A conjecture on the solution existence of the Navier-Stokes equation

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For the solution existence condition of the Navier-Stokes equation, we propose a conjecture as follows: "*The Navier-Stokes equation has a solution if and only if the determinant of flow velocity gradient is not zero, namely* $\det(\boldsymbol{\nabla v}) \neq 0$."

Keywords: solution existence condition,the Navier-Stokes equations, velocity gradient, tensor determinant

I. INTRODUCTION

In continuum physics, there are two ways of describing continuous media or flows, the Lagrangian description and the Eulerian description. In the Eulerian description, the Navier-Stokes equations of incompressible flow can be expressed as follows:

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot (\boldsymbol{\nabla v}) = -\frac{1}{\rho}\boldsymbol{\nabla} p + \nu \boldsymbol{\nabla}^2 \boldsymbol{v}, \tag{1}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0. \tag{2}$$

The Eq. 1 is momentums equation and Eq. 2 is mass conservation equation. In which, $\boldsymbol{v}(\boldsymbol{x}, t)$ is flow velocity field, ρ is constant mass density, $p(\boldsymbol{x}, t)$ is flow pressure, ν is kinematical viscosity, t is time, $\boldsymbol{x} = x^k \boldsymbol{e}_k$ is position coordinates, $\boldsymbol{e}_k$ is a base vector and $\boldsymbol{v}$ is flow velocity, $\boldsymbol{\nabla} = \boldsymbol{e}_k \frac{\partial}{\partial x^k}$ is gradient operator, and $\boldsymbol{\nabla}^2 = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}$.

Applying the divergence operation to both sides of the momentum equation Eq.1 and use the mass conservation leads to a pressure equation: $\boldsymbol{\nabla}^2 \cdot (p\mathbf{1}) = -\rho \boldsymbol{\nabla} \cdot (\boldsymbol{v} \cdot \boldsymbol{\nabla v}) = -\rho\{(\boldsymbol{\nabla v})^2 + [\boldsymbol{\nabla}(\boldsymbol{v\nabla})] \cdot \boldsymbol{v}\}$, where $\mathbf{1} = \boldsymbol{e}_k \boldsymbol{e}_k$ is an identity tensor and $\boldsymbol{v\nabla} = (\boldsymbol{\nabla v})^T$.

The Navier - Stokes existence and smoothness problem is an open problem in mathematics [1], regardless of numerous abstract studies that have been done by mathematicians. Now the question is that, is it possible to propose a simple criteria on the solution existence of the Navier-Stokes equation without complicated mathematics.

II. THE CONVECTIVE ACCELERATION $\boldsymbol{v} \cdot (\boldsymbol{\nabla v})$ IS THE KEY SOURCE OF DIFFICULTY

In order to find some useful information from the Navier-Stokes equation, let's have a look at the meaning of $\boldsymbol{v} \cdot (\boldsymbol{\nabla v})$ in Eq. 1. This term is called convective acceleration that is caused by the flow velocity gradient. It is obvious that the convective acceleration $\boldsymbol{v} \cdot (\boldsymbol{\nabla v})$ is the central point of the Navier-Stokes equations. Without the convective acceleration, the solution existence would not be a problem at all. The understanding on the convective term is quite important for the study on the solution existence should be focus on $\boldsymbol{v} \cdot (\boldsymbol{\nabla v})$. Therefore, we attack the open problem from the $\boldsymbol{v} \cdot (\boldsymbol{\nabla v})$.

III. A FORM-SOLUTION AND SOLUTION EXISTENCE CONJECTURE

Assuming the determinant of the velocity gradient is not zero, namely $\det \boldsymbol{\nabla v} \neq 0$, the form-solution of the Navier-Stokes momentum equation in Eq.(1) can be expressed as follows

$$\boldsymbol{v} = \left[\nu \boldsymbol{\nabla}^2 \boldsymbol{v} - \frac{1}{\rho}\boldsymbol{\nabla} p - \frac{\partial \boldsymbol{v}}{\partial t}\right] \cdot (\boldsymbol{\nabla v})^{-1}, \tag{3}$$

equivalently

$$\boldsymbol{v} = (\boldsymbol{\nabla v})^{-T} \cdot \left[\nu \boldsymbol{\nabla}^2 \boldsymbol{v} - \frac{1}{\rho}\boldsymbol{\nabla} p - \frac{\partial \boldsymbol{v}}{\partial t}\right], \tag{4}$$

equivalently

$$\boldsymbol{v} = \left[\boldsymbol{\nabla} \cdot (\nu \boldsymbol{\nabla v} - \frac{p}{\rho}\mathbf{1}) - \frac{\partial \boldsymbol{v}}{\partial t}\right] \cdot (\boldsymbol{\nabla v})^{-1}, \tag{5}$$

equivalently

$$\boldsymbol{v} = (\boldsymbol{\nabla v})^{-T} \cdot \left[\boldsymbol{\nabla} \cdot (\nu \boldsymbol{\nabla v} - \frac{p}{\rho}\mathbf{1}) - \frac{\partial \boldsymbol{v}}{\partial t}\right]. \tag{6}$$

The new formats of Naver-Stokes equations in Eqs. 3, 4, Eq.(5) and Eq.(6) have never been seen in literature. They are formulated for the first time by Bo-Hua Sun [2]. Those form-solutions provide a rich information on the solution existence.

Therefore we have a conjecture as follows:

Conjecture 1 *The 3D Navier-Stokes equation has a solution if and only if the determinant of flow velocity gradient is not zero, namely*

$$\det(\boldsymbol{\nabla v}) \neq 0.$$

Although we have successfully split the velocity field $\boldsymbol{v}$ from the convective term $\boldsymbol{v} \cdot (\boldsymbol{\nabla v})$, the calculation of the inverse of the velocity gradient $(\boldsymbol{\nabla v})^{-1}$ is still great challenge.

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According to the Cayley-Hamilton theorem [3–6], for the 2nd order tensor $\boldsymbol{\nabla v}$, the following polynomial holds:

$$(\boldsymbol{\nabla v})^3 - I_1(\boldsymbol{\nabla v})^2 + I_2\boldsymbol{\nabla v} - I_3\mathbf{1} = \mathbf{0}, \tag{7}$$

where $I_1 = tr(\boldsymbol{\nabla v}) = \boldsymbol{\nabla}\cdot\boldsymbol{v}$, $I_2 = \frac{1}{2}[(tr\boldsymbol{\nabla v})^2 - tr(\boldsymbol{\nabla v})^2]$ and $I_3 = \det(\boldsymbol{\nabla v})$.

Hence, for the case of $\det(\boldsymbol{\nabla v}) \neq 0$, we have:

$$(\boldsymbol{\nabla v})^{-1} = \frac{(\boldsymbol{\nabla v})^2 - I_1\boldsymbol{\nabla v} + I_2\mathbf{1}}{\det(\boldsymbol{\nabla v})}. \tag{8}$$

For incompressible flow, the divergence of velocity gradient is zero, namely, $I_1 = tr(\boldsymbol{\nabla v}) = \boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$, thus $I_2 = \frac{1}{2}[(tr\boldsymbol{\nabla v})^2 - tr(\boldsymbol{\nabla v})^2] = -\frac{1}{2}tr(\boldsymbol{\nabla v})^2$.

Therefore, the inverse of the velocity gradient for incompressible flow takes a simpler form:

$$(\boldsymbol{\nabla v})^{-1} = \frac{(\boldsymbol{\nabla v})^2 - \frac{1}{2}\mathbf{1}tr(\boldsymbol{\nabla v})^2}{\det(\boldsymbol{\nabla v})}. \tag{9}$$

Therefore, the incompressible flow velocity field in Eq.(5) is then reduced to the following form:

$$\boldsymbol{v} = \frac{\left[\nu\boldsymbol{\nabla}^2\boldsymbol{v} - \frac{1}{\rho}\boldsymbol{\nabla}p - \frac{\partial\boldsymbol{v}}{\partial t}\right]\cdot[(\boldsymbol{\nabla v})^2 - \frac{1}{2}\mathbf{1}tr(\boldsymbol{\nabla v})^2]}{\det(\boldsymbol{\nabla v})}, \tag{10}$$

in which

$$\begin{aligned}
\boldsymbol{\nabla v} &= v_{j,i}\boldsymbol{e}_i\boldsymbol{e}_j,\\
(\boldsymbol{\nabla v})^2 &= \boldsymbol{\nabla v}\cdot\boldsymbol{\nabla v} = v_{j,k}v_{k,i}\boldsymbol{e}_i\boldsymbol{e}_j,\\
tr(\boldsymbol{\nabla v})^2 &= \mathbf{1} : (\boldsymbol{\nabla v})^2 = v_{i,k}v_{k,i},\\
\det(\boldsymbol{\nabla v}) &= \varepsilon_{ijk}v_{1,i}v_{2,j}v_{3,k}
\end{aligned}$$

where ε_{ijk} is permutation symbol.

For steady flow, $\frac{\partial\boldsymbol{v}}{\partial t} = \mathbf{0}$, the Eq. 11 is reduced to

$$\boldsymbol{v} = \frac{\left[\nu\boldsymbol{\nabla}^2\boldsymbol{v} - \frac{1}{\rho}\boldsymbol{\nabla}p\right]\cdot[(\boldsymbol{\nabla v})^2 - \frac{1}{2}\mathbf{1}tr(\boldsymbol{\nabla v})^2]}{\det(\boldsymbol{\nabla v})}, \tag{11}$$

The great challenge to find solution of the Navier-Stokes equation are all from the existence of the velocity gradient $\boldsymbol{\nabla v}$. The form solution in Eq.(11) reveals that the difficulty of finding a solution for N-S equations is because of existence of the nonlinear term $\boldsymbol{v}\cdot(\boldsymbol{\nabla v})$, or in other words, due to the existence of the velocity field gradient $\boldsymbol{\nabla v}$. Accordingly, the solution of the Navier-Stokes equation will be blowup as $\det\boldsymbol{\nabla v}$ tends to an infinitesimal, and has no solution when $\det\boldsymbol{\nabla v} = 0$.

IV. 2D NAVIER-STOKES EQUATIONS

The 2D N-S equations can be written as follows:

$$\frac{\partial v_1}{\partial t} + v_1\frac{\partial v_1}{\partial x_1} + v_2\frac{\partial v_1}{\partial x_2} = \nu(\frac{\partial^2 v_1}{\partial x_1^2} + \frac{\partial^2 v_1}{\partial x_2^2}) - \frac{1}{\rho}\frac{\partial p}{\partial x_1}, \tag{12}$$

$$\frac{\partial v_2}{\partial t} + v_1\frac{\partial v_2}{\partial x_1} + v_2\frac{\partial v_2}{\partial x_2} = \nu(\frac{\partial^2 v_2}{\partial x_1^2} + \frac{\partial^2 v_2}{\partial x_2^2}) - \frac{1}{\rho}\frac{\partial p}{\partial x_1}. \tag{13}$$

where $v_{i,j} = \frac{\partial v_i}{\partial x_j}$. The above equations can be expressed in matrix format

$$\frac{\partial}{\partial t}\begin{pmatrix} v_1\\ v_2\end{pmatrix} + \begin{pmatrix} v_{1,1} & v_{1,2}\\ v_{2,1} & v_{2,2}\end{pmatrix}\begin{pmatrix} v_1\\ v_2\end{pmatrix} \tag{14}$$

$$= -\begin{pmatrix} \frac{1}{\rho}\frac{\partial p}{\partial x_1}\\ \frac{1}{\rho}\frac{\partial p}{\partial x_2}\end{pmatrix} + \nu\nabla^2\begin{pmatrix} v_1\\ v_2\end{pmatrix}, \tag{15}$$

where the 2D Laplace operator $\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2 v_2}{\partial x_2^2}$.

The 2D velocity gradient is

$$\boldsymbol{\nabla v} = \begin{pmatrix} v_{1,1} & v_{1,2}\\ v_{2,1} & v_{2,2}\end{pmatrix}, \tag{16}$$

its determinant is

$$\det(\boldsymbol{\nabla v}) = v_{1,1}v_{2,2} - v_{1,2}v_{2,1}, \tag{17}$$

and the inverse of the 2D velocity gradient is thus

$$\begin{aligned}
(\boldsymbol{\nabla} v)^{-1} &= \frac{1}{\det(\boldsymbol{\nabla} v)}\begin{pmatrix} v_{2,2} & -v_{1,2}\\ -v_{2,1} & v_{1,1}\end{pmatrix}\\
&= \frac{\begin{pmatrix} v_{2,2} & -v_{1,2}\\ -v_{2,1} & v_{1,1}\end{pmatrix}}{v_{1,1}v_{2,2} - v_{1,2}v_{2,1}}.
\end{aligned} \tag{18}$$

Therefore, from Eq. 14, the 2D flow velocity is given by

$$\begin{pmatrix} v_1\\ v_2\end{pmatrix} = \frac{\begin{pmatrix} v_{2,2} & -v_{1,2}\\ -v_{2,1} & v_{1,1}\end{pmatrix}}{v_{1,1}v_{2,2} - v_{1,2}v_{2,1}}\left[(\nu\nabla^2 - \frac{\partial}{\partial t})\begin{pmatrix} v_1\\ v_2\end{pmatrix} - \frac{1}{\rho}\begin{pmatrix} p_{,1}\\ p_{,2}\end{pmatrix}\right]. \tag{19}$$

For the 2D steady flow, Eq.19 is reduced to

$$\begin{pmatrix} v_1\\ v_2\end{pmatrix} = \frac{\begin{pmatrix} v_{2,2} & -v_{1,2}\\ -v_{2,1} & v_{1,1}\end{pmatrix}}{v_{1,1}v_{2,2} - v_{1,2}v_{2,1}}\left[\nu\nabla^2\begin{pmatrix} v_1\\ v_2\end{pmatrix} - \frac{1}{\rho}\begin{pmatrix} p_{,1}\\ p_{,2}\end{pmatrix}\right]. \tag{20}$$

Hence, we have a similar conjecture for 2D flow as follows.

Conjecture 2 *The 2D Navier-Stokes equation has a solution if and only if the determinant of flow velocity gradient is not zero, namely* $\det(\boldsymbol{\nabla v}) \neq 0$, *or equivalently*

$$v_{1,1}v_{2,2} - v_{1,2}v_{2,1} \neq 0.$$

V. A GEOMETRICAL AND PHYSICAL INTERPRETATION OF THE CONJECTURES

Concerning the geometrical and physical meaning of the conjectures, let's try to give a basic interpretation. Assume that $d^2\boldsymbol{X}$ is small surface area of a moving fluid element with volume $d^3\boldsymbol{X}$, they become to $d^2\boldsymbol{x}$ and $d^3\boldsymbol{x}$

due to the velocity gradient $\det(\boldsymbol{\nabla v})$, respectively. Their relations are

$$d^2\boldsymbol{x} = \det(\boldsymbol{\nabla v})(\boldsymbol{\nabla} v)^{-T} \cdot d^2\boldsymbol{X}, \tag{21}$$

and the volume induced by the velocity gradient $\boldsymbol{\nabla v}$ is

$$d^3\boldsymbol{x} = \det(\boldsymbol{\nabla v})d^3\boldsymbol{X}. \tag{22}$$

From the relations in Eqs.21 and 22, both the surface area $d^2\boldsymbol{x}$ and volume $d^3\boldsymbol{x}$ induced by the velocity gradient $\boldsymbol{\nabla v}$ will shrink to a point as $\det(\boldsymbol{\nabla} v) \to 0$. If we image the finite surface area ($d^2\boldsymbol{x} \neq \boldsymbol{0}$) as a window that flow can go through, it means that, if $d^2\boldsymbol{x} = \boldsymbol{0}$, the window is closed and no flow can go through it.

Denoting the momentum flux density tensor in a viscous fluid $\boldsymbol{\Pi} = p\boldsymbol{I} + \rho\boldsymbol{v} \otimes \boldsymbol{v} - \mu\boldsymbol{\nabla v}$, and μ dynamical viscosity, according to Landau and Lifshitz [7], the local form of the equation of motion of the viscous fluid is $\frac{\partial \rho \boldsymbol{v}}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{\Pi} = \boldsymbol{0}$, which can be rewritten in integral form $\int \left(\frac{\partial \rho \boldsymbol{v}}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{\Pi}\right) d^3\boldsymbol{X} = \boldsymbol{0}$, and further simplified to

$$\frac{\partial}{\partial t}\int \rho\boldsymbol{v} d^3\boldsymbol{X} + \oint \boldsymbol{\Pi} \cdot d^2\boldsymbol{X} = \boldsymbol{0}, \tag{23}$$

by Green's formula.

Using Eqs.21 and 22, we can get an induced form of Eq.23 by the velocity gradient $\boldsymbol{\nabla v}$ as follows

$$\frac{\partial}{\partial t}\int \frac{\rho\boldsymbol{v} d^3\boldsymbol{x}}{\det(\boldsymbol{\nabla v})} + \oint \frac{\boldsymbol{\Pi} \cdot (\boldsymbol{\nabla} v)^T \cdot d^2\boldsymbol{x}}{\det(\boldsymbol{\nabla v})} = \boldsymbol{0}. \tag{24}$$

The Eq.24 will be invalid as $\det(\boldsymbol{\nabla} v) \to 0$. These might be viewed as another interpretation of the conjectures on solution existence.

VI. CONCLUSIONS

In conclusion, by taking into account of the importance of the convective term $\boldsymbol{v} \cdot \boldsymbol{\nabla v}$, the conjectures on solution existence condition of Navier-Stokes equation have been proposed, which state that "*The Navier-Stokes equation has a solution if and only if the determinant of flow velocity gradient is not zero, namely* $\det(\boldsymbol{\nabla v}) \neq 0$." [2].

To be honest, this study on the solution existence is still in a primitive stage. From a future perspective, the mathematicians should be invited for comprehensive investigation and proof of the conjectures.

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