

## A study on new type of degenerate poly-Cauchy polynomials and numbers of the second kind

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**Abstract.** Kim-Kim [20] studied the type 2 degenerate poly-Bernoulli numbers and polynomials by using modified degenerate polylogarithm function. In this paper, we construct the new type of degenerate poly-Cauchy polynomials and numbers of the second kind, called the new type of degenerate poly-Cauchy polynomials and numbers of the second kind by using degenerate polylogarithm function and derive several properties on the degenerate poly-Cauchy polynomials and numbers of the second kind. Furthermore, we consider the degenerate unipoly-Cauchy polynomials of the second kind and discuss some properties of them.

**Keywords:** Degenerate polylogarithm functions, Degenerate poly-Cauchy polynomials, Degenerate unipoly-Cauchy polynomials.

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### 1. Introduction

In recent years, many researchers have been studied for various degenerate versions of many special polynomials and numbers which included the degenerate Stirlings numbers of the second kind, degenerate central factorial numbers of the second kind, degenerate Bernoulli numbers of the second kind, degenerate Bernstein polynomials, degenerate Bell numbers and polynomials, degenerate central Bell numbers and polynomials, degenerate complete Bell polynomials and numbers, degenerate Cauchy numbers, and others (see [4-24]). Carlitz [1, 2] initiated a study of degenerate versions of some special polynomials and numbers, namely the degenerate Bernoulli and Euler polynomials and numbers.

As is well known, the Cauchy polynomials  $C_n(x)$  are defined by the means of the following generating function

$$\int_0^1 (1+t)^{x+y} dy = \frac{t}{\log(1+t)} (1+t)^x = \sum_{n=0}^{\infty} C_n(x) \frac{t^n}{n!}, \text{ (see [3, 5, 22]).} \quad (1.1)$$

In case when  $x = 0$ ,  $C_n = C_n(0)$  are called the Cauchy numbers.

The higher-order Bernoulli polynomials are defined by

$$\left( \frac{t}{e^t - 1} \right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{t^n}{n!}, \text{ (see [4, 8]).} \quad (1.2)$$

For  $x = 0$ ,  $B_n^{(\alpha)} = B_n^{(\alpha)}(0)$  are called the higher-order Bernoulli numbers and  $B_n(x) = B_n^{(1)}(x)$  are called the ordinary Bernoulli polynomials.

From (1.1) and (1.2), we note that

$$C_n(x) = B_n^{(n)}(x+1), (n \geq 0), \text{ (see [13]).} \quad (1.3)$$

For  $\lambda \in \mathbb{R}$ , the degenerate exponential functions are defined as

$$e_\lambda^x(t) = (1 + \lambda t)^{\frac{x}{\lambda}}, e_\lambda(t)e_\lambda^1(t) = (1 + \lambda t)^{\frac{1}{\lambda}}, \text{(see [10-15])}. \quad (1.4)$$

Here we note that

$$e_\lambda^x(t) = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}, \quad (1.5)$$

where  $(x)_{0,\lambda} = 1, (x)_{n,\lambda} = x(x - \lambda)(x - 2\lambda) \cdots (x - (n - 1)\lambda), (n \geq 1)$ .

In [1, 2], Carlitz considered the degenerate Bernoulli polynomials which are given by

$$\frac{t}{e_\lambda(t) - 1} e_\lambda^x(t) = \frac{t}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} (1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x) \frac{t^n}{n!}. \quad (1.6)$$

On setting  $x = 0, \beta_{n,\lambda} = \beta_{n,\lambda}(0)$  are called degenerate Bernoulli numbers.

For  $k \in \mathbb{Z}$ , the degenerate modified polyexponential function [20] is defined by Kim-Kim to be

$$\text{Ei}_{k,\lambda}(x) = \sum_{n=1}^{\infty} \frac{(1)_{n,\lambda} x^n}{(n-1)! n^k}, (|x| < 1). \quad (1.7)$$

Note that

$$\text{Ei}_{1,\lambda}(x) = \sum_{n=1}^{\infty} \frac{(1)_{n,\lambda} x^n}{n!} = e_\lambda(x) - 1. \quad (1.8)$$

In [20], Kim-Kim considered the type 2 degenerate poly-Bernoulli polynomials are defined by means of the following generating function

$$\frac{\text{Ei}_{k,\lambda}(\log_\lambda(1+t))}{e_\lambda(t) - 1} e_\lambda^x(t) = \sum_{n=0}^{\infty} B_{n,\lambda}^{(k)}(x) \frac{t^n}{n!}, (k \in \mathbb{Z}). \quad (1.9)$$

In the case when  $x = 0, B_{n,\lambda}^{(k)} = B_{n,\lambda}^{(k)}(0)$  are called the type 2 degenerate poly-Bernoulli numbers.

Kim [13] defined the degenerate Cauchy polynomials  $C_{n,\lambda}(x)$  as follows:

$$\begin{aligned} \int_0^1 (1 + \log(1 + \lambda t)^{\frac{1}{\lambda}})^{x+y} dy &= \frac{\frac{1}{\lambda}(\log(1 + \lambda t))}{\log(1 + \frac{1}{\lambda} \log(1 + \lambda t))} (1 + \log(1 + \lambda t)^{\frac{1}{\lambda}})^x \\ &= \sum_{n=0}^{\infty} C_{n,\lambda}^*(x) \frac{t^n}{n!}, \text{(see [3, 4, 19, 23, 24])}. \end{aligned} \quad (1.10)$$

Letting  $x = 0, C_{n,\lambda} = C_{n,\lambda}(0)$  are the degenerate Cauchy numbers.

The degenerate Cauchy polynomials  $C_{n,\lambda}(x)$  of the second kind are introduced in [13] as follows

$$\frac{t}{\log(1 + \frac{1}{\lambda} \log(1 + \lambda t))} \left(1 + \frac{1}{\lambda} \log(1 + \lambda t)\right)^x = \sum_{n=0}^{\infty} C_{n,\lambda}(x) \frac{t^n}{n!}, \text{(see [9, 10, 25])}. \quad (1.11)$$

In the case when  $x = 0, C_{n,\lambda} = C_{n,\lambda}(0)$  are called the degenerate Cauchy polynomials of the second kind.

In [16], the degenerate Daehee polynomials  $D_{n,\lambda}(x)$  are defined by

$$\frac{\log_\lambda(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_{n,\lambda}(x) \frac{t^n}{n!}, \text{ (see [19])} \quad (1.12)$$

For  $x = 0$ ,  $D_{n,\lambda} = D_{n,\lambda}(0)$  are called the degenerate Daehee numbers.

The degenerate Stirling numbers of the first kind are defined by

$$\frac{1}{k!}(\log_\lambda(1+t))^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k) \frac{t^n}{n!}, \quad (k \geq 0), \text{ (see [10, 16, 19]).} \quad (1.13)$$

Note here that  $\lim_{\lambda \rightarrow 0} S_{1,\lambda}(n,k) = S_1(n,k)$ , where  $S_1(n,k)$  are the Stirling numbers of the first kind given by

$$\frac{1}{k!}(\log(1+t))^k = \sum_{n=k}^{\infty} S_1(n,k) \frac{t^n}{n!}, \quad (k \geq 0), \text{ (see [6, 7]).} \quad (1.14)$$

The degenerate Stirling numbers of the second kind are given by

$$\frac{1}{k!}(e_\lambda(t) - 1)^k = \sum_{n=l}^{\infty} S_{2,\lambda}(n,l) \frac{t^n}{n!}, \text{ (see [15]).} \quad (1.15)$$

Observe here that  $\lim_{\lambda \rightarrow 0} S_{2,\lambda}(n,k) = S_2(n,k)$ , where  $S_2(n,k)$  are the Stirling numbers of the second kind given by

$$\frac{1}{k!}(e^t - 1)^k = \sum_{n=l}^{\infty} S_2(n,l) \frac{t^n}{n!}, \text{ (see [1-26]).} \quad (1.16)$$

By the motivation of the works of Kim-Kim [19], we first define the type 2 degenerate poly-Cauchy polynomials of the second kind by using the degenerate polyexponential functions. We investigate some new properties of these numbers and polynomials and derive some new identities and relations between the new type of degenerate poly-Cauchy polynomials of the second kind. Furthermore, we consider the type 2 degenerate unipoly-Cauchy polynomials of the second kind and discuss some identities of them.

## 2. New type of degenerate poly-Cauchy polynomials of the second kind

For  $k \in \mathbb{Z}$ , by using equations (1.9) and (1.11), we define the new type of degenerate poly-Cauchy polynomials of the second kind as follows:

$$\frac{\text{Ei}_{k,\lambda}(\log_\lambda(1+t))}{\log(1 + \frac{1}{\lambda} \log(1 + \lambda t))} \left(1 + \frac{1}{\lambda} \log(1 + \lambda t)\right)^x = \sum_{n=0}^{\infty} C_{n,\lambda}^{(k)}(x) \frac{t^n}{n!}. \quad (2.1)$$

In case when  $x = 0$  in (2.1),  $C_{n,\lambda}^{(k)} = C_{n,\lambda}^{(k)}(0)$  are called the type 2 degenerate poly-Cauchy numbers of the second kind. For  $k = 1$ , by (2.1), we note that

$$C_{n,\lambda}^{(1)} = C_{n,\lambda}, \quad (n \in \mathbb{N} \cup \{0\}) \quad (2.2)$$

are called the degenerate Cauchy numbers of the second kind.

By using equations (1.7), (1.11) and (2.1), we observe that

$$\sum_{n=0}^{\infty} C_{n,\lambda}^{(k)} \frac{t^n}{n!} = \frac{\text{Ei}_{k,\lambda}(\log_\lambda(1+t))}{\log(1 + \frac{1}{\lambda} \log(1 + \lambda t))}$$

$$\begin{aligned}
&= \frac{t}{\log(1 + \frac{1}{\lambda} \log(1 + \lambda t))} \frac{1}{t} \sum_{m=1}^{\infty} \frac{(\log_{\lambda}(1+t))^m}{(m-1)!m^k} \\
&= \frac{t}{\log(1 + \frac{1}{\lambda} \log(1 + \lambda t))} \frac{1}{t} \sum_{m=0}^{\infty} \frac{(\log_{\lambda}(1+t))^{m+1}}{(m+1)!(m+1)^{k-1}} \\
&= \frac{t}{\log(1 + \frac{1}{\lambda} \log(1 + \lambda t))} \frac{1}{t} \sum_{m=0}^{\infty} \frac{1}{(m+1)^{k-1}} \sum_{l=m+1}^{\infty} S_{1,\lambda}(l, m+1) \frac{t^l}{l!} \\
&= \frac{t}{\log(1 + \frac{1}{\lambda} \log(1 + \lambda t))} \frac{1}{t} \sum_{m=0}^{\infty} \frac{1}{(m+1)^{k-1}} \sum_{l=m}^{\infty} S_{1,\lambda}(l+1, m+1) \frac{t^l}{(l+1)!} \\
&= \left( \sum_{s=0}^{\infty} C_{s,\lambda} \frac{t^s}{s!} \right) \left( \sum_{l=0}^{\infty} \sum_{m=0}^l \frac{1}{(m+1)^{k-1}} \frac{S_{1,\lambda}(l+1, m+1)}{l+1} \frac{t^l}{l!} \right) \\
L.H.S &= \sum_{n=0}^{\infty} \left( \sum_{l=0}^n \sum_{m=0}^l \binom{n}{l} C_{n-l,\lambda} \frac{S_{1,\lambda}(l+1, m+1)}{l+1(m+1)^{k-1}} \right) \frac{t^n}{n!}. \tag{2.3}
\end{aligned}$$

Therefore, by (2.3), we obtain the following theorem.

**Theorem 2.1.** For  $n \geq 0$  and  $k \in \mathbb{Z}$ , we have

$$C_{n,\lambda}^{(k)} = \sum_{l=0}^n \sum_{m=0}^l \binom{n}{l} C_{n-l,\lambda} \frac{S_{1,\lambda}(l+1, m+1)}{l+1(m+1)^{k-1}}. \tag{2.4}$$

**Corollary 2.1.** For  $n \geq 0$  and  $k \in \mathbb{Z}$ , we have

$$C_{n,\lambda}^{(1)} = \sum_{l=0}^n \sum_{m=0}^l \binom{n}{l} C_{n-l,\lambda} \frac{S_{1,\lambda}(l+1, m+1)}{l+1}. \tag{2.5}$$

Moreover,

$$\sum_{l=0}^n \sum_{m=0}^l \binom{n}{l} C_{n-l,\lambda} \frac{S_{1,\lambda}(l+1, m+1)}{l+1} = 0.$$

From (2.1), we observe that

$$\begin{aligned}
&\sum_{n=0}^{\infty} C_{n,\lambda}^{(k)}(x) \frac{t^n}{n!} \\
&= \frac{\text{Ei}_{k,\lambda}(\log_{\lambda}(1+t))}{\log(1 + \frac{1}{\lambda} \log(1 + \lambda t))} \left( 1 + \frac{1}{\lambda} \log(1 + \lambda t) \right)^x \\
&= \left( \sum_{l=0}^{\infty} C_{l,\lambda}^{(k)} \frac{t^l}{l!} \right) \left( \sum_{m=0}^{\infty} \binom{x}{m} \left( \frac{1}{\lambda} \log(1 + \lambda t) \right)^m \right) \\
&= \left( \sum_{l=0}^{\infty} C_{l,\lambda}^{(k)} \frac{t^l}{l!} \right) \left( \sum_{m=0}^{\infty} (x)_m \lambda^{-m} \sum_{s=m}^{\infty} S_1(s, m) \lambda^s \frac{t^s}{s!} \right) \\
&= \left( \sum_{l=0}^{\infty} C_{l,\lambda}^{(k)} \frac{t^l}{l!} \right) \left( \sum_{s=0}^{\infty} \sum_{m=0}^s (x)_m \lambda^{-m} S_1(s, m) \lambda^s \frac{t^s}{s!} \right) \\
L.H.S &= \sum_{n=0}^{\infty} \left( \sum_{l=0}^n \sum_{m=0}^{n-l} \binom{n}{l} C_{l,\lambda}^{(k)} (x)_m \lambda^{n-l-m} S_1(n-l, m) \right) \frac{t^n}{n!}. \tag{2.6}
\end{aligned}$$

By comparing the coefficients on both sides of (2.6), we obtain the following theorem.

**Theorem 2.2.** Let  $n \geq 0$  and  $k \in \mathbb{Z}$ . Then we have

$$C_{n,\lambda}^{(k)}(x) = \sum_{l=0}^n \sum_{m=0}^{n-l} \binom{n}{l} C_{l,\lambda}^{(k)}(x)_m \lambda^{n-l-m} S_1(n-l, m). \quad (2.7)$$

In [26], it is well known that the degenerate Bernoulli polynomials of the second kind are defined by

$$\frac{t}{\log_\lambda(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_{n,\lambda}(x) \frac{t^n}{n!}. \quad (2.8)$$

For  $x = 0$ ,  $b_{n,\lambda} = b_{n,\lambda}(0)$  are called degenerate Bernoulli numbers of the second kind.

From (1.7), we note that

$$\begin{aligned} \frac{d}{dx} \text{Ei}_{k,\lambda}(\log_\lambda(1+x)) &= \frac{d}{dx} \sum_{n=1}^{\infty} \frac{(1)_{n,\lambda}(\log_\lambda(1+x))^n}{(n-1)! n^k} \\ &= \frac{(1+x)^{\lambda-1}}{\log_\lambda(1+x)} \sum_{n=1}^{\infty} \frac{(1)_{n,\lambda}(\log_\lambda(1+x))^n}{(n-1)! n^{k-1}} = \frac{(1+x)^{\lambda-1}}{\log_\lambda(1+x)} \text{Ei}_{k-1,\lambda}(\log_\lambda(1+x)). \end{aligned} \quad (2.9)$$

Thus, from (2.1) and (2.9), we have

$$\begin{aligned} \sum_{n=0}^{\infty} C_{n,\lambda}^{(k)} \frac{x^n}{n!} &= \frac{1}{\log(1 + \frac{1}{\lambda} \log(1 + \lambda x))} \text{Ei}_{k,\lambda}(\log_\lambda(1+x)) \\ &= \frac{1}{\log(1 + \frac{1}{\lambda} \log(1 + \lambda x))} \int_0^x \underbrace{\frac{(1+t)^{\lambda-1}}{\log_\lambda(1+t)} \int_0^t \cdots \frac{(1+t)^{\lambda-1}}{\log_\lambda(1+t)} \int_0^t}_{(k-2)-\text{times}} \frac{(1+t)^{\lambda-1}}{\log_\lambda(1+t)} t dt \cdots dt \\ &= \frac{x}{\log(1 + \frac{1}{\lambda} \log(1 + \lambda x))} \sum_{m=0}^{\infty} \sum_{m_1+\cdots+m_{k-1}=m} \binom{m}{m_1+\cdots+m_{k-1}} \\ &\quad \times \frac{b_{m_1,\lambda}(\lambda-1)}{m_1+1} \frac{b_{m_2,\lambda}(\lambda-1)}{m_1+m_2+1} \cdots \frac{b_{m_{k-1},\lambda}(\lambda-1)}{m_1+\cdots+m_{k-1}+1} \frac{x^m}{m!} \\ &\quad \times \sum_{n=0}^{\infty} C_{n,\lambda}^{(k)} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \sum_{m=0}^n \binom{n}{m} \sum_{m_1+\cdots+m_{k-1}=m} \binom{m}{m_1+\cdots+m_{k-1}} \\ &\quad \times \frac{b_{m_1,\lambda}(\lambda-1)}{m_1+1} \frac{b_{m_2,\lambda}(\lambda-1)}{m_1+m_2+1} \cdots \frac{b_{m_{k-1},\lambda}(\lambda-1)}{m_1+\cdots+m_{k-1}+1} C_{n-m,\lambda} \frac{x^n}{n!}. \end{aligned} \quad (2.10)$$

Therefore, by (2.10), we obtain the following theorem.

**Theorem 2.3.** For  $n \geq 0$ , we have

$$\begin{aligned} C_{n,\lambda}^{(k)} &= \sum_{m=0}^n \binom{n}{m} \sum_{m_1+\cdots+m_{k-1}=m} \binom{m}{m_1+\cdots+m_{k-1}} \\ &\quad \times \frac{b_{m_1,\lambda}(\lambda-1)}{m_1+1} \frac{b_{m_2,\lambda}(\lambda-1)}{m_1+m_2+1} \cdots \frac{b_{m_{k-1},\lambda}(\lambda-1)}{m_1+\cdots+m_{k-1}+1} C_{n-m,\lambda}. \end{aligned} \quad (2.11)$$

**Corollary 2.2.** For  $k \geq 2$ , we have

$$C_{n,\lambda}^{(2)} = \sum_{m=0}^n \binom{n}{m} \frac{b_{m,\lambda}(\lambda-1)}{m+1} C_{n-m,\lambda}.$$

Let  $k \geq 1$ , be an integer. For  $s \in \mathbb{C}$ , we define the function  $\eta_{k,\lambda}(s)$  as

$$\eta_{k,\lambda}(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{\log(1 + \frac{1}{\lambda} \log(1 + \lambda t))} \text{Ei}_{k,\lambda}(\log_\lambda(1+t)) dt$$

$$\begin{aligned}
&= \frac{1}{\Gamma(s)} \int_0^1 \frac{t^{s-1}}{\log(1 + \frac{1}{\lambda} \log(1 + \lambda t))} \text{Ei}_{k,\lambda}(\log_\lambda(1+t)) dt \\
&+ \frac{1}{\Gamma(s)} \int_1^\infty \frac{t^{s-1}}{\log(1 + \frac{1}{\lambda} \log(1 + \lambda t))} \text{Ei}_{k,\lambda}(\log_\lambda(1+t)) dt.
\end{aligned} \tag{2.12}$$

The second integral converges absolutely for any  $s \in \mathbb{C}$  and hence, the second term on the right hand side vanishes at non-positive integers. That is,

$$\lim_{s \rightarrow -m} \left| \frac{1}{\Gamma(s)} \int_1^\infty \frac{t^{s-1}}{\log(1 + \frac{1}{\lambda} \log(1 + \lambda t))} \text{Ei}_{k,\lambda}(\log_\lambda(1+t)) dt \right| \leq \frac{1}{\Gamma(-m)} M = 0. \tag{2.13}$$

On the other hand, for  $\Re(s) > 0$ , the first integral in (2.13) can be written as

$$\frac{1}{\Gamma(s)} \sum_{l=0}^{\infty} \frac{C_{l,\lambda}^{(k)}}{l!} \frac{1}{s+l},$$

which defines an entire function of  $s$ . Thus, we may include that  $\eta_{k,\lambda}(s)$  can be continued to an entire function of  $s$ .

Further, from (2.12) and (2.13), we obtain

$$\begin{aligned}
\eta_{k,\lambda}(-m) &= \lim_{s \rightarrow -m} \frac{1}{\Gamma(s)} \int_0^1 \frac{t^{s-1}}{\log(1 + \frac{1}{\lambda} \log(1 + \lambda t))} \text{Ei}_{k,\lambda}(\log_\lambda(1+t)) dt \\
&= \lim_{s \rightarrow -m} \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} \sum_{l=0}^{\infty} \frac{C_{l,\lambda}^{(k)} t^l}{l!} dt = \lim_{s \rightarrow -m} \frac{1}{\Gamma(s)} \sum_{l=0}^{\infty} \frac{C_{l,\lambda}^{(k)}}{s+l} \frac{1}{l!} \\
&= \dots + 0 + \dots + 0 + \lim_{s \rightarrow -m} \frac{1}{\Gamma(s)} \frac{1}{s+m} \frac{C_{m,\lambda}^{(k)}}{m!} + 0 + 0 + \dots \\
&= \lim_{s \rightarrow -m} \frac{\left(\frac{\Gamma(1-s) \sin \pi s}{\pi}\right) C_{m,\lambda}^{(k)}}{s+m} \frac{1}{m!} = \Gamma(1+m) \cos(\pi m) \frac{C_{m,\lambda}^{(k)}}{m!} \\
&= (-1)^m C_{m,\lambda,2}^{(k)}.
\end{aligned} \tag{2.14}$$

Therefore, by (2.14), we obtain the following theorem.

**Theorem 2.4.** Let  $k \geq 1$  and  $m \in \mathbb{N} \cup \{0\}$ ,  $s \in \mathbb{C}$ , we have

$$\eta_{k,\lambda}(-m) = (-1)^m C_{m,\lambda}^{(k)}.$$

Replacing  $t$  by  $\frac{1}{\lambda} e_\lambda(t) - 1$  in (2.1) and using (1.1), we get

$$\sum_{m=0}^{\infty} C_{m,\lambda}^{(k)}(x) \lambda^{-m} \frac{(e_\lambda(t) - 1)^m}{m!} = \frac{\text{Ei}_{k,\lambda}(\frac{t}{\lambda})}{\log(1+t)} (1+t)^x.$$

Now

$$\begin{aligned}
\frac{\text{Ei}_{k,\lambda}(\frac{t}{\lambda})}{\log(1+t)} (1+t)^x &= \left( \frac{t}{\log(1+t)} (1+t)^x \right) \left( \frac{\text{Ei}_{k,\lambda}(\frac{t}{\lambda})}{t} \right) \\
&= \left( \sum_{n=0}^{\infty} C_n(x) \frac{t^n}{n!} \right) \left( \frac{1}{t} \sum_{m=1}^{\infty} \frac{\lambda^{-m} (1)_{m,\lambda} t^m}{(m-1)! m^k} \right) \\
&= \left( \sum_{n=0}^{\infty} C_n(x) \frac{t^n}{n!} \right) \left( \sum_{m=0}^{\infty} \frac{\lambda^{-m-1} (1)_{m+1,\lambda} t^m}{m! (m+1)^k} \right) \\
L.H.S &= \sum_{n=0}^{\infty} \left( \sum_{m=0}^n \binom{n}{m} C_{n-m}(x) \frac{\lambda^{-m-1} (1)_{m+1,\lambda}}{(m+1)^k} \right) \frac{t^n}{n!}.
\end{aligned} \tag{2.15}$$

On the other hand,

$$\begin{aligned} \sum_{m=0}^{\infty} C_{m,\lambda}^{(k)}(x) \lambda^{-m} \frac{(e_{\lambda}(t) - 1)^m}{m!} &= \sum_{m=0}^{\infty} C_{m,\lambda}^{(k)}(x) \lambda^{-m} \sum_{n=m}^{\infty} S_2(n, m) \lambda^n \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left( \sum_{m=0}^n \lambda^{n-m} C_{m,\lambda}^{(k)}(x) S_2(n, m) \right) \frac{t^n}{n!}. \end{aligned} \quad (2.16)$$

Therefore, by equations (2.15) and (2.16), we obtain the following theorem.

**Theorem 2.5.** Let  $k \geq 1$  and  $m \in \mathbb{N} \cup \{0\}$ ,  $s \in \mathbb{C}$ , we have

$$\begin{aligned} \sum_{m=0}^n \binom{n}{m} C_{n-m}(x) \frac{\lambda^{-m-1} (1)_{m+1,\lambda}}{(m+1)^k} \\ = \sum_{m=0}^n \lambda^{n-m} C_{m,\lambda}^{(k)}(x) S_2(n, m). \end{aligned}$$

From (2.1), we note that

$$\text{Ei}_{k,\lambda}(\log_{\lambda}(1+t)) = \left( \sum_{n=0}^{\infty} C_{n,\lambda}^{(k)} \frac{t^n}{n!} \right) \left( \log(1 + \frac{1}{\lambda} \log(1 + \lambda t)) \right)$$

Now

$$\begin{aligned} &\left( \sum_{n=0}^{\infty} C_{n,\lambda}^{(k)} \frac{t^n}{n!} \right) \left( \log(1 + \frac{1}{\lambda} \log(1 + \lambda t)) \right) \\ &= \left( \sum_{n=0}^{\infty} C_{n,\lambda}^{(k)} \frac{t^n}{n!} \right) \left( \sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{l} \lambda^{-l} (\log(1 + \lambda t))^l \right) \\ &= \left( \sum_{n=0}^{\infty} C_{n,\lambda}^{(k)} \frac{t^n}{n!} \right) \left( (l-1)! (-1)^{l-1} \lambda^{-l} \sum_{m=l}^{\infty} S_1(m, l) \lambda^m \frac{t^m}{m!} \right) \\ &= \left( \sum_{n=0}^{\infty} C_{n,\lambda}^{(k)} \frac{t^n}{n!} \right) \left( \sum_{m=1}^{\infty} \left( \sum_{l=1}^m (l-1)! (-1)^{l-1} \lambda^{m-l} S_1(m, l) \right) \frac{t^m}{m!} \right) \\ L.H.S &= \sum_{n=1}^{\infty} \left( \sum_{m=1}^n \sum_{l=1}^m \binom{n}{m} C_{n-m,\lambda}^{(k)} (l-1)! (-1)^{l-1} \lambda^{m-l} S_1(m, l) \right) \frac{t^n}{n!}. \end{aligned} \quad (2.17)$$

On the other hand,

$$\begin{aligned} \text{Ei}_{k,\lambda}(\log_{\lambda}(1+t)) &= \sum_{m=1}^{\infty} \frac{(1)_{m,\lambda} (\log_{\lambda}(1+t))^m}{(m-1)! m^k} \\ &= \sum_{m=1}^{\infty} \frac{(1)_{m,\lambda} (\log_{\lambda}(1+t))^m}{(m-1)! m^k} \frac{m!}{m!} \\ &= \sum_{m=1}^{\infty} \frac{(1)_{m,\lambda}}{m^{k-1}} \sum_{n=m}^{\infty} S_{1,\lambda}(n, m) \frac{t^n}{n!} \\ L.H.S &= \sum_{n=1}^{\infty} \left( \sum_{m=1}^n \frac{(1)_{m,\lambda} S_{1,\lambda}(n, m)}{m^{k-1}} \right) \frac{t^n}{n!}. \end{aligned} \quad (2.18)$$

Therefore, by (2.17) and (2.18), we obtain the following theorem.

**Theorem 2.6.** Let  $k \geq 1$  and  $m \in \mathbb{N} \cup \{0\}$ ,  $s \in \mathbb{C}$ , we have

$$\begin{aligned} & \sum_{m=1}^n \frac{(1)_{m,\lambda} S_{1,\lambda}(n,m)}{m^{k-1}} \\ &= \sum_{m=1}^n \sum_{l=1}^m \binom{n}{m} C_{n-m,\lambda}^{(k)} (l-1)! (-1)^{l-1} \lambda^{m-l} S_1(m,l). \end{aligned}$$

For  $k = 1$  in Theorem 2.6., we get the following corollary

**Corollary 2.3.** For  $m \in \mathbb{N} \cup \{0\}$ ,  $s \in \mathbb{C}$ , we have

$$\begin{aligned} & \sum_{m=1}^n (1)_{m,\lambda} S_{1,\lambda}(n,m) \\ &= \sum_{m=1}^n \sum_{l=1}^m \binom{n}{m} C_{n-m,\lambda}^{(1)} (l-1)! (-1)^{l-1} \lambda^{m-l} S_1(m,l). \end{aligned}$$

**Corollary 2.4.** For  $m \in \mathbb{N} \cup \{0\}$ ,  $s \in \mathbb{C}$ , we have

$$C_{0,\lambda} = \sum_{m=1}^n \sum_{l=1}^m \binom{n}{m} C_{n-m,\lambda}^{(1)} (l-1)! (-1)^{l-1} \lambda^{m-l} S_1(m,l) = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{if } n > 1. \end{cases}$$

Now, we observe that

$$\begin{aligned} \frac{\text{Ei}_{k,\lambda}(\log_\lambda(1+t))}{\log(1 + \frac{1}{\lambda} \log(1 + \lambda t))} &= \left( \frac{\lambda t}{\log(1 + \lambda t)} \right) \left( \frac{\text{Ei}_{k,\lambda}(\log_\lambda(1+t))}{t} \right) \left( \frac{\frac{1}{\lambda} \log(1 + \lambda t)}{\log(1 + \frac{1}{\lambda} \log(1 + \lambda t))} \right) \\ &= \left( \sum_{n=0}^{\infty} \lambda^l B_n^{(n)} \frac{t^n}{n!} \right) \left( \frac{1}{t} \sum_{m=1}^{\infty} \frac{(1)_{m,\lambda} (\log_\lambda(1+t))^m}{(m-1)! m^k} \right) \left( \sum_{l=0}^{\infty} C_{l,\lambda}^* \frac{t^l}{l!} \right) \\ &= \left( \sum_{n=0}^{\infty} \lambda^l B_n^{(n)} \frac{t^n}{n!} \right) \left( \sum_{m=0}^{\infty} \frac{(1)_{m+1,\lambda} (\log_\lambda(1+t))^{m+1}}{m!(m+1)^k} \frac{(m+1)!}{(m+1)!} \right) \left( \sum_{l=0}^{\infty} C_{l,\lambda}^* \frac{t^l}{l!} \right) \\ &= \left( \sum_{n=0}^{\infty} \lambda^l B_n^{(n)} \frac{t^n}{n!} \right) \left( \sum_{m=0}^{\infty} \frac{(1)_{m+1,\lambda}}{(m+1)^{k-1}} \sum_{r=m+1}^{\infty} S_{1,\lambda}(r, m+1) \frac{t^r}{r!} \right) \left( \sum_{l=0}^{\infty} C_{l,\lambda}^* \frac{t^l}{l!} \right) \\ &= \left( \sum_{n=0}^{\infty} \lambda^n B_n^{(n)} \frac{t^n}{n!} \right) \left( \sum_{r=0}^{\infty} \sum_{m=0}^r \frac{(1)_{m+1,\lambda} S_{1,\lambda}(r+1, m+1)}{(r+1)(m+1)^{k-1}} \frac{t^r}{r!} \right) \left( \sum_{l=0}^{\infty} C_{l,\lambda}^* \frac{t^l}{l!} \right) \\ &= \left( \sum_{n=0}^{\infty} \lambda^n B_n^{(n)} \frac{t^n}{n!} \right) \left( \sum_{l=0}^{\infty} \sum_{r=0}^l \binom{l}{r} \sum_{m=0}^r \frac{(1)_{m+1,\lambda} S_{1,\lambda}(r+1, m+1)}{(r+1)(m+1)^{k-1}} C_{l-r,\lambda}^* \right) \frac{t^l}{l!} \\ L.H.S &= \sum_{n=0}^{\infty} \left( \sum_{l=0}^n \binom{n}{l} \sum_{r=0}^l \binom{l}{r} \sum_{m=0}^r \frac{(1)_{m+1,\lambda} S_{1,\lambda}(r+1, m+1)}{(r+1)(m+1)^{k-1}} C_{l-r,\lambda}^* \lambda^{n-l} B_{n-m}^{(n-m)} \right) \frac{t^n}{n!}. \end{aligned} \tag{2.19}$$

From (2.1), we note that

$$\frac{\text{Ei}_{k,\lambda}(\log_\lambda(1+t))}{\log(1 + \frac{1}{\lambda} \log(1 + \lambda t))} = \sum_{n=0}^{\infty} C_{n,\lambda}^{(k)} \frac{t^n}{n!}. \tag{2.20}$$

Therefore, by (2.19) and (2.20), we obtain the following theorem.

**Theorem 2.7.** Let  $k \geq 1$  and  $m \in \mathbb{N} \cup \{0\}$ ,  $s \in \mathbb{C}$ , we have

$$C_{n,\lambda}^{(k)} = \sum_{l=0}^n \binom{n}{l} \sum_{r=0}^l \binom{l}{r} \sum_{m=0}^r \frac{(1)_{m+1,\lambda} S_{1,\lambda}(r+1, m+1)}{(r+1)(m+1)^{k-1}} C_{l-r,\lambda}^* \lambda^{n-l} B_{n-l}^{(n-l)}.$$

From (2.1), we note that

$$\begin{aligned} \sum_{n=0}^{\infty} C_{n,\lambda}^{(k)} \frac{t^n}{n!} &= \frac{\text{Ei}_{k,\lambda}(\log_{\lambda}(1+t))}{\log(1 + \frac{1}{\lambda} \log(1 + \lambda t))} \left( 1 + \frac{1}{\lambda} \log(1 + \lambda t) \right) \\ &= \frac{\text{Ei}_{k,\lambda}(\log_{\lambda}(1+t))}{\log(1 + \frac{1}{\lambda} \log(1 + \lambda t))} + \frac{\text{Ei}_{k,\lambda}(\log_{\lambda}(1+t)) \frac{1}{\lambda} \log(1 + \lambda t)}{\log(1 + \frac{1}{\lambda} \log(1 + \lambda t))} \\ &= \sum_{n=0}^{\infty} C_{n,\lambda}^{(k)} \frac{t^n}{n!} + t \frac{\text{Ei}_{k,\lambda}(\log_{\lambda}(1+t))}{t} \frac{\frac{1}{\lambda} \log(1 + \lambda t)}{\log(1 + \frac{1}{\lambda} \log(1 + \lambda t))} \\ &\quad \sum_{n=1}^{\infty} \left[ C_{n,\lambda}^{(k)}(1) - C_{n,\lambda}^{(k)} \right] \frac{t^n}{n!} = t \frac{\text{Ei}_{k,\lambda}(\log_{\lambda}(1+t))}{t} \frac{\frac{1}{\lambda} \log(1 + \lambda t)}{\log(1 + \frac{1}{\lambda} \log(1 + \lambda t))} \\ &= \left( \frac{1}{t} \sum_{m=1}^{\infty} \frac{(1)_{m,\lambda}(\log_{\lambda}(1+t))^m}{(m-1)!m^k} \right) \left( \sum_{l=0}^{\infty} C_{l,\lambda}^* \frac{t^l}{l!} \right) \\ &= \left( \sum_{m=0}^{\infty} \frac{(1)_{m+1,\lambda}(\log_{\lambda}(1+t))^{m+1}}{m!(m+1)^k} \frac{(m+1)!}{(m+1)!} \right) \left( \sum_{l=0}^{\infty} C_{l,\lambda}^* \frac{t^l}{l!} \right) \\ &= \left( \sum_{r=0}^{\infty} \sum_{m=0}^r \frac{(1)_{m+1,\lambda} S_{1,\lambda}(r+1, m+1)}{(r+1)(m+1)^{k-1}} \frac{t^r}{r!} \right) \left( \sum_{l=0}^{\infty} C_{l,\lambda}^* \frac{t^l}{l!} \right) \\ R.H.S &= \sum_{n=0}^{\infty} \left( \sum_{r=0}^n \binom{n}{r} \sum_{m=0}^r \frac{(1)_{m+1,\lambda} S_{1,\lambda}(r+1, m+1)}{(r+1)(m+1)^{k-1}} C_{n-r,\lambda}^* \right) \frac{t^n}{n!}. \end{aligned} \quad (2.21)$$

On the other hand,

$$\begin{aligned} \sum_{n=1}^{\infty} \left[ C_{n,\lambda}^{(k)}(1) - C_{n,\lambda,2}^{(k)} \right] \frac{t^{n-1}}{n!} \\ = \sum_{n=0}^{\infty} \left[ \frac{C_{n+1,\lambda,2}^{(k)}(1) - C_{n+1,\lambda,2}^{(k)}}{n+1} \right] \frac{t^n}{n!}. \end{aligned} \quad (2.22)$$

Therefor, by (2.21) and (2.22), we obtain the following theorem.

**Theorem 2.8.** For  $n \geq 0$ , we have

$$\frac{C_{n+1,\lambda}^{(k)}(1) - C_{n+1,\lambda}^{(k)}}{n+1} = \sum_{r=0}^n \binom{n}{r} \sum_{m=0}^r \frac{(1)_{m+1,\lambda} S_{1,\lambda}(r+1, m+1)}{(r+1)(m+1)^{k-1}} C_{n-r,\lambda}^*.$$

### 3. Type 2 degenerate unipoly-Cauchy polynomials of the second kind

Let  $p$  be any arithmetic function which is a real or complex valued function defined on the set of positive integers  $\mathbb{N}$ . Kim-Kim [6] defined the unipoly function attached to polynomials  $p(x)$  by

$$u_k(x|p) = \sum_{n=1}^{\infty} \frac{p(n)}{n^k} x^n, (k \in \mathbb{Z}). \quad (3.1)$$

Moreover,

$$u_k(x|1) = \sum_{n=1}^{\infty} \frac{x^n}{n^k} = \text{Li}_k(x), (\text{see [14]}), \quad (3.2)$$

is the ordinary polylogarithm function.

In this paper, we define the degenerate unipoly function attached to polynomials  $p(x)$  as follows:

$$u_{k,\lambda}(x|p) = \sum_{i=1}^{\infty} p(i) \frac{(1)_{i,\lambda} x^i}{i^k}. \quad (3.3)$$

It is worthy to note that

$$u_{k,\lambda}\left(x \mid \frac{1}{\Gamma}\right) = \text{Ei}_{k,\lambda}(x) \quad (3.4)$$

is the degenerate modified polyexponential function.

By using (3.3), we define the new type of degenerate unipoly-Cauchy polynomials of the second kind as follows:

$$\frac{u_{k,\lambda}(\log_{\lambda}(1+t)|p)}{\log(1 + \frac{1}{\lambda} \log(1 + \lambda t))} \left(1 + \frac{1}{\lambda} \log(1 + \lambda t)\right)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} C_{n,\lambda,p}^{(k)}(x) \frac{t^n}{n!}. \quad (3.5)$$

In the case when  $x = 0$ ,  $C_{n,\lambda,p}^{(k)} = C_{n,\lambda,p}^{(k)}(0)$  are called the new type of degenerate unipoly-Cauchy numbers of the second kind. Let us take  $p(n) = \frac{1}{\Gamma \lambda}$ . Then we have

$$\begin{aligned} \sum_{n=0}^{\infty} C_{n,\lambda,\frac{1}{\Gamma}}^{(k)}(x) \frac{t^n}{n!} &= \frac{u_{k,\lambda}(\log_{\lambda}(1+t)|\frac{1}{\Gamma}p)}{\log(1 + \frac{1}{\lambda} \log(1 + \lambda t))} \left(1 + \frac{1}{\lambda} \log(1 + \lambda t)\right)^{\frac{x}{\lambda}} \\ &= \frac{1}{\log(1 + \frac{1}{\lambda} \log(1 + \lambda t))} \sum_{m=1}^{\infty} \frac{(\log(1+t))^m}{m^k(m+1)!} \left(1 + \frac{1}{\lambda} \log(1 + \lambda t)\right)^{\frac{x}{\lambda}} \\ &= \frac{\text{Ei}_{k,\lambda}(\log_{\lambda}(1+t))}{\log(1 + \frac{1}{\lambda} \log(1 + \lambda t))} \left(1 + \frac{1}{\lambda} \log(1 + \lambda t)\right)^{\frac{x}{\lambda}} \\ &= \sum_{n=0}^{\infty} C_{n,\lambda}^{(k)}(x) \frac{t^n}{n!}. \end{aligned} \quad (3.6)$$

Thus, by (3.6), we have the following theorem.

**Theorem 3.1.** Let  $n \geq 0$  and  $k \in \mathbb{Z}$ , and  $\Gamma n$  be a Gamma function. Then, we have

$$C_{n,\lambda,\frac{1}{\Gamma}}^{(k)}(x) = C_{n,\lambda}^{(k)}(x).$$

From (3.5), we get

$$\begin{aligned} \sum_{n=0}^{\infty} C_{n,\lambda,p}^{(k)} \frac{t^n}{n!} &= \frac{u_{k,\lambda}(\log_{\lambda}(1+t)|p)}{\log(1 + \frac{1}{\lambda} \log(1 + \lambda t))} \\ &= \frac{1}{\log(1 + \frac{1}{\lambda} \log(1 + \lambda t))} \sum_{m=1}^{\infty} \frac{p(m)(1)_{m,\lambda}}{m^k} (\log_{\lambda}(1+t))^m \\ &= \frac{1}{\log(1 + \frac{1}{\lambda} \log(1 + \lambda t))} \sum_{m=0}^{\infty} \frac{p(m+1)(1)_{m+1,\lambda}(m+1)!}{(m+1)^k} \sum_{l=m+1}^{\infty} S_{1,\lambda}(m+1, l) \frac{t^l}{l!} \\ &= \left( \sum_{j=0}^{\infty} C_{j,\lambda} \frac{t^j}{j!} \right) \left( \sum_{m=0}^{\infty} \sum_{l=0}^m \frac{p(m+1)(1)_{m+1,\lambda}(m+1)!}{(m+1)^k} S_{1,\lambda}(m+1, l) \frac{t^l}{l!} \right) \\ &= \sum_{n=0}^{\infty} \left( \sum_{l=0}^n \sum_{m=0}^l \binom{n}{l} \frac{p(m+1)(1)_{m+1,\lambda}(m+1)! S_{1,\lambda}(m+1, l+1) C_{n-l,\lambda}}{(m+1)^k (l+1)} \right) \frac{t^n}{n!}. \end{aligned} \quad (3.8)$$

Therefore, by comparing the coefficients on both sides of (3.8), we obtain the following theorem.

**Theorem 3.2.** Let  $n \in \mathbb{N}$  and  $k \in \mathbb{Z}$ . Then we have

$$C_{n,\lambda,p}^{(k)} = \sum_{l=0}^{\infty} \sum_{m=0}^l \binom{n}{l} \frac{p(m+1)(1)_{m+1,\lambda}(m+1)! S_{1,\lambda}(m+1, l+1) C_{n-l,\lambda}}{(m+1)^k (l+1)}. \quad (3.9)$$

In particular,

$$C_{n,\lambda,\frac{1}{\lambda}}^{(k)} = C_{n,\lambda}^{(k)} = \sum_{l=0}^{\infty} \sum_{m=0}^l \binom{n}{l} \frac{S_{1,\lambda}(m+1, l+1) C_{n-l,\lambda,2}}{(m+1)^{k-1} (l+1)}. \quad (3.10)$$

From (3.5), we observe that

$$\begin{aligned} \sum_{n=0}^{\infty} C_{n,\lambda}^{(k,p)}(x) \frac{t^n}{n!} &= \frac{u_{k,\lambda}(\log_{\lambda}(1+t)|p)}{\log(1 + \frac{1}{\lambda} \log(1 + \lambda t))} \left(1 + \frac{1}{\lambda} \log(1 + \lambda t)\right)^{\frac{x}{\lambda}} \\ &= \frac{u_{k,\lambda}(\log_{\lambda}(1+t)|p)}{\log(1 + \frac{1}{\lambda} \log(1 + \lambda t))} \sum_{m=0}^{\infty} \binom{\frac{x}{\lambda}}{m} \left(\frac{1}{\lambda} \log(1 + \lambda t)\right)^m \\ &= \left(\sum_{l=0}^{\infty} C_{l,\lambda,p}^{(k)} \frac{t^l}{l!}\right) \left(\sum_{m=0}^{\infty} (x)_{m,\lambda} \lambda^{-2m} \sum_{s=m}^{\infty} S_1(s, m) \frac{t^s}{s!}\right) \\ &= \left(\sum_{l=0}^{\infty} C_{l,\lambda,p}^{(k)} \frac{t^l}{l!}\right) \left(\sum_{s=0}^{\infty} \sum_{m=0}^s (x)_{m,\lambda} \lambda^{-2m} S_1(s, m) \frac{t^s}{s!}\right) \\ L.H.S &= \sum_{n=0}^{\infty} \left( \sum_{l=0}^n \sum_{m=0}^{n-l} C_{l,\lambda,p}^{(k)} (x)_{m,\lambda} \lambda^{-2m} S_1(n-l, m) \right) \frac{t^n}{n!}. \end{aligned} \quad (3.11)$$

From (3.11), we obtain the following theorem.

**Theorem 3.3.** Let  $n \geq 0$  and  $k \in \mathbb{Z}$ . Then we have

$$C_{n,\lambda,p}^{(k)}(x) = \sum_{l=0}^n \sum_{m=0}^{n-l} C_{l,\lambda,p}^{(k)} (x)_{m,\lambda} \lambda^{-2m} S_1(n-l, m). \quad (3.12)$$

From (3.5), we observe that

$$\begin{aligned} \sum_{n=0}^{\infty} C_{n,\lambda,p}^{(k)} \frac{t^n}{n!} &= \frac{u_{k,\lambda}(\log_{\lambda}(1+t)|p)}{\log(1 + \frac{1}{\lambda} \log(1 + \lambda t))} \\ &= \frac{1}{\log(1 + \frac{1}{\lambda} \log(1 + \lambda t))} \sum_{m=0}^{\infty} \frac{p(m+1)(1)_{m+1,\lambda} m!}{(m+1)^k m!} (\log_{\lambda}(1+t))^{m+1} \\ &= \frac{\log_{\lambda}(1+t)}{\log(1 + \frac{1}{\lambda} \log(1 + \lambda t))} \sum_{m=0}^{\infty} \frac{p(m+1)(1)_{m+1,\lambda} m!}{(m+1)^k m!} (\log_{\lambda}(1+t))^m \\ &= \frac{\log_{\lambda}(1+t)}{t} \frac{t}{\log(1 + \frac{1}{\lambda} \log(1 + \lambda t))} \sum_{m=0}^{\infty} \frac{p(m+1)(1)_{m+1,\lambda} m!}{(m+1)^k} \sum_{l=m}^{\infty} S_{1,\lambda}(l, m) \frac{t^l}{l!} \\ &= \left( \sum_{s=0}^{\infty} D_{s,\lambda} \frac{t^s}{s!} \right) \left( \sum_{a=0}^{\infty} C_{a,\lambda} \frac{t^a}{a!} \right) \left( \sum_{l=0}^{\infty} \sum_{m=0}^n \frac{p(m+1)(1)_{m+1,\lambda} m!}{(m+1)^k} S_{1,\lambda}(l, m) \frac{t^l}{l!} \right) \end{aligned}$$

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$$\begin{aligned}
&= \left( \sum_{b=0}^{\infty} \sum_{a=0}^b \binom{b}{a} D_{b-a, \lambda} C_{a, \lambda} \frac{t^b}{b!} \right) \left( \sum_{l=0}^{\infty} \sum_{m=0}^n \frac{p(m+1)(1)_{m+1, \lambda} m!}{(m+1)^k} S_{1, \lambda}(l, m) \frac{t^l}{l!} \right) \\
L.H.S &= \sum_{n=0}^{\infty} \left( \sum_{l=0}^n \sum_{a=0}^{n-l} \sum_{m=0}^l \binom{n}{l} D_{n-l-a, \lambda} C_{a, \lambda} \frac{p(m+1)(1)_{m+1, \lambda} m!}{(m+1)^k} S_{1, \lambda}(l, m) \right) \frac{t^n}{n!}.
\end{aligned} \tag{3.13}$$

By comparing coefficients on both sides of (3.13), we obtain the following theorem.

**Theorem 3.4.** Let  $n \geq 0$  and  $k \in \mathbb{Z}$ . Then we have

$$C_{n, \lambda, p}^{(k)} = \sum_{l=0}^n \sum_{a=0}^{n-l} \sum_{m=0}^l \binom{n}{l} D_{n-l-a, \lambda} C_{a, \lambda} \frac{p(m+1)(1)_{m+1, \lambda} m!}{(m+1)^k} S_{1, \lambda}(l, m). \tag{3.14}$$

#### 4. Conclusions

In 2020 Kim-Kim considered the type 2 degenerate poly-Bernoulli polynomials by making use of the modified polylogarithm functions and Kim [13] introduced the degenerate Cauchy numbers of the second kind. By using these functions and polynomials, we defined the new type of degenerate poly-Cauchy polynomials of the second kind and obtained some identities of the degenerate poly-Cauchy numbers of the second kind in Theorems 2.1 and 2.2. In particular, we obtained an identity of the degenerate poly-Cauchy polynomials of the second kind in Theorem 2.3. Furthermore, by using the unipoly functions, we defined the degenerate unipoly-Cauchy polynomials of the second kind (Eq. (3.5)) and obtained some properties of the degenerate unipoly-Cauchy numbers of the second kind (Theorems 3.1, and 3.3). Finally, we obtained an identity of the new type of degenerate unipoly-Cauchy polynomials of the second kind in Theorem 3.3 and gave the identity indicating the relationship of the degenerate unipoly-Cauchy numbers of the second kind and the Daehee numbers and degenerate Cauchy numbers of the second kind in Theorem 3.4.

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