Generation and Application of Nested Entanglement in Matryoshka Quantum Resource-States

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Multipartite entanglement is a resource for application in disparate protocols, of computing, communication and cryptography. Nested entanglement provides resource-states for quantum information processing. In this paper, *Matryoshka* quantum resource-states, which contain nested entanglement patterns, has been studied. A novel scheme for the generation of such quantum states has been proposed using an anisotropic XY spin-spin interaction-based model. The application of the *Matryoshka GHZ-Bell states* for n-qubit teleportation is reviewed and an extension to more general *Matryoshka ExhS-Bell states* is posited. An example of *Matryoshka ExhS-Bell states* is given in the form of the genuinely entangled seven-qubit *Xin-Wei Zha state*. Generation, characterisation and application of this seven-qubit resource state in theoretical schemes for quantum teleportation of arbitrary one, two and three qubits states, bidirectional teleportation of arbitrary two qubit states and probabilistic circular controlled teleportation are presented.

Keywords: Quantum Computation, Multipartite Entanglement, Quantum State Sharing

Introduction. Quantum Entanglement, along with other general non-local quantum correlations, has sparked a revolution in Physics that has led to the emergence of Quantum Information (processing) [1–10]. It has been used in applications such as teleportation and super-dense coding [11–13]. Bennet et al first proposed a scheme for quantum teleportation, wherein a genuinely entangled Bell state was used to transmit an arbitrary single qubit [14]. Many different kinds of entangled quantum states have been used to teleport arbitrary quantum states since then, including W states [15, 16], Bell states [17, 18], GHZ states [19, 20] and multiqubit states [21–23]. There have been hop-by-hop and multi-hop quantum teleportation schemes proposed since then, as well as schemes to teleport GHZ-like states using two types of four-qubit states [24, 25]. Teleportation has been proposed in two-copy quantum teleportation scheme [26], using cluster states [27], in higher dimensions [28] and also shown to be possible over atmosphere channels [29]. More recently, various derivatives of the standard teleportation scheme have been proposed, including those used for controlled teleportation [30, 31], bidirectional teleportation [20, 32–35], quantum secret sharing [36–38], quantum operation sharing [39, 40] and arbitrated quantum teleportation [41, 42]. What has also been of interest has been controlled teleportation of an entangled state in a noisy environment [42, 43].

This broad area of research around entanglement and teleportation in noisy environments is related to another important area of work that centres around decoherence, decoherence-free subspaces and Markovian as well as Non-Markovian quantum noises [44–52].

Entanglement has also been key in quantum communication protocols [53–55], with Quantum Repeaters and Entanglement Purification being the subject of interest lately [56–58]. For information transfer using entanglement between parties, one needs an established entangled channel-state and means of classical communication. For a large number of parties, multipartite entanglement and entangled multiqubit states play the preeminent role, with states varying from GHZ- and W-states to clusters states [59, 60]. Teleportation of an arbitrary single qubit state using a channel comprising of an EPR pair was first demonstrated by Bennett et al [14]. Lately, W-GHZ composite states have been used for teleportation, remote state sharing as well as superdense coding of arbitrary quantum states [61]. These composite (*Matryoshka*) quantum states contain nested entanglement, which can be used for applications in quantum information processing. In this paper, the generation, characterisation and application of *Matryoshka GHZ-Bell* and *GHZ-ExhS* quantum resource-states will be explored.

Generation of Matryoshka GHZ-Bell States. The primary resource state used as a resource in this study is a GHZ-Bell state. The concept of *Matryoshka* states was first given by *Di Franco et al* [62], with the name ‘*Matryoshka*’ coming from the Russian word for ‘nesting doll’. In such states, we have a nested entanglement pattern. In this paper, we will be looking at the *Matryoshka* generation of the GHZ-Bell states over arbitrary number of qubits. We consider N spin- $\frac{1}{2}$ particles, with each spin coupled to its nearest neighbors by the XY Hamiltonian

$$H = \sum_{i=1}^{N-1} (J_{X,i} \hat{X}_i \hat{X}_{i+1} + J_{Y,i} \hat{Y}_i \hat{Y}_{i+1}) \quad (1)$$

where $J_{\sigma,i}$ is the pairwise coupling constant with $\sigma = \hat{X}, \hat{Y}, \hat{Z}$ being the Pauli operators. For the purposes of

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this paper, we take N to be odd. *Franco et al* [62] showed that it is sufficient to state that the information flux between the $\hat{X}$ ($\hat{Y}$) operators of the first and last qubits in the spin-chain depends on an alternating set of coupling strengths. For example, the information flux from $\hat{X}_1$ to $\hat{X}_N$ depends only on the set $\{J_{Y,1}, J_{X,2}, \dots, J_{Y,N-1}\}$ and is independent of any other coupling rate in the spin-chain. *Christandl et al* [63, 64] showed that after a time $t^* = \pi/\lambda$ with λ being a scaling constant (as mentioned in the definition of the case of a perfect state transfer in a linear spin-chain given by weighted coupling strengths: $J_{\sigma,i} = \lambda\sqrt{i(N-i)}$), the state of the first qubit in the spin-chain can be perfectly transferred to the last qubit. We see that by preparing the initial state of this spin-chain in an completely separable eigenstate of the tensorial product of Z_i operators, say $|\Psi(0)\rangle = |000\dots 0\rangle_{12\dots N}$, we obtain an information flux towards symmetric two-site spin operators, and a final state of the form [62]

$$|\psi_0\rangle = |0\rangle_c \otimes_{i=0}^M |\psi_+\rangle_{2i+1, N-2i} \otimes_{i=1}^M |\psi_-\rangle_{2i, N-2i+1} \quad (2)$$

$$|\psi_1\rangle = |1\rangle_c \otimes_{i=0}^M |\psi_-\rangle_{2i+1, N-2i} \otimes_{i=1}^M |\psi_+\rangle_{2i, N-2i+1} \quad (3)$$

where c labels the central site of the spin-chain, $M = \frac{N-3}{4}$ and $|\psi_{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$. An illustration of the setup has been shown in *Figure 1*.

The critical step in the creation of the Matryoshka GHZ-Bell state is the evolution of the central and two neighbouring qubits to the GHZ state, without disturbing the rest of the spin-chain. For this, we need to switch off all the interactions except for those connecting the central qubit to the neighbouring ones. A point to note here is that had we started with $|\Psi(0)\rangle = |111\dots 1\rangle_{12\dots N}$, we would have obtained a final state of the form

$$|\psi_0\rangle = |0\rangle_c \otimes_{i=0}^M |\psi_-\rangle_{2i+1, N-2i} \otimes_{i=1}^M |\psi_+\rangle_{2i, N-2i+1} \quad (4)$$

$$|\psi_1\rangle = |1\rangle_c \otimes_{i=0}^M |\psi_+\rangle_{2i+1, N-2i} \otimes_{i=1}^M |\psi_-\rangle_{2i, N-2i+1} \quad (5)$$

We use this principle and the idea that after evolution over time t^* , the states in equations (2) and (3) transform back to $|000\dots 000\rangle_{12\dots N}$ and states in equations (4) and (5) transform back to $|111\dots 11\rangle_{12\dots N}$. We can utilise this concept, by taking the state in equation (2) and evolving it, for the truncated subsystem comprising of the central qubit and the adjoining qubits. A point to note here is that due to only coupling that connects to the central qubits, the coupling strength ($J'_{\sigma,i} = \lambda'\sqrt{i(3-i)}$) and time of evolution ($t'' = \pi/\lambda'$) vary accordingly. Before carrying out this evolution, we perform a Hadamard operation on the central qubit to give

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle_c + |1\rangle_c) \otimes_{i=0}^M |\psi_+\rangle_{2i+1, N-2i} \otimes_{i=1}^M |\psi_-\rangle_{2i, N-2i+1} \quad (6)$$

We now perform the truncated subsystem time-evolution with the parameters (J', t'') to give us the state

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)_{c-1, c, c+1} \otimes_{i=0}^{M-1} |\psi_+\rangle_{2i+1, N-2i} \otimes_{i=1}^M |\psi_-\rangle_{2i, N-2i+1} \quad (7)$$

Therefore, we can obtain a Matryoshka GHZ-Bell state using nearest spin-spin interactions in a spin-chain.

Teleportation of arbitrary n-qubit quantum states using Matryoshka GHZ-Bell States. The teleportation of an arbitrary n-qubit state can be performed using Matryoshka GHZ-Bell States. *Saha et al* proposed a scheme for teleportation of a multiqubit state using the following resource state [65]:

$$|\psi\rangle = [\frac{1}{\sqrt{2}}(|0_A 0_B\rangle + |1_A 1_B\rangle)] [\frac{1}{\sqrt{2}}(|0_A 0_B\rangle + |1_A 1_B\rangle)] \dots [\frac{1}{\sqrt{2}}(|0_A 0_B\rangle + |1_A 1_B\rangle)] \quad (8)$$

To achieve the teleportation of an n -qubit state, we start with a $2n+1$ qubit state of the form given in equation (8), with Alice having $n+1$ qubits and Bob having n qubits. Let us say the arbitrary n -qubit state Alice wants to teleport to Bob is: $\sum_{i=0}^{2^n-1} \alpha_i |a_i\rangle$, where a_i denotes the binary representation of i . The combined state $|\psi_c\rangle$ can then be written in terms of subsystems possessed by Alice and Bob,

$$|\psi_c\rangle = \sum_{i=0}^{2^{2n}-1} |\omega_i\rangle_A |\eta_i\rangle_B \quad (9)$$

where $|\omega_i\rangle_A \forall i \in [0, 2^{2n}-1]$ constitute a mutually orthogonal basis. It is seen that for even n , $|\omega_0\rangle_{1,4,3,2,7,\dots,2n,2n-3,2n-2,2n+1} = |GHZ_+\rangle |\psi_+\rangle \dots |\psi_+\rangle$ and for odd n , $|\omega_0\rangle_{1,2,5,4,3,\dots,2n,2n-3,2n-2,2n+1} = |GHZ_+\rangle |\psi_+\rangle \dots |\psi_+\rangle$, where $|GHZ_+\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ and $|\psi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ [65]. We can obtain the other $|\omega_i\rangle_A$ from this

$$|\omega_i\rangle_{1,2,3,\dots,2n,2n+1} = \otimes_{k=1}^n (Z_k)^{b_k} (X_k)^{b_{k+n}} |\omega_0\rangle_{1,2,3,\dots,2n,2n+1} \quad (10)$$

where i is the decimal representation of the string $b_{2n}\dots b_2 b_1$ with a general b_k being 0 or 1. After measurement is done using these orthogonal states, Alice's state evolves into one of the states $|\omega_i\rangle'$. If her state is $|\omega_i\rangle$ then Bob must apply $\otimes_{k=1}^n (Z_k)^{b_k} (X_k)^{b_{k+n}}$ on his n -qubit system to obtain the unknown state, where i is the decimal representation of the string $b_{2n}\dots b_2 b_1$ with a general b_k being 0 or 1.

Extension of Matryoshka formalism to ExhS-Bell States. We can extend the idea of nested entanglement (Matryoshka formalism) to the case of *ExhS-Bell* States, which comprise of the exhaustive set over the basis for

the qubit states in the *ExhS* subsystem alongwith the maximally entangled Bell-states for the remainder subsystem. Such states are found to be less entangled in the three-qubit subsystem but have greater nested entangled resources. An example of Matryoshka ExhS-Bell States are the Xin-Wei Zha (XZW) State. Xin-Wei Zha et al [66] discovered a genuinely entangled seven-qubit state through a numerical optimization process, following the path taken by Brown et al [67] and Borrás et al [68] to find genuinely entangled five-qubit and six-qubit states:

$$|\psi_7\rangle = \frac{1}{2\sqrt{2}}(|000\rangle_{135}|\psi_+\rangle_{24}|\psi_+\rangle_{67} + |001\rangle_{135}|\phi_-\rangle_{24}|\phi_+\rangle_{67} + |010\rangle_{135}|\psi_-\rangle_{24}|\phi_-\rangle_{67} + |011\rangle_{135}|\phi_+\rangle_{24}|\psi_-\rangle_{67} + |100\rangle_{135}|\phi_+\rangle_{24}|\phi_+\rangle_{67} + |101\rangle_{135}|\psi_-\rangle_{24}|\psi_+\rangle_{67} + |110\rangle_{135}|\phi_-\rangle_{24}|\psi_-\rangle_{67} + |111\rangle_{135}|\psi_+\rangle_{24}|\phi_-\rangle_{67}) \quad (11)$$

It is seen that this state can be used for teleportation of arbitrary single, double and triple qubit states. The 3-2-2 structure of the resource-state, given in equation (11), helps us in devising a quantum circuit to generate the state, as shown in *Figure 2* and realised on *IBM Quantum Experience*, giving us the state

$$|\psi\rangle = \frac{1}{2\sqrt{2}}(|000\rangle_{135}|\psi_+\rangle_{24}|\psi_+\rangle_{67} + |001\rangle_{135}|\phi_-\rangle_{24}|\phi_+\rangle_{67} + |010\rangle_{135}|\psi_-\rangle_{24}|\phi_-\rangle_{67} + |011\rangle_{135}|\phi_+\rangle_{24}|\psi_-\rangle_{67} + |100\rangle_{135}|\phi_+\rangle_{24}|\phi_+\rangle_{67} + |101\rangle_{135}|\psi_-\rangle_{24}|\psi_+\rangle_{67} + |110\rangle_{135}|\phi_-\rangle_{24}|\psi_-\rangle_{67} + |111\rangle_{135}|\psi_+\rangle_{24}|\phi_-\rangle_{67}) \quad (12)$$

To obtain the resource-state, we apply a unitary operator on qubits 1, 3 and 5: $U = I_{4 \times 4} \oplus (\sigma_z \otimes \sigma_z)$. This state has marginal density matrices for subsystems over one or two qubits that are completely mixed, with

$$\pi_{ij} = Tr_{ij}\rho_{ij}^2 = \frac{1}{4} \quad \forall \quad i, j \in \{1, 2, 3, 4, 5, 6, 7\}, i < j \quad (13)$$

$$\pi_i = Tr_i\rho_i^2 = \frac{1}{2} \quad \forall \quad i \in \{1, 2, 3, 4, 5, 6, 7\} \quad (14)$$

For three-qubit subsystems, some of the partitions have mixed marginal density matrices:

$$\pi_{ijk} = Tr_{ijk}\rho_{ijk}^2 = \frac{1}{8} \quad \forall \quad i, j \in \{1, 2, 3, 4, 5, 6, 7\}, i < j < k \wedge (ijk) \neq (127), (367), (457) \quad (15)$$

$$\pi_{127} = \pi_{367} = \pi_{457} = \frac{1}{4} \quad (16)$$

Linear, Bidirectional and Circular Quantum Teleportation using the XZW Resource State. The seven-qubit genuinely entangled resource state $|\Gamma_7\rangle$ can be used for a number of applications, such as the teleportation of arbitrary one, two and three qubit states. To begin with, an arbitrary single qubit state can be teleported

using the resource state $|\Gamma_7\rangle$ will be considered. In this case Alice possesses qubits 1, 2, 3, 4, 5, 6 and the 7th particle belongs to Bob. Alice wants to transport an arbitrary state $|\psi^{(1)}\rangle = \alpha|0\rangle + \beta|1\rangle$ to Bob. The combined state of the system is $|\Gamma_7^{(1)}\rangle = |\psi^{(1)}\rangle \otimes |\Gamma_7\rangle$. Alice measures the seven qubits in her possession via the seven qubit orthonormal states: $|\xi^\pm\rangle = |0000\rangle|\Psi_{GHZ}^0\rangle - |0001\rangle|\Psi_{GHZ}^3\rangle + |0010\rangle|\Psi_{GHZ}^7\rangle + |0011\rangle|\Psi_{GHZ}^4\rangle - |0100\rangle|\Psi_{GHZ}^5\rangle - |0101\rangle|\Psi_{GHZ}^6\rangle + |0110\rangle|\Psi_{GHZ}^2\rangle + |0111\rangle|\Psi_{GHZ}^1\rangle \pm (|1000\rangle|\Psi_{GHZ}^2\rangle - |1001\rangle|\Psi_{GHZ}^1\rangle - |1010\rangle|\Psi_{GHZ}^5\rangle - |1011\rangle|\Psi_{GHZ}^6\rangle + |1100\rangle|\Psi_{GHZ}^7\rangle + |1101\rangle|\Psi_{GHZ}^4\rangle + |1110\rangle|\Psi_{GHZ}^0\rangle - |1111\rangle|\Psi_{GHZ}^3\rangle)$ and $|\nu^\pm\rangle = |1000\rangle|\Psi_{GHZ}^0\rangle - |1001\rangle|\Psi_{GHZ}^3\rangle + |1010\rangle|\Psi_{GHZ}^7\rangle + |1011\rangle|\Psi_{GHZ}^4\rangle - |1100\rangle|\Psi_{GHZ}^5\rangle - |1101\rangle|\Psi_{GHZ}^6\rangle + |1110\rangle|\Psi_{GHZ}^2\rangle + |1111\rangle|\Psi_{GHZ}^1\rangle \pm (|0000\rangle|\Psi_{GHZ}^2\rangle - |0001\rangle|\Psi_{GHZ}^1\rangle - |0010\rangle|\Psi_{GHZ}^5\rangle - |0011\rangle|\Psi_{GHZ}^6\rangle + |0100\rangle|\Psi_{GHZ}^7\rangle + |0101\rangle|\Psi_{GHZ}^4\rangle + |0110\rangle|\Psi_{GHZ}^0\rangle - |0111\rangle|\Psi_{GHZ}^3\rangle)$, where $|\Psi_{GHZ}^{0,1}\rangle = \frac{1}{\sqrt{2}}[|000\rangle \pm |111\rangle]$, $|\Psi_{GHZ}^{2,3}\rangle = \frac{1}{\sqrt{2}}[|001\rangle \pm |110\rangle]$, $|\Psi_{GHZ}^{4,5}\rangle = \frac{1}{\sqrt{2}}[|010\rangle \pm |101\rangle]$ and $|\Psi_{GHZ}^{6,7}\rangle = \frac{1}{\sqrt{2}}[|100\rangle \pm |011\rangle]$.

Alice then conveys the outcome of the measurement results to Bob via two classical bits. Bob then applies a suitable unitary operation from the set $I, \sigma_x, i\sigma_y, \sigma_z$ to recover the original state, sent by Alice. In this way, one can teleport an arbitrary single-qubit state using the state $|\Gamma_7\rangle$. Similarly, an arbitrary two qubit quantum state can be teleported using the resource-state. In this case Alice possesses qubits 1, 2, 3, 4 and 5, and the 6th and 7th particles belong to Bob. Alice wants to transport an arbitrary state $|\psi^{(2)}\rangle = \alpha|00\rangle + \mu|10\rangle + \gamma|01\rangle + \beta|11\rangle$ to Bob. The combined state of the system is $|\Gamma_7^{(2)}\rangle = |\psi^{(2)}\rangle \otimes |\Gamma_7\rangle$,

$$|\Gamma_7^{(2)}\rangle = \alpha(A_{00}|00\rangle + A_{01}|01\rangle + A_{10}|10\rangle + A_{11}|11\rangle) + \mu(B_{00}|00\rangle + B_{01}|01\rangle + B_{10}|10\rangle + B_{11}|11\rangle) + \gamma(C_{00}|00\rangle + C_{01}|01\rangle + C_{10}|10\rangle + C_{11}|11\rangle) + \beta(D_{00}|00\rangle + D_{01}|01\rangle + D_{10}|10\rangle + D_{11}|11\rangle) \quad (17)$$

where $A_{00} = |0000\rangle|\Psi_{GHZ}^0\rangle + |0001\rangle|\Psi_{GHZ}^4\rangle - |0010\rangle|\Psi_{GHZ}^5\rangle - |0011\rangle|\Psi_{GHZ}^7\rangle$, $A_{11} = |0000\rangle|\Psi_{GHZ}^1\rangle + |0001\rangle|\Psi_{GHZ}^5\rangle - |0010\rangle|\Psi_{GHZ}^2\rangle - |0011\rangle|\Psi_{GHZ}^6\rangle$, $A_{01} = |0000\rangle|\Psi_{GHZ}^6\rangle - |0001\rangle|\Psi_{GHZ}^2\rangle + |0010\rangle|\Psi_{GHZ}^4\rangle + |0011\rangle|\Psi_{GHZ}^0\rangle$, $A_{10} = -|0000\rangle|\Psi_{GHZ}^7\rangle - |0001\rangle|\Psi_{GHZ}^3\rangle + |0010\rangle|\Psi_{GHZ}^5\rangle + |0011\rangle|\Psi_{GHZ}^1\rangle$, $B_{00} = |1000\rangle|\Psi_{GHZ}^0\rangle + |1001\rangle|\Psi_{GHZ}^4\rangle - |1010\rangle|\Psi_{GHZ}^2\rangle + |1011\rangle|\Psi_{GHZ}^6\rangle$, $B_{11} = |1000\rangle|\Psi_{GHZ}^0\rangle + |1001\rangle|\Psi_{GHZ}^5\rangle - |1010\rangle|\Psi_{GHZ}^3\rangle - |1011\rangle|\Psi_{GHZ}^7\rangle$, $B_{01} = |1000\rangle|\Psi_{GHZ}^6\rangle - |1001\rangle|\Psi_{GHZ}^2\rangle + |1010\rangle|\Psi_{GHZ}^5\rangle + |1011\rangle|\Psi_{GHZ}^0\rangle$, $B_{10} = -|1000\rangle|\Psi_{GHZ}^7\rangle - |1001\rangle|\Psi_{GHZ}^3\rangle + |1010\rangle|\Psi_{GHZ}^4\rangle + |1011\rangle|\Psi_{GHZ}^1\rangle$, $C_{00} = |0100\rangle|\Psi_{GHZ}^0\rangle + |0101\rangle|\Psi_{GHZ}^4\rangle - |0110\rangle|\Psi_{GHZ}^3\rangle + |0111\rangle|\Psi_{GHZ}^6\rangle$, $C_{11} = |0100\rangle|\Psi_{GHZ}^1\rangle + |0101\rangle|\Psi_{GHZ}^5\rangle - |0110\rangle|\Psi_{GHZ}^2\rangle - |0111\rangle|\Psi_{GHZ}^7\rangle$, $C_{01} = |0100\rangle|\Psi_{GHZ}^6\rangle - |0101\rangle|\Psi_{GHZ}^2\rangle - |0110\rangle|\Psi_{GHZ}^4\rangle + |0111\rangle|\Psi_{GHZ}^0\rangle$, $C_{10} = |0100\rangle|\Psi_{GHZ}^7\rangle - |0101\rangle|\Psi_{GHZ}^3\rangle + |0110\rangle|\Psi_{GHZ}^5\rangle + |0111\rangle|\Psi_{GHZ}^1\rangle$, $D_{00} = |1100\rangle|\Psi_{GHZ}^0\rangle + |1101\rangle|\Psi_{GHZ}^4\rangle -$

$|1110\rangle|\Psi_{GHZ}^2\rangle + |1111\rangle|\Psi_{GHZ}^6\rangle$, $D_{11} = |1100\rangle|\Psi_{GHZ}^1\rangle + |1101\rangle|\Psi_{GHZ}^5\rangle - |1110\rangle|\Psi_{GHZ}^3\rangle - |1111\rangle|\Psi_{GHZ}^7\rangle$, $D_{01} = |1100\rangle|\Psi_{GHZ}^6\rangle - |1101\rangle|\Psi_{GHZ}^2\rangle + |1110\rangle|\Psi_{GHZ}^4\rangle + |1111\rangle|\Psi_{GHZ}^0\rangle$, $D_{10} = |1100\rangle|\Psi_{GHZ}^6\rangle - |1101\rangle|\Psi_{GHZ}^3\rangle + |1110\rangle|\Psi_{GHZ}^5\rangle + |1111\rangle|\Psi_{GHZ}^1\rangle$, where $|\Psi_{GHZ}^{0,1}\rangle = \frac{1}{\sqrt{2}}[|000\rangle \pm |111\rangle]$, $|\Psi_{GHZ}^{2,3}\rangle = \frac{1}{\sqrt{2}}[|001\rangle \pm |110\rangle]$, $|\Psi_{GHZ}^{4,5}\rangle = \frac{1}{\sqrt{2}}[|010\rangle \pm |101\rangle]$ and $|\Psi_{GHZ}^{6,7}\rangle = \frac{1}{\sqrt{2}}[|100\rangle \pm |011\rangle]$. Now, Bob can carry out a combination of unitary operations, according to the given table, to obtain the original state teleported by Alice.

State Obtained by Alice	Unitary Operation by Bob
$A_{01} + B_{11} + C_{00} + D_{01}$	$I \otimes \sigma_x$
$A_{01} + B_{11} - C_{00} - D_{01}$	$\sigma_z \otimes \sigma_x$
$A_{01} - B_{11} + C_{00} - D_{01}$	$I \otimes i\sigma_y$
$A_{01} - B_{11} - C_{00} + D_{01}$	$\sigma_z \otimes i\sigma_y$
$A_{11} + B_{01} + C_{10} + D_{00}$	$\sigma_x \otimes \sigma_x$
$A_{11} - B_{01} + C_{10} - D_{00}$	$\sigma_x \otimes i\sigma_y$
$A_{11} + B_{01} - C_{10} - D_{00}$	$i\sigma_y \otimes \sigma_x$
$A_{11} - B_{01} - C_{10} + D_{00}$	$i\sigma_y \otimes i\sigma_y$
$A_{00} + B_{10} + C_{01} + D_{11}$	$I \otimes I$
$A_{00} - B_{10} + C_{01} - D_{11}$	$I \otimes \sigma_z$
$A_{00} + B_{10} - C_{01} - D_{11}$	$\sigma_z \otimes I$
$A_{00} - B_{10} - C_{01} + D_{11}$	$\sigma_z \otimes \sigma_z$
$A_{10} + B_{11} + C_{00} + D_{01}$	$\sigma_x \otimes I$
$A_{10} - B_{11} + C_{00} - D_{01}$	$\sigma_x \otimes \sigma_z$
$A_{10} + B_{11} - C_{00} - D_{01}$	$i\sigma_y \otimes I$
$A_{10} - B_{11} - C_{00} + D_{01}$	$i\sigma_y \otimes \sigma_z$

The resource-state can also be used for bidirectional quantum teleportation. Bidirectional Controlled Quantum Teleportation (BCQT) protocols have been proposed for multi-qubit resource states, such as five-qubit [69], six-qubit [70, 71], seven-qubit [34, 72, 73] and eight-qubit states [74]. Bidirectional Controlled Quantum Teleportation can teleport arbitrary states between two users under the supervision of a third party. Zha et al proposed the first scheme for BCQT of single qubit states using a maximally entangled seven-qubit quantum state [34]. There have been schemes proposed for BCQT that utilise states with the same number of qubits as the quantum channel being used, and thereby realise bidirectional teleportation of arbitrary single- and two-qubit states under the controller Charlie [72, 73].

Let us say Alice and Bob would like to teleport two-qubit states to each other by utilizing the seven-qubit genuinely entangled resource state. We assume the form of the two-qubit states to be

$$|\phi\rangle_{A_1A_2} = \alpha_0|00\rangle + \alpha_1|01\rangle + \alpha_2|10\rangle + \alpha_3|11\rangle \quad (18)$$

$$|\phi\rangle_{B_1B_2} = \beta_0|00\rangle + \beta_1|01\rangle + \beta_2|10\rangle + \beta_3|11\rangle \quad (19)$$

For the resource-state, let Alice have the qubits 1,4 and 7, while Bob has the qubits 2, 3 and 6 and Charlie has the qubit 5.

The steps for the scheme are as follows:

- Alice measures qubit 7 of the resource state and A_1 in the bell basis.
- Bob measures qubit 2 of the resource state and B_1 in the bell basis.
- Charlie, Alice and Bob measure their qubits in the Z-basis.
- Alice and Bob measure their qubits A_2 and B_2 in the X-basis.
- We apply unitary transformations to the composite state to now get Alice's initial arbitrary state in Bob's terminal and Bob's initial arbitrary state in Alice's terminal.

Due to the special 3-2-2 form of the resource-state, we can also use entanglement swapping to perform bidirectional teleportation [75].

The seven-qubit resource state can be used for the perfect linear teleportation of an arbitrary three qubit state. In this case, Alice possesses qubits 1, 2, 3, 4 and 5, and the 6th and 7th particles belong to Bob. Alice wants to transport an arbitrary state $|\psi^{(3)}\rangle = a|000\rangle + b|001\rangle + c|010\rangle + d|011\rangle + e|100\rangle + f|101\rangle + g|110\rangle + h|111\rangle$ to Bob. Using the decomposition given in *Supplementary Material*, the states possessed by, and the unitary transforms to be performed by, Bob have been recorded, to accomplish the teleportation of an arbitrary three-qubit state. A point to note here is that we get the GHZ state for $a = h = \frac{1}{\sqrt{2}}, b = c = d = e = f = g = 0$ and the W state for $b = c = e = \frac{1}{\sqrt{3}}, a = d = f = g = h = 0$.

Not only is the seven-qubit resource state useful for linear and bidirectional teleportation but can also facilitate the probabilistic teleportation of an arbitrary single-qubit states in a circular manner between three network-nodes (users). Let us say I have Alice, Bob and Charlie in the system, with the first qubit used as a control qubit, qubits 1 and 4 given to Alice, qubits 2 and 6 given to Bob and qubits 3 and 7 given to Charlie. Let us say the arbitrary states are $|\psi_A\rangle = \alpha_A|0_A\rangle + \beta_A|1_A\rangle$, $|\psi_B\rangle = \alpha_B|0_B\rangle + \beta_B|1_B\rangle$ and $|\psi_C\rangle = \alpha_C|0_C\rangle + \beta_C|1_C\rangle$. Then, the composite state is given by $|\psi_A\rangle \otimes |\psi_B\rangle \otimes |\psi_C\rangle \otimes |\Gamma_7\rangle_{TA_1B_1C_1A_1B_1C_1}$, where $|\Gamma_7\rangle_T$ is the control qubit. We apply a CNOT gate using the qubits A, B and C of the arbitrary states as the control-qubits and the first qubits of each user as the target-qubit. Let us for simplicity only consider the case where $|\Gamma_7\rangle_T = |0\rangle$. Let us now measure the first qubits of Alice, Bob and Charlie in the Z-basis. Let us say $|\Gamma_7\rangle_{A_1B_1C_1} = |010\rangle$, then we have the state $|\psi''\rangle = \frac{1}{4}(|001\rangle - |010\rangle)\alpha_A\alpha_B\alpha_C|0_A0_B0_C\rangle + (|100\rangle - |111\rangle)\alpha_A\alpha_B\beta_C|0_A0_B1_C\rangle + (|000\rangle + |011\rangle)\alpha_A\beta_B\alpha_C|0_A1_B0_C\rangle + (|101\rangle + |110\rangle)\alpha_A\beta_B\beta_C|0_A1_B1_C\rangle + (|100\rangle - |111\rangle)\beta_A\alpha_B\alpha_C|1_A0_B0_C\rangle + (-|001\rangle + |010\rangle)\beta_A\alpha_B\beta_C|1_A0_B1_C\rangle + (-|101\rangle -$

$|110\rangle\rangle\beta_A\beta_B\alpha_C|1_A1_B0_C\rangle + |000\rangle + |011\rangle\rangle\beta_A\beta_B\beta_C|1_A1_B1_C\rangle$). We can now measure the control qubits in the X-basis. So, let us say, I have $|Q_AQ_BQ_C\rangle = |+_A -_B +_C\rangle$, then we obtain the state $|C_1\rangle(|A_1\rangle(-|B_1\rangle + \chi|B_3\rangle - \chi^{-1}|B_4\rangle - |B_2\rangle) + |A_2\rangle(-|B_1\rangle + \chi|B_3\rangle + \chi^{-1}|B_4\rangle + |B_2\rangle)) + |C_2\rangle(|A_1\rangle(-|B_1\rangle - \chi|B_3\rangle - \chi^{-1}|B_4\rangle + |B_2\rangle) + |A_2\rangle(-|B_1\rangle - \chi|B_3\rangle + \chi^{-1}|B_4\rangle - |B_2\rangle)) + |C_3\rangle(|A_1\rangle(|B_4\rangle - \chi|B_2\rangle - \chi^{-1}|B_1\rangle - |B_3\rangle) + |A_2\rangle(|B_4\rangle - \chi|B_2\rangle + \chi^{-1}|B_1\rangle - |B_3\rangle)) + |C_4\rangle(|A_1\rangle(|B_4\rangle + \chi|B_2\rangle - \chi^{-1}|B_1\rangle - |B_3\rangle) + |A_2\rangle(|B_4\rangle + \chi|B_2\rangle + \chi^{-1}|B_1\rangle + |B_3\rangle))$, where $|C_1\rangle = \beta_C|0\rangle + \alpha_C|1\rangle$, $|C_2\rangle = \beta_C|0\rangle - \alpha_C|1\rangle$, $|C_3\rangle = \beta_C|1\rangle + \alpha_C|0\rangle$, $|C_4\rangle = \beta_C|1\rangle - \alpha_C|0\rangle$, $|B_1\rangle = \alpha_B|1\rangle + \beta_B|0\rangle$, $|B_2\rangle = \alpha_B|1\rangle - \beta_B|0\rangle$, $|B_3\rangle = \alpha_B|0\rangle + \beta_B|1\rangle$, $|B_4\rangle = \alpha_B|0\rangle - \beta_B|1\rangle$, $\chi = \frac{a_2}{a_1}$ with $a_1 = \beta_A + \alpha_A$, $a_2 = \alpha_A - \beta_A$, $|A_1\rangle = a_1|0\rangle + a_2|1\rangle$, $|A_2\rangle = a_1|0\rangle - a_2|1\rangle$. Therefore I see that the users can obtain states derived from the original state of the users next to them (Alice $\rightarrow$ Bob $\rightarrow$ Charlie $\rightarrow$ Alice). However, as you can see, this can be done in a probabilistic manner with one of the users not quite obtaining the original state but rather a derivative-state based on the original.

Conclusion. In this paper, the generation and application of nested entanglement in Matryoshka resource-states for quantum information processing was studied. A novel scheme for the generation of such

quantum states has been proposed using an anisotropic XY spin-spin interaction-based model. The application of the *Matryoshka GHZ-Bell states* for n-qubit teleportation is reviewed and an extension to more general *Matryoshka ExhS-Bell states* is posited. An example of *Matryoshka ExhS-Bell states* was given in the form of the genuinely entangled seven-qubit *Xin-Wei Zha state*. Generation, characterisation and application of this seven-qubit resource state in theoretical schemes for quantum teleportation of arbitrary one, two and three qubits states, bidirectional teleportation of arbitrary two qubit states and probabilistic circular controlled teleportation were presented. This work should lay the groundwork for other studies into the area of nested entanglement.

Data Availability Statement. Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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FIG. 1. Scheme for the generation of Matryoshka GHZ-Bell resource-states, where the effective spin-spin XY Hamiltonian is obtained as an effective adiabatic Hamiltonian for a linear chain of optical cavities with each interacting with a three-level atomic system and that are mutually coupled via photon-hopping terms. The ground states of each atomic unit provide the computational space of each spin, and the dipole-forbidden transition between these states is realized as an (adiabatic) Raman transition through the excited state: $|e\rangle_i$ with $i = 1, 2, \dots, N$. The cavity field drives off-resonantly the dipole-allowed channel $|j\rangle_i \leftrightarrow |e\rangle_i$ with the Rabi frequency g_j , $j = 0, 1$. Two lasers are also coupled to these atomic transitions with strength Ω_j and detuning Δ_j . If we assume large detunings, the excited level of each atomic unit as well as the corresponding cavity field's degrees of freedom can be eliminated adiabatically to give an effective spin-spin Hamiltonian

FIG. 2. Quantum Circuit for the generation of the seven-qubit genuinely entangled state, on *IBM Quantum Experience*. Here *CX gate* is the CNOT gate, *cZ gate* is the CPHASE gate and *H gate* is the Hadamard gate

Supplementary Materials

The teleportation of an arbitrary three-qubit state using our resource-state has as the initial composite state,

$$\begin{aligned}
 |\Gamma_7^{(3)}\rangle &= |\psi^{(3)}\rangle \otimes |\Gamma_7\rangle \\
 &= aA_{000}|000\rangle + aA_{001}|001\rangle + aA_{010}|010\rangle + aA_{011}|011\rangle \\
 &+ aA_{100}|100\rangle + aA_{101}|101\rangle + aA_{110}|110\rangle + aA_{111}|111\rangle \\
 &+ bB_{000}|000\rangle + bB_{001}|001\rangle + bB_{010}|010\rangle + bB_{011}|011\rangle \\
 &+ bB_{100}|100\rangle + bB_{101}|101\rangle + bB_{110}|110\rangle + bB_{111}|111\rangle \\
 &+ cC_{000}|000\rangle + cC_{001}|001\rangle + cC_{010}|010\rangle + cC_{011}|011\rangle \\
 &+ cC_{100}|100\rangle + cC_{101}|101\rangle + cC_{110}|110\rangle + cC_{111}|111\rangle \\
 &+ dD_{000}|000\rangle + dD_{001}|001\rangle + dD_{010}|010\rangle + dD_{011}|011\rangle \\
 &+ dD_{100}|100\rangle + dD_{101}|101\rangle + dD_{110}|110\rangle + dD_{111}|111\rangle \\
 &+ eE_{000}|000\rangle + eE_{001}|001\rangle + eE_{010}|010\rangle + eE_{011}|011\rangle \\
 &+ eE_{100}|100\rangle + eE_{101}|101\rangle + eE_{110}|110\rangle + eE_{111}|111\rangle \\
 &+ fF_{000}|000\rangle + fF_{001}|001\rangle + fF_{010}|010\rangle + fF_{011}|011\rangle \\
 &+ fF_{100}|100\rangle + fF_{101}|101\rangle + fF_{110}|110\rangle + fF_{111}|111\rangle \\
 &+ gG_{000}|000\rangle + gG_{001}|001\rangle + gG_{010}|010\rangle + gG_{011}|011\rangle \\
 &+ gG_{100}|100\rangle + gG_{101}|101\rangle + gG_{110}|110\rangle + gG_{111}|111\rangle \\
 &+ hH_{000}|000\rangle + hH_{001}|001\rangle + hH_{010}|010\rangle + hH_{011}|011\rangle \\
 &+ hH_{100}|100\rangle + hH_{101}|101\rangle + hH_{110}|110\rangle + hH_{111}|111\rangle
 \end{aligned} \tag{1}$$

I. THREE QUBIT STATES

$$\begin{aligned}
 |A_{000}\rangle &= |0000000\rangle + |0000101\rangle - |0001011\rangle + |0001110\rangle \\
 |A_{001}\rangle &= |0000010\rangle + |0001001\rangle + |0001100\rangle - |0000111\rangle \\
 |A_{010}\rangle &= |0000111\rangle - |0000010\rangle + |0001001\rangle + |0001100\rangle \\
 |A_{011}\rangle &= |0000000\rangle + |0000101\rangle + |0001011\rangle - |0001110\rangle \\
 |A_{100}\rangle &= |0000011\rangle + |0000110\rangle - |0001000\rangle + |0001101\rangle \\
 |A_{101}\rangle &= |0000001\rangle - |0000100\rangle + |0001010\rangle + |0001111\rangle \\
 |A_{110}\rangle &= |0000001\rangle - |0000100\rangle - |0001010\rangle - |0001111\rangle \\
 |A_{111}\rangle &= |0001101\rangle - |0000011\rangle - |0000110\rangle - |0001000\rangle
 \end{aligned}$$

$$\begin{aligned}
 |B_{000}\rangle &= |0010000\rangle + |0010101\rangle - |0011011\rangle + |0011110\rangle \\
 |B_{001}\rangle &= |0010010\rangle + |0011001\rangle + |0011100\rangle - |0010111\rangle \\
 |B_{010}\rangle &= |0010111\rangle - |0010010\rangle + |0011001\rangle + |0011100\rangle \\
 |B_{011}\rangle &= |0010000\rangle + |0010101\rangle + |0011011\rangle - |0011110\rangle \\
 |B_{100}\rangle &= |0010011\rangle + |0010110\rangle - |0011000\rangle + |0011101\rangle \\
 |B_{101}\rangle &= |0010001\rangle - |0010100\rangle + |0011010\rangle + |0011111\rangle \\
 |B_{110}\rangle &= |0010001\rangle - |0010100\rangle - |0011010\rangle - |0011111\rangle \\
 |B_{111}\rangle &= |0011101\rangle - |0010011\rangle - |0010110\rangle - |0011000\rangle
 \end{aligned}$$

$$\begin{aligned}
 |C_{000}\rangle &= |0100000\rangle + |0100101\rangle - |0101011\rangle + |0101110\rangle \\
 |C_{001}\rangle &= |0100010\rangle + |0101001\rangle + |0101100\rangle - |0100111\rangle \\
 |C_{010}\rangle &= |0100111\rangle - |0100010\rangle + |0101001\rangle + |0101100\rangle \\
 |C_{011}\rangle &= |0100000\rangle + |0100101\rangle + |0101011\rangle - |0101110\rangle \\
 |C_{100}\rangle &= |0100011\rangle + |0100110\rangle - |0101000\rangle + |0101101\rangle
 \end{aligned}$$

$$\begin{aligned}
 |C_{101}\rangle &= |0100001\rangle - |0100100\rangle + |0101010\rangle + |0101111\rangle \\
 |C_{110}\rangle &= |0100001\rangle - |0100100\rangle - |0101010\rangle - |0101111\rangle \\
 |C_{111}\rangle &= |0101101\rangle - |0100011\rangle - |0100110\rangle - |0101000\rangle
 \end{aligned}$$

$$\begin{aligned}
 |D_{000}\rangle &= |0110000\rangle + |0110101\rangle - |0111011\rangle + |0111110\rangle \\
 |D_{001}\rangle &= |0110010\rangle + |0111001\rangle + |0111100\rangle - |0110111\rangle \\
 |D_{010}\rangle &= |0110111\rangle - |0110010\rangle + |0111001\rangle + |0111100\rangle \\
 |D_{011}\rangle &= |0110000\rangle + |0110101\rangle + |0111011\rangle - |0111110\rangle \\
 |D_{100}\rangle &= |0110011\rangle + |0110110\rangle - |0111000\rangle + |0111101\rangle \\
 |D_{101}\rangle &= |0110001\rangle - |0110100\rangle + |0111010\rangle + |0111111\rangle \\
 |D_{110}\rangle &= |0110001\rangle - |0110100\rangle - |0111010\rangle - |0111111\rangle \\
 |D_{111}\rangle &= |0111101\rangle - |0110011\rangle - |0110110\rangle - |0111000\rangle
 \end{aligned}$$

$$\begin{aligned}
 |E_{000}\rangle &= |1000000\rangle + |1000101\rangle - |1001011\rangle + |1001110\rangle \\
 |E_{001}\rangle &= |1000010\rangle + |1001001\rangle + |1001100\rangle - |1000111\rangle \\
 |E_{010}\rangle &= |1000111\rangle - |1000010\rangle + |1001001\rangle + |1001100\rangle \\
 |E_{011}\rangle &= |1000000\rangle + |1000101\rangle + |1001011\rangle - |1001110\rangle \\
 |E_{100}\rangle &= |1000011\rangle + |1000110\rangle - |1001000\rangle + |1001101\rangle \\
 |E_{101}\rangle &= |1000001\rangle - |1000100\rangle + |1001010\rangle + |1001111\rangle \\
 |E_{110}\rangle &= |1000001\rangle - |1000100\rangle - |1001010\rangle - |1001111\rangle \\
 |E_{111}\rangle &= |1001101\rangle - |1000011\rangle - |1000110\rangle - |1001000\rangle
 \end{aligned}$$

$$\begin{aligned}
 |F_{000}\rangle &= |1010000\rangle + |1010101\rangle - |1011011\rangle + |1011110\rangle \\
 |F_{001}\rangle &= |1010010\rangle + |1011001\rangle + |1011100\rangle - |1010111\rangle \\
 |F_{010}\rangle &= |1010111\rangle - |1010010\rangle + |1011001\rangle + |1011100\rangle \\
 |F_{011}\rangle &= |1010000\rangle + |1010101\rangle + |1011011\rangle - |1011110\rangle \\
 |F_{100}\rangle &= |1010011\rangle + |1010110\rangle - |1011000\rangle + |1011101\rangle \\
 |F_{101}\rangle &= |1010001\rangle - |1010100\rangle + |1011010\rangle + |1011111\rangle \\
 |F_{110}\rangle &= |1010001\rangle - |1010100\rangle - |1011010\rangle - |1011111\rangle \\
 |F_{111}\rangle &= |1011101\rangle - |1010011\rangle - |1010110\rangle - |1011000\rangle
 \end{aligned}$$

$$\begin{aligned}
 |G_{000}\rangle &= |1100000\rangle + |1100101\rangle - |1101011\rangle + |1101110\rangle \\
 |G_{001}\rangle &= |1100010\rangle + |1101001\rangle + |1101100\rangle - |1100111\rangle \\
 |G_{010}\rangle &= |1100111\rangle - |1100010\rangle + |1101001\rangle + |1101100\rangle \\
 |G_{011}\rangle &= |1100000\rangle + |1100101\rangle + |1101011\rangle - |1101110\rangle \\
 |G_{100}\rangle &= |1100011\rangle + |1100110\rangle - |1101000\rangle + |1101101\rangle \\
 |G_{101}\rangle &= |1100001\rangle - |1100100\rangle + |1101010\rangle + |1101111\rangle \\
 |G_{110}\rangle &= |1100001\rangle - |1100100\rangle - |1101010\rangle - |1101111\rangle \\
 |G_{111}\rangle &= |1101101\rangle - |1100011\rangle - |1100110\rangle - |1101000\rangle
 \end{aligned}$$

$$\begin{aligned}
 |H_{000}\rangle &= |1110000\rangle + |1110101\rangle - |1111011\rangle + |1111110\rangle \\
 |H_{001}\rangle &= |1110010\rangle + |1111001\rangle + |1111100\rangle - |1110111\rangle \\
 |H_{010}\rangle &= |1110111\rangle - |1110010\rangle + |1111001\rangle + |1111100\rangle \\
 |H_{011}\rangle &= |1110000\rangle + |1110101\rangle + |1111011\rangle - |1111110\rangle \\
 |H_{100}\rangle &= |1110011\rangle + |1110110\rangle - |1111000\rangle + |1111101\rangle \\
 |H_{101}\rangle &= |1110001\rangle - |1110100\rangle + |1111010\rangle + |1111111\rangle \\
 |H_{110}\rangle &= |1110001\rangle - |1110100\rangle - |1111010\rangle - |1111111\rangle \\
 |H_{111}\rangle &= |1111101\rangle - |1110011\rangle - |1110110\rangle - |1111000\rangle
 \end{aligned}$$

II. DECOMPOSITION OF ARBITRARY THREE QUBIT STATE WITH $|\Psi_7\rangle$

$$(a|000\rangle + b|001\rangle + c|010\rangle + d|011\rangle + e|100\rangle + f|101\rangle + g|110\rangle + h|111\rangle)|\Psi_7\rangle \tag{2}$$

$$+ f|101\rangle + g|110\rangle + h|111\rangle)|\Psi_7\rangle \tag{3}$$

$$= \sum_{\text{permutations}} ((-1)^{I_1} A_{a_1 a_2 a_3} + (-1)^{I_2} B_{b_1 b_2 b_3}) \quad (4)$$

$$+ (-1)^{I_3} C_{c_1 c_2 c_3} + (-1)^{I_4} D_{d_1 d_2 d_3} \quad (5)$$

$$+ (-1)^{I_5} E_{e_1 e_2 e_3} + (-1)^{I_6} F_{f_1 f_2 f_3} \quad (6)$$

$$+ (-1)^{I_7} G_{g_1 g_2 g_3} + (-1)^{I_8} H_{h_1 h_2 h_3}) \quad (7)$$

$$(8)$$

$$((-1)^{I_1} a|a_1 a_2 a_3\rangle + (-1)^{I_2} b|b_1 b_2 b_3\rangle) \quad (9)$$

$$+ (-1)^{I_3} c|c_1 c_2 c_3\rangle + (-1)^{I_4} d|d_1 d_2 d_3\rangle \quad (10)$$

$$+ (-1)^{I_5} e|e_1 e_2 e_3\rangle + (-1)^{I_6} f|f_1 f_2 f_3\rangle \quad (11)$$

$$+ (-1)^{I_7} g|g_1 g_2 g_3\rangle + (-1)^{I_8} h|h_1 h_2 h_3\rangle) \quad (12)$$

$$(13)$$

where I_i ($i=1, 2, 3, 4, 5, 6, 7, 8$) can take values 0 or 1 independently, and L_j ($L=a, b, c, d, e, f, g, h; j=1, 2, 3$) can take values 0 or 1 independently. The summation is over all possible permutation states obtained.

III. TRANSFORMATIONS FOR THREE-QUBIT TELEPORTATION

Projection of i^{th} component P_i :

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (14)$$

Flip and Projection of i^{th} component F_i :

$$F_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad (15)$$

-A ₁₀₀ - B ₁₀₁ + C ₁₁₀ + D ₁₁₁ + E ₀₀₀ + F ₀₀₁ + G ₀₁₀ + H ₀₁₁	-I ₂ ⊗ σ _z ⊗ F ₁ + I ₂ ⊗ I ₂ ⊗ F ₂
-A ₁₀₀ + B ₁₀₁ - C ₁₁₀ + D ₁₁₁ + E ₀₀₀ + F ₀₀₁ + G ₀₁₀ + H ₀₁₁	-σ _z ⊗ I ₂ ⊗ F ₁ + I ₁ ⊗ I ₂ ⊗ F ₂
-A ₁₀₀ + B ₁₀₁ + C ₁₁₀ - D ₁₁₁ + E ₀₀₀ + F ₀₀₁ + G ₀₁₀ + H ₀₁₁	-σ _z ⊗ σ _z ⊗ F ₁ + I ₂ ⊗ I ₂ ⊗ F ₂
A ₁₀₀ - B ₁₀₁ - C ₁₁₀ + D ₁₁₁ + E ₀₀₀ + F ₀₀₁ + G ₀₁₀ + H ₀₁₁	σ _z ⊗ σ _z ⊗ F ₁ + I ₂ ⊗ I ₂ ⊗ F ₂
-A ₁₀₀ - B ₁₀₁ - C ₁₁₀ + D ₁₁₁ + E ₀₀₀ + F ₀₀₁ + G ₀₁₀ + H ₀₁₁	-I ₂ ⊗ P ₁ ⊗ F ₁ - σ _z ⊗ P ₂ ⊗ F ₁ + I ₂ ⊗ I ₂ ⊗ F ₂
-A ₁₀₀ - B ₁₀₁ + C ₁₁₀ - D ₁₁₁ + E ₀₀₀ + F ₀₀₁ + G ₀₁₀ + H ₀₁₁	-I ₂ ⊗ P ₁ ⊗ F ₁ + σ _z ⊗ P ₂ ⊗ F ₁ + I ₂ ⊗ I ₂ ⊗ F ₂
-A ₁₀₀ + B ₁₀₁ - C ₁₁₀ - D ₁₁₁ + E ₀₀₀ + F ₀₀₁ + G ₀₁₀ + H ₀₁₁	-σ _z ⊗ P ₁ ⊗ F ₁ - I ₂ ⊗ P ₂ ⊗ F ₁ + I ₂ ⊗ I ₂ ⊗ F ₂
A ₁₀₀ - B ₁₀₁ - C ₁₁₀ - D ₁₁₁ + E ₀₀₀ + F ₀₀₁ + G ₀₁₀ + H ₀₁₁	σ _z ⊗ P ₁ ⊗ F ₁ - I ₂ ⊗ P ₂ ⊗ F ₁ + I ₂ ⊗ P ₁ ⊗ F ₂ + I ₂ ⊗ P ₂ ⊗ F ₂
A ₁₀₁ + B ₁₀₀ + C ₁₁₁ + D ₁₁₀ + E ₀₀₁ + F ₀₀₀ + G ₀₁₁ + H ₀₁₀	σ _x ⊗ I ₂ ⊗ F ₁ + σ _x ⊗ I ₂ ⊗ F ₂
-A ₁₀₁ + B ₁₀₀ + C ₁₁₁ + D ₁₁₀ + E ₀₀₁ + F ₀₀₀ + G ₀₁₁ + H ₀₁₀	Iσ _y ⊗ P ₁ ⊗ F ₁ + σ _x ⊗ P ₂ ⊗ F ₁ + σ _x ⊗ I ₂ ⊗ F ₂
A ₁₀₁ + B ₁₀₀ - C ₁₁₁ + D ₁₁₀ + E ₀₀₁ + F ₀₀₀ + G ₀₁₁ + H ₀₁₀	σ _x ⊗ P ₁ ⊗ F ₁ + Iσ _y ⊗ P ₂ ⊗ F ₁ + σ _x ⊗ I ₂ ⊗ F ₂
A ₁₀₁ + B ₁₀₀ + C ₁₁₁ + D ₁₁₀ - E ₀₀₁ + F ₀₀₀ + G ₀₁₁ + H ₀₁₀	σ _x ⊗ I ₂ ⊗ F ₁ + Iσ _y ⊗ P ₁ ⊗ F ₂ + σ _x ⊗ P ₂ ⊗ F ₂
A ₁₀₁ + B ₁₀₀ + C ₁₁₁ + D ₁₁₀ + E ₀₀₁ + F ₀₀₀ - G ₀₁₁ + H ₀₁₀	σ _x ⊗ I ₂ ⊗ F ₁ + σ _x ⊗ P ₁ ⊗ F ₂ + Iσ _y ⊗ P ₂ ⊗ F ₂
-A ₁₀₁ + B ₁₀₀ - C ₁₁₁ + D ₁₁₀ + E ₀₀₁ + F ₀₀₀ + G ₀₁₁ + H ₀₁₀	Iσ _y ⊗ I ₂ ⊗ F ₁ + σ _x ⊗ I ₂ ⊗ F ₂
-A ₁₀₁ + B ₁₀₀ + C ₁₁₁ + D ₁₁₀ - E ₀₀₁ + F ₀₀₀ + G ₀₁₁ + H ₀₁₀	Iσ _y ⊗ P ₁ ⊗ F ₁ + σ _x ⊗ P ₂ ⊗ F ₁ + Iσ _y ⊗ P ₁ ⊗ F ₂ + σ _x ⊗ P ₂ ⊗ F ₂
-A ₁₀₁ + B ₁₀₀ + C ₁₁₁ + D ₁₁₀ + E ₀₀₁ + F ₀₀₀ - G ₀₁₁ + H ₀₁₀	Iσ _y ⊗ P ₁ ⊗ F ₁ + σ _x ⊗ P ₂ ⊗ F ₁ + σ _x ⊗ P ₁ ⊗ F ₂ + Iσ _y ⊗ P ₂ ⊗ F ₂
A ₁₀₁ + B ₁₀₀ - C ₁₁₁ + D ₁₁₀ - E ₀₀₁ + F ₀₀₀ + G ₀₁₁ + H ₀₁₀	σ _x ⊗ P ₁ ⊗ F ₁ + Iσ _y ⊗ P ₂ ⊗ F ₁ + Iσ _y ⊗ P ₁ ⊗ F ₂ + σ _x ⊗ P ₂ ⊗ F ₂
A ₁₀₁ + B ₁₀₀ - C ₁₁₁ + D ₁₁₀ + E ₀₀₁ + F ₀₀₀ - G ₀₁₁ + H ₀₁₀	σ _x ⊗ P ₁ ⊗ F ₁ + Iσ _y ⊗ P ₂ ⊗ F ₁ + Iσ _y ⊗ P ₁ ⊗ F ₂ + σ _x ⊗ P ₂ ⊗ F ₂
A ₁₀₁ + B ₁₀₀ + C ₁₁₁ + D ₁₁₀ - E ₀₀₁ + F ₀₀₀ - G ₀₁₁ + H ₀₁₀	σ _x ⊗ I ₂ ⊗ F ₁ + Iσ _y ⊗ I ₂ ⊗ F ₂
-A ₁₀₁ + B ₁₀₀ - C ₁₁₁ + D ₁₁₀ - E ₀₀₁ + F ₀₀₀ + G ₀₁₁ + H ₀₁₀	Iσ _y ⊗ I ₂ ⊗ F ₁ + Iσ _y ⊗ P ₁ ⊗ F ₂ + σ _x ⊗ P ₂ ⊗ F ₂
-A ₁₀₁ + B ₁₀₀ + C ₁₁₁ + D ₁₁₀ - E ₀₀₁ + F ₀₀₀ - G ₀₁₁ + H ₀₁₀	Iσ _y ⊗ P ₁ ⊗ F ₁ + σ _x ⊗ P ₂ ⊗ F ₁ + Iσ _y ⊗ I ₂ ⊗ F ₂
-A ₁₀₁ + B ₁₀₀ - C ₁₁₁ + D ₁₁₀ + E ₀₀₁ + F ₀₀₀ - G ₀₁₁ + H ₀₁₀	Iσ _y ⊗ I ₂ ⊗ F ₁ + σ _x ⊗ P ₁ ⊗ F ₂ + Iσ _y ⊗ P ₂ ⊗ F ₂
A ₁₀₁ + B ₁₀₀ - C ₁₁₁ + D ₁₁₀ - E ₀₀₁ + F ₀₀₀ - G ₀₁₁ + H ₀₁₀	σ

[illegible]