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THE MAXIMAL PRIME GAPS SUPREMUM AND THE FIROOZBAKHT'S HYPOTHESIS N^{O} 30

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Abstract.

The maximal prime gaps upper bound problem is one of the major mathematical problems to date. The objective of current research is to develop a standard which will aid in the understanding of the distribution of prime numbers. This paper presents theoretical results which originated with a research in the subject of the maximal prime gaps. The document presents the sharpest upper bound for the maximal prime gaps ever developed. The result becomes the Supremum bound on the maximal prime gaps and subsequently culminates with the conclusive proof of the Firoozbakht's Hypothesis N^o 30. Firoozbakht Hypothesis implies quite a bold conjecture concerning the maximal prime gaps. In fact it imposes one of the strongest maximal prime gaps bounds ever conjectured. Its truth implies the truth of a greater number of known prime gaps conjectures, simultaneously the Firoozbakht's Hypothesis disproves a known heuristic argument of Granville and Maier. This paper is dedicated to a fellow mathematician, the late Farideh Firoozbakht.

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1. The binomial expansion $2^{\left(n+\mathcal{G}s_{(n)}\right)}$

1.1. Preliminaries.

Within the scope of the paper, prime gap of the size $\mathfrak{g} \in \mathbb{N} \mid \mathfrak{g} \geq 2$ is defined as an interval between two primes $(p_i,\ p_{i+1}]$, containing $(\mathfrak{g}-1)$ composite integers. Maximal prime gap of the size $\mathfrak{g}$, is a gap strictly exceeding in size any preceding gap. All calculations and graphing were carried out with the aid of the *Mathematica®* software. For all $n \in \mathbb{N} \mid n \geq 8$, we make the following definitions:

Definition 1.1 (Interval length). $c = \mathcal{G}s_{(n)} = \left\lfloor 5\left(\log_{10}\ n\right)^2 - \frac{15}{8}\left(\log_{10} n\right)\right\rfloor$

Definition 1.2 (Interval endpoint). $t = (n+c)$

The binomial coefficient $\mathcal{M}_{(t)}$ related to the current research is a part of the associated binomial expansion:

$$2^t >> \binom{n+c}{n}$$

Definition 1.3 (Binomial coefficient).

$$\mathcal{M}_{(t)} = \binom{n+c}{n} = \left(\frac{(n+c)!}{(n! \times c!)}\right)$$

Definition 1.4 (Logarithm of the binomial coefficient).

$$\log \mathcal{M}_{(t)} = \log\left(\frac{(n+c)!}{(n! \times c!)}\right) = \log(t!) - \log(n!) - \log(c!) = \sum_{k=1}^{c} \log(n+k) - \sum_{k=1}^{c} \log k$$

Graphs in this paper implement a variant of logarithmic scaling of the horizontal axis given by:

Definition 1.5 (Scaling factor). $\xi = \frac{\log_{10}\left(\frac{n}{24}\right)}{\log_{10}(24)}$

Remark 1.1.

The function $\mathcal{G}s_{(n)}$ is an increasing, weakly monotone function. Due to the floor function used, $\mathcal{G}s_{(n)}$ clearly increases stepwise, producing a sequence in $\mathbb{N}$.

The logarithm of the binomial coefficient $\log \mathcal{M}_{(t)}$ produces an increasing, strictly monotone sequence in $\mathbb{R}$. The sudden increase in value of the function $\mathcal{G}s_{(n)}$, causes an analogous simultaneous jump in value of the associated binomial coefficient $\mathcal{M}_{(t)}$. Because at all other intermediate points of that particular interval the binomial coefficient $\mathcal{M}_{(t)}$ increases steadily, therefore the graph of $\log \mathcal{M}_{(t)}$ forms a distinctive slanted staircase pattern.

1.2. Bounds on the logarithm of the binomial coefficient.

Lemma 1.6 (Upper and Lower bounds on the logarithm of $n!$).
The bounds on the logarithm of $n!$ are given by:

$$n\log(n) - n + 1 \le \log(n!) \le (n+1)\log(n+1) - n \quad \forall n \in \mathbb{N} \mid n \ge 5 \tag{1.1}$$

Proof.
Evidently,

$$\log(n!) = \sum_{k=1}^{n} log(k) \quad \forall n \in \mathbb{N} \mid n \ge 2 \tag{1.2}$$

Hence, the pertinent integrals to consider are:

$$\int_1^n \log(x)\ dx \le \log(n!) \le \int_0^n \log(x+1)\ dx \quad \forall n \in \mathbb{N} \mid n \ge 5 \tag{1.3}$$

Accordingly, evaluating those integrals we obtain:

$$n\log(n) - n + 1 \le \log(n!) \le n\log\left(\frac{(n+1)}{e}\right) + \log\left(\frac{(n+1)}{e}\right) + 1 \tag{1.4}$$
$$= (n+1)\log(n+1) - n$$

Concluding the proof of Lemma 1.6. □

Remark 1.2.
Observe that $\log \mathcal{M}_{(t)}$ is a difference of logarithms of factorial terms:

$$\log \mathcal{M}_{(t)} = (\log(t!) - \log(n!) - \log(c!))$$

Consequently, implementing the lower/upper bounds on the logarithm of $n!$ for the bounds on $\log \mathcal{M}_{(t)}$, results in bounds of the form:

$$\log\left(\frac{(t+k)^{(t+k)}}{(n+k)^{(n+k)}\ (c+k)^{(c+k)}}\right) \quad \text{for } \forall\ k \in \mathbb{N} \cup \{0\} \tag{1.5}$$

Keeping the values of c, n and t constant and letting the variable k to increase unboundedly, results in an unboundedly decreasing function. When implementing the lower/upper bounds on the logarithm of $n!$ for the Supremum/Infimum bounds on $\log \mathcal{M}_{(t)}$, the variable k appears only with values $k = \{0, 1\}$ respectively. The combined effect of the difference of the logarithms of factorial terms in $\log \mathcal{M}_{(t)}$ and the decreasing property of the function 1.5, imposes a reciprocal interchange of the bounds 1.1, when implementing them for the bounds on $\log \mathcal{M}_{(t)}$.

Lemma 1.7 ($\log \mathcal{M}_{(t)}$ Supremum Bound).
The Supremum Bound on the logarithm of the binomial coefficient $\mathcal{M}_{(t)}$ is given by:

$$\log \mathcal{M}_{(t)} \le \log\left(\frac{t^t}{n^n\ c^c}\right) - 1 = \mathcal{UB}_{(t)} \quad \forall n \in \mathbb{N} \mid n \ge 8 \tag{1.6}$$

Proof.
From Lemma1.6 we have:

$$(n\log(n) - n + 1) \leq \log(n!) \tag{1.7}$$

Substituting from the inequality 1.7 into the Definition 1.4 we obtain:

$$\begin{aligned}(\log(t!) - \log(n!) - \log(c!)) \\ \leq ((t\log(t) - t + 1) - (n\log(n) - n + 1) - (c\log(c) - c + 1)) \\ = t\log(t) - n\log(n) - c\log(c) - 1 = \log\left(\frac{t^t}{n^n\ c^c}\right) - 1\end{aligned} \tag{1.8}$$

Consequently,

$$\log\mathcal{M}_{(t)} \leq \log\left(\frac{t^t}{n^n\ c^c}\right) - 1 = \mathcal{UB}_{(t)} \tag{1.9}$$

The Supremum bound $\mathcal{UB}_{(t)}$ produces an increasing, strictly monotone sequence in $\mathbb{R}$. At $n = 8$, the difference $\mathcal{UB}_{(t)} - \log\mathcal{M}_{(t)}$ attains 0.197362 and diverges as $n \to \infty$. Therefore, Lemma 1.7 holds as specified. □

Lemma 1.8 ($\log\mathcal{M}_{(t)}$ Infimum Bound)**.**
The Infimum Bound on the natural logarithm of the binomial coefficient $\mathcal{M}_{(t)}$ for all $n \in \mathbb{N} \mid n \geq 8$ is given by:

$$\log\mathcal{M}_{(t)} \geq \log\left(\frac{(t+1)^{(t+1)}}{(n+1)^{(n+1)}\ (c+1)^{(c+1)}}\right) = \mathcal{LB}_{(t)} \tag{1.10}$$

Proof.
From Lemma1.6 we have:

$$\log(n!) \leq n\log(n+1) - n + \log(n+1) \tag{1.11}$$

Substituting from the inequality 1.11 into the Definition 1.4 we obtain:

$$\begin{aligned}(\log(t!) - \log(n!) - \log(c!)) \\ \geq t\log(t+1) - n\log(n+1) - c\log(c+1) + \log(t+1) - \log(n+1) - \log(c+1) \\ = \log\left(\frac{(t+1)^t}{(n+1)^n\ (c+1)^c}\right) + \log\left(\frac{(t+1)}{(n+1)\ (c+1)}\right)\end{aligned} \tag{1.12}$$

Consequently,

$$\log\mathcal{M}_{(t)} \geq \log\left(\frac{(t+1)^{(t+1)}}{(n+1)^{(n+1)}\ (c+1)^{(c+1)}}\right) = \mathcal{LB}_{(t)} \tag{1.13}$$

The Infimum Bound $\mathcal{LB}_{(t)}$ produces an increasing, strictly monotone sequence in $\mathbb{R}$. At $n = 8$, the difference $\log\mathcal{M}_{(t)} - \mathcal{LB}_{(t)}$ attains 0.500673 and diverges as $n \to \infty$. Therefore, Lemma 1.8 holds as specified. □

Consequently, from Lemma 1.8 and 1.7 we have for all $n \in \mathbb{N} \mid n \geq 8$:

$$\log\left(\frac{(t+1)^{(t+1)}}{(n+1)^{(n+1)}\ (c+1)^{(c+1)}}\right) \leq \log\mathcal{M}_{(t)} \leq \log\left(\frac{t^t}{n^n\ c^c}\right) - 1 \tag{1.14}$$

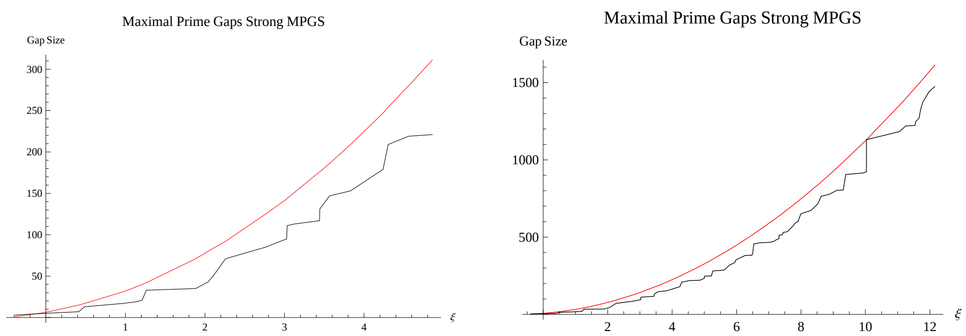

FIGURE 1. shows the graphs of $\mathcal{G}s_{(n)}$ (red) and the actual maximal gaps (black) with respect to ξ as given by Def. 1.5. The graph has been produced on the basis of data obtained from C. Caldwell as well as from T. Nicely tables of maximal prime gaps. The occurrence of the gap 64; $\mathfrak{g} = 1131$, beginning at the prime 1 693 182 318 746 371, is abnormally low (T. Oliveira e Silva, 2006). It is clearly visible on the graph at $\xi \approx 10.03$ with the difference $\mathcal{G}s_{(n)} - \mathfrak{g} = 0.0132662$.

2. MAXIMAL PRIME GAPS

From the Prime Number Theorem we have that an average gap between consecutive primes is given by $\log n$ for any $n \in \mathbb{N}$. There exist however prime gaps much shorter - containing only a single composite number, and gaps which are much longer than average - the maximal prime gaps.

We begin with a preliminary derivation. Since the integers from 1 to n contain $\left\lfloor \frac{n}{p} \right\rfloor$ multiples of the prime number p, $\left\lfloor \frac{n}{p^2} \right\rfloor$ multiples of p^2 etc. Thus it follows that:

$$n! = \prod_{p} p^{u_{(n,p)}}; \; where \; u_{(n,p)} = \sum_{m\geq 1} \left\lfloor \frac{n}{p^m} \right\rfloor$$

In accordance with the definitions 1.1 of $\mathcal{G}s_{(n)}$, 1.2 of t and 1.3 of $\mathcal{M}_{(t)}$ we obtain:

$$\mathcal{M}_{(t)} = \prod_{p\leq t} p^{\mathcal{K}_p}$$

where

$$\mathcal{K}_p = \sum_{m=1}^{\infty} \left(\left\lfloor \frac{t}{p^m} \right\rfloor - \left\lfloor \frac{n}{p^m} \right\rfloor - \left\lfloor \frac{\mathcal{G}s_{(n)}}{p^m} \right\rfloor \right)$$

it follows that

$$\mathcal{K}_p \leq \left\lfloor \frac{\log t}{\log p} \right\rfloor$$

and so by the above, Lemma 1.7 and 1.8 we have:

$$\mathcal{LB}_{(t)} \leq \log \mathcal{M}_{(t)} = \log \prod_{p\leq t} p^{\mathcal{K}_p} = \sum_{p\leq t} \mathcal{K}_p \log p \;\leq\; \mathcal{UB}_{(t)} \quad \forall n \in \mathbb{N} \mid n \geq 8 \tag{2.1}$$

Where p is as usual a prime number. Let's define:

Definition 2.1. $s = \left\lfloor \frac{(n+\mathcal{G}s_{(n)})}{2} \right\rfloor = \left\lfloor \frac{t}{2} \right\rfloor$

Lemma 2.2 (Prime Factors of $\mathcal{M}_{(t)}$).
The case when there does not exist any prime factor p of $\mathcal{M}_{(t)}$ within the interval from n to $(n + \mathcal{G}s_{(n)}) = t$ for any $n \in \mathbb{N} \mid n \geq 11$, imposes an upper limit on all prime factors p of $\mathcal{M}_{(t)}$. Consequently in this particular case, every prime factor p must be less than or equal to $s = \left\lfloor \frac{t}{2} \right\rfloor$.

Proof.
Let p be a prime factor of $\mathcal{M}_{(t)}$ so that $\mathcal{K}_p \geq 1$ and suppose that every prime factor $p \leq n$. If

$$s < p \leq n$$

then,

$$p < (n + \mathcal{G}s_{(n)}) < 2p$$

and

$$p^2 > \left(\frac{(n + \mathcal{G}s_{(n)})}{2} \right)^2 > (n + \mathcal{G}s_{(n)})$$

and so $\mathcal{K}_p = 0$. Therefore $p \leq s$ for every prime factor p of $\mathcal{M}_{(t)}$, for any $n \in \mathbb{N} \mid n \geq 11$. □

2.1. **Maximal Prime Gaps Supremum.** The bounds on the logarithm of $\mathcal{M}_{(t)}$ are given by Lemma 1.7 and 1.8:

$$\mathcal{LB}_{(t)} = \log \left(\frac{(t+1)^{(t+1)}}{(n+1)^{(n+1)} \ (c+1)^{(c+1)}} \right) \tag{2.2}$$

$$\leq \log \mathcal{M}_{(t)} = \sum_{k=1}^{c} \log{(n+k)} - \sum_{k=1}^{c} \log k \leq \log \left(\frac{t^t}{n^n \ c^c} \right) - 1 = \mathcal{UB}_{(t)}$$

$$\forall n \in \mathbb{N} \mid n \geq 11$$

Remark 2.1.

- The proof of the Maximal Prime Gaps Supremum implements the Supremum bound function $\mathcal{UB}_{(t_s)}$. Due to the fact that the function $\mathcal{UB}_{(t)}$ applies values of n, c and t directly, it imposes a requirement to generate a set of pertinent values to correctly approximate the interval s. This is to ascertain that the generated interval is at least equal or greater than s as given by Definition 2.1, as well as the corresponding value of c. Respective definitions follow:

  Definition 2.3. $n_s = \frac{n}{2}$

  Definition 2.4. $c_s = 5\left(\log_{10} n_s\right)^2 - \frac{15}{8}\left(\log_{10} n_s\right) + 1$

  Definition 2.5. $t_s = n_s + c_s$

- The function $\mathcal{G}s_{(n)}$ due to the implementation of the Floor function increases stepwise. The sudden increase in value of the function $\mathcal{G}s_{(n)}$ is mirrored by an analogous, simultaneous increase in both, implemented bounds on the function $\log \mathcal{M}_{(t)}$ as well as the function $\log \mathcal{M}_{(t)}$ itself. This is especially important for any kind of test or computation. The bottom peaks occur at $n - 1$ immediately preceding the sudden increase of value of $c = \mathcal{G}s_{(n)}$ at n.

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Theorem 2.6 (Maximal Prime Gaps Supremum and Infimum for primes)**.**
For any $n \in \mathbb{N} \mid n \geq 11$ *there exists at least one* $p \in \mathbb{N} \mid n < p \leq n + \mathcal{G}s_{(n)} = t$*; where* p *is as usual a prime number and the maximal prime gaps standard measure* $\mathcal{G}s_{(n)}$ *is given by:*

$$(2.3) \quad \mathcal{SUP} = \mathcal{G}s_{(n)} = \left\lfloor 5\left(\log_{10}\ n\right)^2 - \frac{15}{8}\left(\log_{10} n\right)\right\rfloor \quad \forall n \in \mathbb{N} \mid n \geq 11$$

$$\textit{Equivalently}, p_{i+1} - p_i \leq \mathcal{G}s_{(p_i)}$$

Proof.
Suppose that there is no prime within the interval from n to t. Then in accordance with the hypothesis, by Lemma 2.2 we have that every prime factor p of $\mathcal{M}_{(t)}$ must be less than or equal to $s = \left\lfloor \frac{t}{2} \right\rfloor$. Invoking Definitions 2.3, 2.4 and 2.5, Lemma 1.7, 1.8 and the inequality 2.1 we obtain in such a case, for all $n \in \mathbb{N} \mid n \geq 11$:

$$(2.4) \quad \mathcal{LB}_{(t)} = \log\left(\frac{(t+1)^{(t+1)}}{(n+1)^{(n+1)}\ (c+1)^{(c+1)}}\right)$$

$$\leq \log \mathcal{M}_{(t)} = \log \prod_{p \leq t_{(s)}} p^{\mathcal{K}_p} = \sum_{p \leq t_{(s)}} \mathcal{K}_p \log p \leq \log\left(\frac{(t_s)^{t_s}}{(n_s)^{n_s}\ (c_s)^{c_s}}\right) - 1 = \mathcal{UB}_{(t_s)}$$

In accordance with the hypothesis therefore, it must be true that:

$$(2.5) \qquad \log\left(\frac{(t+1)^{(t+1)}}{(n+1)^{(n+1)}\ (c+1)^{(c+1)}}\right) - \log\left(\frac{(t_s)^{t_s}}{(n_s)^{n_s}\ (c_s)^{c_s}}\right) + 1 \leq 0$$

However, at $n = 47$ the difference 2.5 attains ~ 7.69823 and diverges as n increases unboundedly. Since the difference generates a positive sequence in $\mathbb{R}$, we apply therefore the Cauchy's Root Test for $n \geq 47$:

$$(2.6) \qquad \limsup_{c\to\infty} \sqrt[c]{|a_c|} = \limsup_{c\to\infty} \sqrt[c]{\mathcal{LB}_{(t)} - \mathcal{UB}_{(t_s)}} \to 1$$

At $n = 47$ the Cauchy's Root Test attains ≈ 1.20947 and tends asymptotically to 1 decreasing strictly from above. Thus, by the definition of the Cauchy's Root Test, the series formed from the terms of the difference $\mathcal{LB}_{(t)} - \mathcal{UB}_{(t_s)}$ diverges to infinity as c increases unboundedly. This implies that for all $n \in \mathbb{N} \mid n \geq 47$:

$$(2.7) \qquad \mathcal{LB}_{(t)} - \mathcal{UB}_{(t_s)} > 0$$

Hence, we have a contradiction to the initial hypothesis. Necessarily therefore, there must be at least one prime within the interval c for all $n \in \mathbb{N} \mid n \geq 47$. Table 1 lists all values of n s.t. $11 \leq n \leq 53$. Evidently, every possible sub-interval contains at least one prime number. Thus we deduce that Theorem 2.6 holds in this range as well. Consequently Theorem 2.6 holds as stated for all $n \in \mathbb{N} \mid n \geq 11$, thus completing the proof. □

Remark 2.2. We may now slightly relax the function $\mathcal{G}s_{(n)}$ by dropping the Floor function, if needed.

TABLE 1. Low range $\mathcal{G}s_{(n)}$ vs primes within the range

n	$\mathcal{G}s_{(n)}$	*primes*	n	$\mathcal{G}s_{(n)}$	*primes*
11	3	13	31	8	37
13	4	17	37	9	41,43
17	5	19	41	9	43, 47
19	5	23	43	10	47, 53
23	6	29	47	10	53
29	7	31	53	11	59, 61

3. Firoozbakht's Hypothesis N° 30

In 1982, an Iranian mathematician Farideh Firoozbakht formulated a conjecture:

Theorem 3.1 (Firoozbakht's Hypothesis N° 30)**.**
The Firoozbakht Hypothesis defined by the relation:

$$\sqrt[n]{p_{(n)}} > \sqrt[(n+1)]{p_{(n+1)}} \tag{3.1}$$

is valid for all $p_n \in \mathbb{N} \mid p_n \geq 2$. *Where* $n \in \mathbb{N} \mid n \geq 1$ *is the index of the* n*-th prime number.*

Proof.

$$\sqrt[n]{p_{(n)}} > \sqrt[(n+1)]{p_{(n+1)}} \equiv (p_n)^{(n+1)} > \left(p_{(n+1)}\right)^n \tag{3.2}$$

Now, $(p_n)^{(n+1)} = (p_n)^n\, p_n$. Therefore, upon substitution into 3.2 we obtain:

$$(p_n)^n\, p_n > \left(p_{(n+1)}\right)^n \equiv p_n > \left(\frac{p_{(n+1)}}{p_n}\right)^n \tag{3.3}$$

Since the prime gap $\mathfrak{g} = p_{(n+1)} - p_n$ thus, $p_{(n+1)} = p_n + \mathfrak{g}$. Hence, from 3.3 we have:

$$p_n > \left(\frac{p_n + \mathfrak{g}}{p_n}\right)^n \tag{3.4}$$

Taking logs,

$$\frac{\log p_n}{n} > \log\left(\frac{p_n + \mathfrak{g}}{p_n}\right) \equiv \frac{\log p_n}{n} > \log\left(1 + \frac{\mathfrak{g}}{p_n}\right) \tag{3.5}$$

By the Theorem 2.6, maximal prime gaps are bounded above by the Supremum bound:

$$\mathfrak{g} = p_{(n+1)} - p_n \leq \mathcal{SUP} = 5\left(\log_{10} p_{(n)}\right)^2 - \frac{15}{8}\left(\log_{10} p_{(n)}\right)$$
$$\forall p_{(n+1)} \in \mathbb{N} \mid p_{(n+1)} \geq 13$$

Therefore, substituting $\mathcal{SUP}$ for the prime gaps $\mathfrak{g}$ into 3.5, we obtain:

$$\frac{\log p_n}{n} > \log\left(1 + \frac{\mathcal{SUP}}{p_n}\right) \tag{3.6}$$

Clearly,

$$\lim_{n\to\infty}\left(\log\left(1 + \frac{\mathcal{SUP}}{p_n}\right)\right) \to 0 \tag{3.7}$$

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Let's examine in turn, the first term of 3.6. By the PNT we have that:

$$n \log n + n(\log \log n - 1) < p_n < n \log n + n \log \log n \quad \forall n \in \mathbb{N}|\ n \geq 6 \tag{3.8}$$

Hence, from the Inequality 3.8, we have that:

$$p_n < n \log n + n \log \log n \tag{3.9}$$

Clearly, both sides of the inequality 3.9 diverge as $n \to \infty$ therefore, we substitute RHS of 3.9 into the first term of 3.6 thereby obtaining:

$$\frac{\log p_n}{n} \leq \frac{\log(n \log n + n \log \log n)}{n} = \frac{\log n + \log(\log n) + \log\left(1 + \frac{\log \log n}{\log n}\right)}{n} \tag{3.10}$$

Now, for any $n \in \mathbb{N} \mid n \geq 6$, the limit of 3.10 by the L'Hôpital's rule is:

$$\lim_{n\to\infty}\left(\frac{\log p_n}{n}\right) = \lim_{n\to\infty}\left(\frac{\log n + \log(\log n) + \log\left(1 + \frac{\log \log n}{\log n}\right)}{n}\right) \tag{3.11}$$

$$= \lim_{n\to\infty}\left[\frac{1}{n} + \frac{1}{n \log n} + \frac{\frac{1}{n(\log n)^2} - \frac{\log(\log n)}{n(\log n)^2}}{1 + \frac{\log(\log n)}{\log n}}\right] \to 0$$

Clearly, both n and $\log p_n$ for all $n \in \mathbb{N}|\ n \geq 1$ are positive divergent functions. Consequently, due to the fact that:

$$\lim_{n\to\infty}\left(\frac{\log p_n}{n}\right) \to 0 \tag{3.12}$$

this of course implies $n \gg \log p_n$ for all $n \in \mathbb{N} \mid n \geq 6$. Table 2 demonstrates that $n > \log p_n$ for all $n \in \mathbb{N} \mid 1 \leq n \leq 6$. Consequently, $n \gg \log p_n$ for all $n \in \mathbb{N} \mid n \geq 1$. Now,

$$\log\left(1 + \frac{\mathcal{SUP}}{p_n}\right) = \log(p_n + \mathcal{SUP}) - \log(p_n) \tag{3.13}$$

Which implies that Inequality 3.6,

$$\frac{\log p_n}{n} > \log\left(1 + \frac{\mathcal{SUP}}{p_n}\right) \equiv (n+1)\log p_n > n \log(p_n + \mathcal{SUP}) \tag{3.14}$$

Suppose that the inequality 3.14 **is false**. First we exponentiate both sides of the inequality. Consequently, in accordance with the hypothesis for $p_n \geq 11$:

$$(p_n)^{(n+1)} - (p_n + \mathcal{SUP})^n < 0 \tag{3.15}$$

However, at $p_5 = 11$ the difference attains $\sim 1.13722 \times 10^6$ and diverges exponentially. Since the difference of terms is positive, we apply the Cauchy's Root Test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{(p_n)^{(n+1)} - (p_n + \mathcal{SUP})^n} \to \infty \tag{3.16}$$

Hence, the Cauchy's Root Test diverges, with the rate of divergence $\propto k\, p_n \mid k \sim 1$. This implies that a series formulated from the terms of the difference 3.15 diverges. Consequently, we have a contradiction to the initial hypothesis. Hence, it implies that:

$$(p_n)^{(n+1)} > (p_n + \mathcal{SUP})^n \quad \forall p_n \in \mathbb{N} \mid p_n \geq 11 \tag{3.17}$$

A straightforward computer calculation verifies that the relation 3.17 holds in the interval $2 \leq p_n \leq 11$. Now, since $p_{(n+1)}$ is defined as $p_{(n+1)} = p_n + \mathfrak{g}$, therefore, from the inequality 3.4, 3.6 and 3.17 we derive accordingly:

$$(p_n)^{(n+1)} > \left(p_{(n+1)}\right)^n \quad \forall p_n \in \mathbb{N} \mid p_n \geq 2$$

□

TABLE 2. Low range difference $n - \log p_{(n)}$

$n - \log p_{(n)}$	Difference value	$n - \log p_{(n)}$	Difference value
$1 - \log 2$	0.306853	$4 - \log 7$	2.05409
$2 - \log 3$	0.901388	$5 - \log 11$	2.6021
$3 - \log 5$	1.39056	$6 - \log 13$	3.43505

Remark 3.1. The functions implemented in the Theorem 3.1 are of the form:

$$\sqrt[(n+k)]{p_{(n+k)}} \quad \forall\ k \in \mathbb{N} \cup \{0\} \tag{3.18}$$

The property of this function, is in a way analogous to the property of the Supremum/Infimum bounds on $\log \mathcal{M}_{(t)}$; vide Remark 1.2. Keeping n constant and allowing the variable k to increase unboundedly, results in 3.18 being a decreasing function, converging asymptotically, strictly from above to 1. Firoozbakht's Theorem implements 3.18 with the variable $k = \{0, 1\}$ respectively.

From the Firoozbakht Hypothesis N° 30, the weak Firoozbakht's Maximal Prime Gaps Bound Lemma can be derived. Next, we may go one step further and define the strong Firoozbakht's Maximal Prime Gaps Bound. Both Lemmas and their proofs follow next.

Lemma 3.2 (The Weak Firoozbakht's Maximal Prime Gaps Bound)**.**
For every $n \in \mathbb{N} \mid n \geq 11$ the maximal prime gaps satisfy the inequality:

$$\mathfrak{g} = p_{(n+1)} - p_n < (\log p_n)^2 - \log p_n = \mathcal{FW}_n \tag{3.19}$$

where $p_{(n)}$ is the n-th prime number.

Proof.
The weak form of the Firoozbakht's Maximal Prime Gaps Bound, for all $n \in \mathbb{N} \mid n \geq 11$ asserts:

$$\mathfrak{g}_{p_n} = p_{(n+1)} - p_n < (\log p_n)^2 - \log p_n = \mathcal{FW}_n \tag{3.20}$$

The difference of the function $\mathcal{FW}_n$ and the Supremum, defined by Theorem 2.6 is given by:

$$\left[\left((\log p_k)^2 - \log p_k\right) - \left(5\,(\log_{10} p_k)^2 - \frac{15}{8}\,(\log_{10} p_k)\right)\right] = \tag{3.21}$$

$$= \left((\log 10)^2 - 5\right)(\log_{10} p_k)^2 - \left((\log 10) - \frac{15}{8}\right)(\log_{10} p_k) \quad \forall p \in \mathbb{N} \mid p \geq 11$$

The factors:

$$(\log 10)^2 - 5 \sim 0.301898 \text{ as well as } (\log 10) - \frac{15}{8} \sim 0.427585 \tag{3.22}$$

This implies that for all $p \in \mathbb{N} \mid p \geq 7$ the weak Firoozbakht's maximal prime gaps bound $\mathcal{FW}_n$ lies above the Supremum, vide Table 4 in the Appendix,

$$\left((\log p_k)^2 - \log p_k\right) > \left(5\,(\log_{10} p_k)^2 - \frac{15}{8}\,(\log_{10} p_k)\right) \quad \forall p \in \mathbb{N} \mid p \geq 7$$

Consequently by Theorem 2.6, the weak Firoozbakht's maximal prime gaps bound holds for all $p \in \mathbb{N} \mid p \geq 11$:

$$\mathfrak{g} = p_{(n+1)} - p_n < (\log p_n)^2 - \log p_n$$

□

Lemma 3.3 (The Strong Firoozbakht's Maximal Prime Gaps Bound)**.**
For every $n \in \mathbb{N} \mid n \geq 37$ the maximal prime gaps satisfy the inequality:

$$\mathfrak{g} = p_{(n+1)} - p_n < (\log p_n)^2 - 2\log p_n + 1 = \mathcal{FS}_n \tag{3.23}$$

where $p_{(n)}$ is the n-th prime number.

Proof.
The strong form of the Firoozbakht's Maximal Prime Gaps Bound, for all $n \in \mathbb{N} \mid n \geq 37$ asserts:

$$\mathfrak{g}_{p_n} = p_{(n+1)} - p_n < (\log p_n)^2 - 2\log p_n + 1 = \mathcal{FS}_n \tag{3.24}$$

The difference of $\mathcal{FS}_n$ above and the Supremum defined by Theorem 2.6 is given by:

$$\left[\left((\log p_k)^2 - 2\log p_k + 1\right) - \left(5\,(\log_{10} p_k)^2 - \frac{15}{8}\,(\log_{10} p_k)\right)\right] = \tag{3.25}$$
$$= \left((\log 10)^2 - 5\right)(\log_{10} p_k)^2 - \left((2\log 10) - \frac{15}{8}\right)(\log_{10} p_k) \quad \forall p \in \mathbb{N} \mid p \geq 11$$

The factors:

$$(\log 10)^2 - 5 \sim 0.301898 \text{ as well as } 2\,(\log 10) - \frac{15}{8} \sim 2.73017 \tag{3.26}$$

Consequently, it is a matter of time before the leading quadratic term will begin to 'play the first violin'. The point where the difference 3.25 becomes positive, hence the point where the Supremum prime gaps bound intersects the stronger form of the Firoozbakht's prime gaps bound and remains below it, occurs:

$$p_k \in \mathbb{N} \mid p_k > \exp\left(\frac{\sqrt{5(301(\log 10)^2 - 96(\log 10)^3)} + (16\,(\log 10) - 15)\,(\log 10)}{2\left(8\,(\log 10)^2 - 40\right)}\right)$$
$$\text{which means for all } p_k \in \mathbb{N} \mid p_k > 458\,034\,213$$

Consequently, by Theorem 2.6, the stronger form of the Firoozbakht's maximal prime gaps bound holds for all $p \in \mathbb{N} \mid p > 458\,034\,213$. For all $p \in \mathbb{N} \mid 37 \leq p \leq 458\,034\,213$, a computer calculation verifies that the stronger form of the Firoozbakht's maximal prime gaps bound holds in this range, vide Table 3 below and Table 4 in the Appendix. Therefore the stronger form of the Firoozbakht maximal prime gaps bound holds for all $p \in \mathbb{N} \mid p \geq 37$. □

TABLE 3. Low range $\mathcal{FS}_n$ vs primes within the range

n	$\mathcal{FS}_n$	*primes*	n	$\mathcal{FS}_n$	*primes*
31	5.92429	—	61	9.67754	67
37	6.81689	41, 43	67	10.2701	71, 73
41	7.36347	43, 47	71	10.6451	73, 79
43	7.62423	47	73	10.8271	79, 83
47	8.12334	53	79	11.3532	83, 89
53	8.82263	59, 61	83	11.6885	89
59	9.47124	61, 67	89	12.1706	97, 101

Firoozbakht's Hypothesis is consistent with the Shank's Asymptotic Equality of Record Gaps Conjecture although it exposes a flaw and inconsistency in the Maier-Granville argument [14]:

$$\mathfrak{g}_{p_k} = p_{(k+1)} - p_k < \mathcal{M}\left(\log p_k\right)^2 \tag{3.27}$$

with the limit as k tends to infinity:

$$\limsup_{k\to\infty}\left(\frac{p_{(k+1)} - p_k}{\left(\log p_k\right)^2}\right) \geq 2\exp\left(-\gamma\right) \tag{3.28}$$

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4. Appendix

4.1. **Tabular data.**

Table 4. Theorem 2.6

Maximal Prime Gaps				
Gap start p_i	Actual gap $\mathfrak{g}$	Gap estim. $\mathcal{G}s_{(p_i)}$	Firoozb. Strong Gap est.	Firoozb. Weak Gap est.
7	3	1	0.894746	1.84066
23	5	6	4.56034	6.69583
89	7	15	12.1706	15.6592
113	13	17	13.8934	17.6208
523	17	31	27.6632	32.9228
887	19	37	33.4991	39.287
1129	21	40	36.3499	42.379
1327	33	42	38.3245	44.5151
9551	35	71	66.6574	74.8218
15683	43	80	75.0014	83.6617
19609	51	84	78.9209	87.8047
31397	71	92	87.5061	96.8605
155921	85	125	120.058	131.015
360653	95	143	139.138	150.934
370261	111	144	139.759	151.581
492113	113	151	146.566	158.673
1349533	117	176	172.01	185.126
1357201	131	176	172.159	185.28
2010733	147	186	182.628	196.142
4652353	153	209	206.005	220.358
17051707	179	247	244.978	260.629
20831323	209	254	251.285	267.137
47326693	219	280	277.975	294.648
122164747	221	311	310.495	328.116
189695659	233	327	326.197	344.258
191912783	247	327	326.617	344.69
387096133	249	352	352.47	371.244
436273009	281	357	356.975	375.869
1294268491	287	398	399.249	419.23
1453168141	291	402	403.89	423.987
2300942549	319	420	422.573	443.13
3842610773	335	441	443.92	464.99
4302407359	353	445	448.696	469.878
10726904659	381	484	488.234	510.33
20678048297	383	512	517.669	540.421
22367084959	393	516	521.248	544.079
25056082087	455	521	526.445	549.389
Continued ...				

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TABLE 4. Continued

Maximal Prime Gaps (Continued)				
42652618343	463	545	551.139	574.616
127976334671	467	596	603.936	628.511
182226896239	473	612	621.431	646.36
241160624143	485	626	635.48	660.689
297501075799	489	636	646.109	671.528
303371455241	499	637	647.103	672.541
304599508537	513	637	647.309	672.751
416608695821	515	653	663.341	689.097
461690510011	531	658	668.644	694.503
614487453523	533	672	683.512	709.656
738832927927	539	682	693.181	719.51
1346294310749	581	712	725.137	752.066
1408695493609	587	715	727.58	754.553
1968188556461	601	732	745.734	773.042
2614941710599	651	747	761.333	788.925
7177162611713	673	802	818.07	846.672
13829048559701	715	838	856.018	885.276
19581334192423	765	858	876.491	906.097
42842283925351	777	903	923.464	953.852
90874329411493	803	948	969.731	1000.87
171231342420521	805	986	1009.59	1041.36
218209405436543	905	1001	1025.05	1057.07
1189459969825483	915	1108	1136.52	1170.23
1686994940955803	923	1130	1160.2	1194.26
1693182318746371	1131	1131	1160.45	1194.52
43841547845541059	1183	1353	1392.73	1430.05
55350776431903243	1197	1370	1410.19	1447.74
80873624627234849	1219	1397	1438.81	1476.74
203986478517455989	1223	1465	1509.85	1548.71
218034721194214273	1247	1470	1515.03	1553.96
305405826521087869	1271	1495	1541.38	1580.64
352521223451364323	1327	1506	1552.67	1592.07
401429925999153707	1355	1516	1562.92	1602.46
418032645936712127	1369	1519	1566.13	1605.7
804212830686677669	1441	1569	1618.34	1658.57
1425172824437699411	1475	1613	1664.71	1705.51
The End				