ = \sum_{m=1}^{\infty} \frac{(-2)(1)_{m,\lambda}}{(m-1)!} (-2x)^{m-1} &= \sum_{m=0}^{\infty} \frac{(-2)(1)_{m+1,\lambda}}{m!} (-2x)^m \\ = -2 \left(\sum_{m=0}^{\infty} \frac{(1)_{m,\lambda}}{m!} (-2x)^m (1-m\lambda) \right) &= -2e_\lambda(-2x) + \lambda 2x \frac{d}{dx} e_\lambda(-2x) \\ \frac{d}{dx} e_\lambda(-2x) = \frac{-2}{(1-2\lambda x)} &= -2e_\lambda^{1-\lambda}(-2x). \end{aligned} \tag{2.5}$$

Therefore, by (2.5), we obtain the following lemma.

Lemma 2.1. For $\lambda \in \mathbb{R}$, we have

$$\frac{d}{dx} e_\lambda(-2x) = 2e_\lambda^{1-\lambda}(-2x). \tag{2.6}$$

We observe that

$$\frac{d}{dx}l_{k,\lambda}(1-e_\lambda(-2x)) = \frac{d}{dx}\sum_{m=1}^{\infty}\frac{(-\lambda)^{n-1}(1)_{m,1/\lambda}}{(m-1)!m^k}(1-e_\lambda(-2x))^m$$
$$\frac{1}{(1-e_\lambda(-2x))}l_{k-1,\lambda}(1-e_\lambda(-2x)). \tag{2.7}$$

From (2.1), (2.6) and (2.7), for $k \geq 2$, we obtain

$$\sum_{n=0}^{\infty}E_{n,\lambda}^{(k)}\frac{x^n}{n!} = \frac{1}{x(e_\lambda(x)+1)}l_{k-1,\lambda}(1-e_\lambda(-2x))$$
$$= \frac{1}{x(e_\lambda(x)+1)}\int_0^x \frac{2e_\lambda^{1-\lambda}(-2t)}{1-e_\lambda(-2t)}\int_0^t \frac{2e_\lambda^{1-\lambda}(-2t)}{1-e_\lambda(-2t)}\cdots\int_0^t \frac{2e_\lambda^{1-\lambda}(-2t)}{1-e_\lambda(-2t)}2tdtdt\cdots dt$$
$$= \frac{2}{x(e_\lambda(x)+1)}\int_0^x \underbrace{\frac{2e_\lambda^{1-\lambda}(-2t)}{1-e_\lambda(-2t)}\int_0^t \frac{2e_\lambda^{1-\lambda}(-2t)}{1-e_\lambda(-2t)}\cdots\int_0^t \frac{2e_\lambda^{1-\lambda}(-2t)}{1-e_\lambda(-2t)}}_{(k-2)-\text{times}}2tdtdt\cdots dt. \tag{2.8}$$

Therefore, by (2.8), we get the following theorem.

Theorem 2.2. Let $k \geq 2$. Then

$$\sum_{n=0}^{\infty}E_{n,\lambda}^{(k)}\frac{x^n}{n!} = \frac{2}{x(e_\lambda(x)+1)}\int_0^x \underbrace{\frac{2e_\lambda^{1-\lambda}(-2t)}{1-e_\lambda(-2t)}\int_0^t \frac{2e_\lambda^{1-\lambda}(-2t)}{1-e_\lambda(-2t)}\cdots\int_0^t \frac{2e_\lambda^{1-\lambda}(-2t)}{1-e_\lambda(-2t)}}_{(k-2)-\text{times}}2tdtdt\cdots dt. \tag{2.9}$$

For $k=2$, by Theorem 2.2, we get

$$\sum_{n=0}^{\infty}E_{n,\lambda}^{(2)}\frac{x^n}{n!} = \frac{2}{x(e_\lambda(x)+1)}\int_0^x \frac{-2t}{e_\lambda(-2t)-1}e_\lambda^{1-\lambda}(-2t)dt$$
$$= \frac{2}{x(e_\lambda(x)+1)}\int_0^x \sum_{n=0}^{\infty}B_{n,\lambda}(1-\lambda)\frac{(-2t)^n}{n!}dt$$
$$= \frac{2}{e_\lambda(x)+1}\sum_{m=0}^{\infty}(-2)^m\frac{B_{m,\lambda}(1-\lambda)}{m+1}\frac{x^m}{m!}$$
$$= \left(\sum_{n=0}^{\infty}E_{n,\lambda}\frac{x^n}{n!}\right)\left(\sum_{m=0}^{\infty}(-2)^m\frac{B_{m,\lambda}(1-\lambda)}{m+1}\frac{x^m}{m!}\right)$$
$$L.H.S = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}(-2)^m\frac{B_{m,\lambda}(1-\lambda)}{m+1}E_{n-m,\lambda}\right)\frac{x^n}{n!}. \tag{2.10}$$

Therefore, by (2.10), we get the following theorem.

Theorem 2.3. Let $n \geq 0$. Then

$$E_{n,\lambda}^{(2)} = \sum_{m=0}^{n}\binom{n}{m}(-2)^m\frac{B_{m,\lambda}(1-\lambda)}{m+1}E_{n-m,\lambda} = \sum_{m=0}^{n}\binom{n}{m}(-2)^{n-m}\frac{B_{n-m,\lambda}(1-\lambda)}{n-m+1}E_{m,\lambda}.$$

In general, from (2.8), we note that

$$\sum_{n=0}^{\infty}E_{n,\lambda}^{(k)}\frac{x^n}{n!} = \frac{2}{x(e_\lambda(x)+1)}\int_0^x \underbrace{\frac{2e_\lambda^{1-\lambda}(-2t)}{1-e_\lambda(-2t)}\int_0^t \frac{2e_\lambda^{1-\lambda}(-2t)}{1-e_\lambda(-2t)}\cdots\int_0^t \frac{2e_\lambda^{1-\lambda}(-2t)}{1-e_\lambda(-2t)}}_{(k-2)-\text{times}}2tdtdt\cdots dt$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 10 July 2020 doi:10.20944/preprints202007.0202.v1

$$= \frac{2}{e_\lambda(x)+1} \sum_{n_1,n_2,\cdots,n_{k-1}=m}^{\infty} (-2)^m \begin{pmatrix} n \\ n_1,n_2,\cdots,n_k \end{pmatrix} \frac{B_{n_1,\lambda}}{n_1+1} \frac{B_{n_2,\lambda}}{n_1+n_2+1} \cdots \frac{B_{n_{k-1},\lambda}}{n_1+\cdots+n_{k-1}+1}$$

$$L.H.S = \sum_{n=0}^{\infty} \left(\sum_{n_1,n_2,\cdots,n_k=m} \binom{n}{m} (-2)^m \begin{pmatrix} n \\ n_1,n_2,\cdots,n_k \end{pmatrix} \frac{B_{n_1,\lambda}}{n_1+1} \frac{B_{n_2,\lambda}}{n_1+n_2+1} \cdots \frac{B_{n_{k-1},\lambda}}{n_1+\cdots+n_{k-1}+1} E_{n-m,\lambda} \right) \frac{x^n}{n!}. \tag{2.11}$$

Therefore by comparing the coefficients on both sides of (2.11), we obtain the following theorem.

Theorem 2.4. Let $k \in \mathbb{Z}$ and $n \geq 0$, we have

$$E_{n,\lambda}^{(k)} = \sum_{n_1,n_2,\cdots,n_k=m} \binom{n}{m} (-2)^m \begin{pmatrix} n \\ n_1,n_2,\cdots,n_k \end{pmatrix} \frac{B_{n_1,\lambda}}{n_1+1} \frac{B_{n_2,\lambda}}{n_1+n_2+1} \cdots \frac{B_{n_{k-1},\lambda}}{n_1+\cdots+n_{k-1}+1} E_{n-m,\lambda}.$$

From (2.3), we note that

$$\sum_{j=0}^{\infty} E_{j,\lambda}^{(k)}(u) \frac{z^j}{j!} = \frac{l_{k,\lambda}(1-e_\lambda(-2z))}{z(e_\lambda(z)+1)} e_\lambda^u(z)$$

$$= \frac{l_{k,\lambda}(1-e_\lambda(-2z))}{z(e_\lambda(z)+1)(e_\lambda(z)-1)} e_\lambda^u(z)(e_\lambda(z)-1)$$

$$= \frac{1}{z} \left(\frac{l_{k,\lambda}(1-e_\lambda(-2z))}{e_{\frac{\lambda}{2}}(2z)-1} e_\lambda^u \right) (e_\lambda(z)-1)$$

$$= \frac{1}{z} \left(\sum_{m=0}^{\infty} \beta_{m,\frac{\lambda}{2}} (\frac{u}{2}) \frac{2^m z^m}{m!} \right) \left(\sum_{j=1}^{\infty} (1)_{j,\lambda} \frac{z^j}{j!} \right)$$

$$= \frac{1}{z} \left(\sum_{m=0}^{\infty} \beta_{m,\frac{\lambda}{2}} (\frac{u}{2}) \frac{2^m z^m}{m!} \right) \left(\sum_{j=0}^{\infty} (1)_{j+1,\lambda} \frac{z^{j+1}}{(j+1)!} \right)$$

$$= \left(\sum_{m=0}^{\infty} \beta_{m,\frac{\lambda}{2}} (\frac{u}{2}) \frac{2^m z^m}{m!} \right) \left(\sum_{j=0}^{\infty} \frac{(1)_{j+1,\lambda}}{j+1} \frac{z^j}{j!} \right)$$

$$L.H.S = \sum_{j=0}^{\infty} \left(\sum_{m=0}^{j} \binom{m}{j} 2^m \beta_{m,\frac{\lambda}{2}} (\frac{u}{2}) \frac{(1)_{j-m+1,\lambda}}{j-m+1} \right) \frac{z^j}{j!}. \tag{2.12}$$

By comparing the coefficients on both sides of (2.12), we get the following theorem.

Theorem 2.5. Let $k \in \mathbb{Z}$ and $n \geq 0$, we have

$$E_{j,\lambda}^{(k)}(u) = \sum_{m=0}^{j} \binom{m}{j} 2^m \beta_{m,\frac{\lambda}{2}} (\frac{u}{2}) \frac{(1)_{j-m+1,\lambda}}{j-m+1}.$$

From (2.3), () and (), we note that

$$\sum_{j=0}^{\infty} E_{j,\lambda}^{(k)}(u) \frac{z^j}{j!} = \left(\frac{2}{e_\lambda(z)+1} e_\lambda^u(z) \right) \frac{1}{2z} \sum_{m=1}^{\infty} \frac{(-\lambda)^{m-1}(1)_{m,1/\lambda}}{(m-1)! m^k} (1-e_\lambda(-2z))^m$$

$$= \left(\frac{2}{e_\lambda(z)+1} e_\lambda^u(z) \right) \frac{1}{2z} \sum_{l=1}^{\infty} \left(\sum_{m=1}^{l} \frac{(1)_{m,1/\lambda}(-1)^{l-1}}{m^{k-1}} \lambda^{m-1} 2^l S_{2,\lambda}(l,m) \right) \frac{z^l}{l!}$$

$$= \left(\sum_{j=0}^{\infty} E_{j,\lambda}^{(k)} \frac{z^j}{j!}\right) \frac{1}{2z} \sum_{l=0}^{\infty} \left(\sum_{m=1}^{l+1} \frac{(1)_{m,1/\lambda}(-1)^l}{m^{k-1}} \lambda^{m-1} 2^{l+1} S_{2,\lambda}(l+1,m)\right) \frac{z^{l+1}}{(l+1)!}$$

$$= \left(\sum_{j=0}^{\infty} E_{j,\lambda}^{(k)} \frac{z^j}{j!}\right) \sum_{l=0}^{\infty} \left(\sum_{m=1}^{l+1} \frac{(1)_{m,1/\lambda}(-1)^l}{(l+1)m^{k-1}} \lambda^{m-1} 2^{l} S_{2,\lambda}(l+1,m)\right) \frac{z^{l+1}}{(l+1)!}$$

$$L.H.S = \sum_{j=0}^{\infty} \left(\sum_{l=0}^{n} \binom{j}{l} \sum_{m=1}^{l+1} \frac{(1)_{m,1/\lambda}(-1)^l}{(l+1)m^{k-1}} \lambda^{m-1} 2^{l} S_{2,\lambda}(l+1,m) E_{j-l,\lambda}^{(k)}\right) \frac{z^j}{j!}. \quad (2.13)$$

Therefore, by comparing the coefficients on both side of (2.13), we get the following theorem.

Theorem 2.6. Let $j \geq 0$. Then

$$E_{j,\lambda}^{(k)}(u) = \sum_{l=0}^{n} \binom{j}{l} \sum_{m=1}^{l+1} \frac{(1)_{m,1/\lambda}(-1)^l}{(l+1)m^{k-1}} \lambda^{m-1} 2^{l} S_{2,\lambda}(l+1,m) E_{j-l,\lambda}^{(k)}.$$

From (2.3) and ( ), we have

$$\sum_{j=0}^{\infty} E_{j,\lambda}^{(k)}(u) \frac{z^j}{j!} = \frac{l_{k,\lambda}(1-e_\lambda(-2z))}{z(e_\lambda(z)+1)} e_\lambda^u(z)$$

$$= \frac{l_{k,\lambda}(1-e_\lambda(-2z))}{z(e_\lambda(z)+1)} (e_\lambda(z)-1+1)^u$$

$$= \frac{l_{k,\lambda}(1-e_\lambda(-2z))}{z(e_\lambda(z)+1)} \left(\sum_{m=0}^{\infty} \binom{x}{m} (e_\lambda(z)-1)^m\right)$$

$$= \frac{l_{k,\lambda}(1-e_\lambda(-2z))}{z(e_\lambda(z)+1)} \left(\sum_{m=0}^{\infty} (x)_m \sum_{m=l}^{\infty} S_{2,\lambda}(l,m) \frac{z^l}{l!}\right)$$

$$= \left(\sum_{j=0}^{\infty} E_{j,\lambda}^{(k)} \frac{z^j}{j!}\right) \left(\sum_{l=0}^{\infty} \sum_{m=0}^{l} (x)_m S_{2,\lambda}(l,m) \frac{z^l}{l!}\right)$$

$$L.H.S = \sum_{j=0}^{\infty} \left(\sum_{l=0}^{j} \binom{j}{l} \sum_{m=0}^{l} (x)_m S_{2,\lambda}(l,m) E_{j,\lambda}^{(k)}\right) \frac{z^j}{j!}. \quad (2.14)$$

By comparing the coefficients of t^n on both sides, we obtain the following theorem.

Theorem 2.7. Let $k \in \mathbb{Z}$ and $n \geq 0$, we have

$$E_{j,\lambda}^{(k)}(u) = \sum_{l=0}^{j} \binom{j}{l} \sum_{m=0}^{l} (x)_m S_{2,\lambda}(l,m) E_{j,\lambda}^{(k)}.$$

From (2.1), we observe that

$$l_{k,l}(1-e_\lambda(-2z)) = z(e_\lambda(z)+1) \sum_{s=0}^{\infty} E_{s,\lambda}^{(k)} \frac{z^s}{s!}$$

$$= z\left(\sum_{j=1}^{\infty} \frac{(1)_{j,\lambda} z^j}{j!} + 1\right) \sum_{s=0}^{\infty} E_{s,\lambda}^{(k)} \frac{z^s}{s!}$$

$$= z\sum_{j=0}^{\infty}\left(\sum_{s=0}^{j}\binom{j}{s}E_{s,\lambda}^{(k)}(1)_{j-s,\lambda}+E_{j,\lambda}^{(k)}\right)\frac{z^j}{j!}$$
$$=\sum_{j=0}^{\infty}\left(\sum_{s=0}^{j}\binom{j}{s}E_{s,\lambda}^{(k)}(1)_{j-s,\lambda}+E_{j,\lambda}^{(k)}\right)(j+1)\frac{z^{j+1}}{j!}$$
$$=\sum_{j=1}^{\infty}j\left(\sum_{s=0}^{j-1}\binom{j-1}{s}E_{s,\lambda}^{(k)}(1)_{j-s-1,\lambda}+E_{j-1,\lambda}^{(k)}\right)\frac{z^j}{j!}$$
$$L.H.S=\sum_{j=1}^{\infty}j\left(E_{j-1,\lambda}^{(k)}(1)+E_{j-1,\lambda}^{(k)}\right)\frac{z^j}{j!}. \tag{2.15}$$

On the other hand,

$$l_{k,l}(1-e_\lambda(-2z))=\sum_{m=1}^{\infty}\frac{(-\lambda)^{m-1}(1)_{m,1/\lambda}}{(m-1)!m^k}(1-e_\lambda(-2z))^m$$
$$=\sum_{j=1}^{\infty}\left(\sum_{m=1}^{j}\frac{(1)_{m,1/\lambda}(-1)^{j-1}}{m^{k-1}}\lambda^{m-1}2^jS_{2,\lambda}(j,m)\right)\frac{z^j}{j!}, \tag{2.16}$$

Therefore, by equations (2.15) and (2.16), we get the following theorem.

Theorem 2.8. Let $k\in\mathbb{Z}$ and $j\geq 1$. Then

$$E_{j-1,\lambda}^{(k)}(1)+E_{j-1,\lambda}^{(k)}=\frac{2^j}{j}\sum_{m=1}^{j}\frac{(1)_{m,1/\lambda}(-1)^{j-1}}{m^{k-1}}\lambda^{m-1}S_{2,\lambda}(j,m).$$

3. New type degenerate unipoly-Euler numbers and polynomials

Let p be any arithmetic function which is a real or complex valued function defined on the set of positive integers $\mathbb{N}$. Kim-Kim [7] defined the unipoly function attached to polynomials $p(x)$ by

$$u_k(x|p)=\sum_{n=1}^{\infty}\frac{p(n)}{n^k}x^n,(k\in\mathbb{Z}). \tag{3.1}$$

Moreover,

$$u_k(x|1)=\sum_{n=1}^{\infty}\frac{x^n}{n^k}=\mathrm{Li}_k(x),(\text{see [4]}), \tag{3.2}$$

is the ordinary polylogaritm function.

In this paper, we define the degenerate unipoly function attached to polynomials $p(x)$ as follows:

$$u_{k,\lambda}(x|p)=\sum_{i=1}^{\infty}p(i)\frac{(-\lambda)^{i-1}(1)_{i,1/\lambda}}{i^k}x^i. \tag{3.3}$$

It is worthy to note that

$$u_{k,\lambda}\left(x|\frac{1}{\Gamma}\right)=\mathrm{l}_{k,\lambda}(x) \tag{3.4}$$

is the degenerate polylogarithm function.

Now, we define the new type degenerate unipoly-Euler polynomials attached to polynomials $p(x)$ by

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 10 July 2020 doi:10.20944/preprints202007.0202.v1

$$\frac{u_{k,\lambda}\left(1-e_\lambda(-2z)|p\right)}{z(e_\lambda(z)+1)}e_\lambda^u(z)=\sum_{r=0}^{\infty}E_{r,\lambda,p}^{(k)}(u)\frac{z^r}{r!}. \tag{3.5}$$

In the case when $u=0$, $E_{r,\lambda,p}^{(k)}=E_{r,\lambda,p}^{(k)}(0)$ are called the new type degenerate unipoly-Euler numbers attached to p.

From (3.5), we see

$$\begin{aligned}\sum_{n=0}^{\infty}E_{n,\lambda,\frac{1}{\Gamma}}^{(k)}\frac{z^n}{n!}&=\frac{1}{z(e_\lambda(z)+1)}u_{k,\lambda}\left(1-e_\lambda(-2z)|\frac{1}{\Gamma}\right)\\&=\frac{1}{z(e_\lambda(z)+1)}\sum_{r=1}^{\infty}\frac{(-\lambda)^{r-1}(1)_{r,1/\lambda}(1-e_\lambda(-2z))^r}{r^k(r-1)!}\\&=\frac{1}{z(e_\lambda(z)+1)}u_{k,\lambda}\left(1-e_\lambda(-2z)\right)=\sum_{n=0}^{\infty}E_{n,\lambda}^{(k)}\frac{z^n}{n!}.\end{aligned} \tag{3.6}$$

Thus, by (3.6), we have

$$E_{n,\lambda,\frac{1}{\Gamma}}^{(k)}=E_{n,\lambda}^{(k)}. \tag{3.7}$$

From (3.5), we have

$$\begin{aligned}&\sum_{n=0}^{\infty}E_{n,\lambda,p}^{(k)}\frac{z^n}{n!}=\frac{1}{z(e_\lambda(z)+1)}u_{k,\lambda}\left(1-e_\lambda(-2z)|p\right)\\&=\frac{1}{z(e_\lambda(z)+1)}\sum_{m=1}^{\infty}\frac{p(m)(-\lambda)^{m-1}(1)_{m,1/\lambda}}{m^k}(1-e_\lambda(-2z))^m\\&=\left(\frac{2}{z(e_\lambda(z)+1)}\right)\sum_{l=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{m!p(m)(-1)^{l-1}\lambda^{m-1}(1)_{m,1/\lambda}2^{l-1}}{m^k}\right)S_{2,\lambda}(l,m)\frac{z^l}{l!}\\&=\left(\sum_{n=0}^{\infty}E_{n,\lambda}\frac{z^n}{n!}\right)\left(\sum_{l=0}^{\infty}\left(\sum_{m=1}^{l+1}\frac{m!p(m)(-1)^{l}\lambda^{m-1}(1)_{m,1/\lambda}2^{l}}{m^k}\frac{S_{2,\lambda}(l+1,m)}{l+1}\right)\right)\frac{z^l}{l!}\\&L.H.S=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=1}^{l+1}\binom{n}{l}\frac{m!p(m)(-1)^{l}\lambda^{m-1}(1)_{m,1/\lambda}2^{l}}{m^k}\frac{S_{2,\lambda}(l+1,m)}{l+1}E_{n-l,\lambda}\right)\frac{z^n}{n!}.\end{aligned} \tag{3.8}$$

By equation (3.8), we get the following theorem.

Theorem 3.1. Let $n\geq 0$ and $k\in\mathbb{Z}$. Then

$$E_{n,\lambda,p}^{(k)}=\sum_{l=0}^{n}\sum_{m=1}^{l+1}\binom{n}{l}\frac{m!p(m)(-1)^{l}\lambda^{m-1}(1)_{m,1/\lambda}2^{l}}{m^k}\frac{S_{2,\lambda}(l+1,m)}{l+1}E_{n-l,\lambda}.$$

From (3.5), we have

$$\begin{aligned}\sum_{j=0}^{\infty}E_{j,\lambda,p}^{(k)}(x)\frac{z^j}{j!}&=\frac{1}{z(e_\lambda(z)+1)}u_{k,\lambda}(1-e_\lambda(-2z)|p)\left(e_\lambda(z)-1+1\right)^x\\&=\frac{u_{k,\lambda}\left(1-e_\lambda(-2z)|p\right)}{z(e_\lambda(z)+1)}\sum_{i=0}^{\infty}(x)_i\frac{(e_\lambda(z)-1)^i}{i!}\end{aligned}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 10 July 2020 doi:10.20944/preprints202007.0202.v1

$$= \sum_{j=0}^{\infty} E_{j,\lambda,p}^{(k)} \frac{z^j}{j!} \sum_{i=0}^{\infty} (x)_i \sum_{q=i}^{\infty} S_{2,\lambda}(q,i) \frac{z^q}{q!}$$

$$= \sum_{j=0}^{\infty} E_{j,\lambda,p}^{(k)} \frac{z^j}{j!} \sum_{q=0}^{\infty} \sum_{i=0}^{q} (x)_i S_{2,\lambda}(q,i) \frac{z^q}{q!}$$

$$L.H.S = \sum_{j=0}^{\infty} \left(\sum_{q=0}^{j} \sum_{i=0}^{q} \binom{j}{q} (x)_i S_{2,\lambda}(q,i) E_{j-q,\lambda,p}^{(k)} \right) \frac{z^j}{j!}. \tag{3.9}$$

By equation (3.9), we get the following theorem.

Theorem 3.2. Let j be nonnegative integer and $k \in \mathbb{Z}$. Then

$$E_{j,\lambda,p}^{(k)}(x) = \sum_{q=0}^{j} \sum_{i=0}^{q} \binom{j}{q} (x)_i S_{2,\lambda}(q,i) E_{j-q,\lambda,p}^{(k)}.$$

By using (1.3)and (3.5), we get

$$\sum_{n=0}^{\infty} E_{n,\lambda,p}^{(k)}(x) \frac{z^n}{n!} = \frac{u_{k,\lambda}(1 - e_\lambda(-2z)|p)}{z(e_\lambda(z)+1)} e_\lambda^x(z)$$

$$= \frac{u_{k,\lambda}(1 - e_\lambda(-2z)|p)}{z(e_\lambda(z)+1)(e_\lambda(z)-1)} e_\lambda^x(z)(e_\lambda(z)-1)$$

$$= \frac{1}{z} \left(\frac{u_{k,\lambda}(1 - e_\lambda(-2z)|p)}{e_{\frac{\lambda}{2}}(2z) - 1} e_\lambda^x(z) \right) (e_\lambda(z)-1)$$

$$= \frac{1}{z} \left(\sum_{j=0}^{\infty} \beta_{j,\frac{\lambda}{2},p}^{(k)} \left(\frac{x}{2}\right) \frac{2^j z^j}{j!} \right) \left(\sum_{n=1}^{\infty} (1)_{n,\lambda} \frac{z^n}{n!} \right)$$

$$= \frac{1}{z} \left(\sum_{j=0}^{\infty} \beta_{j,\frac{\lambda}{2},p}^{(k)} \left(\frac{x}{2}\right) \frac{2^j z^j}{j!} \right) \left(\sum_{n=0}^{\infty} (1)_{n+1,\lambda} \frac{z^{n+1}}{(n+1)!} \right)$$

$$= \left(\sum_{j=0}^{\infty} \beta_{j,\frac{\lambda}{2},p}^{(k)} \left(\frac{x}{2}\right) \frac{2^j z^j}{j!} \right) \left(\sum_{n=0}^{\infty} \frac{(1)_{n+1,\lambda}}{n+1} \frac{z^n}{n!} \right)$$

$$L.H.S = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \binom{n}{j} \frac{2^j (1)_{n-j+1,\lambda}}{n-j+1} \beta_{j,\frac{\lambda}{2},p}^{(k)} \left(\frac{x}{2}\right) \right) \frac{z^n}{n!}. \tag{3.10}$$

By equation (3.10), we get the following theorem.

Theorem 3.3. Let $n \geq 0$ and $k \in \mathbb{Z}$. Then

$$E_{n,\lambda,p}^{(k)}(x) = \sum_{j=0}^{n} \binom{n}{j} \frac{2^j (1)_{n-j+1,\lambda}}{n-j+1} \beta_{j,\frac{\lambda}{2},p}^{(k)} \left(\frac{x}{2}\right).$$

4. Conclusion

In this paper, we introduced the new type of degenerate poly-Euler polynomials and numbers by using the degenerate polylogarithm function and derive several properties on the degenerate poly-Euler numbers and polynomials. We represented the generating function of the degenerate poly-Euler numbers by iterated integrals in Theorem 2.2, 2.3, and 2.4 and explicit degenerate poly-Euler polynomials in terms of the Bernoulli polynomials and degenerate Stirling numbers of the second kind in Theorem 2.5. We also represented those numbers in terms of the degenerate Stirling numbers of the second kind in Theorem 2.6, 2.7 and 2.8. In the final section, we defined the degenerate unipoly-Euler polynomials by using the degenerate polylogarithm function and obtained the identity degenerate unipoly-Euler polynomials in terms of the degenerate Euler numbers and degenerate Stirling numbers of the second kind in Theorem 3.1, the degenerate unipoly-Euler numbers and the degenerate Stirling numbers of the second kind in Theorem 3.3.

Author Contributions: All authors contributed equally to the manuscript and typed, read, and approved final manuscript.

Funding: None.

Acknowledgements: None.

Conflict of Interest: The authors declare no conflict of interest.

References

[1] Carlitz, L. Degenerate Stirling, Bernoulli and Eulerian numbers. Utilitas Math. **1979**, 15, 51-88.
[2] Carlitz, L. A degenerate Staudt-Clausen theorem, Arch. Math. (Basel) **1956**, 7, 28-33.
[3] Hamahata, Y. Poly-Euler polynomials and Arakwa-Kaneko type zeta function. Funct. Approx. **2014**, 51, 7-22.
[4] Kaneko, M. Poly-Bernoulli numbers. J. Théor Nombres Bordeaux. **1997**, 9(1), 221-228.
[5] Khan, W.A. A new class of degenerate Frobenius-Euler-Hermite polynomials. Adv. Stud. Contemp. Math. (Kyungshang). **2018**, 28(4), 567-576.
[6] Kim, D.S.; Kim, T. Higher-order Bernoulli and poly-Bernoulli mixed type polynomials. Georgian Math. J. bf 2015, 22, 265-272.
[7] Kim, D.S.; Kim, T. A note on polyexponential and unipoly functions. Russ. J. Math. Phys. bf 2019, 26(1), 40-49.
[8] Kim, D.S.; Kim, T.; Seo, J. A note on Changhee polynomials and numbers. Adv. Stud. Theor. Phys. **2013**, 7(1), 993-1003.
[9] Kim, T. A note on degenerate Stirling polynomials of the second kind. Proc. Jangjeon Math. Soc. **2017**, 20(3), 319-331.
[10] Kim, T.; Jang,L.-C.; Kim, D.S.; Kim, H.-Y. Some identities on type 2 degenerate Bernoulli polynomials of the second kind. Symmetry. **2020**, 12(4), Article ID 510(2020).
[11] Kim, T.; Kim, D.S. Degenerate polyexponential functions and degenerate Bell polynomials. J. Math. Anal. Appl. **2020**, 487(2), 124017.
[12] Kim, D.S.; Kim, T. A note on a new type of degenerate Bernoulli numbers. Russ. J. Math. Phys. **2020**, 27(2), 227-235.
[13] Kim, T.; Kim, D.S.; Kim, H.Y.; Kwon, J. Some identities of degenerate Bell polynomials. Mathematics. **2020**, 8(1), Article ID 40.
[14] Kim, T.; Kim, D.S.; Kim, H.Y.; Kwon, J. Degenerate Stirling polynomials of the second kind and some applications. Symmetry. **2019**, 11(8), Article ID 1046.

[15] Kim, T.; Kim, D.S. Degenerate Laplace transform and degenerate gamma function. Russ. J. Math. Phys. **2017**, 24(2), 241-248.
[16] Kim, T.; Khan, W.A.; Sharma, S. K.; Ghayasuddin, M. A note on parametric kinds of the degenerate poly-Bernoulli and poly-Genocchi polynomials. Symmetry. **2020**, 12(4), Article ID 614.
[17] Kim, T.; Kim, D.S.; Kwon, J.; Lee, H. Degenerate polyexpnential functions and type 2 degenerate poly-Bernoulli numbers and polynomials. Adv. Differ. Equ. **2020**, 2020:168, 12pp.
[18] Kim, T.; Kim, D.S.; Dolgy, D.V.; Kwon, J. Some identities on degenerate Genocchi and Euler numbers. Informatica. **2020**, 31(4), 42-51.
[19] Kim, T.; Kim, D.S.; Kwon, J.; Kim, H.Y. A note on degenerate Genocchi and poly-Genocchi numbers and polynomials. J. Ineq. Appl. **2020**, 2020:110, 13pp.
[20] Lewin, L. Polylogarithms and Associated Functions; With a Foreword by A.J. Vander Poorten; North-Holland Publishing Co.: Amsterdam, NY, USA, 1981.
[21] Lee, D.S.; Kim, H.-Y.; Jang, L.-C. type 2 degenerate poly-Euler polynomials. Symmetry. **2020**, 12, Article ID 1011.
[22] Kucukoglu, I.; Simsek, B.; Simsek, Y. An approach to negative hypergeometric distribution by generating function for special numbers and polynomials. Turk. J. Math. **2019**, 43, 2337-2353.