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On Type 2 Degenerate Poly-Frobenius-Genocchi Polynomials and Numbers

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Abstract

In this paper, we consider a class of new generating function for the Frobenius-Genocchi polynomials, called the type 2 degenerate poly-Frobenius-Genocchi polynomials, by means of the polyexponential function. Then, we investigate diverse explicit expressions and some identities for those polynomials.

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1. Introduction

Special polynomials have their origin in the solution of the differential equations (or partial differential equations) under some conditions. Special polynomials can be defined in a various ways such as by generating functions, by recurrence relations, by p-adic integrals in the sense of fermionic and bosonic, by degenerate versions, etc.

Kim-Kim have introduced polyexponential function in [18] and its degenerate version in [20],[21]. By making use of aforementioned function, they have introduced a new class of some special polynomials. This idea provides a powerfool tool in order to define special numbers and polynomials by making use of polyexponential function. One may see that the notion of polyexponential function form a special class of polynomials because of their great applicability, *cf.* [12, 18-22, 26, 27, 29, 31]. The importance of these polynomials would be to find applications in analytic number theory, applications in classical analysis and statistics, *cf.* [1-34].

Throughout of the paper we make use of the following notations: $\mathbb{N} := \{1, 2, 3, \cdots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Here, as usual, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers.

The classical Bernoulli $B_n(x)$, Euler $E_n(x)$ and Genocchi $G_n(x)$ polynomials and the degenerate Bernoulli $B_{n,\lambda}(x)$, Euler $E_{n,\lambda}(x)$ and Genocchi $G_{n,\lambda}(x)$ polynomials are given as follows (*cf.* [5, 8, 10, 11, 14, 16, 18-20, 22, 23, 26-32]):

$$\sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = \frac{t}{e^t - 1} \quad \text{and} \quad \sum_{n=0}^{\infty} B_{n,\lambda}(x) \frac{t^n}{n!} = \frac{t}{e_\lambda(t) - 1} e_\lambda^x(t) \tag{1.1}$$

$$\sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} = \frac{2}{e^t + 1} \quad \text{and} \quad \sum_{n=0}^{\infty} E_{n,\lambda}(x) \frac{t^n}{n!} = \frac{2}{e_\lambda(t) + 1} e_\lambda^x(t) \tag{1.2}$$

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$$\sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!} = \frac{2t}{e^t+1} \quad \text{and} \quad \sum_{n=0}^{\infty} G_{n,\lambda}(x)\frac{t^n}{n!} = \frac{2t}{e_\lambda(t)+1}e_\lambda^x(t). \tag{1.3}$$

One may look at the references [1, 4-13, 15, 17-19, 21, 22, 25-31] to see the various applications of Bernoulli, Euler and Genocchi polynomials.

Frobenius studied the polynomials $F_n(x \mid u)$ given by (*cf.* [2, 3])

$$\frac{1-u}{e^t-u}e^{xt} = \sum_{n=0}^{\infty} F_n(x \mid u)\frac{t^n}{n!} \quad (u \in \mathbb{C}\backslash\{1\}). \tag{1.4}$$

Upon setting $u=-1$, it becomes

$$F_n(x \mid -1) = E_n(x).$$

Owing to relationship with the Euler polynomials as well as their important properties, and in the honor of Frobenius, the aforementioned polynomials denoted by $F_n(x \mid u)$ are called the Frobenius-Euler polynomials, *cf.* [2, 3].

Parallel to (1.4), Yaşar and Özarslan [34] introduced the Frobenius-Genocchi polynomials $G_n^F(x;u)$ given by

$$\frac{(1-u)t}{e^t-u}e^{xt} = \sum_{n=0}^{\infty} G_n^F(x;u)\frac{t^n}{n!}, \tag{1.5}$$

since

$$G_n^F(x;-1) = G_n(x).$$

The case $x=0$ in (1.5), $G_n^F(0;u) := G_n^F(u)$ stands for the Frobenius-Genocchi numbers. Several recurrence relations and differential equations are also investigated in [34].

Khan and Srivastava [17] introduced a new class of the generalized Apostol type Frobenius-Genocchi polynomials and investigated some properties and relations including implicit summation formulae and various symmetric identities. Moreover a relation in between Array-type polynomials, Apostol-Bernoulli polynomials and generalized Apostol-type Frobenius-Genocchi polynomials is also given in [17]. Wani *et al.* [33] considered Gould-Hopper based Frobenius-Genocchi polynomials and then, summation formulae and operational rule for these polynomials.

The Bernoulli polynomials of the second kind are defined by means of the following generating function

$$\sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!} = \frac{t}{\log(1+t)}(1+t)^x. \tag{1.6}$$

When $x=0$, $b_n(0) := b_n$ are called the Bernoulli numbers of the second kind, *cf.* [20].

It is well-known from (1.6) that

$$\left(\frac{t}{\log(1+t)}\right)^r (1+t)^{x-1} = \sum_{n=0}^{\infty} B_n^{(n-r+1)}(x)\frac{t^n}{n!}, \tag{1.7}$$

where $B_n^{(r)}(x)$ are the Bernoulli polynomials of order r, see [20].

For $\lambda \in \mathbb{C}$, the λ-falling factorial $(x)_{n,\lambda}$ is defined by (see [10, 11, 20-22, 24-27, 29-31])

$$(x)_{n,\lambda} = \begin{cases} x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda), & n=1,2,\ldots \\ 1 & n=0. \end{cases} \tag{1.8}$$

In the case $\lambda=1$, the λ-falling factorial reduces to the familiar falling factorial as follows

$$(x)_{n,1} := (x)_n = x(x-1)\cdots(x-n+1) \text{ and } (x)_0 = 1.$$

The Δ_λ difference operator is defined by (see [10, 11])

$$\Delta_\lambda f(x) = \frac{1}{\lambda}(f(x+\lambda)-f(x)), \quad \lambda \neq 0. \tag{1.9}$$

The degenerate exponential function $e_{\lambda}^{x}(t)$ is defined as follows

$$e_{\lambda}^{x}(t)=(1+\lambda t)^{\frac{x}{\lambda}} \text{ and } e_{\lambda}^{1}(t)=e_{\lambda}(t). \tag{1.10}$$

It is readily seen that $\lim_{\lambda\to 0} e_{\lambda}^{x}(t)=e^{xt}$ (*cf.* [10, 11, 20-22, 24-27, 29-31]). From (1.8) and (1.10), we obtain the following relation

$$e_{\lambda}^{x}(t)=\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^{n}}{n!}, \tag{1.11}$$

which satisfies the following difference rule

$$\Delta_{\lambda}e_{\lambda}^{x}(t)=te_{\lambda}^{x}(t). \tag{1.12}$$

The Stirling numbers of the first kind $S_1(n,k)$ and the Stirling numbers of the second kind $S_2(n,k)$ are defined (*cf.* [2, 4, 5, 12]) by means of the following generating functions:

$$\frac{(\log(1+t))^{k}}{k!}=\sum_{n=0}^{\infty}S_1(n,k)\frac{t^{n}}{n!} \text{ and } \frac{(e^{t}-1)^{k}}{k!}=\sum_{n=0}^{\infty}S_2(n,k)\frac{t^{n}}{n!}. \tag{1.13}$$

From (1.13), we get the following relations for $n\geq 0$:

$$(x)_{n}=\sum_{k=0}^{n}S_1(n,k)x^{k} \text{ and } x^{n}=\sum_{k=0}^{n}S_1(n,k)(x)_{k}. \tag{1.14}$$

Very recently, Kim-Kim [22] performed to generalize the degenerate Bernoulli polynomials by using poly-exponential function

$$\mathrm{Ei}_{k}(t)=\sum_{n=1}^{\infty}\frac{t^{n}}{(n-1)!n^{k}} \tag{1.15}$$

as inverse to the polylogarithm function

$$Li_{k}(t)=\sum_{n=1}^{\infty}\frac{t^{n}}{n^{k}} \qquad (|t|<1;k\in\mathbb{Z}) \tag{1.16}$$

given by

$$\frac{\mathrm{Ei}_{k}(\log(1+t))}{e_{\lambda}(t)-1}e_{\lambda}^{x}(t)=\sum_{n=0}^{\infty}\beta_{n,\lambda}^{(k)}(x)\frac{t^{n}}{n!}. \tag{1.17}$$

Upon setting $x=0$ in (1.17), $\beta_{n,\lambda}^{(k)}(0):=\beta_{n,\lambda}^{(k)}$ are called the degenerate poly-Bernoulli numbers. Kim et al. [22] studied the degenerate poly-Bernoulli polynomials and also gave some explicit expressions and several formulas for those polynomials.

For $k\in\mathbb{Z}$, the type 2 degenerate poly-Euler polynomials $\mathfrak{E}_{n,\lambda}^{(k)}(x)$ are defined, *cf.* [29], as follows:

$$\frac{\mathrm{Ei}_{k}(\log(1+2t))}{t(e_{\lambda}(t)+1)}e_{\lambda}^{x}(t)=\sum_{n=0}^{\infty}\mathfrak{E}_{n,\lambda}^{(k)}(x)\frac{t^{n}}{n!}.$$

When $x=0$, $\mathfrak{E}_{n,\lambda}^{(k)}(0):=\mathfrak{E}_{n,\lambda}^{(k)}$ are called the type 2 degenerate poly-Euler numbers. Lee et al. [29] studied the type 2 degenerate poly-Euler polynomials and provided multifarious explicit formulas and identities.

Since $\mathrm{Ei}_{1}(t)=e^{t}-1$, it is worthy to note that

$$\beta_{n,\lambda}^{(1)}(x):=B_{n,\lambda}(x) \text{ and } \mathfrak{E}_{n,\lambda}^{(1)}(x):=E_{n,\lambda}(x).$$

2. The type 2 Degenerate Poly-Frobenius-Genocchi Polynomials

Now, we consider the following Definition 1 by means of the polyexponential function.

Definition 1. *Let $k \in \mathbb{Z}$. The type 2 degenerate poly-Frobenius-Genocchi polynomials are defined via the following exponential generating function (in a suitable neigbourhood of $t = 0$) including the polyexponential function as given below:*

$$\frac{\mathrm{Ei}_k\left(\log\left(1+(1-u)t\right)\right)}{e_\lambda(t)-u}e_\lambda^x(t)=\sum_{n=0}^{\infty}G_{n,\lambda}^{(F,k)}(x;u)\frac{t^n}{n!}. \tag{2.1}$$

At the value $x = 0$ in (2.1), $G_{n,\lambda}^{(F,k)}(0;u) := G_{n,\lambda}^{(F,k)}(u)$ will be called type 2 degenerate poly-Frobenius-Genocchi numbers.

Remark 1. *Taking $k = 1$ in (2.1) yields $G_{n,\lambda}^{(F,1)}(x;u) := G_{n,\lambda}^{F}(x;u)$ are the degenerate Frobenius-Genocchi polynomials $G_{n,\lambda}^{F}(x;u)$ (cf. [15]) as follows*

$$\frac{(1-u)t}{e_\lambda(t)-u}e_\lambda^x(t)=\sum_{n=0}^{\infty}G_{n,\lambda}^{F}(x;u)\frac{t^n}{n!}.$$

Remark 2. *Upon setting $\lambda \to 0$ in (2.1) gives $\lim_{\lambda\to 0}G_{n,\lambda}^{(F,k)}(x;u) := G_{n}^{(F,k)}(x;u)$ are type 2 poly-Frobenius-Genocchi polynomials $G_{n}^{(F,k)}(x;u)$ (cf. [12]) as follows*

$$\frac{\mathrm{Ei}_k\left(\log\left(1+(1-u)t\right)\right)}{e^t-u}e^{xt}=\sum_{n=0}^{\infty}G_{n}^{(F,k)}(x;u)\frac{t^n}{n!}.$$

Remark 3. *Taking $k = 1$ and $\lambda \to 0$ in (2.1) yields $G_{n,\lambda}^{(F,1)}(x;-1) := G_{n,\lambda}(x)$ are the Frobenius-Genocchi polynomials in (1.5).*

A difference operator rule of type 2 degenerate poly-Frobenius-Genocchi polynomials is given as follows

$$\Delta_\lambda G_{n,\lambda}^{(F,k)}(x;u)=G_{n-1,\lambda}^{(F,k)}(x;u).$$

Now, we give the following theorem.

Theorem 1. *The following relation*

$$G_{n,\lambda}^{(F,k)}(x;u)=\sum_{l=0}^{n}\binom{n}{l}G_{n-l,\lambda}^{(F,k)}(u)(x)_{l,\lambda} \tag{2.2}$$

is valid for $k \in \mathbb{Z}$ and $n \geq 0$.

Proof. By Definition 1, we consider that

$$\begin{aligned}\sum_{n=0}^{\infty}G_{n,\lambda}^{(F,k)}(x;u)\frac{t^n}{n!} &= \frac{\mathrm{Ei}_k\left(\log\left(1+(1-u)t\right)\right)}{e_\lambda(t)-u}e_\lambda^x(t)\\ &= \left(\sum_{n=0}^{\infty}G_{n,\lambda}^{(F,k)}(u)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}G_{n-l,\lambda}^{(F,k)}(u)(x)_{l,\lambda}\right)\frac{t^n}{n!},\end{aligned}$$

which implies the asserted result in (2.2). □

Now, we give the following theorem.

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Theorem 2. *The following relation*

$$\frac{d}{dx}G_{n,\lambda}^{(F,k)}(x;u)=n!\sum_{u=1}^{\infty}G_{n-u,\lambda}^{(F,k)}(x;u)\frac{(-1)^{u+1}}{(n-u)!u}\lambda^{u-1} \tag{2.3}$$

is valid for $k\in\mathbb{Z}$ *and* $n\geq 0$.

Proof. By Definition 1, we consider that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{d}{dx}G_{n,\lambda}^{(F,k)}(x;u)\frac{t^n}{n!} &= \frac{\mathrm{Ei}_k(\log(1+(1-u)t))}{e_\lambda(t)-u}\frac{d}{dx}e_\lambda^x(t)\\ &= \sum_{n=0}^{\infty}G_{n,\lambda}^{(F,k)}(x;u)\frac{t^n}{n!}\frac{1}{\lambda}\ln(1+\lambda t)\\ &= \left(\sum_{n=0}^{\infty}G_{n,\lambda}^{(F,k)}(x;u)\frac{t^n}{n!}\right)\sum_{u=1}^{\infty}\frac{(-1)^{u+1}}{u}\lambda^{u-1}t^u\\ &= \sum_{n=0}^{\infty}\sum_{u=1}^{\infty}G_{n,\lambda}^{(F,k)}(x;u)\frac{(-1)^{u+1}}{u}\lambda^{u-1}\frac{t^{n+u}}{n!},\end{aligned}$$

which implies the asserted result in (2.2). □

A relation between the type 2 degenerate poly-Frobenius-Genocchi polynomials and the degenerate Frobenius-Genocchi polynomials is stated in the following theorem.

Theorem 3. *For* $k\in\mathbb{Z}$ *and* $n\geq 0$, *we have*

$$G_{n,\lambda}^{(F,k)}(x;u)=\sum_{m=0}^{n}\sum_{l=1}^{m+1}\binom{n}{m}\frac{S_1(m+1,l)(1-u)^m}{l^{k-1}(m+1)}G_{n-m,\lambda}^{F}(x;u). \tag{2.4}$$

Proof. From (1.15), we observe that

$$\begin{aligned}\mathrm{Ei}_k(\log(1+(1-u)t)) &= \sum_{l=1}^{\infty}\frac{(\log(1+(1-u)t))^l}{(l-1)!l^k}\\ &= \sum_{l=1}^{\infty}\frac{1}{l^{k-1}}\sum_{m=l}^{\infty}S_1(m,l)(1-u)^m\frac{t^m}{m!}\\ &= \sum_{m=0}^{\infty}\sum_{l=1}^{m+1}\frac{S_1(m+1,l)(1-u)^{m+1}}{l^{k-1}(m+1)}\frac{t^{m+1}}{m!}.\end{aligned} \tag{2.5}$$

Then, by (2.1), we get

$$\begin{aligned}\frac{t(1-u)}{e_\lambda(t)-u}e_\lambda^x(t)\frac{1}{t(1-u)}\mathrm{Ei}_k(\log(1+(1-u)t)) &= \sum_{n=0}^{\infty}G_{n,\lambda}^{(F)}(x;u)\frac{t^n}{n!}\\ &\times\sum_{m=0}^{\infty}\sum_{l=1}^{m+1}\frac{S_1(m+1,l)(1-u)^m}{l^{k-1}(m+1)}\frac{t^m}{m!}\\ &=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=1}^{m+1}\binom{n}{m}\frac{S_1(m+1,l)(1-u)^m}{l^{k-1}(m+1)}G_{n-m,\lambda}^{F}(x;u)\right)\frac{t^n}{n!}.\end{aligned}$$

which means the asserted result in (2.4). □

The immediate results of the Theorem 3 are stated below.

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Corollary 1. *For $k \in \mathbb{Z}$ and $n \geq 0$, we have*

$$G_{n,\lambda}^{(F,k)}(u) = \sum_{m=0}^{n} \sum_{l=1}^{m+1} \binom{n}{m} \frac{S_1(m+1,l)(1-u)^m}{l^{k-1}(m+1)} G_{n-m,\lambda}^{F}(u). \tag{2.6}$$

Corollary 2. *Taking $k = 1$ in Theorem 3 gives*

$$G_{n,\lambda}^{(F,1)}(x;u) = \sum_{m=0}^{n} \sum_{l=1}^{m+1} \binom{n}{m} \frac{S_1(m+1,l)(1-u)^m}{(m+1)} G_{n-m,\lambda}^{F}(x;u).$$

Corollary 3. *Taking $k = 1$ and $u = -1$ in Theorem 3 reduces*

$$G_{n,\lambda}(x) = \sum_{m=0}^{n} \sum_{l=1}^{m+1} \binom{n}{m} \frac{S_1(m+1,l)(1-u)^m}{(m+1)} G_{n-m,\lambda}(x).$$

Here, we give the following lemma.

Lemma 1. *For $k \in \mathbb{Z}$ and $n \geq 0$, we have*

$$\frac{d}{dx}\mathrm{Ei}_k(\log(1+(1-u)x)) = \frac{1-u}{(1+(1-u)x)\log(1+(1-u)x)}\mathrm{Ei}_{k-1}(\log(1+(1-u)x)). \tag{2.7}$$

Proof. From (1.15), we observe that

$$\begin{aligned}\frac{d}{dx}\mathrm{Ei}_k(\log(1+(1-u)x)) &= \frac{d}{dx}\sum_{l=1}^{\infty}\frac{(\log(1+(1-u)x))^l}{(l-1)!l^k} \\ &= \frac{1-u}{(1+(1-u)x)\log(1+(1-u)x)}\sum_{l=1}^{\infty}\frac{(\log(1+(1-u)x))^l}{(l-1)!l^{k-1}} \\ &= \frac{1-u}{(1+(1-u)x)\log(1+(1-u)x)}\mathrm{Ei}_{k-1}(\log(1+(1-u)x)),\end{aligned}$$

which is the claimed result in (2.7). □

Theorem 4. *Let $k \geq 2$. We have*

$$\begin{aligned}G_{n,\lambda}^{(F,k)}(u) &= \sum_{m=0}^{n}\binom{n}{m}\sum_{m_1+m_2+\cdots+m_{k-1}=m}^{\infty}\binom{m}{m_1,m_2,\cdots,m_{k-1}}(1-u)^{m_1+m_2+\cdots m_{k-1}} \\ &\quad\times G_{n-m,\lambda}^{F}(u)\frac{B_{m_1}^{(m_1)}(0)}{m_1+1}\frac{B_{m_2}^{(m_2)}(0)}{m_1+m_2+1}\cdots\frac{B_{m_{k-1}}^{(m_{k-1})}(0)}{m_1+m_2+\cdots+m_{k-1}+1}.\end{aligned}$$

Proof. By (2.7), we consider

$$\begin{aligned}\mathrm{Ei}_k(\log(1+(1-u)x)) &= \int_0^x \frac{1-u}{(1+(1-u)x)\log(1+(1-u)x)}\mathrm{Ei}_{k-1}(\log(1+(1-u)x))\,dt \\ &= \int_0^x \frac{1-u}{(1+(1-u)x)\log(1+(1-u)x)} \\ &\times \underbrace{\int_0^t \frac{1-u}{(1+(1-u)x)\log(1+(1-u)x)}\cdots\int_0^t \frac{(1-u)^2 t}{(1+(1-u)x)\log(1+(1-u)x)}}_{(k-2)\text{ times}}dtdt\cdots dt.\end{aligned}$$

Then, we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} G_{n,\lambda}^{(F,k)}(u)\frac{t^n}{n!} = \frac{\mathrm{Ei}_k(\log(1+(1-u)t))}{e_\lambda(t)-u} \\
&= \frac{1}{e_\lambda(t)-u}\int_0^x \frac{1-u}{(1+(1-u)x)\log(1+(1-u)x)} \\
&\times \underbrace{\int_0^t \frac{1-u}{(1+(1-u)x)\log(1+(1-u)x)}\cdots\int_0^t \frac{(1-u)^2 t}{(1+(1-u)x)\log(1+(1-u)x)}}_{(k-2)\text{ times}} dtdt\cdots dt \\
&= \frac{(1-u)x}{e_\lambda(t)-u}\sum_{m=0}^{\infty}\sum_{m_1+m_2+\cdots+m_{k-1}=m}^{\infty}\binom{m}{m_1,m_2,\cdots,m_{k-1}}(1-u)^{m_1+m_2+\cdots m_{k-1}} \\
&\times \frac{B_{m_1}^{(m_1)}(0)}{m_1+1}\frac{B_{m_2}^{(m_2)}(0)}{m_1+m_2+1}\cdots\frac{B_{m_{k-1}}^{(m_{k-1})}(0)}{m_1+m_2+\cdots+m_{k-1}+1}\frac{x^m}{m!} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\sum_{m_1+m_2+\cdots+m_{k-1}=m}^{\infty}\binom{m}{m_1,m_2,\cdots,m_{k-1}}(1-u)^{m_1+m_2+\cdots m_{k-1}} \\
&\times G_{n-m,\lambda}^{F}(u)\frac{B_{m_1}^{(m_1)}(0)}{m_1+1}\frac{B_{m_2}^{(m_2)}(0)}{m_1+m_2+1}\cdots\frac{B_{m_{k-1}}^{(m_{k-1})}(0)}{m_1+m_2+\cdots+m_{k-1}+1}\frac{x^n}{n!}
\end{aligned}$$

This finalizes the proof of the theorem. □

Now, we give the following theorem.

Theorem 5. *For $n \in \mathbb{N}_0$, we have*

$$\sum_{m=0}^{n}\frac{S_2(n,m)}{(1-u)^m}G_{m,\lambda}^{(F,k)}(u) = \sum_{m=0}^{n}\sum_{k=0}^{m}\sum_{j=0}^{k}\binom{n}{m}\binom{m}{k}\frac{G_{j,\lambda}^{(F)}(u)\,S_2(k,j)}{(1-u)^j}\frac{B_{m-k}}{(n-m+1)^k}.$$

Proof. Replacing t by $\frac{e^t-1}{1-u}$ in (2.1), we attain

$$\begin{aligned}
\frac{\mathrm{Ei}_k(t)}{e_\lambda\left(\frac{e^t-1}{1-u}\right)-u} &= \sum_{m=0}^{\infty}(1-u)^{-m}G_{m,\lambda}^{(F,k)}(u)\frac{(e^t-1)^m}{m!} \\
&= \sum_{m=0}^{\infty}(1-u)^{-m}G_{m,\lambda}^{(F,k)}(u)\sum_{n=0}^{\infty}S_2(n,m)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{S_2(n,m)}{(1-u)^m}G_{m,\lambda}^{(F,k)}(u)\frac{t^n}{n!}.
\end{aligned}$$

Also, we investigate

$$\frac{(1-u)\left(\frac{e^t-1}{1-u}\right)}{e_\lambda\left(\frac{e^t-1}{1-u}\right)-u}\frac{1}{e^t-1}\sum_{l=1}^{\infty}\frac{t^l}{(l-1)!l^k}=\frac{(1-u)\left(\frac{e^t-1}{1-u}\right)}{e_\lambda\left(\frac{e^t-1}{1-u}\right)-u}\frac{t}{e^t-1}\sum_{l=0}^{\infty}\frac{t^l}{l!\,(l+1)^k}$$
$$=\sum_{j=0}^{\infty}(1-u)^{-j}G_{j,\lambda}^{(F)}(u)\frac{(e^t-1)^j}{j!}\sum_{i=0}^{\infty}B_i\frac{t^i}{i!}\sum_{l=0}^{\infty}\frac{t^l}{l!\,(l+1)^k}$$
$$=\sum_{k=0}^{\infty}\sum_{j=0}^{k}(1-u)^{-j}G_{j,\lambda}^{(F)}(u)\,S_2(k,j)\frac{t^k}{k!}\sum_{i=0}^{\infty}B_i\frac{t^i}{i!}\sum_{l=0}^{\infty}\frac{t^l}{l!\,(l+1)^k}$$
$$=\sum_{m=0}^{\infty}\sum_{k=0}^{m}\sum_{j=0}^{k}\binom{m}{k}\frac{G_{j,\lambda}^{(F)}(u)\,S_2(k,j)}{(1-u)^j}B_{m-k}\frac{t^m}{m!}\sum_{l=0}^{\infty}\frac{t^l}{l!\,(l+1)^k}$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{k=0}^{m}\sum_{j=0}^{k}\binom{n}{m}\binom{m}{k}\frac{G_{j,\lambda}^{(F)}(u)\,S_2(k,j)}{(1-u)^j}\frac{B_{m-k}}{(n-m+1)^k}\frac{t^n}{n!}.$$

This completes the proof of the theorem. □

Theorem 6. *For $k\in\mathbb{Z}$ and $n\geq 0$, we have*

$$G_{n,\lambda}^{(F,k)}(x+1;u)-uG_{n,\lambda}^{(F,k)}(x;u)=\sum_{m=0}^{n}\sum_{l=1}^{m+1}\binom{n}{m}\frac{S_1(m+1,l)\,(1-u)^{m+1}}{l^{k-1}(m+1)}(x)_{n-m,\lambda}\,. \tag{2.8}$$

Proof. By Definition 1 and formula (2.5), we see that

$$\sum_{n=0}^{\infty}\left(G_{n,\lambda}^{(F,k)}(x+1;u)-uG_{n,\lambda}^{(F,k)}(x;u)\right)\frac{t^n}{n!}=\frac{\mathrm{Ei}_k(\log(1+(1-u)\,t))}{e_\lambda(t)-u}e_\lambda^x(t)\,(e_\lambda(t)-u)$$
$$=\mathrm{Ei}_k(\log(1+(1-u)\,t))\,e_\lambda^x(t)$$
$$=\sum_{m=0}^{\infty}\sum_{l=1}^{m+1}\frac{S_1(m+1,l)\,(1-u)^{m+1}}{l^{k-1}(m+1)}\frac{t^{m+1}}{m!}\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{l=1}^{m+1}\binom{n}{m}\frac{S_1(m+1,l)\,(1-u)^{m+1}}{l^{k-1}(m+1)}(x)_{n-m,\lambda}\frac{t^n}{n!},$$

which gives the asserted result in (2.8). □

3. Conclusion

In the present paper, we have considered type 2 degenerate poly-Frobenius-Genocchi polynomials and numbers by means of the polylogaritm function. Then, we have investigated diverse explicit expressions and some identities for those numbers and polynomials.

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