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Self-Inductance of the Circular Coils of the Rectangular Cross Section with the Radial and the Azimuthal Current Densities

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Abstract- In this paper we give the new formulas for calculating the self-inductance for the circular coils of the rectangular cross sections with the radial and the azimuthal current densities. These formulas are given by the single integration of the elementary functions which are integrable on the interval of the integration. From these new expressions we can obtain the special cases for the self-inductance of the thin disk pancake and the thin wall solenoid that confirm the validity of this approach. For the asymptotic cases, the new formula for the self-inductance of the thin wall solenoid is obtained for the first time in the literature. In this paper we do not use special functions such as the elliptical integrals of the first, second and third kind, Struve and Bessel functions because that is very tedious work. The results of this work are compared with already different known methods and all results are in the excellent agreement. This is way we consider this approach as the novelty because of its simplicity in the self -inductance calculation of the previously mentioned configurations.

Keywords: Self inductance, radial current, azimuthal current, thick coils, disk coils, solenoids.

1. INTRODUCTION

Several monographs and papers are devoted to calculating the self and the mutual inductance for the circular coils of the rectangular cross section with the azimuthal current density, [1-18]. They are very known the conventional coils used in many applications such as all ranges of transformers, generators, motors, current reactors, magnetic resonance applications, antennas, coil guns, medical electronic devices, superconducting magnets, tokamaks, electronic and printed circuit board design, plasma science, etc. Today, with the availability of powerful and general numerical methods, such as finite element method (FEM) and boundary element method (BEM), it is possible to calculate accurately and rapidly the self and mutual inductance of almost any practical 3-D geometric arrangement of conductors. However, in many circumstances, there is still an interest to address this problem using analytic and semi-analytic methods because they considerably simplify the mathematical procedures, and often lead to a significant reduction of the computational effort. The analytical and semi-analytical methods have been used where these important electromagnetic quantities are obtained in the form of the simple, double and triple integrals, the elliptic integrals, the converge series, the Bessel functions, Struve functions [1-18]. Also, there are the circular coils of the rectangular cross section with the radial current density which are interesting from the engineering aspect. These coils are well known the Bitter coils which supply extremely high magnetic fields up to 45 T, (19-27). In this paper our goal was to solve analytically the four integrals in the basic formulas for the self-inductance of the circular coils of the rectangular cross section with the radial and the azimuthal current densities namely L_R (radial current) and L_A (azimuthal current), respectively. We obtained all results of these four integrations in the form of the elementary analytical functions. By the single integration of these expressions, we obtained the simplest formulas for calculating the self-inductance of L_R and L_A without using the special functions. All expressions are arranged in the suitable form for the numerical integration where the possible singularities are treated at the proper manner. Numerous tests are made in MATLAB and Mathematica programing which show that the numerical integration given in Mathematica programming can be used for any range of parameters (very small or very large values of $\alpha = \frac{R_2}{R_1}$ and $b = \frac{l}{R_1}$), where R_1 and R_2 are inner and outer radius of the coil and l is its high [27]. Many examples confirm the validity of the presented method. With

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the presented method all possible cases for the circular coils with the finite cross section and with negligible cross section (thin coils and circular filamentary coils) are covered.

2. BASIC EXESSIONS

Let us take into consideration the circular coil o the rectangular cross section (Fig.1) where,

R_1 – inner radius (m)

R_2 – outer radius (m)

l - high of the coil (m)

I – current in coil (A)

J_R – radial current density (A/ m^2)

J_A – azimuthal current density (A/ m^2)

r_1, r_2 – coordinates which determine any radial position inside the coil (m)

z_1, z_2 – coordinates which determine any axial position inside the coil (m)

N – number of turns

$\mu_0 = 4\pi \cdot 10^{-7} \left(\frac{H}{m}\right)$ – permeability of the free space

A) Radial current

The radial current density and the corresponding self-inductance of the coil of the rectangular cross section are given by [15-20].,

$$J_R = \frac{NI}{l \ln\frac{R_2}{R_1}} \frac{1}{r} \tag{1}$$

$$L_R = \frac{\mu_0 N^2}{l^2 \ln^2\frac{R_2}{R_1}} \int_0^l \int_0^l \int_{R_1}^{R_2} \int_{R_1}^{R_2} \int_0^{\pi} \frac{\cos(\theta) dz_1 dz_2 dr_1 dr_2 d\theta}{R_{12}} \tag{2}$$

where,

$$R_{12} = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta) + (z_2 - z_1)^2}$$

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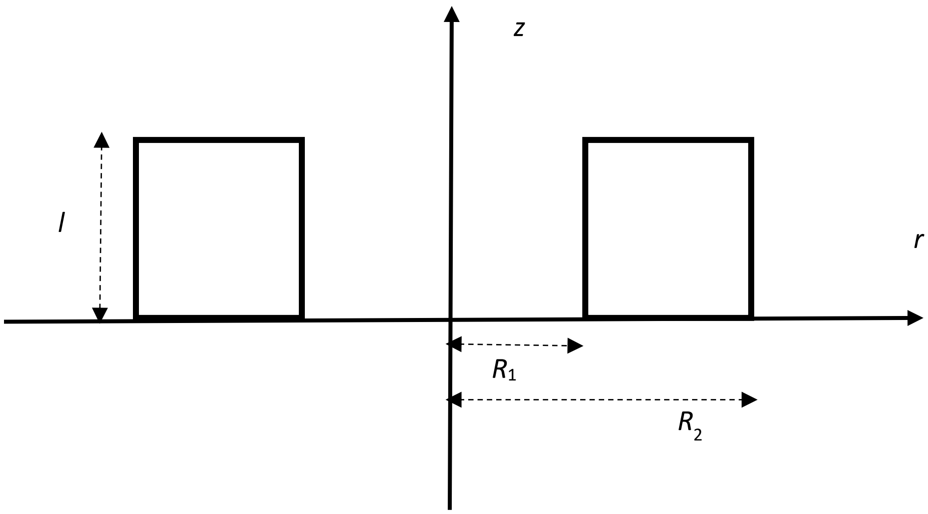

Figure 1. Circular thick coil of the rectangular cross section

B) Azimuthal current

The azimuthal current density and the corresponding self-inductance of the coil of the rectangular cross section are given by [15-20].,

$$J_A = \frac{NI}{l(R_2 - R_1)} \tag{3}$$

$$L_A = \frac{\mu_0 N^2}{l^2 (R_2 - R_1)^2} \int_0^l \int_0^l \int_{R_1}^{R_2} \int_{R_1}^{R_2} \int_0^{\pi} \frac{\cos(\theta) dz_1 dz_2 r_1 dr_1 r_2 dr_2 d\theta}{R_{12}} \tag{4}$$

where,

$$R_{12} = \sqrt{r_1^2 + r_2^2 - 2 r_1 r_2 \cos(\theta) + (z_2 - z_1)^2}$$

3. CALCULATION METHOD

A) The Self-Inductance L_R caused by the radial current density

Introducing the substitution $\theta = \pi - 2\beta$, $r_1 = xR_1$, $r_2 = yR_1, z_1 = vR_1$, $z_2 = zR_1$, $l = bR_1$, $R_2 = \alpha R_1, l = bR_1$ in (2) and making the first four integrations in order to the variables z_2, z_1, r_2 and r_1, (APPENDIX I) we obtained the self-inductance L_R in the following form,

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$$L_R = -\frac{4\mu_0 N^2 R_1}{3b^2 ln^2\alpha}\sum_{n=1}^{7}\int_0^{\frac{\pi}{2}}\cos(2\beta)\,T_n\,d\beta \tag{5}$$

where T_n, ($n = 1,2, \dots ,8$) are the analytical functions integrable on the interval of integration $\beta \in \left[0; \frac{\pi}{2}\right]$. These functions are as follows:

$$T_1 = \frac{b^3}{2\sin(2\beta)}[2\text{arctg}(q) - \text{arctg}(q_{11}) - \text{arctg}(q_{22})]$$

$$T_2 = [(\alpha^2+1)\cos(2\beta) + 2\alpha][r - r_0]$$

$$T_3 = -2\alpha^3\cos^2(\beta)r_2 - 2\cos^2(\beta)r_1 + 4(\alpha^3+1)\cos^3(\beta)$$

$$T_4 = \sin^2(2\beta)[\alpha^3\,\text{atanh}(m_2) + \text{atanh}(m_1) - \alpha^3\,\text{atanh}(m_{22}) - \text{atanh}(m_{11}) -$$
$$-\alpha^3\,\text{atanh}\left(\frac{r_2}{2\alpha\cos^2(\beta)}\right) - \text{atanh}\left(\frac{r_1}{2\cos^2(\beta)}\right) + \alpha^3\,\text{atanh}\left(\frac{1}{\cos(\beta)}\right) + \text{atanh}\left(\frac{1}{\cos(\beta)}\right) \quad]$$

$$T_5 = 3b[(\alpha^2+1)\cos(2\beta) + 2\alpha]\text{asinh}\frac{b}{r_0} + 6\alpha^2 b\cos^2(\beta)\,\text{asinh}\left(\frac{b}{2\alpha\cos(\beta)}\right) +$$
$$+6b\cos^2(\beta)\,\text{asinh}\left(\frac{b}{2\cos(\beta)}\right)$$

$$T_6 = 3\alpha^2 b\sin(2\beta)artg(p_2) + 3b\sin(2\beta)artg(p_1) - 3\alpha^2 b\sin(2\beta)artg(p_{22})$$
$$-3b\sin(2\beta)artg(p_{11})$$

$$T_7 = 3\alpha^2 b\,\text{asinh}(v_{22}) + 3b\,\text{asinh}(v_{11}) - 3\alpha^2 b\,\text{asinh}(v_2) - 3b\,\text{asinh}(v_1)$$

where

$$r = \sqrt{b^2+\alpha^2+1+2\alpha\cos(2\beta)}, \quad r_0 = \sqrt{\alpha^2+1+2\alpha\cos(2\beta)}$$

$$r_{11} = \sqrt{l^2+4cos^2(\beta)}, \qquad r_{22} = \sqrt{l^2+4\alpha^2cos^2(\beta)}$$

$$v_1 = \frac{\alpha+\cos(2\beta)}{\sqrt{b^2+sin^2(2\beta)}}, \qquad v_2 = \frac{1+\alpha\cos(2\beta)}{\sqrt{b^2+\alpha^2sin^2(2\beta)}}$$

$$v_{11} = \frac{1+\cos(2\beta)}{\sqrt{b^2+sin^2(2\beta)}}, \qquad v_{22} = \frac{\alpha+\alpha\cos(2\beta)}{\sqrt{b^2+\alpha^2sin^2(2\beta)}}$$

$$p_1 = \frac{b[\alpha+\cos(2\beta)]}{\sin(2\beta)r}, \qquad p_2 = \frac{b[1+\alpha\cos(2\beta)]}{\alpha\sin(2\beta)r}$$

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$$p_{11} = \frac{b[1+\cos(2\beta)]}{\sin(2\beta)r_{11}}, \qquad p_{22} = \frac{b[\alpha+\alpha\cos(2\beta)]}{\alpha\sin(2\beta)r_{22}}$$

$$q = \frac{\alpha\sin^2(2\beta) - b^2\cos(2\beta)}{b\sin(2\beta)r}$$

$$q_{11} = \frac{\alpha^2 sin^2(2\beta) - b^2\cos(2\beta)}{b\sin(2\beta)r_{22}}, q_{22} = \frac{sin^2(2\beta) - b^2\cos(2\beta)}{b\sin(2\beta)r_{11}}$$

$$m_1 = \frac{r}{\alpha+\cos(2\beta)}, m_2 = \frac{r}{1+\alpha\cos(2\beta)}$$

$$m_{11} = \frac{r_0}{\alpha+\cos(2\beta)}, \qquad m_{22} = \frac{r_0}{1+\alpha\cos(2\beta)}$$

Thus, the new formula for the self-inductance of the circular coil with the rectangular cross section and the radial current density can be obtained by (5) using the simple integration of the previous elementary functions. In this paper we use the Gaussian numerical integration in MATLAB programming and the numerical integration by default in Mathematica programing.

The special case of equation (5) is the self-inductance of the thin disk coil with the radial current [23]. This self-inductance can be obtained from (5) finding the limit when $b \to 0$, or doing three integrations such as in [25].

The self-inductance L_{R-DISK} is obtained in the analytical form as follows:

$$L_{R-DISK} = \frac{8\mu_0 N^2 R_1(\alpha+1)}{ln^2\alpha}[E(k_0) - 1] \tag{6}$$

where,

$$k_0^2 = \frac{4\alpha}{(\alpha+1)^2}$$

and $E(k_0)$ is the elliptic integral of the second kind $[31-33]$.

B) The Self-Inductance L_A caused by the azimuthal current density

Introducing the substitution $\theta = \pi - 2\beta, r_1 = xR_1, r_2 = yR_1, z_1 = vR_1, z_2 = zR_1, l = bR_1, R_2 = \alpha R_1, l = bR_1$ in (4) and making the first four integration (APPENDIX II) in order to the variables r_2, r_1, z_2 and z_1 we obtained the self-inductance L_A in the following form,

$$L_A = -\frac{2\mu_0 N^2 R_1}{15b^2(\alpha-1)^2}\sum_{n=1}^{12}\int_0^{\frac{\pi}{2}}\cos(2\beta)\, S_n\, d\beta \tag{7}$$

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where $S_n, (n = 1,2,\dots,12)$ are the analytical functions integrable on the interval of integration $\beta \in \left[0; \frac{\pi}{2}\right]$. These functions are as follows:

$$S_1 = \frac{b^5 \cos(2\beta)}{\cos^3(2\beta)}[2\mathrm{arctg}(q) - \mathrm{arctg}(q_{11}) - \mathrm{arctg}(q_{22})]$$

$$S_2 = 20b^2 \cos(2\beta)[-\alpha^3 \operatorname{asinh}(v_{22}) - \operatorname{asinh}(v_{11}) + \alpha^3 \operatorname{asinh}(v_2) + \operatorname{asinh}(v_1)]$$

$$S_3 = 12\cos(2\beta)\sin^2(2\beta)[-\alpha^5 \operatorname{atanh}(m_2) - \operatorname{atanh}(m_1) + \alpha^5 \operatorname{atanh}(m_{22}) + \operatorname{atanh}(m_{11})]$$

$$S_4 = 12\cos(2\beta)\, sin^2(2\beta)\left[\alpha^5 \operatorname{atanh}\left(\frac{r_2}{2\alpha cos^2(\beta)}\right) + \operatorname{atanh}\left(\frac{r_1}{2cos^2(\beta)}\right) - \right.$$

$$\left. -(\alpha^5+1)\operatorname{atanh}\left(\frac{1}{\cos(\beta)}\right)\right]$$

$$S_5 = 30b\cos(2\beta)\sin(2\beta)[-\mathrm{arctg}(p_1) - \alpha^4\mathrm{arctg}(p_2) + \mathrm{arctg}(p_{11}) + \alpha^4\mathrm{arctg}(p_{22})]$$

$$S_7 = 15b\{2\alpha^4\sin^2(2\beta)\operatorname{asinh}\left(\frac{b}{2\alpha\cos(\beta)}\right) + 2\sin^2(2\beta)\operatorname{asinh}\left(\frac{b}{2\cos(\beta)}\right) +$$
$$+[2(\alpha^4+1)\cos^2(2\beta) - (\alpha^2+1)^2]\operatorname{asinh}\left(\frac{b}{r_0}\right)\}$$

$$S_8 = -4(\alpha^5+1)\cos(2\beta)[6\cos^2(2\beta) - 3\cos(2\beta) - 8]$$

$$S_9 = -\frac{2b^4 r}{\sin^2(2\beta)} - 9b^2(\alpha^2+1)r - [(20\alpha - 12\alpha^4 - 32)\cos^2(2\beta) + (4\alpha^3 - 16\alpha -$$

$$-20)\cos(2\beta) + 8(\alpha^2+1)^2]r$$

$$S_{10} = [(20\alpha - 12\alpha^4 - 32)\cos^2(2\beta) + (4\alpha^3 - 16\alpha + 20)\cos(\beta) + 8(\alpha^2+1)^2]r_0$$

$$S_{11} = \frac{b^4 r_2}{\sin^2(2\beta)} - 9b^2\alpha^2 r_2 + \alpha^4(12\cos^2(2\beta) - 4\cos(\beta) - 16)r_2$$

$$S_{12} = \frac{b^4 r_1}{\sin^2(2\beta)} - 9b^2 r_1 + (12\cos^2(2\beta) - 4\cos(\beta) - 16)r_1$$

Thus, the self-inductance of the circular coil of the rectangular cross section with the azimuthal current density can be obtained by (7) using the simple integration of the previous elementary functions.

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The special case of this calculation is the self-inductance of the thin disk coil (pancake) with the azimuthal current [26-27]. This self-inductance can be obtained from (7) finding the limit when $\alpha \to 0$ or doing the three integration such as in [26].

The self-inductance L_{A-DISK} is obtained in the analytical form as follows:

$$L_{A-DISK} = \frac{2\mu_0 N^2 R_1}{3(\alpha-1)^2} V \tag{8}$$

where,

$$V = \alpha(\alpha+1)E(k_0) + (\alpha^3+1)(2G-1) + \frac{\pi}{2} ln\frac{k_0}{2} + S_{10} + (\alpha^3-1)S_2 \tag{9}$$

$$k_0^2 = \frac{4\alpha}{(\alpha+1)^2}$$

$$S_1 = \int_0^{\frac{\pi}{2}} ln[1+\Delta]d\beta\,, \qquad S_2 = \int_0^{\frac{\pi}{2}} ln\left[\sqrt{1-k_0^2}+\Delta\right]d\beta, \quad \Delta = \sqrt{1-k_0^2 sin^2(\beta)}$$

$G = 0.915965594\ldots$ - Catalan's constant, $[31-33]$.

The self-inductance is obtained as the combinations of the elementary functions, the elliptical integral of the second kind [31-33], and the single integrals (the semi-analytical solution).

In [27] the new expression for V is given by,

$$V = \alpha(\alpha+1)E(k_0) + (\alpha^3+1)(2G-1) - \frac{\pi}{2} ln2 - S_{10} - \alpha^3 S_{20} \tag{10}$$

$$S_{10} = \int_0^{\frac{\pi}{2}} ln\left[\alpha + \cos(2\beta) + \sqrt{\alpha^2 + 2\alpha\cos(2\beta)+1}\right] d\beta$$

$$S_{20} = \frac{1}{2}\int_0^{\frac{\pi}{2}} ln\frac{\alpha\cos(\beta)+1+\sqrt{\alpha^2+2\alpha\cos(\beta)+1}}{\alpha\sin(\beta)-1+\sqrt{\alpha^2-2\alpha\sin(\beta)+1}} d\beta$$

This expression is also very friendly for the numerical integration as (9).

For the full disk ($R_1 = 0,\ R_2 = R$), the self-inductance is,

$$L_{A-FULL-DISK} = \frac{2\mu_0 N^2 R_1}{3}[2G-1] \tag{11}$$

This formula can be found in [10], [27] and [29].

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There is also one special case when $R_1 \to R_2 \to R$ (Thin wall solenoid of the radius R and the high l). Finding the limit in (7) or solving the three integrals in [28], we obtain the well-known Lorentz's formula (1879).

$$L_{A-WALL-SOLEN} = \frac{2\mu_0 N^2 R^2}{3l}\left[\frac{l}{kR}K(k) - \frac{l^2 - 4R^2}{klR}E(k) - 4\frac{R}{l}\right] \quad (12)$$

where,

$$k^2 = \frac{4R^2}{4R^2 + l^2}$$

and $K(k)$, $E(k)$ is the elliptic integrals of the first and the second kind $[31 - 33]$.,

From previous formulas for L_R and L_A it is obvious that there are the similar terms in them and all expressions are elementary function which are very friendly for the single numerical integration. The special cases are obtained as the analytical and semi-analytical expressions for these important electromagnetic quantities (6,8,9,10,11 and 12).

Asymptotic Behaviors of Disk Coils and Thin-Wall Solenoids

At first, we analyze the disk coil.

For $R_1 = R_2$ we have well-known singular case which gives

$$L = \infty \quad (13)$$

For $R_1 \to R_2 \to R$ (Inner radius tends toward the outer radius)

This case leads to the well-known formula [29],

$$L = \mu_0 N^2 R\left[ln\frac{8R}{a} - 2 + Y\right] \quad (14)$$

where R is the turn radius, a is the radius of the circular wire from which the turn is constructed. If the current flows only on the wire surface (due to the skin effect) Y = 0, and the current flow is homogeneous in the wire then Y = 0.25.

For $R_1 \approx R_2$ (the case of a logarithmic singularity), Conway [29] gives the analogous formula,

$$L = \mu_0 N^2 R_2\left[ln\frac{8R_2}{R_2 - R_1} - \frac{1}{2}\right] \quad (15)$$

From Kirchhoff's formula for the self-inductance of a circular ring of the radius R and the circular section of the radius a with one turn [13]

$$L = \mu_0 R\left[ln\frac{8R}{a} - 1.75\right] \quad (16)$$

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Y. Luo and B. Chan [13] obtained for this asymptotic case,

$$L = \mu_0 \frac{(R_2 + R_1)}{R_1}\left[ln \frac{4(R_2 + R_1)}{(R_2 - R_1)} - 0.5\right] \quad (\frac{H}{m}) \tag{17}$$

The asymptotic case for the thin wall solenoid can be calculated from [13],

$$L = \mu_0 \left[ln \frac{4R}{h} - 0.5\right] \quad (\frac{H}{m}) \tag{18}$$

where R is the wall solenoid's radius and $2h$ is its high.

From this approach the self-inductance of thin wall solenoid in asymptotic case is obtained for the first time in the literature.

Let us put in (12),

$$b = \frac{l}{2R}, \quad k^2 = \frac{1}{1 + b^2}$$

so that the self-inductance of the thin wall solenoid is,

$$L_{A-WALL-SOLEN} = \frac{2\mu_0 N^2 R}{3k}\left[K(k) + \frac{1 - b^2}{b^2}E(k) - \frac{k}{b^2}\right] \tag{19}$$

To find the self-inductance of thin wall solenoid for $b \to 0$ the asymptotic behavior of $K(k)$ and $K(k)$ near the singularity at $k = 1$ are given by the following expression [30],

$$K(k) \sim ln \frac{4}{k'} = ln \frac{4}{\sqrt{1 - k^2}} = ln \frac{4}{b} + ln\sqrt{1 + b^2} \tag{20}$$

$$E(k) \sim 1 + \frac{1}{2}(1 - k^2)[ln \frac{4}{k'} - 0.5] = 1 + \frac{b^2}{2(1 + b^2)}[ln \frac{4}{k'} - 0.5] \tag{21}$$

The approximations (20) and (21) are the first members of the convergent series [30]. We calculate the normalized self-inductance of the extremely short wall solenoid ($b \to 0$) as,

$$L_N = \frac{L_{A-WALL-SOLEN}}{N^2 R} \tag{22}$$

From (19), (20), (21) and (22) we finally have,

$$L_N = \frac{\mu_0}{3}\left[\frac{(3 + b^2)}{\sqrt{1 + b^2}} ln \frac{4\sqrt{1 + b^2}}{b} + \frac{2\left[\sqrt{1 + b^2}(1 - b^2) - 1\right]}{b^2} - \frac{(1 - b^2)}{2\sqrt{1 + b^2}}\right] \tag{23}$$

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For b extremely near at zero, we find using the l'Hospital's Rule from (23) that the first term tends to $ln\frac{4}{b}$, the second to -1 and the third to -0.5. Finally, the self-inductance from this range of the parameter b is,

$$L_N = \mu_0 \left[ln\frac{4}{b} - \frac{1}{2} \right] \tag{24}$$

This formula has been obtained by the ansatz in [13].

As we know the formula (23) appears for the first time in the literature.

Thus, we cover all possible cases with the new formulas and the already well-known or the improved formulas in the calculation of the self-inductance of the previously mentioned circular coils.

4. NUMERICAL VALIDATION

To verify the validity of the new formulas for the self-inductances L_R and L_A we apply the following set of examples. Also, the special cases are discussed. We compare the results of the presented approach with those which are known in the literature.

Example 1.

Calculate the self-inductance of the thick Bitter circular coil of rectangular cross section. The coil dimensions and the number of turns is as follows:

$$R_1 = 1\ (m), R_2 = 2\ (m), l = 2\ (m), N = 100$$

Applying the new formula (5) the self-inductance is,

$$L_R = 17.815333 (mH)$$

By using the Conway's method [18] the self-inductance is,

$$L_{Conway} = 17.815333\ (mH)$$

The results are in an excellent agreement.

Example 2.

Calculate the self-inductance of the thick Bitter circular coil of rectangular cross section. The coil dimensions and the number of turns is as follows:

$$R_1 = 0.025\ (m), R_2 = 0.035\ (m), l = 0.04\ (m), N = 100$$

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By (5) we obtain,

$$L_R = 0.4383980\ (mH)$$

By using the Ren's method [19-20] the self-inductance is,

$$L_{Ren} = 0.4383978\ (mH)$$

This self-inductance is obtained by the double integration.

Using the software ANSYS (FEM) [19-20] the self-inductance is,

$$L_{Ren} = 0.44528\ (mH)$$

All results are in good agreement.

Example 3.

Calculate the self-inductance of the thin Bitter disk (pancake), [23]. The coil dimensions and the number of turns is as follows:

$$R_1 = 1\ (m), R_2 = 2\ (m), N = 1000$$

The formula (6) gives,

$$L_{R-DISK} = 0.5342299\ (H)$$

Example 4.

In Table 1 we compare the results obtained by this work (7), [9] and [13].

In these calculations we take $R_1 = 1\ (m)$ and $\alpha = \frac{R_2}{R_1},\ b = \frac{2l}{R_1}$, Table 3.

Table 1. The accuracy and the computational time for the self-inductance calculation [13], [9], L_A

α	b	$L_{LUO}\ (\frac{\mu H}{m})$[13]	$L_{KAJIKAWA}\ (\frac{\mu H}{m})$[9]	$L_{This\ Work}\ (\frac{\mu H}{m})$
1.5	0.5	2.8693036	2.8693	2.8693035
3.0	2.0	2.5330065	2.533	2.5330065
4.0	6.0	1.9012958	1.9012	1.9012958
7.0	12.0	2.4472979	2.4473	2.4472979
9.0	8.0	4.2674018	4.2661	4.2674018

From Table 1 we can see the excellent agreement between this work (7) and [13], and good agreement with [9].

Example 5.

Let us compare the results of the formula (7) by those obtained using [9] and [13], Table 2.

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In these calculations we take $R_1 = 1\ (m)$ and $\alpha = \frac{R_2}{R_1}$, $b = \frac{2l}{R_1}$. In [13] the high of the coil is $2h$ and $b = \frac{h}{R_1}$. In [9] we also use the same parameters as in this paper, Table 3.

Table 2. The comparison with Luo and Kajikawa formulas

α	b	$L_{LUO}\ (\frac{\mu H}{m})$[13]	$L_{KAJIKAWA}\ (\frac{\mu H}{m})$[9]	$L_{This\ Work}\ (\frac{\mu H}{m})$
1.2	20.0	0.2142821	0.21428	0.2142821
5.0	20.0	1.0456844	1.0457	1.0456844
20.0	20.0	7.8764442	7.867	7.8764442
40.0	20.0	19.950453	19.951	19.950453
1.2	2.0	1.4613306	1.4618	1.4613306
5.0	2.0	3.8343885	3.8343	3.8343885
20.0	2.0	14.120116	14.112	14.120116
40.0	2.0	28.015984	27.992	28.015984
1.2	0.2	3.5880363	3.588	3.5880363
5.0	0.2	5.0682989	5.0681	5.0682989
20.0	0.2	15.288175	15.28	15.288175
40.0	0.2	29.185174	29.161	29.185174

From Table 2. one can see that the results of this work and those in [13] are in an excellent agreement and in particularly good agreement whit Kajikawa results [9] in which the number of significant figures in the calculation is about 3.

Example 6.

In this example we compare the results of the-self-inductance (7) with the self-inductance obtained by Bessel functions [10].

Table 3. The comparison with Conway's formula and Kajikawa's formula

α	b	$L_{CONWAY}\ (\frac{\mu H}{m})$[10]	$L_{KAJIKAWA}\ (\frac{\mu H}{m})$[9]	$L_{This\ Work}\ (\frac{\mu H}{m})$
1.5	0.5	2.8693035	2.8693	2.8693035
3.0	2.0	2.5330065	2.533	2.5330065
4.0	6.0	1.9012858	1.9012	1.9012858
7.0	12.0	2.4472979	2.4473	2.4472979
9.0	8.0	4.2676018	4.2661	4.2676018

There is an excellent agreement between Conway's method and this work, Table 3, and particularly good agreement by Kajikawa's method.

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Example 7.

In this example we calculate the normalized self-inductance of the thin disk coil (pancake) regarding the inner radius and the number of turns for the different shape factor $\alpha = \frac{R_2}{R_1}$ (Table 4). We compare the results of formulas (8-10) with [4] and [29].
From presented results obtained by formulas (8-10), and from Spielrein's and Conway's approaches, we can see that all of them are in an excellent agreement. There is the negligible disagreement with the Spielrein's approach where the self-inductance is calculated by the series which does not converge quickly.

Table 4. Comparison of Computational Accuracy for the different shape factor α

α	$L_{[4]}$ (μH/m)	$L_{[29]}$ (μH/m)	$L_{(8),(9)}$ (μH/m)	$L_{(8),(10)}$ (μH/m)
5.0	36.282205	36.282205	36.282205	36.282205
10.0	8.5558079	8.5558079	8.5558079	8.5558079
3.0	4.1202479	4.1202478	4.1202478	4.1202478
1.5	3.9375566	3.9375570	3.9375569	3.9375569
1.1	5.1875898	5.1875898	5.1875898	5.1875898
1.01	7.8169836	7.8169836	7.8169836	7.8169836
1.001	10.671287	10.671287	10.671287	10.671287
1.00001	16.452442	16.452442	16.452442	16.452442
1.000001	19.345878	19.345878	19.345878	19.345878
1.0000001	22.239382	22.239382	22.239382	22.239382

Example 8.

In this example, the -self-inductance of the disk will be calculated when α is remarkably close to 1 (Table 5) until the extreme case $\alpha = 1$ for which the self-inductance is ∞.

Table 5. Comparison of computational accuracy for the shape factor α close to 1.

$\alpha - 1$	$L_{[29]}$ (μH/m)	$L_{(8),(10)}$ (μH/m)
10^{-1}	5.1875898	5.1875898
10^{-2}	7.8169836	7.8169836
10^{-3}	10.671287	10.671287
10^{-4}	19.345878	19.345878
10^{-8}	25.132895	25.132895
10^{-10}	25.132895	25.132895
10^{-12}	36.706950	36.706950
10^{-15}	45.387491	45.387491
10^{-16}	45.387491	45.387491

From Table 5, where α is extremely close to 1, expressions (8-10) give the same results as Conway's approach.

Example 9.

In this example we compare the results for the self-inductance (23) by those obtained in [13] (Formulas (27) and (46)) and [9].

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Table 6. Comparison of Computational Accuracy for the different shape factor b

b	$L_{[13](27)}$ or $L_{(12)}$ (μH/m)	$L_{[13](46)}$ (μH/m)	$L_{This\ work(23)}$ (μH/m)	$L_{[9]}$ (μH/m)
10^{-1}	4.0133453	4.0072641	4.0037786	4.0134
10^{-2}	6.9008759	6.9007779	6.9006943	6.9009
10^{-3}	9.7942930	9.7942916	9.7942903	9.7942
10^{-6}	18.474833	18.474833	18.474833	-
10^{-9}	25.155374	25.155374	25.155374	-
10^{-12}	35.835916	35.835916	35.835916	-

As we can see all results are in an excellent agreement. It is obvious that the formula (23) gives the more precise results for the range of $10^{-6} \leq b \leq 10^{-1}$ then (46) in [13] and for $b \leq 10^{-6}$ the formula (24) is the same as the formula (46) in [13]. Also, the K. Kajikawa's method gives particularly good results for $10^{-3} \leq b \leq 10^{-1}$. With this calculation we confirm the validity of the new developed formula (23) which also leads to formula (24).

Example 10.

Calculate the self-inductance of the full disk coil of the radius $R_1 = 0.5\ (m)$ and the number of turns $N = 100$.

This work, formula (11) gives,

$$L_{A-FULL-DISK} = 3.4847852\ (mH)$$

The same result is obtained in [29].

From [9] the self-inductance is,

$$L_{A-KAJIKAWA} = 3.4848\ (mH)$$

5. CONCLUSION

The new accurate self-inductance formulas for the circular thick coils of the rectangular cross section with the radial and the azimuthal current densities are given. The formulas are obtained in the form of the single integral which kernel function on the interval of the integration is the sum of the elementary functions. The special cases of these formulas give the self-inductance for thin disk coil and the thin wall solenoid in the closed and the semi-analytical form. For the asymptotic case, the self-inductance of the thin wall solenoid with extremely small height is developed for the first time in the literature. Thus, all cases for the circular coils with and without cross section are given. The presented method can be helpful for engineers, physicist and people who work in this domain so that they can easily make all formulas in Mathematica or MATLAB programming.

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APPENDIX I

The first integration in (2) gives,

$$I_1 = \int_1^{\alpha} \frac{dy}{R_{12}} = \operatorname{asinh}\frac{\alpha + x\cos(2\beta)}{\sqrt{x^2\sin^2(2\beta) + t^2}} - \operatorname{asinh}\frac{1 + x\cos(2\beta)}{\sqrt{x^2\sin^2(2\beta) + t^2}}\ , \qquad t = v - z$$

The second integration in (2) gives,

$$I_2 = \int_1^{\alpha} I_1 dx = 2\left[\alpha\ \operatorname{asinh}\frac{\alpha + \alpha\cos(2\beta)}{\sqrt{\alpha^2\sin^2(2\beta) + t^2}} + \operatorname{asinh}\frac{1 + \alpha\cos(2\beta)}{\sqrt{\sin^2(2\beta) + t^2}} - \right.$$

$$-\alpha\ \operatorname{asinh}\frac{1 + \alpha\cos(2\beta)}{\sqrt{\alpha^2\sin^2(2\beta) + t^2}} - \operatorname{asinh}\frac{1 + \cos(2\beta)}{\sqrt{\sin^2(2\beta) + t^2}} +$$

$$+\frac{t}{\sin(2\beta)} arctg \frac{\alpha\sin^2(2\beta) - t^2\cos(2\beta)}{t\sin(2\beta)\sqrt{t^2 + \alpha^2 + 2\alpha\cos(2\beta) + 1}} -$$

$$-\frac{t}{2\sin(2\beta)} arctg \frac{\alpha^2\sin^2(2\beta) - t^2\cos(2\beta)}{t\sin(2\beta)\sqrt{t^2 + 4\alpha^2\cos^2(\beta)}} -$$

$$\left. -\frac{t}{2\sin(2\beta)} arctg \frac{\sin^2(2\beta) - t^2\cos(2\beta)}{t\sin(2\beta)\sqrt{t^2 + 4\cos^2(\beta)}}\right]$$

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The third integration in (2) gives,

$$I_3 = \int_0^b \frac{I_2}{2} dv = \Big[\frac{t^2}{2\sin(2\beta)} \text{arctg} \frac{\alpha\sin^2(2\beta) - t^2\cos(2\beta)}{t\sin(2\beta)\sqrt{t^2+\alpha^2+2\alpha\cos(2\beta)+1}} -$$

$$-\frac{t^2}{4\sin(2\beta)} \text{arctg} \frac{\alpha^2\sin^2(2\beta) - t^2\cos(2\beta)}{t\sin(2\beta)\sqrt{t^2+4\alpha^2\cos^2(\beta)}} -$$

$$-\frac{t^2}{4\sin(2\beta)} \text{arctg} \frac{\sin^2(2\beta) - t^2\cos(2\beta)}{t\sin(2\beta)\sqrt{t^2+4\cos^2(\beta)}} - \frac{[(\alpha^2+1)\cos(2\beta)+2\alpha]}{2} \text{asinh} \frac{t}{r_0} +$$

$$+\frac{\alpha^2[\cos(2\beta)+1]}{2} \text{asinh} \frac{t}{2\alpha\cos(\beta)} + \frac{[\cos(2\beta)+1]}{2} \text{asinh} \frac{t}{2\cos(\beta)} +$$

$$+\alpha t \text{asinh} \frac{\alpha+\alpha\cos(2\beta)}{\sqrt{\alpha^2\sin^2(2\beta)+t^2}} + t \text{asinh} \frac{1+\alpha\cos(2\beta)}{\sqrt{\sin^2(2\beta)+t^2}} - \alpha t \text{asinh} \frac{1+\alpha\cos(2\beta)}{\sqrt{\alpha^2\sin^2(2\beta)+t^2}} -$$

$$-t \text{asinh} \frac{\alpha+\cos(2\beta)}{\sqrt{\sin^2(2\beta)+t^2}} - \frac{\alpha^2\sin(2\beta)}{2} \text{arctg} \frac{t\cos(\beta)}{\sin(2\beta)\sqrt{t^2+4\alpha^2\cos^2(\beta)}} -$$

$$-\frac{\sin(2\beta)}{2} \text{arctg} \frac{t\cos(\beta)}{\sin(2\beta)\sqrt{t^2+4\cos^2(\beta)}} + \frac{\alpha^2\sin(2\beta)}{2} \text{arctg} \frac{t\cos(\beta)}{\sin(2\beta)\sqrt{t^2+4\alpha^2\cos^2(\beta)}} +$$

$$+\frac{\alpha^2\sin(2\beta)}{2} \text{arctg} \frac{t(1+\alpha\cos(\beta))}{\alpha\sin(2\beta)\sqrt{t^2+\alpha^2+2\alpha\cos(2\beta)+1}} +$$

$$+\frac{\sin(2\beta)}{2} \text{arctg} \frac{t(\alpha+\cos(\beta))}{\alpha\sin(2\beta)\sqrt{t^2+\alpha^2+2\alpha\cos(2\beta)+1}}\Big]_{-z}^{b-z}$$

Finally, the fourth integration in (2)

$$I_4 = \int_0^b (2I_{3)} dz$$

leads to T_n, ($n = 1,2,\ldots,8$) which appear in the expression (5) for the self-inductance L_R of the circular coil with the rectangular cross section and the radial current density.

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APPENDIX II

The first integration in (4) gives,

$$I_1 = \int_1^{\alpha} \frac{ydy}{R_{12}} = \left[\sqrt{x^2 + 2x\alpha\cos(2\beta) + \alpha^2 + t^2} - \sqrt{x^2 + 2x\cos(2\beta) + 1 + t^2} - \right.$$

$$\left. -x\cos(2\alpha)\frac{\alpha + x\cos(2\beta)}{\sqrt{x^2\sin^2(2\beta) + t^2}} + x\cos(2\alpha)\frac{1 + x\cos(2\beta)}{\sqrt{x^2\sin^2(2\beta) + t^2}}\right]$$

where $t = v - z$

The second integration in (4) gives,

$$I_2 = \int_1^{\alpha} zI_1\, dx = \frac{2}{3}\left[-\alpha^3\cos(2\beta)\,\mathrm{asinh}\frac{\alpha + \alpha\cos(2\beta)}{\sqrt{\alpha^2\sin^2(2\beta) + t^2}} - \right.$$

$$-\cos(2\beta)\,\mathrm{asinh}\frac{1 + \cos(2\beta)}{\sqrt{\sin^2(2\beta) + t^2}} + \alpha^3\cos(2\beta)\,\mathrm{asinh}\frac{1 + \alpha\cos(2\beta)}{\sqrt{\alpha^2\sin^2(2\beta) + t^2}} +$$

$$+\cos(2\beta)\,\mathrm{asinh}\frac{\alpha + \cos(2\beta)}{\sqrt{\sin^2(2\beta) + t^2}} + \left[\alpha^2 + \frac{t^2}{2\sin^2(2\beta)}\right]\sqrt{4\alpha^2\cos^2(\beta) + t^2} +$$

$$+\left[1 + \frac{t^2}{2\sin^2(2\beta)}\right]\sqrt{4\cos^2(\beta) + t^2} - \left[\alpha^2 + 1 + \frac{t^2}{\sin^2(2\beta)}\right]\sqrt{t^2 + \alpha^2 + 2\alpha\cos(2\beta) + 1} +$$

$$+\frac{t^3\cos(2\beta)}{\sin^3(2\beta)}\mathrm{arctg}\frac{\alpha\sin^2(2\beta) - t^2\cos(2\beta)}{t\sin(2\beta)\sqrt{t^2 + \alpha^2 + 2\alpha\cos(2\beta) + 1}} -$$

$$-\frac{t^3\cos(2\beta)}{2\sin^3(2\beta)}\mathrm{arctg}\frac{\alpha^2\sin^2(2\beta) - t^2\cos(2\beta)}{t\sin(2\beta)\sqrt{4\alpha^2\cos^2(\beta) + t^2}} -$$

$$\left. -\frac{t^3\cos(2\beta)}{2\sin^3(2\beta)}\mathrm{arctg}\frac{\sin^2(2\beta) - t^2\cos(2\beta)}{t\sin(2\beta)\sqrt{4\cos^2(\beta) + t^2}}\right]$$

The third integration in (2) gives,

$$I_3 = \int_0^{b} \left(\frac{3}{2}I_2\right) dv = \left[-8t\alpha^3\cos(2\beta)\,\mathrm{asinh}\frac{\alpha + \alpha\cos(2\beta)}{\sqrt{\alpha^2\sin^2(2\beta) + t^2}} - \right.$$

$$-8t\cos(2\beta)\,\mathrm{asinh}\frac{1 + \cos(2\beta)}{\sqrt{\sin^2(2\beta) + t^2}} + 8t\alpha^3\cos(2\beta)\,\mathrm{asinh}\frac{1 + \alpha\cos(2\beta)}{\sqrt{\alpha^2\sin^2(2\beta) + t^2}} +$$

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$$+8t\cos(2\beta)\,\mathrm{asinh}\frac{\alpha+\cos(2\beta)}{\sqrt{\sin^2(2\beta)+t^2}}-\frac{t^4\cos(2\beta)}{\sin^3(2\beta)}\mathrm{arctg}\frac{\alpha^2\sin^2(2\beta)-t^2\cos(2\beta)}{t\sin(2\beta)\sqrt{4\alpha^2\cos^2(\beta)+t^2}}-$$

$$-\frac{t^4\cos(2\beta)}{\sin^3(2\beta)}\mathrm{arctg}\frac{\sin^2(2\beta)-t^2\cos(2\beta)}{t\sin(2\beta)\sqrt{4\cos^2(\beta)+t^2}}+$$

$$+\frac{2t^4\cos(2\beta)}{\sin^3(2\beta)}\mathrm{arctg}\frac{\alpha\sin^2(2\beta)-t^2\cos(2\beta)}{t\sin(2\beta)\sqrt{t^2+\alpha^2+2\alpha\cos(2\beta)+1}}+6\alpha^2\sin^2(2\beta)\mathrm{asinh}\frac{t}{2\alpha\cos(\beta)}+$$

$$+6\sin^2(2\beta)\mathrm{asinh}\frac{t}{2\cos(\beta)}+3[2(\alpha^4+1)\cos(2\beta)-(\alpha^2+1)^2]\mathrm{asinh}\frac{t}{r_0}+$$

$$+6\alpha^4\sin(2\beta)\cos(2\beta)\,\mathrm{arctg}\frac{t\cos(\beta)}{\sin(2\beta)\sqrt{t^2+4\alpha^2\cos^2(\beta)}}+$$

$$+6\sin(2\beta)\cos(2\beta)\,\mathrm{arctg}\frac{t\cos(\beta)}{\sin(2\beta)\sqrt{t^2+4\cos^2(\beta)}}-$$

$$-6\alpha^4\sin(2\beta)\cos(2\beta)\,\mathrm{arctg}\frac{t\big(1+\alpha\cos(\beta)\big)}{\alpha\sin(2\beta)\sqrt{t^2+\alpha^2+2\alpha\cos(2\beta)+1}}-$$

$$-6\sin(2\beta)\cos(2\beta)\,\mathrm{arctg}\frac{t\big(\alpha+\cos(\beta)\big)}{\sin(2\beta)\sqrt{t^2+\alpha^2+2\alpha\cos(2\beta)+1}}+$$

$$+t\left[5\alpha^2+\frac{t^2}{\sin^2(2\beta)}\right]\sqrt{4\alpha^2\cos^2(\beta)+t^2}+t\left[5+\frac{t^2}{\sin^2(2\beta)}\right]\sqrt{4\cos^2(\beta)+t^2}-$$

$$-t\left[5(\alpha^2+1)+\frac{2t^2}{\sin^2(2\beta)}\right]\sqrt{t^2+\alpha^2+2\alpha\cos(2\beta)+1}\quad\Big]_{-z}^{l-z}$$

Finally, the fourth integration in (4),

$$I_4=\int_0^b I_3\,dz$$

leads to $S_n, (n=1,2,\ldots,12)$ which appear in the expression (7) for the self-inductance L_A of the circular coil with the rectangular cross section and the azimuthal current density.