THE ELEMENTARY PROOF OF THE RIEMANN'S HYPOTHESIS

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ABSTRACT. This research paper aims to explicate the complex issue of the Riemann's Hypothesis and ultimately presents its elementary proof. The method implements one of the binomial coefficients, to demonstrate the maximal prime gaps bound. Maximal prime gaps bound constitutes a comprehensive improvement over the Bertrand's result, and becomes one of the key elements of the theory. Subsequently, implementing the theory of the primorial function and its error bounds, an improved version of the Gauss' offset logarithmic integral is developed. The integral serves as the Supremum bound of the prime counting function $\pi_{(n)}$. Due to its very high precision, it permits to verify the relationship between the prime counting function π_n and the offset logarithmic integral of Carl Gauss'. The collective mathematical theory, via the Niels F. Helge von Koch [20] equation:

$$\pi_{(n)} = Li_{(n)} + \mathcal{O}\left(\sqrt{n}\ \log(n)\right)$$

enables to prove the Riemann's Hypothesis conclusively.

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1. Definitions section

Within the scope of the paper, prime gap of the size $\mathfrak{g} \in \mathbb{N} \mid \mathfrak{g} \geq 2$ is defined as an interval between two primes $(p_i,\ p_{i+1}]$, containing $(\mathfrak{g} - 1)$ composite integers. Maximal prime gap of the size $\mathfrak{g}$, is a gap strictly exceeding in size any preceding gap. In this document, all computations pertaining to the logarithmic integral, were carried out using the Gauss' offset logarithmic integral : $\int_2^n \frac{dt}{\log t}$.

All calculations and graphing were carried out with the aid of $Mathematica^{®}$ software.

1.1. **Mathematical constants definitions.**

Definition 1.1 (Golden Mean).

$$\mathcal{GM} = \frac{\sqrt{5} - 1}{2} \approx 0.6180339887498948482045868343656381177720309180$$

Definition 1.2 (Lambda constant).

$$\lambda = \left(\frac{14\pi}{29} + \mathcal{GM} \right) \approx 2.1346649249656571012555181228453981307810116485$$

Definition 1.3 (Khinchin's constant).

$$K = \exp\left(\frac{1}{\log 2} \sum_{i=2}^{\infty} \frac{(-1)^i \left(2 - 2^i\right)}{i} \zeta'_{(i)} \right) \approx 2.6854520010653064453097148354820$$

where ζ' is the derivative of the Riemann zeta function.

Definition 1.4 (Beta constant).

$$\beta = (1 + (\exp(1) - K)) \approx 1.0328298273937387900505726358708668039369$$

where K is the Khinchin's constant.

Definition 1.5 (Glaisher - Kinkelin constant).

$$\mathcal{A} = \exp\left(\frac{1}{12} - \zeta'(-1) \right) \approx 1.2824271291006226368753425688697917277676893$$

Definition 1.6 (Double Twin primes constant).

$$\mathcal{TC} = 2 \prod_{p>2} \left(1 - \frac{1}{(p-1)^2} \right) \approx 1.3203236316937391478556242200291155686525$$

2. The binomial expansion $2^{\left(n+\mathcal{G}_{(n)}\right)}$

2.1. Preliminaries.

Bertrand's Conjecture is a well known mathematical theorem concerning the size of the prime gaps. The first elementary proof of the Bertrands Conjecture regarding the existence of at least one prime within the interval from n to $2n$ was due to Srinivasa Ramanujan, who in 1919 presented his elegant proof. Paul Erdös at the age of 19 improved Ramanujan's proof in 1932. In his proof of the Bertrand's conjecture Paul Erdös utilized the largest binomial coefficient of the binomial expansion 2^{2n}:

$$\mathcal{N} = \binom{2n}{n} = \left(\frac{(2n)!}{(n! \times n!)}\right) = \left(\frac{(n+1)(n+2)\cdots(2n)}{n!}\right)$$

The problem of existence of at least one prime within the interval from n to $n + c = t$ is substantially more difficult than the Bertrand's Conjecture. The issue pertains to the considerably shorter interval length of the function $\mathcal{G}_{(n)}$, as compared to the length of the interval n, pertinent to the research that both Srinivasa Ramanujan and Paul Erdös worked on.

One of the major step-stones of this paper is the comprehensively improved bound on the maximal prime gaps. This goal is achieved by an implementation of a binomial expansion coefficient pertinent to the function $\mathcal{G}_{(n)}$. For all $n \in \mathbb{N} \mid n \geq 5$, we make the following definitions:

Definition 2.1 (Interval length). $c = \mathcal{G}_{(n)} = \left\lfloor 5 \left(\log_{10} n\right)^2 \right\rfloor$

Definition 2.2 (Interval endpoint). $t = (n + \mathcal{G}_{(n)}) = (n + c)$

Definition 2.3 (Inverse definition of n). $n = \lceil 10^a \rceil \quad where \; a = \sqrt{\frac{c}{5}}$

The binomial coefficient $\mathcal{M}_{(t)}$ related to the current research is a part of the associated binomial expansion:

$$2^t >> \binom{n+c}{n}$$

Definition 2.4 (Binomial coefficient).

$$\mathcal{M}_{(t)} = \binom{n+c}{n} = \left(\frac{(n+c)!}{(n! \; c!)}\right)$$

Definition 2.5 (Logarithm of the binomial coefficient).

$$\log \mathcal{M}_{(t)} = \log\left(\frac{(n+c)!}{(n! \times c!)}\right) = \log(t!) - \log(n!) - \log(c!) = \sum_{k=1}^{c} \log(n+k) - \sum_{k=1}^{c} \log k$$

2.2. Bounds on the logarithm of the binomial coefficient.

Lemma 2.6 (Upper and Lower bounds on the log of $n!$).
The bounds on the logarithm of $n!$ are given by:

$$(2.1) \qquad n \log(n) - n + 1 \leq \log(n!) \leq (n+1) \log(n+1) - n \quad \forall n \in \mathbb{N} \mid n \geq 5$$

Proof.

Evidently,

$$(2.2) \qquad \log(n!) = \sum_{k=1}^{n} log(k) \quad \forall n \in \mathbb{N} \mid n \ge 2$$

Now, the pertinent integrals to consider are:

$$(2.3) \qquad \int_{1}^{n} \log(x)\ dx \le \log(n!) \le \int_{0}^{n} \log(x+1)\ dx \quad \forall n \in \mathbb{N} \mid n \ge 5$$

Accordingly, evaluating those integrals we obtain:

$$(2.4) \quad n\log(n) - n + 1 \le \log(n!) \le n\log\left(\frac{(n+1)}{e}\right) + \log\left(\frac{(n+1)}{e}\right) + 1$$
$$= (n+1)\log(n+1) - n$$

Concluding the proof of Lemma 2.6. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 2.1.

Observe that $\log\mathcal{M}_{(t)}$ is a difference of logarithms of factorial terms:

$$\log\mathcal{M}_{(t)} = (\log(t!) - \log(n!) - \log(c!))$$

Consequently, implementing the lower/upper bounds on the logarithm of $n!$ for the bounds on $\log\mathcal{M}_{(t)}$, results in bounds of the form:

$$(2.5) \qquad \log\left(\frac{(t+k)^{(t+k)}}{(n+k)^{(n+k)}\ (c+k)^{(c+k)}}\right) \quad \text{for } \forall\ k \in \mathbb{N} \cup \{0\}$$

Keeping the values of c, n and t constant and letting the variable k to increase unboundedly, results in an unbounded monotonically decreasing function. When implementing the lower/upper bounds on the logarithm of $n!$ for the Supremum/Infimum bounds on $\log\mathcal{M}_{(t)}$, the variable k appears only with values $k = \{0, 1\}$ respectively. The combined effect of the difference of the logarithms of factorial terms in $\log\mathcal{M}_{(t)}$ and the decreasing property of the function 2.5, imposes a reciprocal interchange of the bounds 2.1, when implementing them for the bounds on $\log\mathcal{M}_{(t)}$.

Lemma 2.7 (log $\mathcal{M}_{(t)}$ Supremum Bound)**.**

The Supremum Bound on the logarithm of the binomial coefficient $\mathcal{M}_{(t)}$ is given by:

$$(2.6) \qquad \log\mathcal{M}_{(t)} \le \log\left(\frac{t^t}{n^n\ c^c}\right) - 1 = \mathcal{UB}_{(t)} \quad \forall n \in \mathbb{N} \mid n \ge 5$$

Proof.

Evidently, by Lemma 2.6 we have:

$$(2.7) \qquad\qquad\qquad (n\log(n) - n + 1) \le \log(n!)$$

Substituting from the inequality 2.7 into the Definition 2.5 we obtain:

$$(2.8) \quad (\log(t!) - \log(n!) - \log(c!))$$
$$\le ((t\log(t) - t + 1) - (n\log(n) - n + 1) - (c\log(c) - c + 1))$$
$$= t\log(t) - n\log(n) - c\log(c) - 1 = \log\left(\frac{t^t}{n^n\ c^c}\right) - 1$$

Consequently,

$$(2.9) \qquad \log \mathcal{M}_{(t)} \leq \log \left(\frac{t^t}{n^n \ c^c} \right) - 1 = \mathcal{UB}_{(t)}$$

The Supremum bound $\mathcal{UB}_{(t)}$ produces an increasing, strictly monotone sequence in $\mathbb{R}$. At $n = 5$, the difference $\mathcal{UB}_{(t)} - \log \mathcal{M}_{(t)}$ attains 0.143365 and diverges as $n \to \infty$. Therefore, Lemma 2.7 holds as specified. $\qquad \square$

Lemma 2.8 ($\log \mathcal{M}_{(t)}$ Infimum bound)**.**

The Infimum Bound on the natural logarithm of the binomial coefficient $\mathcal{M}_{(t)}$ for all $n \in \mathbb{N} \mid n \geq 5$ is given by:

$$(2.10) \qquad \log \mathcal{M}_{(t)} \geq \log \left(\frac{(t+1)^{(t+1)}}{(n+1)^{(n+1)} \ (c+1)^{(c+1)}} \right) = \mathcal{LB}_{(t)}$$

Proof.

From Lemma 2.6 we have:

$$(2.11) \qquad \log (n!) \leq n \log (n+1) - n + \log (n+1)$$

Substituting from the inequality 2.11 into the Definition 2.5 we obtain:

$$(2.12) \quad (\log (t!) - \log (n!) - \log (c!))$$
$$\geq t \log (t+1) - n \log (n+1) - c \log (c+1) + \log (t+1) - \log (n+1) - \log (c+1)$$

$$= \log \left(\frac{(t+1)^t}{(n+1)^n \ (c+1)^c} \right) + \log \left(\frac{(t+1)}{(n+1) \ (c+1)} \right)$$

Consequently,

$$(2.13) \qquad \log \mathcal{M}_{(t)} \geq \log \left(\frac{(t+1)^{(t+1)}}{(n+1)^{(n+1)} \ (c+1)^{(c+1)}} \right) = \mathcal{LB}_{(t)}$$

The Infimum bound $\mathcal{LB}_{(t)}$ produces an increasing, strictly monotone sequence in $\mathbb{R}$. At $n = 5$, the difference $\log \mathcal{M}_{(t)} - \mathcal{LB}_{(t)}$ attains 0.455384 and diverges as $n \to \infty$. Therefore, Lemma 2.8 holds as specified. $\qquad \square$

Consequently, from Lemma 2.8 and 2.7 we have:

$$(2.14) \qquad \log \left(\frac{(t+1)^{(t+1)}}{(n+1)^{(n+1)} \ (c+1)^{(c+1)}} \right) \leq \log \mathcal{M}_{(t)} \leq \log \left(\frac{t^t}{n^n \ c^c} \right) - 1$$

Inequality 2.14 presents very well streamlined Supremum/Infimum bounds on the $\log \mathcal{M}_{(t)}$.

3. Maximal prime gaps

From the Prime Number Theorem we have that an average gap between consecutive primes is given by $\log n$ for any $n \in \mathbb{N}$. There exist however prime gaps much shorter - containing only a single composite number, and gaps which are much longer than average - the maximal prime gaps. In 1929 R. Backlund [1] published a paper in which he proved the lower bound on the maximal prime gaps:

$$p_{(n+1)} - p_{(n)} > (2 - \epsilon) \log p_{(n)} \text{ for any } \epsilon > 0$$
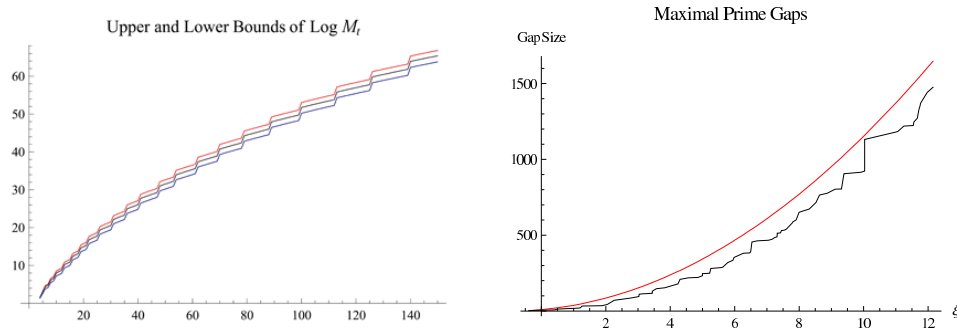
FIGURE 1. The left drawing shows the graphs of the lower (blue) and upper (red) bounds vs $\log \mathcal{M}_{(t)}$ (black). The right drawing shows the graph of $\mathcal{G}_{(n)}$ (red) and the actual maximal gaps (black) with respect to ξ as given by the Definition 5.3. The graph has been produced on the basis of data obtained from C. Caldwell as well as from T. Nicely tables of maximal prime gaps.

This was the first major result in this area. It had been improved upon in 1935 by Paul Erdös [14] who proved that:

$$p_{(n+1)} - p_{(n)} > \frac{c(\log p_{(n)}) \log(\log p_{(n)})}{(\log(\log(\log p_{(n)})))^2}$$

However, it was the pioneering work of H. Cramér [12] using sophisticated probabilistic techniques, who attempted to establish the upper bound on the maximal prime gaps:

$$p_{(n+1)} - p_{(n)} \leq (\log p_{(n)})^2$$

We begin with a preliminary derivation. Since the integers from 1 to n contain $\left\lfloor \frac{n}{p} \right\rfloor$ multiples of the prime number p, $\left\lfloor \frac{n}{p^2} \right\rfloor$ multiples of p^2 etc. Thus it follows that:

$$n! = \prod_p p^{u_{(n,p)}}; \ where \ u_{(n,p)} = \sum_{m \geq 1} \left\lfloor \frac{n}{p^m} \right\rfloor$$

In accordance with the definitions 2.1 of $\mathcal{G}_{(n)}$, 2.2 of t and 2.4 of $\mathcal{M}_{(t)}$ we obtain:

$$\mathcal{M}_{(t)} = \prod_{p \leq t} p^{\mathcal{K}_p}$$

where

$$\mathcal{K}_p = \sum_{m=1}^{\infty} \left(\left\lfloor \frac{t}{p^m} \right\rfloor - \left\lfloor \frac{n}{p^m} \right\rfloor - \left\lfloor \frac{\mathcal{G}_{(n)}}{p^m} \right\rfloor \right)$$

it follows that

$$\mathcal{K}_p \leq \left\lfloor \frac{\log t}{\log p} \right\rfloor$$

and so by the above, Lemma 2.7 and 2.8 we have:

$$(3.1) \quad \mathcal{LB}_{(t)} \leq \log \mathcal{M}_{(t)} = \log \prod_{p \leq t} p^{\mathcal{K}_p} = \sum_{p \leq t} \mathcal{K}_p \log p \ \leq \ \mathcal{UB}_{(t)} \quad \forall n \in \mathbb{N} \mid n \geq 5$$

Where p is as usual a prime number. Let's define:

Definition 3.1. $s = \left\lfloor \frac{t}{2} \right\rfloor$

Lemma 3.2 (Prime Factors of $\mathcal{M}_{(t)}$)**.**

The case when there does not exist any prime factor p of $\mathcal{M}_{(t)}$ within the interval from n to $(n + \mathcal{G}_{(n)}) = t$ for any $n \in \mathbb{N} \mid n \geq 8$, imposes an upper limit on all prime factors p of $\mathcal{M}_{(t)}$. Consequently in this particular case, every prime factor p must be less than or equal to $s = \left\lfloor \frac{t}{2} \right\rfloor$.

Proof.

Let p be a prime factor of $\mathcal{M}_{(t)}$ so that $\mathcal{K}_p \geq 1$ and suppose that every prime factor $p \leq n$. If

$$s < p \leq n$$

then,

$$p < (n + \mathcal{G}_{(n)}) < 2p$$

and

$$p^2 > \left(\frac{(n + \mathcal{G}_{(n)})}{2} \right)^2 > (n + \mathcal{G}_{(n)})$$

and so $\mathcal{K}_p = 0$. Therefore $p \leq s$ for every prime factor p of $\mathcal{M}_{(t)}$, for any $n \in \mathbb{N} \mid n \geq 8$.   □

3.1. Maximal prime gaps standard measure.

The binomial coefficient $\mathcal{M}_{(t)}$:

$$2^{t/2} < n^{\frac{c}{2}} < \exp\left(\mathcal{LB}_{(t)}\right) \leq \mathcal{M}_{(t)} = \left(\frac{(n+c)!}{(n! \times c!)} \right) \leq \exp\left(\mathcal{UB}_{(t)}\right) < n^{\frac{2c}{3}} < 2^t$$

$$\forall n \in \mathbb{N} \mid n \geq 22$$

The bounds on the logarithm of $\mathcal{M}_{(t)}$ are given by Lemma 2.7 and 2.8:

$$(3.2) \quad \mathcal{LB}_{(t)} = \log\left(\frac{(t+1)^{(t+1)}}{(n+1)^{(n+1)} (c+1)^{(c+1)}} \right)$$

$$\leq \log \mathcal{M}_{(t)} = \sum_{k=1}^{c} \log(n+k) - \sum_{k=1}^{c} \log k \leq \log\left(\frac{t^t}{n^n \, c^c} \right) - 1 = \mathcal{UB}_{(t)}$$

$$\forall n \in \mathbb{N} \mid n \geq 5$$

Remark 3.1.

- The proof of the Maximal Gaps Theorem implements the Supremum bound function $\mathcal{UB}_{(t_s)}$. Due to the fact that the Supremum function $\mathcal{UB}_{(t)}$ applies values of n, c and t directly, it imposes a technical requirement to generate a set of pertinent values, to correctly approximate the interval s. This is to ascertain that the generated interval is at least equal or greater than s as given by Definition 3.1, as well as the corresponding value of c. Respective definitions follow:

Definition 3.3. $n_s = \frac{n}{2}$

Definition 3.4. $c_s = 5 \left(\log_{10} (n_s)\right)^2 + 1$

Definition 3.5. $t_s = n_s + c_s$

- The function $\mathcal{G}s_{(n)}$ due to the implementation of the Floor function increases stepwise. The sudden increase in value of the function $\mathcal{G}s_{(n)}$ is mirrored by an analogous, simultaneous increase in both, implemented bounds on the function $\log \mathcal{M}_{(t)}$ as well as the function $\log \mathcal{M}_{(t)}$ itself.

Theorem 3.6 (Maximal Prime Gaps Bound and Infimum for primes)**.**

For any $n \in \mathbb{N} \mid n \geq 8$ there exists at least one $p \in \mathbb{N} \mid n < p \leq n + \mathcal{G}_{(n)} = t$; where p is as usual a prime number and the maximal prime gaps standard measure $\mathcal{G}_{(n)}$ is given by:

$$(3.3) \quad \mathcal{G}_{(n)} = \left\lfloor 5 \left(\log_{10} n\right)^2 \right\rfloor \qquad \forall n \in \mathbb{N} \mid n \geq 8$$

$$Equivalently, p_{i+1} - p_i \leq \mathcal{G}_{(p_i)}$$

Proof.

Suppose that there is no prime within the interval from n to t. Then in accordance with the hypothesis, by Lemma 3.2 we have that, every prime factor p of $\mathcal{M}_{(t)}$ must be less than or equal to $s = \left\lfloor \frac{t}{2} \right\rfloor$. Invoking Definitions 3.3, 3.4 and 3.5, Lemma 2.7, 2.8 and the inequality 3.1 we obtain for all $n \in \mathbb{N} \mid n \geq 8$:

$$(3.4) \quad \mathcal{LB}_{(t)} = \log \left(\frac{(t+1)^{(t+1)}}{(n+1)^{(n+1)} \, (c+1)^{(c+1)}} \right)$$

$$\leq \log \mathcal{M}_{(t)} = \log \prod_{p \leq t_{(s)}} p^{\mathcal{K}_p} = \sum_{p \leq t_{(s)}} \mathcal{K}_p \log p \ \leq \ \log \left(\frac{(t_s)^{t_s}}{(n_s)^{n_s} \, (c_s)^{c_s}} \right) - 1 = \mathcal{UB}_{(t_s)}$$

In accordance with the hypothesis therefore, it must be true that:

$$(3.5) \qquad \log \left(\frac{(t+1)^{(t+1)}}{(n+1)^{(n+1)} \, (c+1)^{(c+1)}} \right) - \log \left(\frac{(t_s)^{t_s}}{(n_s)^{n_s} \, (c_s)^{c_s}} \right) + 1 < 0$$

However, at $n = 43$ the difference 3.5 attains ~ 9.45885151 and diverges as n increases unboundedly. Since it generates a positive sequence in $\mathbb{R}$, we may therefore apply the Cauchy's Root Test for $n \geq 43$:

$$(3.6) \qquad \limsup_{c \to \infty} \sqrt[c]{|a_c|} = \limsup_{c \to \infty} \sqrt[c]{\left| \mathcal{LB}_{(t)} - \mathcal{UB}_{(t_s)} \right|} \to 1$$

At $n = 43$ the Cauchy's Root Test attains ≈ 1.17851 and tends asymptotically to 1 decreasing strictly from above. Thus, by the definition of the Cauchy's Root Test, the series formed from the terms of the difference $\mathcal{LB}_{(t)} - \mathcal{UB}_{(t_s)}$ diverges to infinity as c increases unboundedly. This implies that for all $n \in \mathbb{N} \mid n \geq 43$:

$$(3.7) \qquad \mathcal{LB}_{(t)} - \mathcal{UB}_{(t_s)} > 0$$

Hence, we have a contradiction to the initial hypothesis. Necessarily therefore, there must be at least one prime within the interval c for all $n \in \mathbb{N} \mid n \geq 43$. Table 1 lists all values of n s.t. $8 \leq n \leq 47$. Evidently, every possible sub-interval contains at least one prime number. Thus we deduce that Theorem 3.6 holds in this range as well. Consequently Theorem 3.6 holds as stated for all $n \in \mathbb{N} \mid n \geq 8$, thus completing the proof. $\qquad \square$

TABLE 1. Low range $\mathcal{G}_{(n)}$ vs primes within the range

n	$\mathcal{G}_{(n)}$	$primes$	n	$\mathcal{G}_{(n)}$	$primes$
8	4	11	29	10	31, 37
11	5	13	31	11	37, 41
13	6	17, 19	37	12	41, 43, 47
17	7	19, 23	41	13	43, 47, 53
19	8	23	43	13	47, 53
23	9	29, 31	47	13	53, 59

Remark 3.2.

From now on, we may relax the function $\mathcal{G}_{(n)}$, by dropping the floor function.

Corollary 3.7 (Cramér's Conjecture)**.**

There exist at least one prime $p \in \mathbb{N} \mid n < p \leq (n + (\log n)^2) \ \forall n \in \mathbb{N} \mid n \geq 8$.

Proof.

By Theorem 3.6 we have that there exist at least one prime $p \in \mathbb{N} \mid n < p \leq t$. Since,

$$\mathcal{G}_{(n)} = 5 \left(\log_{10} n\right)^2 < \left((\log 10)\left(\log_{10} n\right)\right)^2 \quad \forall n \in \mathbb{N} \mid n \geq 8$$

Therefore the Cramér's Maximal Gaps Conjecture follows ipso facto. $\square$

Corollary 3.8 (Legendre's Conjecture)**.**

There exist at least one prime $p \in \mathbb{N} \mid n^2 < p \leq (n+1)^2 \ \forall n \in \mathbb{N} \mid n \geq 2$

Proof.

Suppose that Theorem 3.8 is false for some $n \in \mathbb{N} \mid n > 10$. This implies that

$$\pi_{n^2} = \pi_{(n+1)^2}$$

By Theorem 3.6 we have that

$$\pi_{n^2} < \pi_{\left(n^2 + \mathcal{G}_{(n^2)}\right)} \quad \forall n \in \mathbb{N} \mid n \geq 8$$

Therefore, in accordance with the hypothesis it must be true that:

$$(n+1)^2 < \left(n^2 + \mathcal{G}_{(n^2)}\right)$$

Thus,

$$(3.8) \qquad \frac{2n+1}{5\left(\log_{10} n^2\right)^2} < 1$$

However, for any $n \in \mathbb{N} \mid n > 10$, the limit of 3.8 by the L'Hôpital's rule is:

$$\lim_{n \to \infty} \frac{2n+1}{5\left(\log_{10} n^2\right)^2} = \lim_{n \to \infty} \frac{n(\log 10)^2}{20} \to \infty$$

Hence the ratio 3.8 increases unboundedly as n tends to infinity. At $n = 10$, the value of the inequality 3.8 equals 1.05. It implies that:

$$(n+1)^2 > \left(n^2 + \mathcal{G}_{(n^2)}\right) \quad \forall n \in \mathbb{N} \mid n \geq 10$$

Hence we have a contradiction to the initial hypothesis. Consequently, Theorem 3.8 is satisfied for all $n \in \mathbb{N} \mid n \geq 10$. For all $n \in \mathbb{N} \mid 2 \leq n < 10$ a simple computer verification shows that Theorem 3.8 holds in this range as well, thus concluding the proof. $\square$

4. Theory of the Primorial Function

4.1. Upper Bound on the logarithm of the primorial function.

The natural logarithm of the primorial function is a key element of the definition of the tailored logarithmic integral. It paves the way for the estimation of the prime counting function $\pi_{(n)}$ with unparalleled accuracy. First, we define the primorial function for all $k \in \mathbb{N}$:

Definition 4.1. $p_k\sharp = \prod_{i=1}^{k} (p_i)$

Definition 4.2. $\log (p_k\sharp) = \log \left(\prod_{i=1}^{k} (p_i)\right) = \sum_{i=1}^{k} (\log p_i)$

Lemma 4.3 (Upper Bound on the logarithm of the primorial)**.**

The natural logarithm of the primorial function is strictly less than the respective prime number $p \in \mathbb{N}$:

$(4.1) \quad \log p_{(n)}\sharp < p_{(n)} \le n \quad \forall n \in \mathbb{N} \mid n \ge 2,$ where p_n is the largest prime $p \le n$

In particular the natural logarithm of the primorial function is asymptotic (from below) to the respective prime number:

$$(4.2) \qquad\qquad \log p_{(n)}\sharp \sim p_{(n)}$$

For the purpose of the proof we may assume that the twin primes continue indefinitely, the proof validity will not be affected by this. This issue will be expounded on in the Remark 4.2 **below.**

Proof.

From the inequality 4.1 we have:

$$(4.3) \qquad p_{(n)}\sharp < \exp\left(p_{(n)}\right) \le \exp(n) \quad \forall n \in \mathbb{N} \mid n \ge 2$$

Since prime numbers continue indefinitely, both $p_{(n)}\sharp$ and $\exp\left(p_{(n)}\right)$, are monotonically increasing divergent sequences of positive real numbers for all $n \in \mathbb{N} \mid n \ge 2$ Suppose that Lemma 4.3 is false, in accordance with the hypothesis it implies that:

$$(4.4) \qquad\qquad \exp\left(p_{(n)}\right) - p_{(n)}\sharp < 0$$

However, at $p_n = 13$ the difference 4.4 attains ~ 412383.39201 and further diverges exponentially. Therefore, we apply the d'Alemberts Ratio Test. Define a sequence for all prime numbers $p_{(n-1)}, p_{(n)} \in \mathbb{N} \mid p_{(n)} \ge 13$:

Definition 4.4. $a_{(n)} = \dfrac{\exp\left(p_{(n)}\right) - p_{(n)}\sharp}{\exp\left(p_{(n-1)}\right) - p_{(n-1)}\sharp}$

Remark 4.1. The sequence $a_{(n)}$ given by the Definition 4.4, has the least value at the twin primes as the difference $p_{(n)} - p_{(n-1)} = 2$. Consequently, it is therefore both necessary and sufficient, to consider the sequence $a_{(n)}$ at the twin prime numbers only, with $p_{(n)} = 6i + 7 \mid i \in \mathbb{N}, \ i \ge 1$.

At the twin primes:

$$(4.5) \qquad \exp\left(p_{(n)}\right) = \exp\left(p_{(n-1)} + 2\right) = \exp\left(p_{(n-1)}\right) \times \exp(2)$$

Further,

$$(4.6) \qquad\qquad p_{(n)}\sharp = p_{(n-1)}\sharp \left(p_{(n)}\right)$$

Thus, at the twin primes the sequence $a_{(n)}$ equals:
(4.7)
$$a_{(n)} = \frac{\exp\left(p_{(n)}\right) - p_{(n)}\sharp}{\exp\left(p_{(n-1)}\right) - p_{(n-1)}\sharp} = \exp\left(2\right) \times \left[\frac{\left(\exp\left(p_{(n-1)}\right)\right) - \left(p_{(n-1)}\sharp\right)\frac{p_{(n)}}{\exp\left(2\right)}}{\exp\left(p_{(n-1)}\right) - p_{(n-1)}\sharp}\right]$$

The bracketed expression on the RHS, at the twin primes approaches the limit:

(4.8)
$$\lim_{n \to \infty} \left[\frac{\left(\exp\left(p_{(n-1)}\right)\right) - \left(p_{(n-1)}\sharp\right)\frac{p_{(n)}}{\exp\left(2\right)}}{\exp\left(p_{(n-1)}\right) - p_{(n-1)}\sharp}\right] \to 1$$

at the twin primes therefore, the sequence $a_{(n)}$ must clearly approach the limit:
(4.9)
$$\lim_{n \to \infty} a_{(n)} = \lim_{n \to \infty} \left[\exp\left(2\right) \times \left(\frac{\left(\exp\left(p_{(n-1)}\right)\right) - \left(p_{(n-1)}\sharp\right)\frac{p_{(n)}}{\exp\left(2\right)}}{\exp\left(p_{(n-1)}\right) - p_{(n-1)}\sharp}\right)\right] \to \exp\left(2\right)$$

Consequently by the d'Alemberts Ratio Test, the series formed from the terms of the difference $\exp\left(p_{(n)}\right) - p_{(n)}\sharp$, diverges as $p_{(n)}$ increases unboundedly. Thus, it logically follows that:

(4.10)
$$p_{(n)}\sharp \le \exp\left(p_{(n)}\right) \quad \forall \ p_{(n)} \ \in \ \mathbb{N}| \ p_{(n)} \ \ge \ 13$$

Necessarily therefore, we have a contradiction to the initial hypothesis. Since at the twin primes the sequence $\exp\left(p_{(n)}\right) - p_{(n)}\sharp$ approaches:

(4.11)
$$\left[\exp\left(p_{(n)}\right) - p_{(n)}\sharp\right] \ \sim \ \exp\left(2\right)\left[\exp\left(p_{(n-1)}\right) - p_{(n-1)}\sharp\right]$$

Rearranging the above, we obtain that at the twin primes the primorial approaches:

(4.12)
$$p_{(n)}\sharp \ \sim \ \exp\left(p_{(n)}\right) - \exp\left(2\right)\left[\exp\left(p_{(n-1)}\right) - p_{(n-1)}\sharp\right]$$

This in turn implies that a strict inequality holds:

(4.13)
$$p_{(n)}\sharp \ < \ \exp\left(p_{(n)}\right)$$

Since increasing the gap between the consecutive primes has the effect of exponentially increasing the value that the sequence $a_{(n)}$ attains, therefore this result holds for all $p_{(n)} \ \in \ \mathbb{N}| \ p_{(n)} \ \ge \ 13$. By taking the logarithms, we obtain:

(4.14)
$$\log\left(p_{(n)}\sharp\right) \ < \ p_{(n)} \quad \forall \ p_{(n)} \ \in \ \mathbb{N}| \ p_{(n)} \ \ge \ 13$$

Thus, Lemma 4.3 holds for all $p_{(n)} \ \in \ \mathbb{N}| \ p_{(n)} \ \ge \ 13$. Direct computation verifies that Lemma 4.3 holds for all $p_{(n)} \ \in \ \mathbb{N}| \ 2 \le \ p_{(n)} \ \le \ 13$. Therefore, Lemma 4.3 holds as stated:

(4.15)
$$\log\left(p_{(n)}\sharp\right) \ < \ p_{(n)} \ \le \ n \quad \forall \ n \ \in \ \mathbb{N}| \ n \ \ge \ 2$$

Consequently, this implies that the sequence of the natural logarithm of the primorial function is asymptotic from below:

(4.16)
$$\log\left(p_{(n)}\sharp\right) \ \sim \ p_{(n)}$$

Concluding the proof of Lemma 4.3.   □

Lemma 4.3 also implies that:

(4.17)
$$\sqrt[\left(p_{(n)}\right)]{\left(p_{(n)}\sharp\right)} < \ \sqrt[\left(p_{(n)}\right)]{\exp\left(p_{(n)}\right)} = \exp\left(1\right) \quad \forall \ p_{(n)} \ \in \ \mathbb{N}$$

By the PNT, (Ruiz, 1997; Finch, 2003), and Lemma 4.3 we obtain therefore:

$$(4.18) \qquad \lim_{n \to \infty} \left(\sqrt[{(p_{(n)})}]{\left(p_{(n)} \sharp \right)} \right) \to \exp(1)$$

Remark 4.2. The sequence:

$$(4.19) \qquad a_{(n)} = \frac{\exp\left(p_{(n)}\right) - p_{(n)}\sharp}{\exp\left(p_{(n-1)}\right) - p_{(n-1)}\sharp}$$

as it has been demonstrated for the twin primes example in the proof of Lemma 4.3; for primes such that $\left(p_{(n)} - p_{(n-1)} = d \mid d \in \mathbb{N} \right)$ for some given particular d, the sequence $a_{(n)}$ at the respective prime pairs, converges to the limit:

$$(4.20) \qquad \lim_{n \to \infty} a_{(n)} = \lim_{n \to \infty} \left[\frac{\exp\left(p_{(n)}\right) - p_{(n)}\sharp}{\exp\left(p_{(n-1)}\right) - p_{(n-1)}\sharp} \right] \to \exp(d)$$

The approximation improves rapidly as $p_{(n)}$ increases. This is the reason why the validity of the twin primes conjecture is not essential.

4.2. **The estimation error bounds on the difference of** $\left(p_{(n)} - \log p_{(n)} \sharp \right)$**.**

Lemma 4.5 (Lower Estimation Error Bound On The Difference $p_n - \log p_n \sharp$)**.**
   The error of estimation of the primorial function by the use of the value of $p_{(n)}$ imposes the following lower bound:

$$(4.21) \quad \mathcal{LB}_{p_{(n)}} = \left(\sqrt{5} - 1 \right)\left(4\gamma^2 - 2\gamma \right)\left(\log p_{(n)} \right)\sqrt[3]{p_{(n)}} < \left(p_{(n)} - \log p_{(n)}\sharp \right)$$

$$\forall p_{(n)} \in \mathbb{N} \mid p_{(n)} \geq 2$$

   where $\gamma \approx 0.57721566490153286060651209$ is the Euler-Mascheroni constant.

Proof.
   Both $\exp\left(p_{(n)}\right)$ and $p_{(n)}\sharp$ as well as $\exp\left(\mathcal{LB}_{p_{(n)}}\right)$ are monotone, divergent sequences of positive real numbers for all $p_{(n)} \in \mathbb{N} \mid p_{(n)} \geq 2$. Suppose that Lemma 4.5 is false. From inequality 4.21 therefore, in accordance with the hypothesis we derive:

$$(4.22) \qquad \exp\left(p_{(n)}\right) - \left(p_{(n)}\sharp\right)\exp\left(\mathcal{LB}_{p_{(n)}}\right) < 0$$

However, at $p_n = 13$ the difference 4.22 attains ~ 328977.240182 and further diverges exponentially. Therefore we apply the d'Alemberts Ratio Test. Define a sequence for all prime numbers $p_{(n-1)}, p_{(n)} \in \mathbb{N} \mid p_{(n)} \geq 13$:

Definition 4.6. $a_{(n)} = \dfrac{\exp\left(p_{(n)}\right) - \left(p_{(n)}\sharp\right)\exp\left(\mathcal{LB}_{p_{(n)}}\right)}{\exp\left(p_{(n-1)}\right) - \left(p_{(n-1)}\sharp\right)\exp\left(\mathcal{LB}_{p_{(n-1)}}\right)}$ $\forall p_{(n)} \in \mathbb{N} \mid p_{(n)} \geq 13$

Remark 4.3. The terms of the sequence $a_{(n)}$ given by the Definition 4.6 have the least value at the twin primes since the difference $p_{(n)} - p_{(n-1)} = 2$. Consequently, it is both necessary and sufficient, to consider the sequence 4.6 **at the twin primes only**, with $p_{(n)} = 6i + 7 \mid i \in \mathbb{N}, i \geq 1$.

   At the twin primes:

$$(4.23) \qquad \exp\left(p_{(n)}\right) = \exp\left(p_{(n-1)} + 2\right) = \exp\left(p_{(n-1)}\right) \times \exp(2)$$

Further:

$$(4.24) \qquad p_{(n)}\sharp \;=\; p_{(n-1)}\sharp\left(p_{(n)}\right)$$

Thus, at the twin primes the sequence $a_{(n)}$ equals:

$$(4.25) \qquad a_{(n)} = \exp\left(2\right) \times \left[\frac{\left(\exp\left(p_{(n-1)}\right)\right) - \left(p_{(n-1)}\sharp\right)\exp\left(\mathcal{LB}_{p_{(n)}}\right)\frac{p_{(n)}}{\exp\left(2\right)}}{\exp\left(p_{(n-1)}\right) - p_{(n-1)}\sharp\exp\left(\mathcal{LB}_{p_{(n-1)}}\right)}\right]$$

The bracketed expression on the RHS, at the twin primes approaches the limit:

$$(4.26) \qquad \lim_{n\to\infty}\left[\frac{\left(\exp\left(p_{(n-1)}\right)\right) - \left(p_{(n-1)}\sharp\right)\exp\left(\mathcal{LB}_{p_{(n)}}\right)\frac{p_{(n)}}{\exp\left(2\right)}}{\exp\left(p_{(n-1)}\right) - p_{(n-1)}\sharp\exp\left(\mathcal{LB}_{p_{(n-1)}}\right)}\right] \;\to\; 1$$

at the twin primes therefore, the sequence $a_{(n)}$ must clearly approach the limit:

$$(4.27) \quad \lim_{n\to\infty} a_{(n)}$$

$$= \lim_{n\to\infty}\left[\exp\left(2\right) \times \left(\frac{\left(\exp\left(p_{(n-1)}\right)\right) - \left(p_{(n-1)}\sharp\right)\exp\left(\mathcal{LB}_{p_{(n)}}\right)\frac{p_{(n)}}{\exp\left(2\right)}}{\exp\left(p_{(n-1)}\right) - p_{(n-1)}\sharp\exp\left(\mathcal{LB}_{p_{(n-1)}}\right)}\right)\right] \;\to\; \exp\left(2\right)$$

Consequently, by d'Alemberts Ratio Test, the series formed from the terms of the difference $\exp\left(p_{(n)}\right) - p_{(n)}\sharp\exp\left(\mathcal{LB}_{p_{(n)}}\right)$ diverges, as $p_{(n)}$ increases unboundedly. Thus, it logically follows that:

$$(4.28) \qquad p_{(n)}\sharp\exp\left(\mathcal{LB}_{p_{(n)}}\right) \le \exp\left(p_{(n)}\right) \quad \forall\, p_{(n)} \in \mathbb{N}\,|\, p_{(n)} \ge 13$$

Necessarily therefore, we have a contradiction to the initial hypothesis. Since at the twin primes the sequence $\exp\left(p_{(n)}\right) - p_{(n)}\sharp\exp\left(\mathcal{LB}_{p_{(n)}}\right)$ approaches:

$$(4.29)$$
$$\left[\exp\left(p_{(n)}\right) - p_{(n)}\sharp\exp\left(\mathcal{LB}_{p_{(n)}}\right)\right] \;\sim\; \exp\left(2\right)\left[\exp\left(p_{(n-1)}\right) - p_{(n-1)}\sharp\exp\left(\mathcal{LB}_{p_{(n-1)}}\right)\right]$$

Rearranging the above we obtain that, at the twin primes the primorial approaches:

$$(4.30)$$
$$p_{(n)}\sharp\exp\left(\mathcal{LB}_{p_{(n)}}\right) \;\sim\; \exp\left(p_{(n)}\right) - \exp\left(2\right)\left[\exp\left(p_{(n-1)}\right) - p_{(n-1)}\sharp\exp\left(\mathcal{LB}_{p_{(n-1)}}\right)\right]$$

This in turn implies that a strict inequality holds:

$$(4.31) \qquad p_{(n)}\sharp\exp\left(\mathcal{LB}_{p_{(n)}}\right) \;<\; \exp\left(p_{(n)}\right)$$

Since increasing the gap between the consecutive primes has the effect of exponentially increasing the value that the sequence $a_{(n)}$ attains, therefore this result holds for all $p_{(n)} \in \mathbb{N}\,|\, p_{(n)} \ge 13$. By taking the logarithms, we obtain:

$$(4.32) \qquad \mathcal{LB}_{p_{(n)}} < \left(p_{(n)} - \log p_{(n)}\sharp\right) \quad \forall\, p_{(n)} \in \mathbb{N}\,|\, p_{(n)} \ge 13$$

Thus, Lemma 4.5 holds for all $p_{(n)} \in \mathbb{N}\,|\, p_{(n)} \ge 13$. Direct computation verifies that Lemma 4.5 holds for all $p_{(n)} \in \mathbb{N}\,|\, 2 \le p_{(n)} \le 13$. Therefore, Lemma 4.5 holds as stated, concluding the proof. $\square$

Lemma 4.7 (Upper Estimation Error Bound On The Difference $p_n - \log p_n\sharp$). *The error of estimation of the primorial function by the use of the value of $p_{(n)}$ imposes the following upper bound:*

$$(4.33) \quad \left(p_{(n)} - \log p_{(n)}\sharp\right) < 2\sqrt{p_{(n)}} = \mathcal{UB}_{p_{(n)}} \quad \forall p_{(n)} \in \mathbb{N} \mid p_{(n)} \ge 2$$

where $\gamma \approx 0.5772156649015328606065120 9$ is the Euler-Mascheroni constant.

Proof.

Both $\exp\left(p_{(n)}\right)$ and $p_{(n)}\sharp$ as well as $\exp\left(\mathcal{UB}_{p_{(n)}}\right)$ are monotone, divergent sequences of positive real numbers for all $p_{(n)} \in \mathbb{N} \mid p_{(n)} \geq 2$. Suppose that Lemma 4.7 is false. From inequality 4.33, in accordance with the hypothesis we derive:

$$(4.34) \qquad \left(p_{(n)}\sharp\right)\exp\left(\mathcal{UB}_{p_{(n)}}\right) - \exp\left(p_{(n)}\right) \; < \; 0$$

However, at $p_n = 13$ the difference 4.34 attains $\sim 4.02297598 \times 10^7$ and further diverges exponentially. Therefore we apply the d'Alemberts Ratio Test. Define a sequence for all prime numbers $p_{(n-1)}, p_{(n)} \in \mathbb{N} \mid p_{(n)} \geq 13$:

Definition 4.8. $a_{(n)} = \dfrac{\left(p_{(n)}\sharp\right)\exp\left(\mathcal{UB}_{p_{(n)}}\right) - \exp\left(p_{(n)}\right)}{\left(p_{(n-1)}\sharp\right)\exp\left(\mathcal{UB}_{p_{(n-1)}}\right) - \exp\left(p_{(n-1)}\right)} \quad \forall p_{(n)} \in \mathbb{N} \mid p_{(n)} \geq 13$

Remark 4.4. The terms of the sequence $a_{(n)}$ given by the Definition 4.8 have the least value at the twin primes since the difference $p_{(n)} - p_{(n-1)} = 2$. Consequently, it is both necessary and sufficient, to consider the sequence 4.8 **at the twin primes only**, with $p_{(n)} = 6i + 7 \mid i \in \mathbb{N}, i \geq 1$.

At the twin primes:

$$(4.35) \qquad \exp\left(p_{(n)}\right) = \exp\left(p_{(n-1)} + 2\right) = \exp\left(p_{(n-1)}\right) \times \exp(2)$$

Further,

$$(4.36) \qquad p_{(n)}\sharp \; = \; p_{(n-1)}\sharp\left(p_{(n)}\right)$$

Thus, at the twin primes the sequence $a_{(n)}$ equals:

$$(4.37) \qquad a_{(n)} = p_{(n)} \times \left[\frac{\left(p_{(n-1)}\sharp\right)\exp\left(\mathcal{UB}_{p_{(n)}}\right) - \frac{\exp\left(p_{(n)}\right)}{\left(p_{(n)}\right)}}{\left(p_{(n-1)}\sharp\right)\exp\left(\mathcal{UB}_{p_{(n-1)}}\right) - \exp\left(p_{(n-1)}\right)} \right]$$

The bracketed expression on the RHS, at the twin primes approaches the limit:

$$(4.38) \qquad \lim_{n\to\infty} \left[\frac{\left(p_{(n-1)}\sharp\right)\exp\left(\mathcal{UB}_{p_{(n)}}\right) - \frac{\exp\left(p_{(n)}\right)}{\left(p_{(n)}\right)}}{\left(p_{(n-1)}\sharp\right)\exp\left(\mathcal{UB}_{p_{(n-1)}}\right) - \exp\left(p_{(n-1)}\right)} \right] \to 1$$

at the twin primes therefore, the sequence $a_{(n)}$ must clearly approach the limit:

$$(4.39) \quad \lim_{n\to\infty} a_{(n)}$$

$$= \lim_{n\to\infty} \left[\left[\frac{\left(p_{(n-1)}\sharp\right)\exp\left(\mathcal{UB}_{p_{(n)}}\right) - \frac{\exp\left(p_{(n)}\right)}{\left(p_{(n)}\right)}}{\left(p_{(n-1)}\sharp\right)\exp\left(\mathcal{UB}_{p_{(n-1)}}\right) - \exp\left(p_{(n-1)}\right)} \right] p_{(n)} \right] \to p_{(n)}$$

Consequently by the d'Alemberts Ratio Test, the series formed from the terms of the difference $\left(p_{(n)}\sharp\right)\exp\left(\mathcal{UB}_{p_{(n)}}\right) - \exp\left(p_{(n)}\right)$ diverges, as $p_{(n)}$ increases unboundedly. Thus, it logically follows that:

$$(4.40) \qquad p_{(n)}\sharp\exp\left(\mathcal{UB}_{p_{(n)}}\right) \geq \exp\left(p_{(n)}\right) \quad \forall \, p_{(n)} \in \mathbb{N} \mid p_{(n)} \geq 13$$

Necessarily therefore, we have a contradiction to the initial hypothesis. Since at the twin primes the sequence $\left(p_{(n)}\sharp\right)\exp\left(\mathcal{UB}_{p_{(n)}}\right) - \exp\left(p_{(n)}\right)$ approaches:
$$(4.41)$$
$$\left[\left(p_{(n)}\sharp\right)\exp\left(\mathcal{UB}_{p_{(n)}}\right) - \exp\left(p_{(n)}\right)\right] \; \sim \; \left(p_{(n)}\right)\left[\left(p_{(n-1)}\sharp\right)\exp\left(\mathcal{UB}_{p_{(n-1)}}\right) - \exp\left(p_{(n-1)}\right)\right]$$

Rearranging the above we obtain, that at the twin primes the primorial approaches:
(4.42)
$$p_{(n)} \sharp \exp \left(\mathcal{UB}_{p_{(n)}} \right) \ \sim \ \exp \left(p_{(n)} \right) + \left(p_{(n)} \right) \left[\left(p_{(n-1)} \sharp \right) \exp \left(\mathcal{UB}_{p_{(n-1)}} \right) - \exp \left(p_{(n-1)} \right) \right]$$

This in turn implies that a strict inequality holds:

$$(4.43) \qquad\qquad p_{(n)} \sharp \exp \left(\mathcal{UB}_{p_{(n)}} \right) \ > \ \exp \left(p_{(n)} \right)$$

Since increasing the gap between the consecutive primes has the effect of exponentially increasing the value that the sequence $a_{(n)}$ attains, therefore this result holds for all $p_{(n)} \in \mathbb{N} \mid p_{(n)} \geq 13$. By taking the logarithms, we obtain:

$$(4.44) \qquad \mathcal{UB}_{p_{(n)}} > \left(p_{(n)} - \log p_{(n)} \sharp \right) \quad \forall \, p_{(n)} \in \mathbb{N} \mid p_{(n)} \geq 13$$

Thus, Lemma 4.7 holds for all $p_{(n)} \in \mathbb{N} \mid p_{(n)} \geq 13$. Direct computation verifies that Lemma 4.7 holds for all $p_{(n)} \in \mathbb{N} \mid 2 \leq p_{(n)} \leq 13$. Therefore, Lemma 4.7 holds as stated, concluding the proof. □

5. The prime counting function $\pi_{(n)}$ Supremum

5.1. The Tailored logarithmic integral definitions.

Definition 5.1 (Upper integration limit).

$$\theta_{(n)} = \log p_{(n)} \sharp \quad \text{where } p_{(n)} \text{ is the largest prime } p \in \mathbb{N} \mid p \leq n$$

Definition 5.2 (Tailored logarithmic integral definition).

$$TLi_{(n)} = \int_2^{\theta_{(n)}} \frac{dt}{\log t} \quad \forall n \in \mathbb{N} \mid n \geq 5$$

Graphs in this section implement a variant of logarithmic scaling of the horizontal axis given by:

Definition 5.3 (Scaling factor). $\xi = \dfrac{\log_{10}\left(\frac{n}{24} \right)}{\log_{10}(24)}$

5.2. Preliminary theory.

Tailored logarithmic integral given by the definition 5.2, presents a significantly improved accuracy of estimation of the function $\pi_{(n)}$. Lemma 4.3 states:

$$\log p_{(n)} \sharp < p_{(n)} \leq n \quad \forall n \in \mathbb{N} \mid n \geq 2$$

Consequently,

$$(5.1) \qquad\qquad \frac{\log p_{(n)} \sharp}{\log n} < \frac{n}{\log n} \quad \forall n \in \mathbb{N} \mid n \geq 2$$
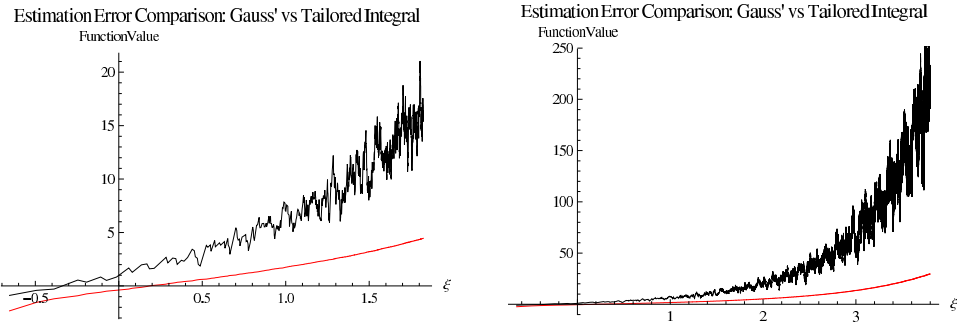
FIGURE 2. The drawings contrast the difference in estimation error made by the Gauss' Logarithmic Integral (black) vs the Tailored Integral (red). The figures drawn with respect to ξ, give the range up to $n = 7\,919$ (left), and up to $n = 4\,256\,233$ (right).

Remark 5.1. The classical offset logarithmic integral $Li_{(n)}$ of C.F. Gauss, is an improvement of the estimate of the number of primes, up to some $n \in \mathbb{N}$ given by $n/\log n$. Therefore, since the left side of the inequality 5.1 increases only at the primes as π_n does, it constitutes an improvement in $\pi_{(n)}$ estimation. Numerical comparison of the performance of the Carl F. Gauss offset $Li_{(n)}$ vs the $TLi_{(n)}$ is given in Table 2, and graphically presented in Fig. 2.

The graph of the tailored integral is below that of $\pi_{(n)}$ for all $n \in \mathbb{N} \mid n < 43$, please refer to Fig. 4a. Since the primorial function increases only at the primes, necessarily therefore, the estimation error of the tailored integral increases at the primes only. Hence, if the relation $TLi_{(n)} \geq \pi_{(n)}$ holds at the primes, it therefore holds at every other point. This contrasts strongly with the Gauss' logarithmic integral $Li_{(n)}$ in which, the estimation error term increases over the intervals between the primes and decreases at the primes. As a result, it produces large estimation error oscillations. On the other hand $TLi_{(n)}$, accurately duplicates the pattern of the curve of $\pi_{(n)}$, with minimal error increase.

TABLE 2. Comparison: Gauss' $Li_{(n)}$ vs $TLi_{(n)}$

n	$\pi_{(n)}$	$Li_{(n)}$	$Li_{(n)} - \pi_{(n)}$	$TLi_{(n)}$	$TLi_{(n)} - \pi_{(n)}$
47	15	16.6506001	1.6506001	15.0538449	0.0538449
97	25	28.427383	3.42738	25.4798052	0.4798052
997	168	176.1301053	8.130105	170.2098996	2.2098996
9973	1229	1242.160134	13.160134	1233.7930553	4.7930553
99991	9592	9627.982104	35.982104	9601.4333493	9.4333493
999983	78498	78625.273494	127.273494	78516.778831	18.7788305
9999991	664579	664916.801506	337.801506	664618.2703	39.27032
99999989	5761455	5762207.73313	752.733129	5761542.23	87.22988
999999937	50847534	50849230.8718	1696.87178	50847736.926	202.925523
9999999967	455052511	455055612.108	3101.10829	455052999.99	488.985131
99999999977	4118054813	4118066398.67	11585.6684	4118056037.8	1224.81062
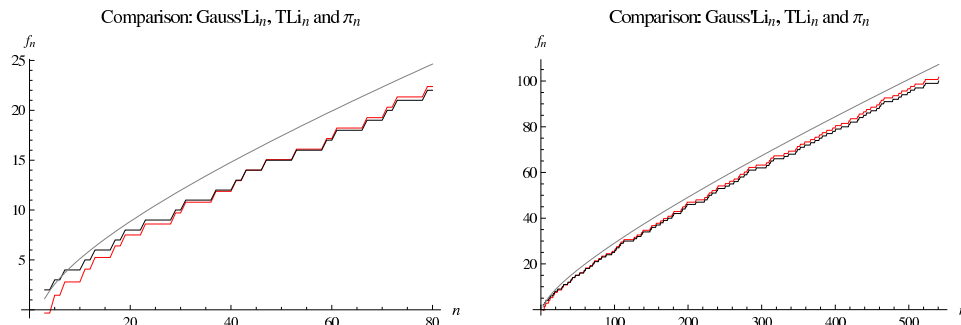
FIGURE 3. The figures drawn at every $n \in \mathbb{N}$ in the range, show the graphs of $Li_{(n)}$ (grey), $\pi_{(n)}$ (black) and $TLi_{(n)}$ (red).
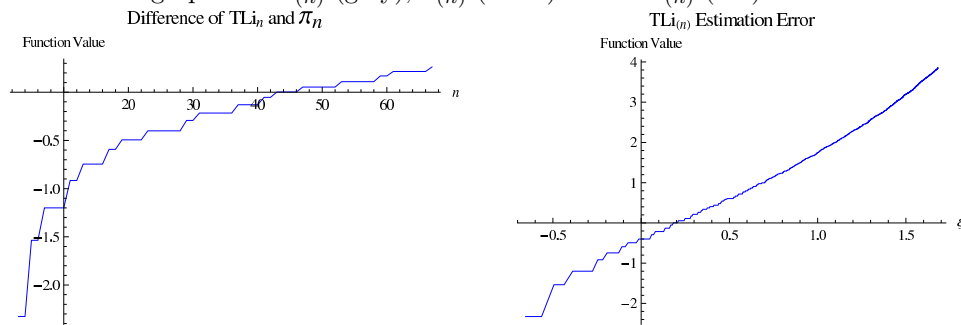
FIGURE 4. The drawings show the estimation error $TLi_{(n)} - \pi_{(n)}$ curve drawn at every $n \in \mathbb{N}$ within the pertinent range. The right figure drawn with respect to ξ, corresponding to: $3 \leq n \leq 5,000$.

5.3. **Tailored integral $TLi_{(n)}$ step sequence.**

Due to the fact that $TLi_{(n)}$ increases stepwise at the primes, the analysis of the step size and its limit as n approaches infinity forms the core of the proof of the tailored integral. Thus, for a pair of consecutive primes: p_i and p_{i+1} we define the stepwise limits of integration for all $p_i \in \mathbb{N} \mid p_i \geq 2$:

Definition 5.4 (Stepwise lower integration limit). $\theta_1 = \log p_{(i)} \sharp$

Definition 5.5 (Stepwise upper integration limit). $\theta_2 = \log p_{(i+1)} \sharp$

Definition 5.6 (Step sequence).

$$\mathcal{SSQ}_{\left(p_{(i+1)}\right)} = \left(TLi_{\left(p_{(i+1)}\right)} - TLi_{(p_i)}\right) = \int_{\theta_1}^{\theta_2} \frac{dt}{\log t} \quad \forall p_i \in \mathbb{N} \mid p_i \geq 3$$

Definition 5.7 (Step estimation error sequence).

$$\mathcal{SER}_{\left(p_{(i+1)}\right)} = \left(TLi_{\left(p_{(i+1)}\right)} - TLi_{(p_i)} - 1\right) = \int_{\theta_1}^{\theta_2} \frac{dt}{\log t} - 1 \quad \forall p_i \in \mathbb{N} \mid p_i \geq 3$$

Remark 5.2. Both the tailored integral $TLi_{(n)}$ and π_n are weakly monotone functions, increasing unboundedly, hence producing a positive sequence of numbers

which diverges to infinity. The initial estimates of the $TLi_{(n)}$ step size indicate that the step sequence quickly approaches the value of 1 from above. Table 3 presents some of the values that the step sequence takes at the powers of 10.

It is obvious that the numerical value attained by the step sequence at various points fluctuates as well, as a consequence of the size of the gap between the two consecutive primes (as well as the distance to the preceding prime pair). The effect however, of the gap interval length rapidly decreases as p_i increases, since by Theorem 3.6 the gaps Supremum is given by $\mathcal{G}_{(p_i)} = 5 \left(\log_{10} p_i \right)^2$.

Lemma 5.8 (Stepwise Convergence Of The Error of Estimation of the $TLi_{(n)}$)**.** *The step sequence of the tailored logarithmic integral $TLi_{(n)}$ is Cauchy and converges asymptotically from above to the limit:*

$$(5.2) \qquad \lim_{p_i \to \infty} \left(\int_{\theta_1}^{\theta_2} \frac{dt}{\log t} \right) = 1$$

Furthermore, the difference of the step integral $TLi_{(n)}$ and its approximation has the following bounds:

(5.3)

$$LDB_{p_{(i+1)}} = \frac{1}{5(p_{(i+1)})} \leq \left[\int_{\theta_1}^{\theta_2} \frac{dt}{\log t} - \frac{\log p_{(i+1)}}{\log \left(\log p_{(i+1)} \sharp \right)} \right] \leq \frac{1}{p_{(i+1)}} = UDB_{p_{(i+1)}}$$

$$\text{for all } p \in \mathbb{N} \mid p \geq 13$$

with θ_1 and θ_2 given by the Definitions 5.4 and 5.5 respectively.

Proof.

By the Prime Number Theorem we may estimate the integral $TLi_{(n)}$ step sequence for any prime number $p \in \mathbb{N} \mid p \geq 3$:

$$(5.4) \quad \int_{\theta_1}^{\theta_2} \frac{dt}{\log t} \sim \frac{\theta_2 - \theta_1}{\log \theta_2} = \frac{\left(\log p_{(i)} \sharp + \log p_{(i+1)} \right) - \log p_{(i)} \sharp}{\log \left(\log p_{(i+1)} \sharp \right)} = \frac{\log p_{(i+1)}}{\log \left(\log p_{(i+1)} \sharp \right)}$$

Thus by the PNT we have,

$$(5.5) \qquad \int_{\theta_1}^{\theta_2} \frac{dt}{\log t} \sim \frac{\log p_{(i+1)}}{\log \left(\log p_{(i+1)} \sharp \right)}$$

The logarithm of the primorial function is clearly a monotone function increasing unboundedly, hence, producing a sequence of positive real numbers which diverges to infinity. From Lemma 4.3 we have that $\log p_{(i+1)} \sharp$ is asymptotic from below to $p_{(i+1)}$, as well as:

$$\log p_{(i+1)} \sharp < p_{(i+1)} \leq n \quad \forall n \in \mathbb{N} \mid n \geq 2,$$

$$\text{where } p_{(i+1)} \text{ is the greatest prime } p \in \mathbb{N} \mid p \leq n$$

Hence, for a prime number $p \in \mathbb{N}$,

$$\log \left(\log p_{(i+1)} \sharp \right) \leq \log p_{(i+1)}$$

This implies that the estimating sequence converges asymptotically from above to the limit:

$$(5.6) \qquad \lim_{p_{(i+1)} \to \infty} \left(\frac{\log p_{(i+1)}}{\log \left(\log p_{(i+1)} \sharp \right)} \right) \to 1$$

Therefore it is Cauchy. The step integral $TLi_{(n)}$ at $p_6 = 13$ attains ~ 1.13056 and the step sequence values decrease, asymptotically approaching 1 as p_n increases unboundedly. Please, also refer to the Table 3. Consequently,

$$(5.7) \qquad \lim_{p_{(i+1)} \to \infty} \left(\int_{\theta_1}^{\theta_2} \frac{dt}{\log t} \right) = \lim_{p_{(i+1)} \to \infty} \left(\frac{\log p_{(i+1)}}{\log \left(\log p_{(i+1)} \sharp \right)} \right) \to 1$$

Thus the step integral $TLi_{(n)}$ is Cauchy as well. Both $LDB_{p_{(i+1)}}$ and $UDB_{p_{(i+1)}}$ are clearly strictly monotone decreasing Cauchy sequences. Suppose that the following assertion is false:

$$(5.8) \qquad \frac{1}{5(p_{(i+1)})} \leq \left[\int_{\theta_1}^{\theta_2} \frac{dt}{\log t} - \frac{\log p_{(i+1)}}{\log \left(\log p_{(i+1)} \sharp \right)} \right]$$

This implies that:

$$(5.9) \qquad 5(p_{(i+1)}) - \left[1 / \left(\int_{\theta_1}^{\theta_2} \frac{dt}{\log t} - \frac{\log p_{(i+1)}}{\log \left(\log p_{(i+1)} \sharp \right)} \right) \right] < 0$$

However, at $p_{(n)} = 13$ the inequality 5.9 attains ~ 48.6109 and diverges as $p_{(n)}$ increases unboundedly with the rate of divergence $\propto k \, p_{(n)}$ s.t. $k \sim 3$ for larger primes $p_{(n)}$. Consequently, we have a contradiction to the hypothesis. Inequality 5.8 therefore, is valid for all $p_n \in \mathbb{N} \mid p_n \geq 13$.

Suppose now, that the following inequality is false:

$$(5.10) \qquad \left[\int_{\theta_1}^{\theta_2} \frac{dt}{\log t} - \frac{\log p_{(i+1)}}{\log \left(\log p_{(i+1)} \sharp \right)} \right] \leq \frac{1}{p_{(i+1)}}$$

This implies that:

$$(5.11) \qquad \left[1 / \left(\int_{\theta_1}^{\theta_2} \frac{dt}{\log t} - \frac{\log p_{(i+1)}}{\log \left(\log p_{(i+1)} \sharp \right)} \right) \right] - p_{(i+1)} < 0$$

However, at $p_{(n)} = 13$ the inequality 5.11 attains ~ 3.38914 and diverges as $p_{(n)}$ increases unboundedly with the rate of divergence $\propto k \, p_{(n)}$ s.t. $k \sim 1$ for larger primes $p_{(n)}$. Consequently, we have a contradiction to the hypothesis. Inequality 5.10 therefore, is valid for all $p_n \in \mathbb{N} \mid p_n \geq 13$. Necessarily this implies that the Inequality 5.3 holds as stated. This demonstrates therefore, that since $UDB_{p_{(i+1)}}$ is strictly monotone decreasing Cauchy sequence with a limit $\mathcal{L} = 0$:

$$\lim_{(p_{(i+1)}) \to \infty} \left(\frac{1}{p_{(i+1)}} \right) = \lim_{(p_{(i+1)}) \to \infty} \left[\int_{\theta_1}^{\theta_2} \frac{dt}{\log t} - \frac{\log p_{(i+1)}}{\log \left(\log p_{(i+1)} \sharp \right)} \right] \to 0$$

Thus, from above we have that the estimating sequence 5.6 converges asymptotically from above to its limit $\mathcal{L} = 1$. Since the step integral at $p_{(i+1)} = 11$ attains ~ 1.2171 necessarily therefore the step integral tends asymptotically from above:

$$(5.12) \qquad \lim_{p_{(i+1)} \to \infty} \left(\frac{\log p_{(i+1)}}{\log \left(\log p_{(i+1)} \sharp \right)} \right) = \lim_{p_{(i+1)} \to \infty} \left(\int_{\theta_1}^{\theta_2} \frac{dt}{\log t} \right) \to 1$$

This implies, that the sequence of the step estimation errors asymptotically converges from above (also refer to Table 3):

$$\lim_{p_{(i+1)} \to \infty} \left(\left(\int_{\theta_1}^{\theta_2} \frac{dt}{\log t} \right) - 1 \right) = \lim_{p_{(i+1)} \to \infty} \left(\frac{\log p_{(i+1)}}{\log \left(\log p_{(i+1)} \sharp \right)} - 1 \right) \to 0$$

Thus concluding the proof of Lemma 5.8.   □

TABLE 3. Step sequence values

p_i	p_{i+1}	Actual step	Est. step Eq. 5.4	Difference
7	11	1.284296315549	1.17139190927	0.112904406279
97	101	1.036036760682	1.029877441266	0.006159319416
997	1009	1.007295189211	1.006767382731	0.000527806480
9973	10007	1.001161895309	1.001111285430	0.000050609879
99991	100003	1.000271361223	1.000266343124	$5.0180995783 \times 10^{-6}$
999983	1000003	1.000109530674	1.00010902981	$5.00863978704 \times 10^{-7}$
9999991	10000019	1.000029984472	1.000029934445	$5.0027059524 \times 10^{-8}$
99999989	100000007	1.000006660234	1.000006655234	$5.00067955427 \times 10^{-9}$

Remark 5.3. The integral part of the step size clearly accounts for the prime number found. Comparing each fractional part of the step (please refer to Table 3) at p_i with the corresponding term of the harmonic series $(\frac{1}{p_i})$, it becomes obvious that it is greater than the term of the series. Since the error of estimation is the sum of $\pi_{(n)}$ of such individual terms, comparing its sum with the divergent sum of reciprocals of successive prime numbers leads to a conjecture, that the sum of the estimation error terms diverges as p_i tends to infinity.

5.4. **Step sequence estimation error bounds.**

Corollary 5.9 (Infimum and Supremum Step Sequence Estimation Error Bounds)**.**
The step sequence error of estimation of the prime counting function $\pi_{(n)}$ by the application of the tailored logarithmic integral $TLi_{(n)}$ $\forall p_{(i)} \in \mathbb{N} \mid p_{(i)} \geq 13$, is bounded below/above by:

$$(5.13) \quad \mathcal{INF}_{p_{(i+1)}} = \frac{\log p_{(i+1)}}{\log \left(\log p_{(i+1)} \sharp \right)} + \frac{1}{5(p_{(i+1)})} - 1 \leq \left[\int_{\theta_1}^{\theta_2} \frac{dt}{\log t} - 1 \right]$$

$$\leq \frac{\log p_{(i+1)}}{\log \left(\log p_{(i+1)} \sharp \right)} + \frac{1}{p_{(i+1)}} - 1 = \mathcal{SUP}_{p_{(i+1)}} \quad \text{for all } p \in \mathbb{N} \mid p \geq 13$$

where $p_{(i)}$ and $p_{(i+1)}$ are associated with lower/upper limits of integration and θ_1, θ_2 are given by the Definitions 5.4 and 5.5 respectively.

Proof. From Lemma 5.8 we have:

$$(5.14)$$
$$\frac{1}{5(p_{(i+1)})} \leq \left[\int_{\theta_1}^{\theta_2} \frac{dt}{\log t} - \frac{\log p_{(i+1)}}{\log \left(\log p_{(i+1)} \sharp \right)} \right] \leq \frac{1}{p_{(i+1)}} \quad \text{for all } p \in \mathbb{N} \mid p \geq 13$$

Which is equivalent to say that:

$$(5.15) \quad \frac{\log p_{(i+1)}}{\log \left(\log p_{(i+1)} \sharp \right)} + \frac{1}{5(p_{(i+1)})} - 1 \leq \left[\int_{\theta_1}^{\theta_2} \frac{dt}{\log t} - 1 \right]$$

$$\leq \frac{\log p_{(i+1)}}{\log \left(\log p_{(i+1)} \sharp \right)} + \frac{1}{p_{(i+1)}} - 1 \quad \text{for all } p \in \mathbb{N} \mid p \geq 13$$

thus completing the proof. $\qquad \square$

The Infimum and Supremum error bounds $\mathcal{ISE}_{\left(p_{(n)} \right)}$ and $\mathcal{SSE}_{\left(p_{(n)} \right)}$ for the tailored integral step estimation error are computationally very demanding. Therefore, Theorems: 5.10 and 5.11 that follow, establish simpler bounds.

Theorem 5.10 (The Step Sequence Estimation Error Lower Bound)**.**
 The estimation error of the tailored logarithmic integral $TLi_{(n)}$ at every step exceeds the value of the inverse of the pertinent prime number hence, it is bounded below by $1/p \ \forall p \in \mathbb{N} \mid p_i \geq 13$:

$$(5.16) \qquad \mathcal{SER}_{\left(p_{(i+1)} \right)} = \left(\int_{\theta_1}^{\theta_2} \frac{dt}{\log t} \right) - 1 > \frac{1}{p_{(i+1)}}$$

where $p_{(i)}$ and $p_{(i+1)}$ are associated with the lower/upper limit of integration θ_1 and θ_2 respectively.

Proof.
 By Lemma 5.8 the sequence $\mathcal{SER}_{\left(p_{(i+1)} \right)}$ is Cauchy and it converges from above to the limit $\mathcal{L} = 0$. The sequence of the reciprocals of prime numbers is clearly Cauchy and converges to the limit $\mathcal{L} = 0$. By Lemma 5.8 we have that:

$$(5.17) \quad \frac{1}{5(p_{(i+1)})} + \frac{\log p_{(i+1)}}{\log \left(\log p_{(i+1)} \sharp \right)} - 1 \leq \int_{\theta_1}^{\theta_2} \frac{dt}{\log t} - 1 \quad \text{for all } p \in \mathbb{N} \mid p \geq 13$$

Consequently Theorem 5.10 is valid if and only if:

$$(5.18) \qquad \frac{1}{p_{(i+1)}} \leq \frac{1}{5(p_{(i+1)})} + \frac{\log p_{(i+1)}}{\log \left(\log p_{(i+1)} \sharp \right)} - 1$$

Now,

$$(5.19) \quad \frac{1}{5(p_{(i+1)})} + \frac{\log p_{(i+1)}}{\log \left(\log p_{(i+1)} \sharp \right)} - 1 - \frac{1}{p_{(i+1)}}$$

$$= \frac{\left(5(p_{(i+1)}) \left(\log \left(p_{(i+1)} \right) \right) \right) - \left(4 + 5(p_{(i+1)}) \right) \left(\log \left(\log \left(p_{(i+1)} \sharp \right) \right) \right)}{\left(5(p_{(i+1)}) \right) \left(\log \left(\log \left(p_{(i+1)} \sharp \right) \right) \right)}$$

From Lemma 4.3 we have that $\log p_{(i+1)} \sharp$ is asymptotic (from below):

$$\log p_{(i+1)} \sharp \sim p_{(i+1)}$$

as well as:

$$\log p_{(i+1)} \sharp < p_{(i+1)} \leq n \quad \forall n \in \mathbb{N} \mid n \geq 2$$

Hence,

$$\log \left(\log p_{(i+1)} \sharp \right) \leq \log p_{(i+1)}$$

From Lemma 4.5 we have for all $p_{(i+1)} \in \mathbb{N} \mid p_{(i+1)} \geq 2$:
(5.20)
$$\mathcal{LB}_{p_{(i+1)}} = \left(\sqrt{5} - 1\right)\left(4\gamma^2 - 2\gamma\right)\left(\log p_{(i+1)}\right)\sqrt[3]{p_{(i+1)}} < \left(p_{(i+1)} - \log p_{(i+1)}\sharp\right)$$

Consequently, from the above we obtain:

(5.21)
$$\left(p_{(i+1)} - \mathcal{LB}_{p_{(i+1)}} > \log p_{(i+1)}\sharp\right)$$

Bearing in mind that for all positive $a, b \in \mathbb{R} \mid a > b$:

(5.22)
$$\log\left(a + b\right) = \log\left(a\left(1 + b/a\right)\right) = \log\left(a\right) + \log\left(1 + \frac{b}{a}\right)$$

Thus, by Lemma 4.3 we have:

(5.23)
$$5p_{(i+1)}\left(\log\left(p_{(i+1)} - \mathcal{LB}_{p_{(i+1)}}\right)\right) \geq 5p_{(i+1)}\left(\log\left(\log\left(p_{(i+1)}\sharp\right)\right)\right)$$

Hence,

(5.24)
$$5p_{(i+1)}\left(\log\left(p_{(i+1)}\right)\right) = 5p_{(i+1)}\left(\log\left(p_{(i+1)} - \mathcal{LB}_{p_{(i+1)}} + \mathcal{LB}_{p_{(i+1)}}\right)\right)$$
$$= 5p_{(i+1)}\left[\log\left(p_{(i+1)} - \mathcal{LB}_{p_{(i+1)}}\right) + \log\left(1 + \frac{\mathcal{LB}_{p_{(i+1)}}}{\left(p_{(i+1)} - \mathcal{LB}_{p_{(i+1)}}\right)}\right)\right]$$
$$\geq 5p_{(i+1)}\left(\log\left(\log\left(p_{(i+1)}\sharp\right)\right)\right)$$

Suppose that the Theorem 5.10 **is false**. Then it must be true that the numerator of equation 5.19 is less than zero. From inequality 5.24 we see that without loss of generality, upon substitution into the numerator of the inequality 5.19, we can drop the common terms obtaining:

(5.25)
$$\left(5(p_{(i+1)})\left(\log\left(p_{(i+1)}\right)\right)\right) - \left(4 + 5(p_{(i+1)})\right)\left(\log\left(\log\left(p_{(i+1)}\sharp\right)\right)\right)$$
$$\geq 5p_{(i+1)}\left[\log\left(1 + \frac{\mathcal{LB}_{p_{(i+1)}}}{\left(p_{(i+1)} - \mathcal{LB}_{p_{(i+1)}}\right)}\right)\right] - 4\left(\log\left(\log\left(p_{(i+1)}\sharp\right)\right)\right) < 0$$

However at $p_{(i+1)} = 37$ the difference 5.25 attains ~ 0.20084385349345676 and diverges. Hence we have a contradiction to our hypothesis which implies that the inequality is true:

(5.26)
$$\frac{1}{p_{(i+1)}} \leq \frac{1}{5(p_{(i+1)})} + \frac{\log p_{(i+1)}}{\log\left(\log p_{(i+1)}\sharp\right)} - 1$$

Consequently this implies that Theorem 5.10 is satisfied for all $p_i \in \mathbb{N} \mid p_i \geq 37$, a simple computer calculation verifies that this inequality also holds within the interval $13 \leq p_i \leq 37$. This necessarily means that Theorem 5.10 is satisfied for all $p_i \in \mathbb{N} \mid p_i \geq 13$, thus completing the proof.   □

Theorem 5.11 (The Step Sequence Estimation Error Upper Bound)**.**

   The inverse of a root of the pertinent prime number at every step exceeds the value of the estimation error of the tailored logarithmic integral $TLi_{(n)}$ step sequence $\forall p_i \in \mathbb{N} \mid p_i \geq 13$:

(5.27)
$$\mathcal{SER}_{\left(p_{(i+1)}\right)} = \left(\int_{\theta_1}^{\theta_2} \frac{dt}{\log t}\right) - 1 < \frac{1}{\sqrt[e]{p_{(i+1)}}} \qquad \text{where } a = \frac{\pi}{2}$$

where $p_{(i)}$ and $p_{(i+1)}$ are associated with the lower/upper limit of integration θ_1 and θ_2 respectively.

Proof.

By Lemma 5.8 the sequence $\mathcal{SER}_{\left(p_{(i+1)}\right)}$ is Cauchy and it converges from above to the limit $\mathcal{L} = 0$. The sequence of the reciprocals of the root of prime numbers is clearly Cauchy and converges to the limit $\mathcal{L} = 0$. By Lemma 5.8 we have that:

$$(5.28) \quad \int_{\theta_1}^{\theta_2} \frac{dt}{\log t} - 1 \leq \frac{1}{p_{(i+1)}} + \frac{\log p_{(i+1)}}{\log\left(\log p_{(i+1)}\sharp\right)} - 1$$

$$\text{for all } p \in \mathbb{N} \mid p \geq 13$$

Consequently Theorem 5.11 is valid if and only if:

$$(5.29) \quad \frac{1}{p_{(i+1)}} + \frac{\log p_{(i+1)}}{\log\left(\log p_{(i+1)}\sharp\right)} - 1 \leq \frac{1}{\sqrt[a]{p_{(i+1)}}} \quad \text{where } a = \frac{\pi}{2}$$

Now,

$$(5.30) \quad \frac{1}{\left(p_{(i+1)}\right)} + \frac{\log p_{(i+1)}}{\log\left(\log p_{(i+1)}\sharp\right)} - 1 - \frac{1}{\sqrt[a]{p_{(i+1)}}} =$$

$$\frac{\left(p_{(i+1)}\sqrt[a]{p_{(i+1)}}\left(\log\left(p_{(i+1)}\right)\right)\right) - \left(p_{(i+1)} - \sqrt[a]{p_{(i+1)}} + p_{(i+1)}\sqrt[a]{p_{(i+1)}}\right)\left(\log\left(\log\left(p_{(i+1)}\sharp\right)\right)\right)}{\left(p_{(i+1)}\sqrt[a]{p_{(i+1)}}\right)\left(\log\left(\log\left(p_{(i+1)}\sharp\right)\right)\right)}$$

From Lemma 4.3 we have that $\log p_{(i+1)}\sharp$ is asymptotic (from below):

$$\log p_{(i+1)}\sharp \sim p_{(i+1)}$$

as well as:

$$\log p_{(i+1)}\sharp < p_{(i+1)} \leq n \quad \forall n \in \mathbb{N} \mid n \geq 2$$

Hence,

$$\log\left(\log p_{(i+1)}\sharp\right) \leq \log p_{(i+1)}$$

From Lemma 4.5 we have for all $p_{(i+1)} \in \mathbb{N} \mid p_{(i+1)} \geq 2$:

$$(5.31) \quad \mathcal{UB}_{p_{(i+1)}} = 2\sqrt{p_{(i+1)}} > \left(p_{(i+1)} - \log p_{(i+1)}\sharp\right)$$

Consequently, from the above we obtain:

$$(5.32) \quad \left(p_{(i+1)} - \mathcal{UB}_{p_{(i+1)}}\right) < \log p_{(i+1)}\sharp$$

Bearing in mind that for all positive $a, b \in \mathbb{R} \mid a > b$:

$$(5.33) \quad \log\left(a + b\right) = \log\left(a\left(1 + b/a\right)\right) = \log\left(a\right) + \log\left(1 + \frac{b}{a}\right)$$

Thus, by Lemma 4.3 we have:

$$(5.34) \quad p_{(i+1)}\sqrt[a]{p_{(i+1)}}\left(\log\left(p_{(i+1)} - \mathcal{UB}_{p_{(i+1)}}\right)\right) \leq p_{(i+1)}\sqrt[a]{p_{(i+1)}}\log\left(\log\left(p_{(i+1)}\sharp\right)\right)$$

Hence,

(5.35)

$$p_{(i+1)} \sqrt[a]{p_{(i+1)}} \left(\log \left(p_{(i+1)} \right) \right) = p_{(i+1)} \sqrt[a]{p_{(i+1)}} \left(\log \left(p_{(i+1)} - \mathcal{UB}_{p_{(i+1)}} + \mathcal{UB}_{p_{(i+1)}} \right) \right)$$

$$= p_{(i+1)} \sqrt[a]{p_{(i+1)}} \left[\log \left(p_{(i+1)} - \mathcal{UB}_{p_{(i+1)}} \right) + \log \left(1 + \frac{\mathcal{UB}_{p_{(i+1)}}}{\left(p_{(i+1)} - \mathcal{UB}_{p_{(i+1)}} \right)} \right) \right]$$

$$\leq p_{(i+1)} \sqrt[a]{p_{(i+1)}} \left(\log \left(\log \left(p_{(i+1)} \sharp \right) \right) \right)$$

Suppose that the Theorem 5.11 is false. Then it must be true that the numerator of equation 5.30 is greater than zero. From inequality 5.35 we see that without loss of generality, upon substitution into the numerator of the inequality 5.30, we can drop the common terms obtaining:

(5.36)

$$\left(p_{(i+1)} \sqrt[a]{p_{(i+1)}} \left(\log \left(p_{(i+1)} \right) \right) \right) - \left(p_{(i+1)} - \sqrt[a]{p_{(i+1)}} + p_{(i+1)} \sqrt[a]{p_{(i+1)}} \right) \left(\log \left(\log \left(p_{(i+1)} \sharp \right) \right) \right)$$

$$\leq p_{(i+1)} \sqrt[a]{p_{(i+1)}} \log \left(1 + \frac{\mathcal{UB}_{p_{(i+1)}}}{\left(p_{(i+1)} - \mathcal{UB}_{p_{(i+1)}} \right)} \right) - \left(p_{(i+1)} - \sqrt[a]{p_{(i+1)}} \right) \left(\log \left(\log \left(p_{(i+1)} \sharp \right) \right) \right)$$

However at $p_{(i+1)} = 197$ the difference 5.36 attains ~ -1.20860443 and diverges. Hence we have a contradiction to our hypothesis which implies that the inequality is true:

(5.37) $$\frac{1}{p_{(i+1)}} + \frac{\log p_{(i+1)}}{\log \left(\log p_{(i+1)} \sharp \right)} - 1 \leq \frac{1}{\sqrt[a]{p_{(i+1)}}} \quad \text{where } a = \frac{\pi}{2}$$

Consequently this implies that Theorem 5.11 is satisfied for all $p_i \in \mathbb{N} \mid p_i \geq 197$, a simple computer calculation verifies that this inequality also holds within the interval $13 \leq p_i \leq 197$. This necessarily means that Theorem 5.11 is satisfied for all $p_i \in \mathbb{N} \mid p_i \geq 13$, thus completing the proof. $\square$

Hence, by Theorems 5.10 and 5.11, for the largest prime number $p_{(i+1)}$ that satisfies the condition $p_{(i+1)} \leq n \in \mathbb{N}$, we have:

$$\frac{1}{p_{(i+1)}} < \left(\int_{\theta_1}^{\theta_2} \frac{dt}{\log t} \right) - 1 < \frac{1}{\sqrt[a]{p_{(i+1)}}} \quad \forall p_{(i)} \geq 13 \quad \text{where } a = \frac{\pi}{2}$$

Remark 5.4.

We need to re-define the lower/upper limits of integration to conform with the summation limits. The computation of the sum of step errors of the integral TLi_n begins at $p_2 = 3$, irrespective of the fact that the computation of the sums pertinent to the bounds (Infimum, Supremum, Lower and Upper) begins first at $p_{15} = 47$.

Definition 5.12 (Theta applicable for summation). $\theta_1 = \log \left(p_{(2+(k-1))} \sharp \right)$

Definition 5.13 (Theta applicable for summation). $\theta_2 = \log \left(p_{(2+k)} \sharp \right)$

Theorem 5.14 ($TLi_{(n)}$ Estimation Error Divergence)**.**

The error arising in the estimation of the prime counting function $\pi_{(n)}$ by the application of the tailored logarithmic integral $TLi_{(n)}$, diverges to infinity:

(5.38)

$$\lim_{n\to\infty}\left(TLi_{(n)}-\pi_{(n)}\right)=\lim_{p_{(n)}\to\infty}\left(\sum_{(k=1)}^{\left(\pi_{(n)}-2\right)}\left\{\left(\int_{\theta_1}^{\theta_2}\frac{dt}{\log t}\right)-1\right\}+\left(\int_2^{\log 6}\frac{dt}{\log t}-\pi_3\right)\right)\to\infty$$

Where the limits of integration θ_1 and θ_2 are given by the Definitions 5.12 and 5.13 respectively. Besides, the prime number $p_{(n)}$ is defined as being the biggest prime $p\leq n$. Furthermore,

(5.39)
$$\left(TLi_{(n)}-\pi_{(n)}\right)>\sum_{p\leq n}\frac{1}{p}\quad\forall\ n\in\mathbb{N}\mid n\geq 983$$

(5.40)
$$TLi_{(n)}=\int_2^{\theta_{(n)}}\frac{dt}{\log t}>\pi_{(n)}\quad\forall n\in\mathbb{N}\mid n\geq 43$$

Proof.

By Lemma 5.8 for all $p_i\in\mathbb{N}\mid p_i\geq 13$ the relation holds at every step:

(5.41)
$$\left\{\left(\int_{\theta_1}^{\theta_2}\frac{dt}{\log t}\right)-1\right\}>\frac{1}{p_{(i+1)}}$$

Because the sum of reciprocals of successive prime numbers diverges, consequently, the sum:

(5.42)
$$\lim_{p_{(n)}\to\infty}\left(\sum_{(k=1)}^{\left(\pi_{(n)}-2\right)}\left\{\left(\int_{\theta_1}^{\theta_2}\frac{dt}{\log t}\right)-1\right\}\right)\to\infty$$

must necessarily diverge, by comparison with the divergent sum of reciprocals of successive prime numbers. The complete estimation error of the tailored integral is given by:

(5.43)
$$\left(TLi_{(n)}-\pi_{(n)}\right)=\left(\sum_{(k=1)}^{\left(\pi_{(n)}-2\right)}\left\{\left(\int_{\theta_1}^{\theta_2}\frac{dt}{\log t}\right)-1\right\}+\left(\int_2^{\log 6}\frac{dt}{\log t}-\pi_3\right)\right)$$

Consequently therefore, by the divergence of the sum 5.42, the estimation error of the tailored integral 5.43 must necessarily diverge:

$$\lim_{n\to\infty}\left(TLi_{(n)}-\pi_{(n)}\right)=\lim_{p_{(n)}\to\infty}\left(\sum_{(k=1)}^{\left(\pi_{(n)}-2\right)}\left\{\left(\int_{\theta_1}^{\theta_2}\frac{dt}{\log t}\right)-1\right\}+\left(\int_2^{\theta_3}\frac{dt}{\log t}-\pi_3\right)\right)\to\infty$$

In fact the sum of reciprocals of successive prime numbers and 5.43 intersect. Direct calculation at $n=983$ shows that the difference:

$$\left(TLi_{(n)}-\pi_{(n)}\right)-\sum_{p\leq n}\frac{1}{p}\approx 0.0004199365947656908$$

Because by Lemma 5.8 the inequality 5.41 holds for all $p_i \in \mathbb{N} \mid p_i \geq 13$, this implies that:

$$\left(TLi_{(n)} - \pi_{(n)}\right) > \sum_{p \leq n} \frac{1}{p} \quad \forall \, n \in \mathbb{N} \mid n \geq 983$$

Direct computation verifies that the estimation error of $TLi_{(n)}$:

$$\left(TLi_{(n)} - \pi_{(n)}\right) > 0 \quad \text{for } n \in \mathbb{N} \mid 43 < n \geq 983$$

At $n = 43$ the estimation error of $TLi_{(n)}$ attains the value of 0.002993180461560385. Therefore, by the divergence of the tailored integral estimation error, it must be true that:

$$TLi_{(n)} = \int_2^{\theta_{(n)}} \frac{dt}{\log t} > \pi_{(n)} \quad \forall n \in \mathbb{N} \mid n \geq 43$$

This concludes the proof of Theorem 5.14. $\qquad\qquad\qquad\qquad\qquad\qquad$ □

5.5. **Supremum and Infimum estimation error bounds on** $TLi_{(n)} - \pi_{(n)}$.

Remark 5.5.

The estimation error of the $TLi_{(n)}$ increases stepwise at the primes. Since however the magnitude of the increase is small, as a result the curve presents itself as a rising virtually smooth slope. It absolutely lacks the large amplitude variation, which is the intrinsic characteristic of the Gauss' offset $Li_{(n)}$. Please refer to Fig. 2. The Infimum and Supremum estimation error bounds, in conjunction with the tailored logarithmic integral, give us both the most accurate estimate of $\pi_{(n)}$ and the best estimation error bounds. The drawback is, that the formulae are computationally quite demanding.

Theorem 5.15 (Infimum Estimation Error Bound)**.**

The error of estimation of the prime counting function $\pi_{(n)}$ by the application of the tailored logarithmic integral $TLi_{(n)}$, for all $n \in \mathbb{N} \mid n \geq 47$ is bounded below by a divergent sum:

(5.44)

$$\mathcal{LB}_{(n)} = \sum_{k=1}^{\left(\pi_{(n)}-14\right)} \frac{1}{p_{(14+k)}} < \mathcal{INF}_{(n)} = \sum_{k=1}^{\left(\pi_{(n)}-14\right)} \left\{ \frac{\log p_{(14+k)}}{\log\left(\log p_{(14+k)}\sharp\right)} + \frac{1}{5(p_{(14+k)})} - 1 \right\}$$

$$< \left(\sum_{(k=1)}^{\left(\pi_{(n)}-2\right)} \left\{ \int_{\theta_1}^{\theta_2} \frac{dt}{\log t} - 1 \right\} + \left(\int_2^{\log 6} \frac{dt}{\log t} - \pi_3 \right) \right) = \left(TLi_{(n)} - \pi_{(n)}\right)$$

Further, the limits of integration θ_1, θ_2 implement Definition 5.12 and 5.13 respectively, while $p_{(n)}$ is the greatest prime $p \in \mathbb{N} \mid p \leq n$.

Proof.

Let's consider both $\mathcal{LB}_{(n)}$ and $\mathcal{INF}_{(n)}$ stepwise first. Evidently, both of them are Cauchy, convergent to zero sequences, while the lower bound function $\mathcal{LB}_{(n)}$ is strictly monotonic. Further, by Theorem 5.10 we have that:

$$\frac{1}{p_{(i+1)}} \leq \frac{1}{5(p_{(i+1)})} + \frac{\log p_{(i+1)}}{\log\left(\log p_{(i+1)}\sharp\right)} - 1$$

The relation holds stepwise at every step for all $p_{(n)} \in \mathbb{N} \mid p_{(n)} \geq 13$. Adding simultaneously the terms of both sequences and forming two respective sums, does not invalidate the relation. Considering in turn stepwise the functions $\mathcal{INF}_{(n)}$ and the step sequence $TLi_{(n)}$, evidently both are Cauchy, convergent to zero sequences. Further by Lemma 5.8 and Corollary 5.9 we have that at every step:

$$(5.45) \quad \frac{\log p_{(i+1)}}{\log \left(\log p_{(i+1)}\sharp\right)} + \frac{1}{5(p_{(i+1)})} - 1 \leq \left[\int_{\theta_1}^{\theta_2} \frac{dt}{\log t} - 1 \right] \quad \text{for all } p \in \mathbb{N} \mid p \geq 13$$

The inclusion of the additional, first step term:

$$(5.46) \quad \int_2^{\log 6} \frac{dt}{\log t} - \pi_3 = \int_2^{\log 6} \frac{dt}{\log t} - 2 \approx -2.3266013098834977$$

has clearly the effect of shifting the curve of the $\left(TLi_{(n)} - \pi_n\right)$ significantly down, thereby upsetting the stepwise inter-relationships. However, by Theorem 5.14 we have that:

$$(5.47) \quad \left(TLi_{(n)} - \pi_{(n)}\right) =$$
$$\left(\sum_{(k=1)}^{\left(\pi_{(n)}-2\right)} \left\{ \int_{\theta_1}^{\theta_2} \frac{dt}{\log t} - 1 \right\} + \left(\int_2^{\log 6} \frac{dt}{\log t} - \pi_3 \right) \right) = \left(\int_2^{\theta_{(n)}} \frac{dt}{\log t} \right) - \pi_{(n)} > 0$$
$$\forall n \in \mathbb{N} \mid n \geq 43$$

with $\theta_{(n)}$ given by the Definition 5.1. Thus, at $p_n = 43$ the tailored integral concludes the stage of recovery instigated by the addition of the initial term, vide equation 5.46 above. The difference:

$$(5.48) \quad \left(\sum_{k=1}^{\left(\pi_{(n)}-14\right)} \frac{1}{p_{(14+k)}} \right) - \left(\sum_{k=1}^{\left(\pi_{(n)}-14\right)} \left\{ \frac{\log p_{(14+k)}}{\log \left(\log p_{(14+k)}\sharp\right)} + \frac{1}{5(p_{(14+k)})} - 1 \right\} \right)$$

at $p_{15} = 47$ attains ~ -0.0200272 and diverges decisively. Consequently, by Theorem 5.10 it must be true that the relation 5.48 holds for all $p_{(n)} \in \mathbb{N} \mid p_{(n)} \geq 47$. The difference:

$$(5.49) \quad \left(\sum_{k=1}^{\left(\pi_{(n)}-14\right)} \left\{ \frac{\log p_{(14+k)}}{\log \left(\log p_{(14+k)}\sharp\right)} + \frac{1}{5(p_{(14+k)})} - 1 \right\} \right) - \left(TLi_{(n)} - \pi_{(n)}\right)$$

at $p_{15} = 47$ the difference attains ~ -0.0125411 and further diverges. Since the relation 5.45 holds stepwise for all $p_{(n)} \in \mathbb{N} \mid p_{(n)} \geq 13$, therefore, necessarily it must be true that the relation 5.49 holds for all $p_{(n)} \in \mathbb{N} \mid p_{(n)} \geq 47$. Consequently, for all $p_{(n)} \in \mathbb{N} \mid p_{(n)} \geq 47$ we obtain:

$$(5.50) \quad \sum_{k=1}^{\left(\pi_{(n)}-14\right)} \frac{1}{p_{(14+k)}} < \sum_{k=1}^{\left(\pi_{(n)}-14\right)} \left\{ \frac{\log p_{(14+k)}}{\log \left(\log p_{(14+k)}\sharp\right)} + \frac{1}{5(p_{(14+k)})} - 1 \right\}$$
$$< \left(\sum_{(k=1)}^{\left(\pi_{(n)}-2\right)} \left\{ \int_{\theta_1}^{\theta_2} \frac{dt}{\log t} - 1 \right\} + \left(\int_2^{\log 6} \frac{dt}{\log t} - \pi_3 \right) \right) = \left(\int_2^{\theta_{(n)}} \frac{dt}{\log t} \right) - \pi_{(n)}$$

This concludes the proof of Theorem 5.15.   □

Remark 5.6.

Inclusion of the additional, initial term:

$$\int_2^{\log 6} \frac{dt}{\log t} - \pi_3 = \int_2^{\log 6} \frac{dt}{\log t} - 2 \approx -2.3266013098834977$$

to complete the domain of integration, when summing the step terms of the TLi_n up to some predetermined $n \in \mathbb{N}$, has a drawback. It upsets the established stepwise balance with all its bounds. To resolve the issue decisively with the bounds, we have to drop 14 of the initial terms of the sum, for each bound. This way, the balance in their inter-relationships is restored, as $\big(TLi_{(n)} - \pi_{(n)}\big) > 0 \quad \forall n \in \mathbb{N} \mid n \geq 43$.

Theorem 5.16 (Supremum Estimation Error Bound)**.**

The error of estimation of the prime counting function $\pi_{(n)}$ by the application of the tailored logarithmic integral $TLi_{(n)}$, for all $n \in \mathbb{N} \mid n \geq 47$ is bounded above by a divergent sum:

$$(5.51) \quad TLi_{(n)} - \pi_{(n)} < \mathcal{SUPR}_{(n)}$$

$$= \sum_{(k=1)}^{\big(\pi_{(n)} - 14\big)} \left\{ \frac{\log p_{(14+k)}}{\log \big(\log p_{(14+k)}\sharp\big)} + \frac{1}{p_{(14+k)}} - 1 \right\} < \sum_{(k=1)}^{\big(\pi_{(n)} - 14\big)} \frac{1}{\sqrt[a]{p_{(14+k)}}} = \mathcal{UB}_{(n)}$$

where $a = \frac{\pi}{2}$, while $p_{(n)}$ is the biggest prime number $p \leq n$.

Proof.

Let's consider both $\mathcal{UB}_{(n)}$ and $\mathcal{SUPR}_{(n)}$ stepwise first. Evidently, both of them are Cauchy, convergent to zero sequences, while the upper bound function $\mathcal{UB}_{(n)}$ is strictly monotonic. Further, by Theorem 5.11 we have that:

$$\frac{1}{\sqrt[a]{p_{(i+1)}}} \geq \frac{1}{(p_{(i+1)})} + \frac{\log p_{(i+1)}}{\log \big(\log p_{(i+1)}\sharp\big)} - 1 \quad where \ a = \frac{\pi}{2}$$

the relation holds stepwise at every step for all $p_{(n)} \in \mathbb{N} \mid p_{(n)} \geq 13$. Adding simultaneously the terms of both sequences and forming two respective sums, does not invalidate the relation. Considering in turn stepwise the functions $\mathcal{INF}_{(n)}$ and the step sequence $TLi_{(n)}$, evidently both are Cauchy, convergent to zero sequences. Further by Lemma 5.8 and Corollary 5.9 we have that at every step:

$$(5.52) \quad \frac{\log p_{(i+1)}}{\log \big(\log p_{(i+1)}\sharp\big)} + \frac{1}{(p_{(i+1)})} - 1 \geq \left[\int_{\theta_1}^{\theta_2} \frac{dt}{\log t} - 1 \right] \quad \text{for all } p \in \mathbb{N} \mid p \geq 13$$

The inclusion of the additional, first step term:

$$(5.53) \quad \int_2^{\log 6} \frac{dt}{\log t} - \pi_3 = \int_2^{\log 6} \frac{dt}{\log t} - 2 \approx -2.3266013098834977$$

has clearly the effect of shifting the curve of the $\big(TLi_{(n)} - \pi_n\big)$ significantly down, thereby upsetting the stepwise inter-relationships. However, by Theorem 5.14 we

have that:

$$(5.54) \quad \left(TLi_{(n)} - \pi_{(n)}\right) =$$

$$\left(\sum_{(k=1)}^{(\pi_{(n)}-2)}\left\{\int_{\theta_1}^{\theta_2}\frac{dt}{\log t}-1\right\}+\left(\int_2^{\log 6}\frac{dt}{\log t}-\pi_3\right)\right) = \left(\int_2^{\theta_{(n)}}\frac{dt}{\log t}\right)-\pi_{(n)} > 0$$

$$\forall n \in \mathbb{N} \mid n \geq 43$$

with $\theta_{(n)}$ given by the Definition 5.1. Thus, at $p_n = 43$ the tailored integral concludes the stage of recovery instigated by the addition of the initial term, vide equation 5.53 above. The difference:

$$(5.55) \quad \left(\sum_{k=1}^{(\pi_{(n)}-14)}\frac{1}{\sqrt[6]{p_{(14+k)}}}\right) - \left(\sum_{k=1}^{(\pi_{(n)}-14)}\left\{\frac{\log p_{(14+k)}}{\log\left(\log p_{(14+k)}\sharp\right)}+\frac{1}{(p_{(14+k)})}-1\right\}\right)$$

at $p_{15} = 47$ attains ~ 0.0278753 and diverges decisively. Consequently, by Theorem 5.11 it must be true that the relation 5.55 holds for all $p_{(n)} \in \mathbb{N} \mid p_{(n)} \geq 47$. The difference:

$$(5.56) \quad \left(\sum_{k=1}^{(\pi_{(n)}-14)}\left\{\frac{\log p_{(14+k)}}{\log\left(\log p_{(14+k)}\sharp\right)}+\frac{1}{(p_{(14+k)})}-1\right\}\right) - \left(TLi_{(n)} - \pi_{(n)}\right)$$

at $p_{15} = 47$ the difference attains ~ 0.00448015 and further diverges. Since the relation 5.52 holds stepwise for all $p_{(n)} \in \mathbb{N} \mid p_{(n)} \geq 13$, therefore, necessarily it must be true that the relation 5.56 holds for all $p_{(n)} \in \mathbb{N} \mid p_{(n)} \geq 47$. Consequently, for all $p_{(n)} \in \mathbb{N} \mid p_{(n)} \geq 47$ we obtain:

$$(5.57) \quad \sum_{k=1}^{(\pi_{(n)}-14)}\frac{1}{\sqrt[a]{p_{(14+k)}}} > \sum_{k=1}^{(\pi_{(n)}-14)}\left\{\frac{\log p_{(14+k)}}{\log\left(\log p_{(14+k)}\sharp\right)}+\frac{1}{(p_{(14+k)})}-1\right\}$$

$$> \left(\sum_{(k=1)}^{(\pi_{(n)}-2)}\left\{\int_{\theta_1}^{\theta_2}\frac{dt}{\log t}-1\right\}+\left(\int_2^{\log 6}\frac{dt}{\log t}-\pi_3\right)\right) = \left(\int_2^{\theta_{(n)}}\frac{dt}{\log t}\right)-\pi_{(n)}$$

Thus, concluding the proof.   □

Therefore on the basis of Theorems 5.15 and 5.16, the relation holds for all prime numbers $p \in \mathbb{N} \mid p \geq 47$:

$$(5.58) \quad \sum_{k=1}^{(\pi_{(n)}-14)}\frac{1}{p_{(14+k)}} < \sum_{k=1}^{(\pi_{(n)}-14)}\left\{\frac{\log p_{(14+k)}}{\log\left(\log p_{(14+k)}\sharp\right)}+\frac{1}{5(p_{(14+k)})}-1\right\}$$

$$< \left(\sum_{(k=1)}^{(\pi_{(n)}-2)}\left\{\int_{\theta_1}^{\theta_2}\frac{dt}{\log t}-1\right\}+\left(\int_2^{\log 6}\frac{dt}{\log t}-\pi_3\right)\right) = \left(\int_2^{\theta_{(n)}}\frac{dt}{\log t}\right)-\pi_{(n)}$$

$$< \sum_{k=1}^{(\pi_{(n)}-14)}\left\{\frac{\log p_{(14+k)}}{\log\left(\log p_{(14+k)}\sharp\right)}+\frac{1}{(p_{(14+k)})}-1\right\} < \sum_{k=1}^{(\pi_{(n)}-14)}\frac{1}{\sqrt[a]{p_{(14+k)}}}$$

$$where\ a = \frac{\pi}{2}$$

The tailored logarithmic integral yields the best estimate of $\pi_{(n)}$, while the Infimum and Supremum estimation error bounds give us the tightest bounds.

5.6. The estimate of the $TLi_{(n)}$.

The calculation of the integral $TLi_{(n)}$ may become computationally quite demanding for larger values of $n \in \mathbb{N}$. This attribute of computation of the tailored integral $TLi_{(n)}$ pertains to the sequential, exhaustive process of calculation of the value of $\log p_{(n)}\sharp$, which has exponential time complexity. This section therefore, presents a method to obtain the approximate value of the integral $TLi_{(n)}$ in an efficient manner.

Definition 5.17 ($TLi_{(n)}$ estimate).

$$EstTLi_{(n)} = \int_2^{est} \frac{dt}{\log t} \quad \forall n \in \mathbb{N} \mid n \geq 3$$
$$\text{where } est = \left(p_{(n)} - \left(\sqrt{5} - 1 \right) \left(4\gamma^2 - 2\gamma \right) \left(\log p_{(n)} \right) \sqrt[3]{p_{(n)}} \right)$$

Definition 5.18 ($TLi_{(n)}$ sub-estimate).

$$SEstTLi_{(n)} = \int_2^{Sest} \frac{dt}{\log t} \quad \forall n \in \mathbb{N} \mid n \geq 11 \quad \text{where } Sest = \left(p_{(n)} - 2\sqrt{p_{(n)}} \right)$$

By Lemmas 4.5 and 4.7 we have:

(5.59)
$$\left(p_{(n)} - 2\sqrt{p_{(n)}} \right) < \log p_{(n)}\sharp < \left(p_{(n)} - \left(\sqrt{5} - 1 \right) \left(4\gamma^2 - 2\gamma \right) \left(\log p_{(n)} \right) \sqrt[3]{p_{(n)}} \right)$$
$$\text{where } p_n \text{ is the greatest prime number } p \in \mathbb{N} \mid p \leq n$$

At the cost of an increased estimation error we may estimate the true value of the tailored integral quite easily.

Theorem 5.19 (Divergence Of The Difference $Li_{(n)}$ - $EstTLi_{(n)}$).

The difference in values taken by the logarithmic integral $Li_{(n)}$ and the estimate of the tailored logarithmic integral $EstTLi_{(n)}$ given by the definition 5.17, for all $n \in \mathbb{N} \mid n \geq 3$, increases without a bound as n tends to infinity:

(5.60)
$$\lim_{n \to \infty} \left(Li_{(n)} - EstTLi_{(n)} \right) = \lim_{n \to \infty} \left(\int_{est}^n \frac{dt}{\log t} \right) \to \infty$$

Consequently, the value of the $EstTLi_{(n)}$ will always remain less than the value obtained by the Gauss' $Li_{(n)}$:

(5.61)
$$Li_{(n)} = \int_2^n \frac{dt}{\log t} > EstTLi_{(n)} = \int_2^{est} \frac{dt}{\log t} \quad \forall n \in \mathbb{N} \mid n \geq 3$$

Proof.
Clearly,

(5.62) $\left(p_{(n)} - \left(\sqrt{5} - 1 \right) \left(4\gamma^2 - 2\gamma \right) \left(\log p_{(n)} \right) \sqrt[3]{p_{(n)}} \right) < p_{(n)} \leq n \quad \forall n \in \mathbb{N} \mid n \geq 3$

where $p_{(n)}$ is the largest prime $p \leq n$. Since the limit diverges:

$$(5.63) \qquad \lim_{n \to \infty} \left(\left(\sqrt{5} - 1 \right) \left(4\gamma^2 - 2\gamma \right) \left(\log p_{(n)} \right) \sqrt[3]{p_{(n)}} \right) \to \infty$$

the difference between the pertinent intervals of computation of $Li_{(n)}$ and $EstTLi_{(n)}$ increases unboundedly. Due to the fact that,

$$\left(\sqrt{5} - 1 \right) \left(4\gamma^2 - 2\gamma \right) \left(\log n \right) > 1 \quad \forall n \in \mathbb{N} \mid n \geq 94$$

the difference between the intervals of computation increases at a rate proportional to $k \left(\log p_{(n)} \right) \sqrt[3]{p_{(n)}}$ where $k \approx 0.220367$ for all $n \in \mathbb{N} \mid n \geq 94$. The exact difference between the intervals of computation of $Li_{(n)}$ and $EstTLi_{(n)}$ is given by:

$$(5.64) \qquad (n - est) = \left(\sqrt{5} - 1 \right) \left(4\gamma^2 - 2\gamma \right) \left(\log p_{(n)} \right) \sqrt[3]{p_{(n)}} + \left(n - p_{(n)} \right)$$

However, because the difference $\left(n - p_{(n)} \right)$ is bounded above by the maximal prime gaps Supremum given by Theorem 3.6, its contribution for large n is negligible. Therefore, we may drop the difference $\left(n - p_{(n)} \right)$ and by an application of the PNT we obtain an estimate of the true value of the minimum difference between the two integrals:

$$(5.65)$$
$$\int_{est}^{n} \frac{dt}{\log t} \gtrapprox \frac{\left(\sqrt{5} - 1 \right) \left(4\gamma^2 - 2\gamma \right) \left(\log p_{(n)} \right) \sqrt[3]{p_{(n)}}}{\log p_{(n)}} = \left(\sqrt{5} - 1 \right) \left(4\gamma^2 - 2\gamma \right) \sqrt[3]{p_{(n)}}$$

The estimate 5.65 clearly increases monotonically without a bound. Consequently,

$$\lim_{n \to \infty} \left(\int_{est}^{n} \frac{dt}{\log t} \right) = \lim_{n \to \infty} \left(\frac{(n - est)}{\log n} \right) \to \infty$$

At $n = 3$ the difference:

$$\left(Li_{(n)} - EstTLi_{(n)} \right) = \int_{est}^{n} \frac{dt}{\log t} \approx 0.3367757764496686$$

and increases as n tends to infinity. This implies,

$$Li_{(n)} = \int_{2}^{n} \frac{dt}{\log t} > EstTLi_{(n)} = \int_{2}^{est} \frac{dt}{\log t} \quad \forall n \in \mathbb{N} \mid n \geq 3$$

Thus completing the proof. $\square$

Remark 5.7. Due to the uneven distribution of primes, both the difference $n - p_{(n)}$ and $p_{(n)} - \log p_{(n)} \sharp$ are inherently highly oscillatory. Approximating $\log p_{(n)} \sharp$ by the difference:

$$(5.66) \qquad p_{(n)} - \left(\sqrt{5} - 1 \right) \left(4\gamma^2 - 2\gamma \right) \left(\log p_{(n)} \right) \sqrt[3]{p_{(n)}}$$

as the upper integration limit for the $EstTLi_{(n)}$, incorporates this effect into the estimation error of the $EstTLi_{(n)}$. Another consequence of the application of the estimates of $\log p_{(n)} \sharp$ instead of the exact values, is that $EstTLi_{(n)}$ also exhibits the tendency to follow the Gauss' $Li_{(n)}$. This is clearly visible over the intervals where both $\pi_{(n)}$ and $TLi_{(n)}$ tend to "sag", the estimate $EstTLi_{(n)}$ keeps on going relatively unaffected. Please refer to the graph 14 in the Appendix. Because $EstTLi_{(n)}$ uses the value of the greatest prime $p \in \mathbb{N} \mid p \leq n$ in the calculation of the upper limit of integration, as a result its graph continues in a straight level line across every prime gap.
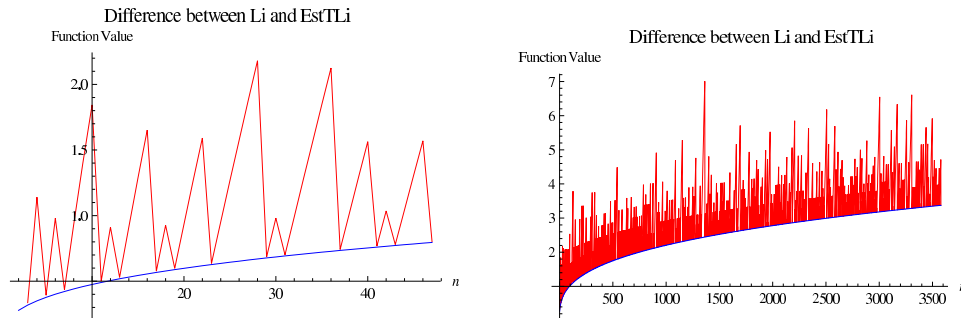
FIGURE 5. The drawings show the graph of the true difference $(Li_{(n)} - EstTLi_{(n)})$ (red), and the estimate of this value given by 5.67 (blue). The figures are drawn with respect to $n \in \mathbb{N}$ at every n within the respective range.
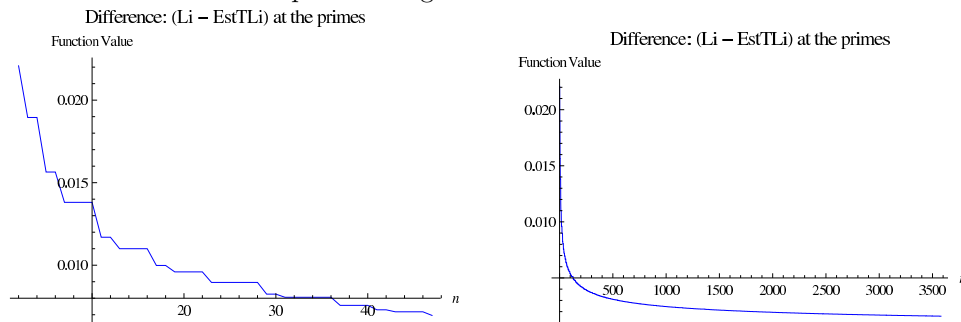
FIGURE 6. The drawings show the graph of the difference of $\left(Li_{(n)} - EstTLi_{(n)}\right)$ and the estimate 5.67, computed at the primes only. The figures are drawn with respect to $n \in \mathbb{N}$ at every n within the respective range.

The omission of the term $n - p_{(n)}$ from the estimate of the difference of $Li_{(n)}$ and $EstTLi_{(n)}$:

$$(5.67) \qquad \int_{est}^{n} \frac{dt}{\log t} \gtrapprox \left(\sqrt{5} - 1\right)\left(4\gamma^2 - 2\gamma\right) \sqrt[3]{p_{(n)}}$$

caused the results to be accurate at the primes only; since $n = p_{(n)}$ at such point. In the intermittent space, the true value of $Li_{(n)} - EstTLi_{(n)}$ increases.

The greater the gap between the primes, the greater the difference between the estimate 5.67 and the true value of the difference $Li_{(n)} - EstTLi_{(n)}$. Consequently, for a given prime gap, we have that the locally biggest difference occurs at every $n \in \mathbb{N} \mid n = p_{(n)} - 1$.

However, by Theorem 3.6, the difference $n - p_{(n)}$ is bound to be less than the maximal gaps Supremum:

$$\frac{n - p_{(n)}}{\log p_{(n)}} < \frac{\mathcal{G}_{(p_{(n)})}}{\log p_{(n)}} = \frac{5}{\log 10}\left(\log_{10} p_{(n)}\right) \approx 2.17147\left(\log_{10} p_{(n)}\right)$$

hence the relative contribution of the difference $n - p_{(n)}$ decreases as n increases, consequently, gradually losing significance. In fact at 10^{15} the ratio:

$$\frac{\frac{5}{\log 10} \left(\log_{10} n \right)}{\left(\left(\sqrt{5} - 1 \right) \left(4\gamma^2 - 2\gamma \right) \sqrt[3]{n} \right)} \approx 0.00147809$$

thus, the error made in estimation by omission of the term $(n - est)$, is less than 0.15 percent at that point.

Theorem 5.20 (Infimum Of The Difference $Li_{(p)} - EstTLi_{(p)}$)**.**

In an instance when n ***is a prime number***, *the Infimum bound on the difference* $Li_{(p)} - EstTLi_{(p)}$ *computed at the primes* $p \in \mathbb{N} \mid p \geq 3$*, is given by:*

$$(5.68) \qquad \mathcal{LTI}_{(p)} = \left(\sqrt{5} - 1 \right) \left(4\gamma^2 - 2\gamma \right) \sqrt[3]{p} \leq Li_{(p)} - EstTLi_{(p)}$$

In this case, the upper estimation error bound, for all $p \in \mathbb{N} \mid p \geq 3$ *is given by,*

$$(5.69) \quad \left[Li_{\left(p_{(n)}\right)} - EstTLi_{\left(p_{(n)}\right)} - \mathcal{LTI}_{\left(p_{(n)}\right)} \right] \leq \frac{1}{(\log(10)(\log(p_n)))^2} \equiv$$

$$\equiv \left[Li_{\left(p_{(n)}\right)} - EstTLi_{\left(p_{(n)}\right)} \right] \leq \frac{1}{(\log(10)(\log(p_n)))^2} + \mathcal{LTI}_{\left(p_{(n)}\right)}$$

Similarly, the lower estimation error bound, for all $p \in \mathbb{N} \mid p \geq 263$ *is given by:*

$$(5.70) \quad \frac{1}{p_{(n)}} \leq \left[Li_{\left(p_{(n)}\right)} - EstTLi_{\left(p_{(n)}\right)} - \mathcal{LTI}_{\left(p_{(n)}\right)} \right] \equiv$$

$$\equiv \frac{1}{p_{(n)}} + \mathcal{LTI}_{\left(p_{(n)}\right)} \leq \left[Li_{\left(p_{(n)}\right)} - EstTLi_{\left(p_{(n)}\right)} \right]$$

Proof.

From the Definition 5.17, the length of the interval separating the $Li_{(n)}$ and $EstTLi_{(n)}$ is given by:

$$(n - est) = \left(\sqrt{5} - 1 \right) \left(4\gamma^2 - 2\gamma \right) \left(\log p_{(n)} \right) \sqrt[3]{p_{(n)}} + \left(n - p_{(n)} \right)$$

where $p_{(n)}$ is the biggest prime $p \leq n$. When n is a prime however, $\left(n - p_{(n)} \right) = 0$. By the application of the PNT therefore, we may approximate the true value of the difference $Li_{\left(p_{(n)}\right)} - EstTLi_{\left(p_{(n)}\right)}$ at the primes within the range:

$$\int_{est}^{p_{(n)}} \frac{dt}{\log t} \sim \frac{\left(\sqrt{5} - 1 \right) \left(4\gamma^2 - 2\gamma \right) \left(\log p_{(n)} \right) \sqrt[3]{p_{(n)}}}{\log p_{(n)}}$$

$$= \left(\sqrt{5} - 1 \right) \left(4\gamma^2 - 2\gamma \right) \sqrt[3]{p_{(n)}} = \mathcal{LTI}_{\left(p_{(n)}\right)}$$

$\mathcal{LTI}_{\left(p_{(n)}\right)}$ is a positive, monotone, increasing without a bound function. By Theorem 5.19, for all $n \in \mathbb{N} \mid n \geq 3$ the difference $Li_{(n)} - EstTLi_{(n)} > 0$ and diverges. The Gauss' offset integral $Li_{(n)}$ clearly increases monotonically, as well as the estimate of the tailored integral $EstTLi_{(n)}$. Suppose that Theorem 5.20 is false, thus, in accordance with the hypothesis the difference:

$$(5.71) \qquad\qquad Li_{\left(p_{(n)}\right)} - EstTLi_{\left(p_{(n)}\right)} - \mathcal{LTI}_{\left(p_{(n)}\right)} < 0$$

However, the inequality 5.71 at $p_n = 3$ attains ~ 0.0189521 and asymptotically tends to zero as p_n increases unboundedly. Therefore, it is a positive decreasing

sequence of real numbers. We implement therefore the Second Ratio Test. Define the test sequence:

$$(5.72) \qquad a_{\left(p_{(n)}\right)} = \left\{ Li_{\left(p_{(n)}\right)} - EstTLi_{\left(p_{(n)}\right)} - \mathcal{LTI}_{\left(p_{(n)}\right)} \right\}$$

the Second Ratio Test, given by $\max\left\{ \dfrac{a_{(2p_n)}}{a_{p_n}}, \dfrac{a_{((2p_n)+1)}}{a_{p_n}} \right\}$:

$$\lim_{p_n \to \infty} \frac{a_{2p_n}}{a_{p_n}} \sim \frac{\pi}{4}$$

Thus, the test sequence converges from below asymptotically to $\sim \frac{\pi}{4} > \frac{1}{2}$. Therefore, by the definition of the Second Ratio Test we conclude that, the series formed by summing the terms of 5.72 diverges, as p_n increases unboundedly. Consequently, we have a contradiction to the initial hypothesis. Therefore, for all $p \in \mathbb{N} \mid p \geq 3$ the relation is valid:

$$(5.73) \qquad \mathcal{LTI}_{\left(p_{(n)}\right)} \leq Li_{\left(p_{(n)}\right)} - EstTLi_{\left(p_{(n)}\right)}$$

Suppose now that $\forall n \in \mathbb{N} \mid n \geq 347$ the following inequality is false:

$$(5.74) \qquad \left[Li_{\left(p_{(n)}\right)} - EstTLi_{\left(p_{(n)}\right)} - \mathcal{LTI}_{\left(p_{(n)}\right)} \right]^{-1} - p_{(n)} < 0$$

However, the relation at $p_n = 347$ attains ~ -60.9818 and diverges with a rate of divergence $\propto k\, p_n \mid for\ k \sim 1$ for larger p_n. Hence, we have a contradiction. Therefore, define a positive valued test sequence:

$$(5.75) \qquad a_{(n)} = \left\{ p_{(n)} - \left[Li_{\left(p_{(n)}\right)} - EstTLi_{\left(p_{(n)}\right)} - \mathcal{LTI}_{\left(p_{(n)}\right)} \right]^{-1} \right\}$$

Subsequently, applying the Cauchy's Root Test:

$$\lim_{n \to \infty} \sqrt[n]{a_n} =$$

$$\lim_{n \to \infty} \sqrt[n]{ \left(p_{(n)} - \left[Li_{\left(p_{(n)}\right)} - EstTLi_{\left(p_{(n)}\right)} - \mathcal{LTI}_{\left(p_{(n)}\right)} \right]^{-1} \right) } \sim 1.06138$$

The Root Test, at $p_n = 347$ attains ~ 1.06138 and converges asymptotically, strictly from above to 1. Consequently, by the definition of the Cauchy's Root Test, the series formed from the terms of the sequence 5.75 diverges. Therefore, we have a contradiction to the initial hypothesis. Hence, $\forall n \in \mathbb{N} \mid n \geq 347$ the inequality is valid:

$$(5.76) \qquad \left[Li_{\left(p_{(n)}\right)} - EstTLi_{\left(p_{(n)}\right)} - \mathcal{LTI}_{\left(p_{(n)}\right)} \right]^{-1} < p_{(n)}$$

Suppose now in turn, that $\forall n \in \mathbb{N} \mid n \geq 3$ the following inequality is false:

$$(5.77) \qquad \left[Li_{\left(p_{(n)}\right)} - EstTLi_{\left(p_{(n)}\right)} - \mathcal{LTI}_{\left(p_{(n)}\right)} \right]^{-1} - \left(\log(10)\left(\log(p_n)\right) \right)^2 > 0$$

However, inequality 5.77, at $p_n = 3$ attains ~ 46.3656 and diverges. Hence, we have a contradiction. Therefore, define a positive valued test sequence:

$$(5.78) \quad a_{(n)} = \left\{ \left[Li_{\left(p_{(n)}\right)} - EstTLi_{\left(p_{(n)}\right)} - \mathcal{LTI}_{\left(p_{(n)}\right)} \right]^{-1} - \left(\log(10)\left(\log(p_n)\right) \right)^2 \right\}$$

Applying the Cauchy's Root Test:

$$\lim_{n \to \infty} \sqrt[n]{a_n} =$$

$$\lim_{n \to \infty} \sqrt[n]{\left\{ \left[Li_{\left(p_{(n)}\right)} - EstTLi_{\left(p_{(n)}\right)} - \mathcal{LTI}_{\left(p_{(n)}\right)} \right]^{-1} - (\log(10)(\log(p_n)))^2 \right\}} \to 1$$

The Root Test of the sequence 5.78, at $p_n = 3$ attains ~ 6.80923 and converges asymptotically, strictly from above to 1. Consequently, by the definition of the Cauchy's Root Test, the series formed from the terms of the sequence 5.78 diverges. Therefore, we have a contradiction to the initial hypothesis. This implies that $\forall n \in \mathbb{N} \mid n \geq 3$ the inequality is valid:

$$(5.79) \qquad \left[Li_{\left(p_{(n)}\right)} - EstTLi_{\left(p_{(n)}\right)} - \mathcal{LTI}_{\left(p_{(n)}\right)} \right]^{-1} > (\log(10)(\log(p_n)))^2$$

Therefore, from the Inequalities 5.73, 5.76 and 5.79 for all $p \in \mathbb{N} \mid p \geq 3$ we have:

$$(5.80) \qquad \mathcal{LTI}_{(p)} = \left(\sqrt{5} - 1 \right)\left(4\gamma^2 - 2\gamma \right) \sqrt[3]{p} \leq Li_{(p)} - EstTLi_{(p)}$$

as well as,

$$(5.81) \quad \left[Li_{\left(p_{(n)}\right)} - EstTLi_{\left(p_{(n)}\right)} - \mathcal{LTI}_{\left(p_{(n)}\right)} \right] \leq \frac{1}{(\log(10)(\log(p_n)))^2} \equiv$$

$$\equiv \left[Li_{\left(p_{(n)}\right)} - EstTLi_{\left(p_{(n)}\right)} \right] \leq \frac{1}{(\log(10)(\log(p_n)))^2} + \mathcal{LTI}_{\left(p_{(n)}\right)}$$

Furthermore, for all $p \in \mathbb{N} \mid p \geq 347$ we have:

$$(5.82) \quad \frac{1}{p_{(n)}} \leq \left[Li_{\left(p_{(n)}\right)} - EstTLi_{\left(p_{(n)}\right)} - \mathcal{LTI}_{\left(p_{(n)}\right)} \right] \equiv$$

$$\equiv \frac{1}{p_{(n)}} + \mathcal{LTI}_{\left(p_{(n)}\right)} \leq \left[Li_{\left(p_{(n)}\right)} - EstTLi_{\left(p_{(n)}\right)} \right]$$

A straightforward computer calculation verifies that the Inequality 5.82 holds for all $p \in \mathbb{N} \mid 263 \leq p \leq 347$ as well. Consequently, Inequality 5.82 holds for all primes $p \in \mathbb{N} \mid p \geq 263$ as stated, thus concluding the proof. $\qquad\square$

Theorem 5.20 implies that we have a very good approximation of the difference $Li_{(n)} - EstTLi_{(n)}$ at the primes within the range, hence the least difference $Li_{(n)} - EstTLi_{(n)}$.

5.7. **Supremum of $\pi_{(n)}$ and the Skewes' $\pi_{(n)} > Li_{(n)}$ problem appraisal.**
About 1792 Carl F. Gauss postulated the PNT on the basis of empirical evidence. He thought that:

$$\pi_{(n)} < Li_{(n)} \quad \forall n \in \mathbb{N} \mid n \geq 2$$

Gauss' belief relied on his observations made, of the tables of primes up to $n = 3,000,000$. Many of those he constructed by hand himself. His belief was shared by Bernhard Riemann and indeed many other mathematicians of the 19-th century. In 1914 John E. Littlewood presented the proof that:

$$(5.83) \qquad\qquad \pi_{(n)} - Li_{(n)} > \frac{k\sqrt{n}}{\log n} \log(\log(\log n))$$

Littlewood's proof of 1914 [21], [18], depends upon the size of the $\log(\log(\log n))$ for large n. Since however the Littlewood's proof was not constructive [24], Stanley

Skewes in 1933 presented a proof (assuming Riemann's hypothesis), that there exist values of n such that:

$$\pi_{(n)} > Li_{(n)} \quad \text{for } n \in \mathbb{N} \mid n < 10^{10^{10^{34}}}$$

In 1955 S. Skewes re-appraised the problem, this time without the assumption of Riemann's hypothesis he produced a different bound:

$$\pi_{(n)} > Li_{(n)} \quad \text{for } n \in \mathbb{N} \mid n < 10^{10^{10^{10^{3}}}}$$

This legendary bound has since been lowered very significantly, however, it still remains out of reach of direct verification [2]. The theory of the tailored integral developed up to this point permits us to attack Skewes' problem and to prove conclusively that $Li_{(n)} > \pi_{(n)} \quad \forall n \in \mathbb{N} \mid n \geq 11$.

Theorem 5.21 (The Supremum Bound Of Estimation Of $\pi_{(n)}$).
The Tailored Integral is less or at most equal in value to the estimate $EstTLi_{(n)}$:

$$(5.84) \qquad EstTLi_{(n)} - TLi_{(n)} = \int_{\theta_{(n)}}^{est} \frac{dt}{\log t} \quad \geq 0 \ \ \forall n \in \mathbb{N} \mid n \geq 3$$

Further, the tailored logarithmic integral $TLi_{(n)}$ constitutes the Supremum estimation bound of the prime counting function $\pi_{(n)}$:

$$(5.85) \qquad \pi_{(n)} < \int_{2}^{\theta_{(n)}} \frac{dt}{\log t} \leq \int_{2}^{est} \frac{dt}{\log t} < \int_{2}^{n} \frac{dt}{\log t} \quad \forall n \in \mathbb{N} \mid n \geq 43$$

consequently,

$$(5.86) \qquad Li_{(n)} = \int_{2}^{n} \frac{dt}{\log t} > \pi_{(n)} \quad \forall n \in \mathbb{N} \mid n \geq 11$$

Proof.
By Theorem 5.19 we have that:

$$(5.87) \qquad Li_{(n)} = \int_{2}^{n} \frac{dt}{\log t} > EstTLi_{(n)} = \int_{2}^{est} \frac{dt}{\log t} \quad \forall n \in \mathbb{N} \mid n \geq 3$$

where the difference in values taken by the estimate of the tailored logarithmic integral and the Gauss' logarithmic integral diverges as n tends to infinity:

$$(5.88) \qquad \lim_{n \to \infty} \left(Li_{(n)} - EstTLi_{(n)} \right) = \lim_{n \to \infty} \left(\int_{est}^{n} \frac{dt}{\log t} \right) \to \infty$$

On the other hand, by Theorem 5.14 we have:

$$(5.89) \qquad TLi_{(n)} = \int_{2}^{\theta_{(n)}} \frac{dt}{\log t} > \pi_{(n)} \ \ \forall n \in \mathbb{N} \mid n \geq 43$$

With the estimation error increasing unboundedly as n tends to infinity:

$$(5.90)$$

$$\lim_{n \to \infty} \left(TLi_{(n)} - \pi_{(n)} \right) = \lim_{n \to \infty} \left(\sum_{p_{(i+1)}=47}^{p_{(n)}} \left\{ \left(\int_{\theta_1}^{\theta_2} \frac{dt}{\log t} \right) - 1 \right\} + \left(\int_{2}^{\theta_{43}} \frac{dt}{\log t} - \pi_{43} \right) \right) \to \infty$$

$$\text{where} \ \left(\int_{2}^{\theta_{43}} \frac{dt}{\log t} - 14 \right) \approx 0.002993180461560385$$

Now, by Lemma 4.5 we have for all $p_{(n)} \in \mathbb{N} \mid p_{(n)} \geq 2$ where p_n is the greatest prime number $p \in \mathbb{N} \mid p \leq n$:

(5.91) $\; Sest = \left(p_{(n)} - 2\sqrt{p_{(n)}} \right)$

$$< \log p_{(n)}\sharp < \left(p_{(n)} - \left(\sqrt{5}-1\right)\left(4\gamma^2 - 2\gamma\right)\left(\log p_{(n)}\right)\sqrt[3]{p_{(n)}} \right) = est$$

Which gives the interval containing the true value of $\theta_{(n)} = \log p_{(n)}\sharp$:

(5.92) $\quad 2\sqrt{p_{(n)}} - \left(\sqrt{5}-1\right)\left(4\gamma^2 - 2\gamma\right)\left(\log p_{(n)}\right)\sqrt[3]{p_{(n)}}$

$$= \left(\frac{2}{\sqrt{p_{(n)}}\left(\log p_{(n)}\right)} - \frac{\left(\sqrt{5}-1\right)\left(4\gamma^2 - 2\gamma\right)}{p_{(n)}^{2/3}} \right) \times p_{(n)}\left(\log p_{(n)}\right)$$

The upper endpoint of the interval 5.92 is the *est*. In the case that the value of $\log p_{(n)}\sharp$ is located close to the upper endpoint of the interval 5.91, this implies that $\theta_{(n)} \lessapprox est$. Consequently,

$$EstTLi_{(n)} - TLi_{(n)} = \int_{\theta_{(n)}}^{est} \frac{dt}{\log t} \gtrapprox 0$$

Hence, the Infimum of the difference $EstTLi_{(n)} - TLi_{(n)}$ is:

$$\inf\left(EstTLi_{(n)} - TLi_{(n)} \right) = \inf\left(\int_{\theta_{(n)}}^{est} \frac{dt}{\log t} \right) = 0 \quad \forall\, p_{(n)} \in \mathbb{N} \mid p_{(n)} \geq 3$$

The maximum possible value of the difference $EstTLi_{(n)} - TLi_{(n)}$ may be estimated by using the length of the interval 5.92 and applying the PNT:

(5.93) $\quad EstTLi_{(n)} - TLi_{(n)} = \int_{\theta_{(n)}}^{est} \frac{dt}{\log t}$

$$\lessapprox \left(\frac{2}{\sqrt{p_{(n)}}\left(\log p_{(n)}\right)} - \frac{\left(\sqrt{5}-1\right)\left(4\gamma^2 - 2\gamma\right)}{p_{(n)}^{2/3}} \right) \times p_{(n)}$$

clearly, the limit of the estimated maximum value of the difference 5.93 diverges, as n tends to infinity. Hence we have that:

(5.94) $\quad 0 \leq EstTLi_{(n)} - TLi_{(n)} = \int_{\theta_{(n)}}^{est} \frac{dt}{\log t}$

$$\leq \left(\frac{2}{\sqrt{p_{(n)}}\left(\log p_{(n)}\right)} - \frac{\left(\sqrt{5}-1\right)\left(4\gamma^2 - 2\gamma\right)}{p_{(n)}^{2/3}} \right) \times p_{(n)} \quad \forall\, p_{(n)} \in \mathbb{N} \mid p_{(n)} \geq 3$$

as well as by Theorem 5.14 we have:

$$TLi_{(n)} - \pi_{(n)} = \int_{2}^{\theta_{(n)}} \frac{dt}{\log t} - \pi_{(n)} > 0 \;\; \text{for } n \in \mathbb{N} \mid n \geq 43$$

Since $EstTLi_{(n)} - TLi_{(n)} \geq 0$ for all $p_{(n)} \in \mathbb{N} \mid p_{(n)} \geq 3$, $TLi_{(n)} - \pi_{(n)} > 0$ for all $n \in \mathbb{N} \mid n \geq 43$, this implies that $EstTLi_{(n)} - \pi_{(n)} > 0$ for all $n \in \mathbb{N} \mid n \geq 43$.

Direct computation shows that $EstTLi_{(n)} - \pi_{(n)} > 0$ for all $n \in \mathbb{N} \mid 23 \leq n \leq 43$. Please refer to Table 4 below. Consequently, from 5.87, 5.89 and 5.94 we have that:

$$\pi_{(n)} < \int_2^{\theta_{(n)}} \frac{dt}{\log t} \leq \int_2^{est} \frac{dt}{\log t} < \int_2^{n} \frac{dt}{\log t} \quad \text{for } n \in \mathbb{N} \mid n \geq 43$$

This shows that $TLi_{(n)}$ constitutes the Supremum estimation bound on the prime counting function $\pi_{(n)}$ for all $n \in \mathbb{N} \mid n \geq 43$. Direct computation confirms that the difference $EstTLi_{(n)} - TLi_{(n)} > 0$ holds for all $n \in \mathbb{N} \mid 3 \leq n \leq 43$. Please refer to Fig. 7. Therefore, Theorem 5.21 holds as stated, concluding the proof. □

TABLE 4. Low range difference $EstTLi_{(n)} - \pi_{(n)}$

n	$EstTLi_{(n)} - \pi_{(n)}$	n	$EstTLi_{(n)} - \pi_{(n)}$
23	0.202729	37	1.23064
29	0.9967	41	1.29738
31	0.569752	43	0.820459

Corollary 5.22 (Littlewood - Skewes' problem)**.**

   Theorem 5.21 implies that we have a strong contradiction to the statement, that infinitely often:

(5.95) $$\pi_{(n)} - Li_{(n)} > 0$$

Necessarily, this implies that the Littlewood's theorem of 1914 and hence the relation 5.95 above are both false, disproving them for every $n \in \mathbb{N} \mid n \geq 11$. Since the relation 5.95 is obviously false within the range $n \in \mathbb{N} \mid 2 \leq n \leq 11$, consequently this implies that the Littlewood's theorem of 1914 is false for every $n \in \mathbb{N}$.

   Theorem 5.21 implies that Carl F. Gauss' belief, shared by Bernhard Riemann and indeed many other mathematicians of the 19-th century, was correct thereby proving their historical guess.
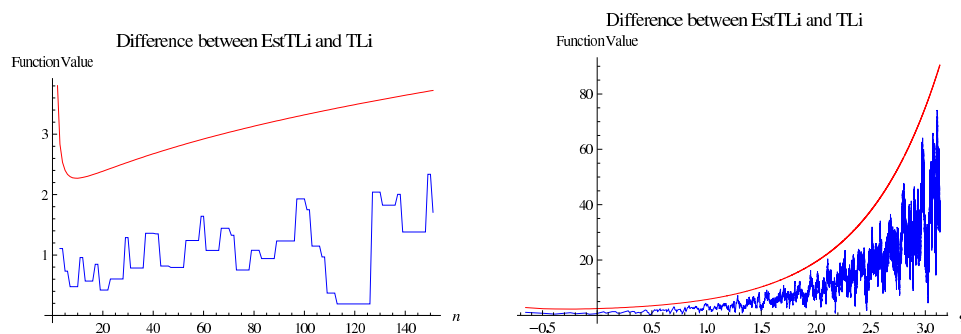
FIGURE 7. The drawings show the graph of the true difference $(EstTLi_{(n)} - TLi_{(n)})$ (blue), and the estimate of the maximal value of the difference given by 5.93 (red). The left figure is drawn at every $n \in \mathbb{N} \mid 3 \leq n \leq 151$, while the right figure drawn with respect to ξ at every $n \in \mathbb{N} \mid 3 \leq n \leq 500\,009$.

5.8. **Estimation error bounds on the difference** $TLi_{(n)} - \pi_{(n)}$**.**

Because the Infimum and Supremum estimation error bounds are inherently difficult to compute for large $n \in \mathbb{N}$, this section presents alternative lower and upper estimation error bounds.

Theorem 5.23 (Lower Estimation Error Bound)**.**

The error of estimation of the prime counting function $\pi_{(m)}$ by the application of the tailored logarithmic integral $TLi_{(m)}$, for all $m \in \mathbb{N} \mid m \geq 11$ is bounded below by:

$$(5.96) \quad TLi_{(m)} - \pi_{(m)} \geq$$

$$\mathcal{LEB}_{(m)} = \frac{\mathcal{A}\,(\log 10)}{2\gamma(\log_{10} m)^{(\sqrt{3}-1)}}\left(\frac{\sqrt{m}}{(\log_{10} m)^2} + \frac{5}{\log 10}\,(\log_{10} m) - 6\right) - (\exp(2))$$

Where γ is the Euler-Mascheroni constant and $\mathcal{A}$ is the Glaisher-Kinkelin constant given by definition 1.5.

Proof.

Evidently, $\pi_{(m)}$ defines a weakly monotone, divergent function. By Theorem 5.14 the estimation error of the tailored logarithmic integral $TLi_{(m)}$ defines a monotone divergent sequence. Also, the lower estimation error bound $\mathcal{LEB}_{(m)}$ clearly defines a monotone divergent sequence. Suppose that Theorem 5.23 is false for $m \in \mathbb{N} \mid m \geq 1\,000\,000\,007$, then it has to be true that:

$$(5.97) \qquad\qquad TLi_{(m)} < \left(\mathcal{LEB}_{(m)} + \pi_{(m)}\right)$$

Which is equivalent to say,

$$(5.98) \qquad\qquad \exp\left(TLi_{(m)}\right) - \exp\left(\mathcal{LEB}_{(m)} + \pi_{(m)}\right) < 0$$

However, at $m = 1\,000\,000\,007$ the difference 5.98 attains $\sim 9.9890903 * 10^{22082891}$ and rapidly diverges as m increases unboundedly. Therefore, the difference 5.98 generates positive numerical sequence in $\mathbb{R}$. Thus, we implement the Cauchy's Root Test[1]:

$$C = \lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} \sqrt[n]{\exp\left(TLi_{(m)}\right) - \exp\left(\mathcal{LEB}_{(m)} + \pi_{(m)}\right)} \to$$

$$\to \exp(1) \approx 2.718281828 > 1$$

The test at $m = 1\,000\,000\,007$ attains ~ 2.7182926769186047 and converges to $\sim \exp(1)$ strictly from above. By the definition of the Cauchy's Root Test this implies that a series formed by the terms of the Inequality 5.98 necessarily diverges. Consequently, the difference:

$$TLi_{(m)}$$

$$- \left(\frac{\mathcal{A}\,(\log 10)}{2\gamma(\log_{10} m)^{(\sqrt{3}-1)}}\left(\frac{\sqrt{m}}{(\log_{10} m)^2} + \frac{5}{\log 10}\,(\log_{10} m) - 6\right) - (\exp(2)) + \pi_{(m)}\right) > 0$$

for all $m \in \mathbb{N} \mid m \geq 1\,000\,000\,007$. This implies that we have a contradiction to the hypothesis. Direct computer calculation confirms that Theorem 5.23 also holds within the range for all $m \in \mathbb{N} \mid 11 \leq m \leq 1\,000\,000\,007$. The pertinent data

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[1]The degree n of the root pertains to the prime number p_n being the n-th prime number, the largest one that satisfies the relation $p_n \leq m$.

had been rendered in graphical form, please refer to Fig. 9 to 13 in the Appendix. Therefore, Theorem 5.23 holds for all $m \in \mathbb{N} | m \geq 11$; thus concluding the proof. $\square$

Theorem 5.24 (Upper Estimation Error Bound)**.**

The error of estimation of the prime counting function $\pi_{(m)}$ by the application of the tailored logarithmic integral $TLi_{(m)}$, for all $m \in \mathbb{N} \mid m \geq 3$ is bounded above by:

$$(5.99) \quad TLi_{(m)} - \pi_{(m)}$$

$$\leq \; \mathcal{UEB}_{(m)} = \mathcal{TC}\left(\frac{\sqrt{m}}{(\log_{10} m)^{(\sqrt{2}+1)}} + \frac{5\,(\log_{10} m)^2}{(\log 10)\,(\log_{10} m)^{(\sqrt{2}+1)}} + \frac{7}{(\log_{10} m)^{\sqrt{2}}} \right)$$

Where $\mathcal{TC}$ is given by the Definition 1.6.

Proof.

Evidently, $\pi_{(m)}$ defines a weakly monotone, divergent function. By Theorem 5.14 the estimation error of the tailored logarithmic integral $TLi_{(m)}$ defines a monotone divergent sequence. Also, the upper estimation error bound $\mathcal{UEB}_{(m)}$ clearly defines a monotone divergent sequence. Suppose that Theorem 5.24 is false for $m \in \mathbb{N} \mid m \geq 1\,000\,000\,007$, then it has to be true that:

$$(5.100) \qquad\qquad TLi_{(m)} > \mathcal{UEB}_{(m)} + \pi_{(m)}$$

Which is equivalent to say,

$$(5.101) \qquad\qquad \exp\left(\mathcal{UEB}_{(m)} + \pi_{(m)}\right) - \exp\left(TLi_{(m)}\right) < 0$$

However, at $m = 1\,000\,000\,007$ the difference 5.101 attains $\sim 3.81666351 * 10^{29156538}$ and rapidly diverges as m increases unboundedly. Therefore, the difference 5.101 generates positive numerical sequence in $\mathbb{R}$. Thus, we implement the Cauchy's Root Test[2]:

$$C = \lim_{n \to \infty} \sqrt[n]{|a_n|}$$

$$= \lim_{n \to \infty} \sqrt[n]{\exp\left(\mathcal{UEB}_{(m)} + \pi_{(m)}\right) - \exp\left(TLi_{(m)}\right)} \to$$

$$\to \exp(1) \approx 2.718281828 > 1$$

The test at $m = 1\,000\,000\,007$ attains ~ 2.71829300286192 and converges to $\sim \exp(1)$ strictly from above. By the definition of the Cauchy's Root Test this implies that a series formed by the terms of the Inequality 5.101 necessarily diverges. Consequently, the difference:

$$\left(\mathcal{TC}\left(\frac{\sqrt{m}}{(\log_{10} m)^{(\sqrt{2}+1)}} + \frac{5\,(\log_{10} m)^2}{(\log 10)\,(\log_{10} m)^{(\sqrt{2}+1)}} + \frac{7}{(\log_{10} m)^{\sqrt{2}}} \right) + \pi_{(m)} \right)$$

$$- TLi_{(m)} > 0$$

for all $m \in \mathbb{N} | \; m \geq 1\,000\,000\,007$. This implies that we have a contradiction to the hypothesis. Direct computation $\forall m \in \mathbb{N} \mid 3 \leq m \leq 1\,000\,000\,007$ verifies that Theorem 5.24 also holds within this range. Pertinent data had been rendered in graphical form, please refer to Fig. 9 to 13 in the Appendix. Therefore, Theorem 5.24 holds for all $m \in \mathbb{N} \mid m \geq 3$, concluding the proof. $\square$

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[2]The degree n of the root pertains to the prime number p_n being the n-th prime number, the largest one that satisfies the relation $p_n \leq m$.

Remark 5.8. Both the lower and upper estimation error bound follow the $TLi_{(n)}$ estimation error curve very closely. This situation extends over a prolonged interval. Please refer to Table 6 in the Appendix for a listing of the local minima.

6. Gauss' logarithmic integral $Li_{(n)}$ and Riemann's hypothesis

This section develops the mathematical basis, to link unambiguously the tailored integral theory with the Gauss' logarithmic integral error term.

6.1. Divergence of the estimation error of $Li_{(n)}$.

Theorem 6.1 (Divergence of the estimation error of $Li_{(n)}$)**.**
   The estimation error of the Gauss' logarithmic integral $Li_{(n)}$ diverges as n tends to infinity:

$$(6.1) \qquad \lim_{n \to \infty} \left(Li_{(n)} - \pi_{(n)} \right) \to \infty$$

Proof.
   By Theorems: 5.14, 5.19 and 5.21 we have that:

$$\pi_{(n)} < \int_2^{\theta_{(n)}} \frac{dt}{\log t} \leq \int_2^{est} \frac{dt}{\log t} < \int_2^n \frac{dt}{\log t} \quad \text{for } n \in \mathbb{N} \mid n \geq 43$$

and clearly:

$$\pi_{(n)} < Li_{(n)} \quad \forall n \in \mathbb{N} \mid 11 \leq n \leq 43$$

The estimation error of the tailored integral $TLi_{(n)}$ by Theorem 5.14 diverges:

(6.2)

$$\lim_{n \to \infty} \left(TLi_{(n)} - \pi_{(n)} \right) = \lim_{p_{(n)} \to \infty} \left(\sum_{p_{(i+1)}=5}^{p_{(n)}} \left\{ \left(\int_{\theta_1}^{\theta_2} \frac{dt}{\log t} \right) - 1 \right\} + \left(\int_2^{\theta_3} \frac{dt}{\log t} - \pi_3 \right) \right) \to \infty$$

$$\text{where} \quad \left(\int_2^{\theta_3} \frac{dt}{\log t} - \pi_3 \right) \approx -2.3266013098834977$$

By Theorem 5.19 the difference in values between the estimate of the tailored integral and the Gauss' logarithmic integral diverges:

$$(6.3) \qquad \lim_{n \to \infty} \left(Li_{(n)} - EstTLi_{(n)} \right) = \lim_{n \to \infty} \left(\int_{est}^n \frac{dt}{\log t} \right) \to \infty$$

Consequently therefore:

$$(6.4) \qquad \lim_{n \to \infty} \left(Li_{(n)} - \pi_{(n)} \right) \to \infty$$

Thus concluding the proof of Theorem 6.1.   □
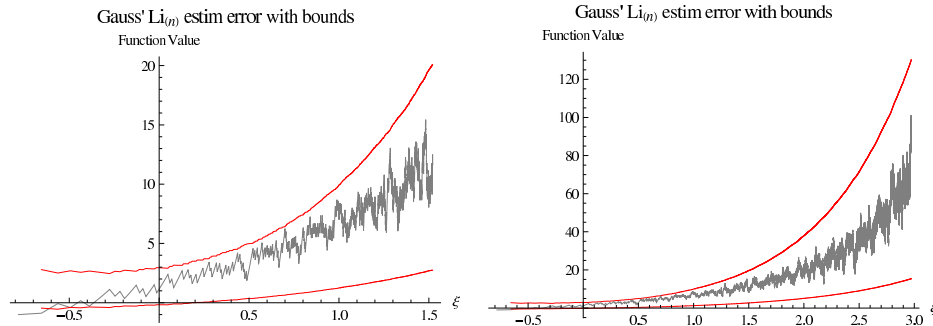
FIGURE 8. The figures show the estimation error $Li_n - \pi_n$ (gray) and the upper and lower bounds 6.10 and 6.6 (red), the figure is drawn with respect to ξ at every $n \in \mathbb{N} \mid 11 \leq n \leq 3000$ in left figure and $n \in \mathbb{N} \mid 11 \leq n \leq 300000$ in right figure.

6.2. $Li_{(n)}$ estimation error bounds.

The estimation error of the tailored logarithmic integral $TLi_{(n)}$ produces virtually oscillation free curve which by Theorem 5.14 diverges:

$$\lim_{n \to \infty} \left(TLi_{(n)} - \pi_{(n)} \right) \to \infty \tag{6.5}$$

The smooth characteristic of the curve makes it possible to establish the estimation error bounds for the $Li_{(n)}$ by converting the estimation error bounds of the tailored logarithmic integral $TLi_{(n)} - \pi_{(n)}$ to the upper and lower bounds for the estimation error of the logarithmic integral $Li_{(n)} - \pi_{(n)}$, by the use of a specific multiplier. The upper bound implements the multiplier:

Definition 6.2. $M_1 = \frac{2\gamma}{\mathcal{TC}} \left(\log_{10} m \right)^{\sqrt{2}}$ for $m \in \mathbb{N}$

The lower bound applies the multiplier:

Definition 6.3. $M_2 = \frac{(\log_{10} m)^{(\sqrt{3}-1)}}{\mathcal{A}(\log 10)}$ for $m \in \mathbb{N}$

Where $\mathcal{TC}$ and $\mathcal{A}$ are given by the definitions: 1.6 and 1.5 respectively, and γ is the Euler-Mascheroni constant gamma.

Theorem 6.4 (Lower $Li_{(m)}$ Estimation Error Bound)**.**
For any $m \in \mathbb{N} \mid m \geq 11$, where p_n is the n-th prime, the largest one satisfying the relation: $p_n \leq m$, the following relation holds:
(6.6)

$$M_2 \left(TLi_{(m)} - \pi_{(m)} \right) = \left(\frac{(\log_{10} m)^{(\sqrt{3}-1)}}{\mathcal{A} \left(\log 10 \right)} \right) \left(TLi_{(m)} - \pi_{(m)} \right) \leq \left(Li_{(m)} - \pi_{(m)} \right)$$

Where $\mathcal{A}$ is given by the definition 1.5.

Proof.
The estimation error of $TLi_{(m)}$ as given by Theorem 5.14, and the estimation error of $Li_{(m)}$ as given by Theorem 6.1, both diverge as m tends to infinity. Due to the fact that the $TLi_{(m)}$ estimation error increases only at the primes and remains constant between them, consequently, for all $m \in \mathbb{N} \mid m \geq 3$ it defines a weakly

monotone, divergent function. Clearly, $\pi_{(m)}$ defines a weakly monotone, divergent function as well. Necessarily therefore,

$$\left(\frac{(\log_{10} m)^{\left(\sqrt{3}-1\right)}}{\mathcal{A}\left(\log 10\right)}\right)\left(TLi_{(m)} - \pi_{(m)}\right) + \pi_{(m)}$$

defines a monotone, divergent function. The Gauss' logarithmic integral $Li_{(m)}$ clearly is strictly monotone, divergent function. Suppose therefore, that Theorem 6.4 is false. Hence in accordance with the hypothesis we have that:

$$(6.7) \qquad \left(\frac{(\log_{10} m)^{\left(\sqrt{3}-1\right)}}{\mathcal{A}\left(\log 10\right)}\right)\left(TLi_{(m)} - \pi_{(m)}\right) + \pi_{(m)} > Li_{(m)}$$

The difference $Li_{(m)} - \pi_m$ clearly is highly oscillatory, which obviously applies equally well to the difference $Li_{(m)} - TLi_{(m)}$. Therefore to smooth out the characteristics of the difference of the terms of Inequality 6.7 we take the exponential:

$$(6.8) \qquad 0 > \exp\left(Li_{(m)}\right) - \exp\left(\left(\frac{(\log_{10} m)^{\left(\sqrt{3}-1\right)}}{\mathcal{A}\left(\log 10\right)}\right)\left(TLi_{(m)} - \pi_{(m)}\right) + \pi_{(m)}\right)$$

However at $p_n = 11$, the difference attains ~ 153.504313 and rapidly diverges. Since the difference produces positive numerical output, we apply the Cauchy's Root Test[3]:

$$C = \lim_{n\to\infty} \sqrt[n]{|a_n|} =$$

$$\lim_{n\to\infty} \sqrt[n]{\exp\left(Li_{(m)}\right) - \exp\left(\left(\frac{(\log_{10} m)^{\left(\sqrt{3}-1\right)}}{\mathcal{A}\left(\log 10\right)}\right)\left(TLi_{(m)} - \pi_{(m)}\right) + \pi_{(m)}\right)} \to$$

$$\to \exp(1) \approx 2.718281828 > 1$$

The test at $p_n = 47$ attains ~ 2.79234 and converges strictly from above to $\sim \exp(1)$. By the definition of the Cauchy's Root Test this implies that a series formed by the terms of the Inequality 6.8 necessarily diverges. Consequently, the difference:

$$(6.9) \qquad 0 < Li_{(m)} - \left(\left(\frac{(\log_{10} m)^{\left(\sqrt{3}-1\right)}}{\mathcal{A}\left(\log 10\right)}\right)\left(TLi_{(m)} - \pi_{(m)}\right) + \pi_{(m)}\right)$$

for all $m \in \mathbb{N}|\ m \geq 47$. This implies that we have a contradiction to the hypothesis. Computer calculation confirms that Theorem 6.4 also holds within the range for all $m \in \mathbb{N}|\ 11 \leq m \leq 47$. Therefore, Theorem 6.4 holds for all $m \in \mathbb{N}|m \geq 11$; thus concluding the proof. □

Theorem 6.5 (Upper $Li_{(n)}$ Estimation Error Bound).

For any $m \in \mathbb{N} \mid m \geq 3$, where p_n is the n-th prime, the largest one satisfying the relation: $p_n \leq m$, the following relation holds:

$$(6.10) \qquad \frac{2\gamma}{\mathcal{TC}}\left(\log_{10} m\right)^{\sqrt{2}}\left(TLi_{(m)} - \pi_{(m)}\right) + 3.5 \geq \left(Li_{(m)} - \pi_{(m)}\right)$$

Where $\mathcal{TC}$ is given by the definition 1.6.

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[3]The degree n of the root pertains to the prime number p_n being the n-th prime number, the largest one that satisfies the relation $p_n \leq m$.

Proof.

The estimation error of $TLi_{(m)}$ as given by Theorem 5.14, and the estimation error of $Li_{(m)}$ as given by Theorem 6.1, both diverge as m tends to infinity. Due to the fact that the $TLi_{(m)}$ estimation error increases only at the primes and remains constant between them, consequently, for all $m \in \mathbb{N} \mid m \geq 3$ it defines a weakly monotone, increasing without bound function. Clearly, $\pi_{(m)}$ defines a weakly monotone, divergent function as well, as is $(\log_{10} m)^{\sqrt{2}}$ function. Necessarily therefore,

$$\frac{2\gamma}{\mathcal{TC}} (\log_{10} m)^{\sqrt{2}} \left(TLi_{(m)} - \pi_{(m)} \right) + 3.5 + \pi_{(m)}$$

defines a monotone, divergent function. The Gauss' logarithmic integral $Li_{(m)}$ clearly is monotone and divergent function. Suppose therefore, that Theorem 6.5 is false. Hence in accordance with the hypothesis we have that:

$$(6.11) \qquad \frac{2\gamma}{\mathcal{TC}} (\log_{10} m)^{\sqrt{2}} \left(TLi_{(m)} - \pi_{(m)} \right) + 3.5 + \pi_{(m)} < Li_{(m)}$$

The difference $Li_{(m)} - \pi_m$ clearly is highly oscillatory, which obviously applies equally well to the difference $Li_{(m)} - TLi_{(m)}$. Therefore to smooth out the characteristics of the difference of the terms of Inequality 6.11 we take the exponential:

$$(6.12) \quad \exp\left(\frac{2\gamma}{\mathcal{TC}} (\log_{10} m)^{\sqrt{2}} \left(TLi_{(m)} - \pi_{(m)} \right) + 3.5 + \pi_{(m)} \right) - \exp\left(Li_{(m)} \right) < 0$$

However at $p_n = 541$, the difference attains $\sim 3.57932 \times 10^{47}$ and rapidly diverges. Since the difference produces positive numerical output, we apply the Cauchy's Root Test[4]:

$$C = \lim_{n \to \infty} \sqrt[n]{|a_n|} =$$

$$\lim_{n \to \infty} \sqrt[n]{\exp\left(\frac{2\gamma}{\mathcal{TC}} (\log_{10} m)^{\sqrt{2}} \left(TLi_{(m)} - \pi_{(m)} \right) + 3.5 + \pi_{(m)} \right) - \exp\left(Li_{(m)} \right)} \rightarrow$$

$$\rightarrow \exp(1) \approx 2.718281828 > 1$$

The test at $p_n = 541$ attains ~ 2.95685 and converges strictly from above to $\sim \exp(1)$. By the definition of the Cauchy's Root Test this implies that a series formed by the terms of the Inequality 6.12 necessarily diverges. Consequently, the difference:

$$(6.13) \qquad \left(\frac{2\gamma}{\mathcal{TC}} (\log_{10} m)^{\sqrt{2}} \left(TLi_{(m)} - \pi_{(m)} \right) + 3.5 + \pi_{(m)} \right) - Li_{(m)} > 0$$

for all $m \in \mathbb{N} \mid m \geq 541$. This implies that we have a contradiction to the hypothesis. Direct computation $\forall m \in \mathbb{N} \mid 3 \leq m \leq 541$ verifies that Theorem 6.5 also holds within this range. Therefore Theorem 6.5 holds for all $m \in \mathbb{N} \mid m \geq 3$, concluding the proof. $\square$

Remark 6.1. Determining the $Li_{(n)}$ estimation error bounds by the application of $TLi_{(n)}$ estimation error curve is computationally very inefficient process, applicable to a relatively small values of $n \in \mathbb{N}$ only, a different more efficient method will be presented shortly.

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[4]The degree n of the root pertains to the prime number p_n being the n-th prime number, the largest one that satisfies the relation $p_n \leq m$.

6.3. Primary estimation error bound of $Li_{(n)}$.

The equation arising in estimation of the true value of $\pi_{(n)}$ by the application of the logarithmic integral

$$(6.14) \qquad\qquad \pi_{(n)} = Li_{(n)} + \mathcal{O}\left(\sqrt{n}\ \log(n)\right)$$

has been shown in 1901 by Niels F. Helge von Koch, to be equivalent to the Riemann's Hypothesis. The size of the estimation error term which depends on the gaps between primes, is intimately connected with the location of the zeroes of the Riemann zeta function.

Theorem 6.6 (Primary Lower Bound Of $Li_{(n)}$ Estimation Error)**.**

For any $n \in \mathbb{N} \mid n \geq 53$, the error made in estimation of the true value of $\pi_{(n)}$ by the application of the logarithmic integral is bounded below by:

$$(6.15) \qquad\qquad Gauss\mathcal{ELB}_{(n)} = \sum_{p \leq n} \frac{1}{p} \leq \left(Li_{(n)} - \pi_{(n)}\right)$$

Proof. By Theorems: 5.14, 5.19 and 5.21 we have that:

$$(6.16) \qquad \pi_{(n)} < \int_2^{\theta_{(n)}} \frac{dt}{\log t} \leq \int_2^{est} \frac{dt}{\log t} < \int_2^n \frac{dt}{\log t} \quad \text{for } n \in \mathbb{N} \mid n \geq 43$$

The estimation error of the tailored integral $TLi_{(n)}$ by Theorem 5.14 diverges:

$$\lim_{n \to \infty} \left(TLi_{(n)} - \pi_{(n)}\right) = \lim_{p_{(n)} \to \infty} \left(\sum_{p_{(i+1)}=5}^{p_{(n)}} \left\{ \left(\int_{\theta_1}^{\theta_2} \frac{dt}{\log t} \right) - 1 \right\} + \left(\int_2^{\theta_3} \frac{dt}{\log t} - \pi_3 \right) \right) \to \infty$$

$$\text{where} \quad \left(\int_2^{\theta_3} \frac{dt}{\log t} - \pi_3 \right) \approx -2.3266013098834977$$

By Theorem 5.19 the difference in values attained between the estimate of the tailored integral and the Gauss' logarithmic integral diverges:

$$(6.17) \qquad\qquad \lim_{n \to \infty} \left(Li_{(n)} - EstTLi_{(n)}\right) = \lim_{n \to \infty} \left(\int_{est}^n \frac{dt}{\log t} \right) \to \infty$$

Consequently therefore:

$$\lim_{n \to \infty} \left(Li_{(n)} - \pi_{(n)}\right) \to \infty$$

Both $Li_{(n)} > \pi_{(n)}$ and $TLi_{(n)} > \pi_{(n)}$ for $n \in \mathbb{N} \mid n \geq 43$, from 6.16 we obtain:

$$(6.18) \qquad\qquad TLi_{(n)} - \pi_{(n)} < Li_{(n)} - \pi_{(n)} \quad \forall n \in \mathbb{N} \mid n \geq 43$$

by Theorem 5.14 we have:

$$\left(TLi_{(n)} - \pi_{(n)}\right) > \sum_{p \leq n} \frac{1}{p} \quad \forall\ n \in \mathbb{N} \mid n \geq 983$$

Consequently,

$$(6.19) \qquad \left(Li_{(n)} - \pi_{(n)}\right) > \left(TLi_{(n)} - \pi_{(n)}\right) > \sum_{p \leq n} \frac{1}{p} \quad \forall n \in \mathbb{N} \mid n \geq 983$$

Direct evaluation confirms that:

$$(6.20) \qquad \left(Li_{(n)} - \pi_{(n)}\right) > \sum_{p \leq n} \frac{1}{p} \quad \forall n \in \mathbb{N} \mid 53 \leq n \leq 983$$

Please refer to the Table 7 in the Appendix. Therefore, the sum of reciprocals of prime numbers is for all $n \in \mathbb{N} \mid n \geq 53$, the primary lower bound of the estimation error, made by the application of the Gauss' logarithmic integral. Concluding the proof of Theorem 6.6. $\qquad \square$

6.4. **Lower bound on the error term of the Gauss' logarithmic integral.**
     By Theorem 6.6 we have that the sum of reciprocals of prime numbers for all $n \in \mathbb{N} \mid n \geq 53$ is the primary lower bound on the estimation error made by the application of the Gauss' logarithmic integral. However, despite the fact that this bound may well serve its purpose, it is far from being optimal and evidently does not prove equation 6.21.

Theorem 6.7 (Lower Estimation Error Bound on The Logarithmic Integral).
     For any $n \in \mathbb{N} \mid n \geq 4$, the error made in estimation of the true value of $\pi_{(n)}$ by the application of the logarithmic integral is bounded below by:

$$(6.21)$$

$$Gauss\mathcal{LEB}_{(n)} = \frac{1}{2\gamma}\left(\frac{\sqrt{n}}{(\log_{10} n)^2} + \frac{5}{\log 10}\left(\log_{10} n\right) - 6\right) - \left(\frac{(\exp(2))\left(\log_{10} n\right)^{\left(\sqrt{3}-1\right)}}{\mathcal{A}\left(\log 10\right)}\right)$$

$$\leq \left(Li_{(n)} - \pi_{(n)}\right)$$

Proof.
     The Theorem 6.4 states that:

$$(6.22) \qquad \left(\frac{(\log_{10} n)^{\left(\sqrt{3}-1\right)}}{\mathcal{A}\left(\log 10\right)}\right)\left(TLi_{(n)} - \pi_{(n)}\right) \leq \left(Li_{(n)} - \pi_{(n)}\right) \quad \forall n \in \mathbb{N} \mid n \geq 11$$

Theorem 5.23 states that the lower estimation error bound on the tailored logarithmic integral is given by:

$$(6.23) \quad \mathcal{LEB}_{(n)} = \frac{\mathcal{A}\left(\log 10\right)}{2\gamma(\log_{10} n)^{\left(\sqrt{3}-1\right)}}\left(\frac{\sqrt{n}}{(\log_{10} n)^2} + \frac{5}{\log 10}\left(\log_{10} n\right) - 6\right) - (\exp(2))$$

$$\leq TLi_{(n)} - \pi_{(n)} \text{ for any } n \in \mathbb{N} \text{ s.t. } n \geq 11$$

Therefore for all $n \in \mathbb{N} \mid n \geq 11$,

$$(6.24) \qquad \left(\frac{(\log_{10} n)^{\left(\sqrt{3}-1\right)}}{\mathcal{A}\left(\log 10\right)}\right) \times \left(\mathcal{LEB}_{(n)}\right)$$

$$= \frac{1}{2\gamma}\left(\frac{\sqrt{n}}{(\log_{10} n)^2} + \frac{5}{\log 10}\left(\log_{10} n\right) - 6\right) - \left(\frac{(\exp(2))\left(\log_{10} n\right)^{\left(\sqrt{3}-1\right)}}{\mathcal{A}\left(\log 10\right)}\right)$$

$$\leq \left(\frac{(\log_{10} n)^{\left(\sqrt{3}-1\right)}}{\mathcal{A}\left(\log 10\right)}\right)\left(TLi_{(n)} - \pi_{(n)}\right) \leq \left(Li_{(n)} - \pi_{(n)}\right)$$

Direct computation verifies that the lower bound is strictly less than $\left(Li_{(n)} - \pi_{(n)}\right)$ for all $n \in \mathbb{N} \mid 4 \leq n \leq 11$ with the difference at $n = 4$:

$$\left(Li_{(n)} - \pi_{(n)}\right) - \mathcal{LEB}_{(n)} \approx 0.933759$$

Consequently, Theorem 6.7 holds for all $n \in \mathbb{N} \mid n \geq 4$, concluding the proof. $\quad\square$

6.5. **Upper bound on the error term of the Gauss' logarithmic integral.**

Theorem 6.8 (Upper Estimation Error Bound on The Gauss' Logarithmic Integral)**.**

For any $n \in \mathbb{N} \mid n \geq 3$, the error made in estimation of the true value of $\pi_{(n)}$ by the application of the logarithmic integral is bounded above by:

$$(6.25) \quad Gauss\mathcal{UEB}_{(n)} = 2\gamma\left(\frac{\sqrt{n}}{(\log_{10} n)} + \frac{5\left(\log_{10} n\right)}{(\log 10)} + 7\right) + 3.5 \geq \left(Li_{(n)} - \pi_{(n)}\right)$$

Proof.

Theorem 6.5 states that for all $n \in \mathbb{N} \mid n \geq 3$:

$$(6.26) \quad \frac{2\gamma}{\mathcal{TC}}\left(\log_{10} n\right)^{\sqrt{2}}\left(TLi_{(n)} - \pi_{(n)}\right) + 3.5 \geq \left(Li_{(n)} - \pi_{(n)}\right)$$

Theorem 5.24 states that the upper error bound on the tailored logarithmic integral estimation error is given by:

$$(6.27) \quad TLi_{(n)} - \pi_{(n)}$$

$$\leq \mathcal{UEB}_{(n)} = \mathcal{TC}\left(\frac{\sqrt{n}}{(\log_{10} n)^{\left(\sqrt{2}+1\right)}} + \frac{5\left(\log_{10} n\right)^2}{(\log 10)\left(\log_{10} n\right)^{\left(\sqrt{2}+1\right)}} + \frac{7}{\left(\log_{10} n\right)^{\sqrt{2}}}\right)$$

$$\text{for all } n \in \mathbb{N} \mid n \geq 3$$

Consequently for all $n \in \mathbb{N} \mid n \geq 3$,

$$(6.28) \quad \frac{2\gamma}{\mathcal{TC}}\left(\log_{10} n\right)^{\sqrt{2}} \times \left(\mathcal{UEB}_{(n)}\right) + 3.5 = 2\gamma\left(\frac{\sqrt{n}}{(\log_{10} n)} + \frac{5\left(\log_{10} n\right)}{(\log 10)} + 7\right) + 3.5$$

$$\geq \frac{2\gamma}{\mathcal{TC}}\left(\log_{10} n\right)^{\sqrt{2}}\left(TLi_{(n)} - \pi_{(n)}\right) + 3.5 \geq \left(Li_{(n)} - \pi_{(n)}\right)$$

Direct computation verifies that the upper bound is strictly greater than $\left(Li_{(n)} - \pi_{(n)}\right)$ at $n = 2$:

$$\mathcal{UEB}_{(n)} - \left(Li_{(n)} - \pi_{(n)}\right) \approx 18.7591$$

Therefore, Theorem 6.8 holds for all $n \in \mathbb{N} \mid n \geq 2$, concluding the proof. $\quad\square$

6.6. **Estimation error bounds $\left(Li_{(n)} - \pi_{(n)}\right)$ and $\left(TLi_{(n)} - \pi_{(n)}\right)$ summary.**

The $\left(Li_{(n)} - \pi_{(n)}\right)$ estimation error bounds are given by Theorems 6.7 and 6.8:

$$\frac{1}{2\gamma}\left(\frac{\sqrt{n}}{\left(\log_{10} n\right)^2} + \frac{5}{\log 10}\left(\log_{10} n\right) - 6\right) - \left(\frac{(\exp(2))\left(\log_{10} n\right)^{\left(\sqrt{3}-1\right)}}{\mathcal{A}\left(\log 10\right)}\right)$$

$$\leq \left(Li_{(n)} - \pi_{(n)}\right) \leq 2\gamma\left(\frac{\sqrt{n}}{(\log_{10} n)} + \frac{5\left(\log_{10} n\right)}{(\log 10)} + 7\right) + 3.5 \quad \forall n \in \mathbb{N} \mid n \geq 4$$

In an analogous way by Theorems 5.24 and 5.23 we have:

$$\frac{\mathcal{A}\left(\log 10\right)}{2\gamma(\log_{10} n)^{\left(\sqrt{3}-1\right)}}\left(\frac{\sqrt{n}}{\left(\log_{10} n\right)^2} + \frac{5}{\log 10}\left(\log_{10} n\right) - 6\right) - \left(\exp(2)\right) \leq \left(TLi_{(n)} - \pi_{(n)}\right)$$

$$\leq \frac{\mathcal{TC}}{\left(\log_{10} n\right)^{\sqrt{2}}}\left(\frac{\sqrt{n}}{\log_{10} n} + \frac{5\left(\log_{10} n\right)^2}{\log n} + 7\right)$$

$$\forall n \in \mathbb{N} \mid n \geq 11$$

The Infimum and Supremum estimation error bounds are given by Theorems 5.15 and 5.16:

$$\sum_{k=1}^{\left(\pi_{(n)}-14\right)}\left\{\frac{\log p_{(14+k)}}{\log\left(\log p_{(14+k)}\sharp\right)} + \frac{1}{5\left(p_{(14+k)}\right)} - 1\right\}$$

$$< TLi_{(n)} - \pi_{(n)} < \sum_{k=1}^{\left(\pi_{(n)}-14\right)}\left\{\frac{\log p_{(14+k)}}{\log\left(\log p_{(14+k)}\sharp\right)} + \frac{1}{p_{(14+k)}} - 1\right\}$$

$$\forall n \in \mathbb{N} \mid n \geq 47 \quad \text{where } p_{(n)} \text{ is the greatest prime number } p \in \mathbb{N} \mid p \leq n$$

6.7. Riemann's hypothesis.

Riemann's hypothesis is the final major objective of this research.

Theorem 6.9 (The Riemann's Hypothesis).

The prime counting function $\pi_{(n)}$ is given by:

$$(6.29) \qquad \pi_{(n)} = Li_{(n)} + \mathcal{O}\left(\sqrt{n}\,\log n\right) \quad \forall n \in \mathbb{N} \mid n \geq 2$$

Proof.

By Theorem 6.8 we have that:

$$\pi_{(n)} = Li_{(n)} + \mathcal{O}\left(Gauss\mathcal{UEB}_{(n)}\right) \quad \forall n \in \mathbb{N} \mid n \geq 2$$

Where:

$$(6.30) \qquad Gauss\mathcal{UEB}_{(n)} = 2\gamma\left(\frac{\sqrt{n}}{\left(\log_{10} n\right)} + \frac{5\left(\log_{10} n\right)}{\left(\log 10\right)} + 7\right) + 3.5$$

Suppose that:

$$Gauss\mathcal{UEB}_{(n)} > \sqrt{n}\log n$$

Necessarily therefore,

$$(6.31) \qquad \frac{\sqrt{n}\log n}{Gauss\mathcal{UEB}_{(n)}} < 1$$

By L'Hôpital's rule we obtain:

$$(6.32) \quad \lim_{n \to \infty}\left(\frac{\sqrt{n}\,\log n}{2\gamma\left(\frac{\sqrt{n}}{\log_{10} n} + \frac{5(\log_{10} n)}{\log 10} + 7\right) + 3.5}\right) =$$

$$= \lim_{n \to \infty}\left(\frac{(2 + \log n)}{2\gamma\left(\left(\frac{1}{\log n} - \frac{2}{(\log n)^2}\right)\log 10 + \frac{10}{\sqrt{n}(\log 10)^2}\right)}\right)$$

Every term in the denominator of 6.32 tends to zero as n increases unboundedly, while the numerator diverges. Necessarily therefore,

$$(6.33) \qquad \lim_{n \to \infty} \left(\frac{\sqrt{n} \, \log n}{2\gamma \left(\frac{\sqrt{n}}{\log_{10} n} + \frac{5(\log_{10} n)}{\log 10} + 7 \right) + 3.5} \right) \to \infty$$

Direct computation verifies that the ratio 6.31 attains the value ≈ 0.0551976 at $n = 2$ and further increases. At $n = 33$ it exceeds the value of 1 and decisively diverges. For all $n \in \mathbb{N} \mid n \geq 967$ the ratio 6.31 diverges at a rate exceeding $\log(n)$, while further accelerating. Thus we have a contradiction to the initial hypothesis. Since the ratio $\frac{\sqrt{n} \, \log(n)}{Gauss\mathcal{UEB}_{(n)}}$ diverges, this implies that the ratio $\frac{Gauss\mathcal{UEB}_{(n)}}{\sqrt{n} \log n}$ asymptotically converges to zero, as n increases unboundedly. Analogous numerical computation confirms that at $n = 2$ the difference:

$$\sqrt{n} \log n - (Li_{(n)} - \pi_{(n)}) \approx 1.98026$$

and diverges. Therefore, for all $n \in \mathbb{N} \mid n \geq 2$:

$$Gauss\mathcal{UEB}_{(n)} = o\left(\sqrt{n} \, \log n \right)$$

Since

$$(6.34) \qquad \pi_{(n)} = Li_{(n)} + \mathcal{O}\left(Gauss\mathcal{UEB}_{(n)} \right)$$

this of course implies that

$$\pi_{(n)} = Li_{(n)} + \mathcal{O}\left(\sqrt{n} \, \log n \right)$$

Concluding the proof of the Riemann's hypothesis. $\square$

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7. Appendix

7.1. **Graphical comparison of functions.**

The drawings 9 to 13 show the graphs of the upper and lower estimation error bounds (red) and the estimation error $TLi_{(p_n)} - \pi_{(p_n)}$ (blue). Due to the fact that the estimation error curve $TLi_{(p_n)} - \pi_{(p_n)}$ is relatively smooth, to reduce the size of the resultant database, a technique had been implemented in construction of the figures: 9 to 12. The figures were produced by computing the distance from the curve $TLi_{(p_n)} - \pi_{(p_n)}$ to each bound, at every prime within a sub-interval. Two individual points exhibiting the least distance were stored per interval. The interval widths for all $n \in \mathbb{N} \mid n \leq 10^9$ were computed in accordance with the criteria:

TABLE 5. Interval widths specifications

n	Interval width
$< 10^3$	50
$< 10^5$	200
$< 10^6$	10^3
$< 10^7$	10^4
$< 10^8$	$5 * 10^4$
$< 10^9$	10^5
$< 10^{10}$	10^6
above	10^7

The graphs had been constructed in such a way that their ranges slightly overlap. The points exhibiting locally/globally least distance from either estimation error bound are indicated on the graphs by a black dot located on the respective bound curve. Table 6, specifies the coordinates of such points.

TABLE 6. Critical points - minima

n	ξ	$\mathcal{UEB}_{(n)} - (TLi_n - \pi_n)$	attribute
1 866 373	3.5435062790146286	0.4783128416250477	local
3 189 553	3.71212639526346	0.42763599962528389	local
4 890 913	3.8466421193071674	0.3706127200088041	local
6 862 489	3.953214011532586	0.3374931519939466	global
10 675 969	4.092269278529259	0.42793313427111457	local
16 957 337	4.237863185491645	0.4720328466920449	local
n	ξ	$(TLi_n - \pi_n) - \mathcal{LEB}_{(n)}$	attribute
34 189 626	4.45850818707939	0.4403051640949798	local
84 620 646	4.743668868936391	0.39131309508213974	local
118 888 260	4.850654400251596	0.3233165712131978	local
197 430 720	5.010250084067886	0.14750273586641072	global
319 028 868	5.161252515015835	0.5455210759252225	local
502 777 012	5.304379467814857	0.2490621490423166	local
729 743 022	5.421603942627343	1.1090294068878563	local
876 080 592	5.479112609478988	1.8641473315881	local
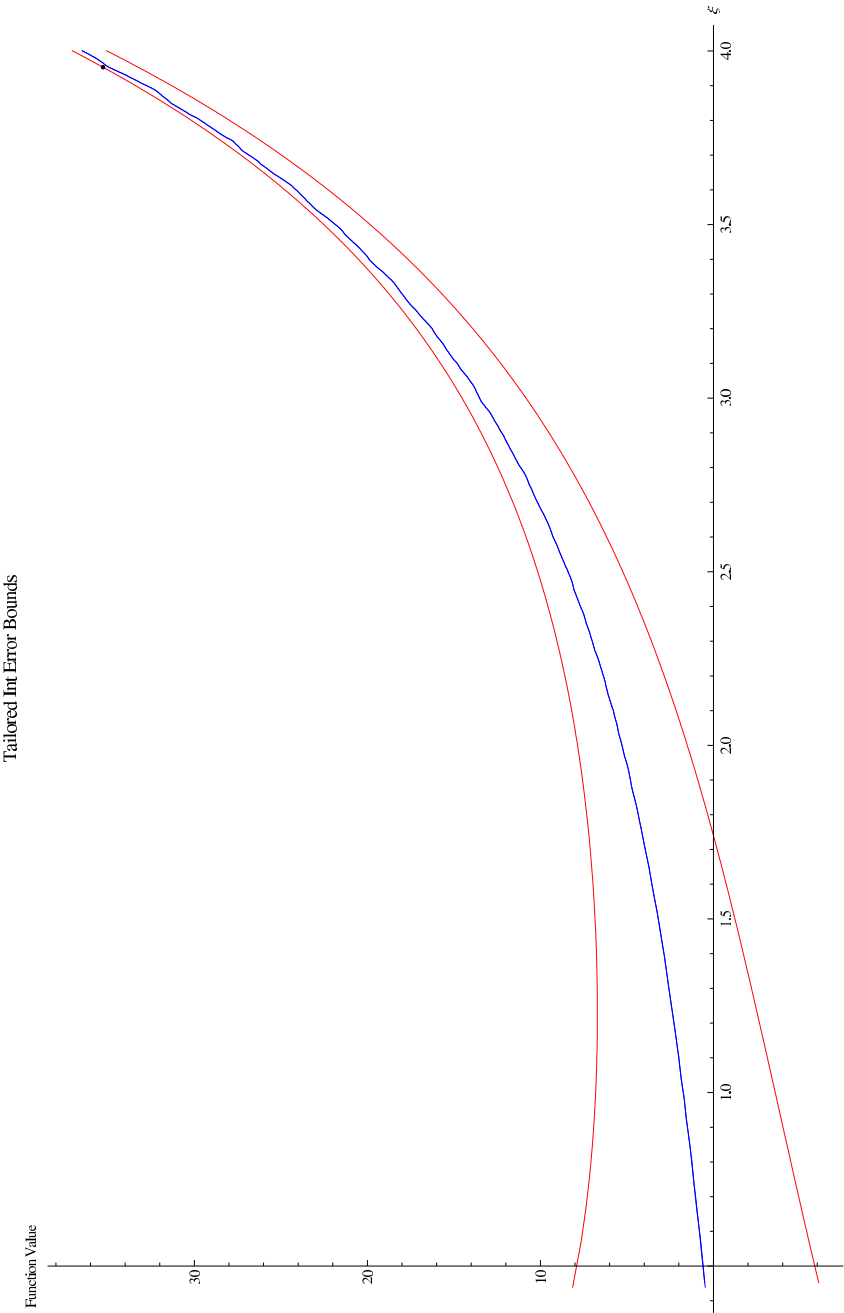
FIGURE 9. The figure is drawn with respect to ξ, which gives the range $n \in \mathbb{N} \mid 11 \leq n \leq 7\,975\,013$, which includes the global minimum point at $n = 6\,862\,489$, $\xi \approx 3.9532140115325856$. Please refer to Table 6.
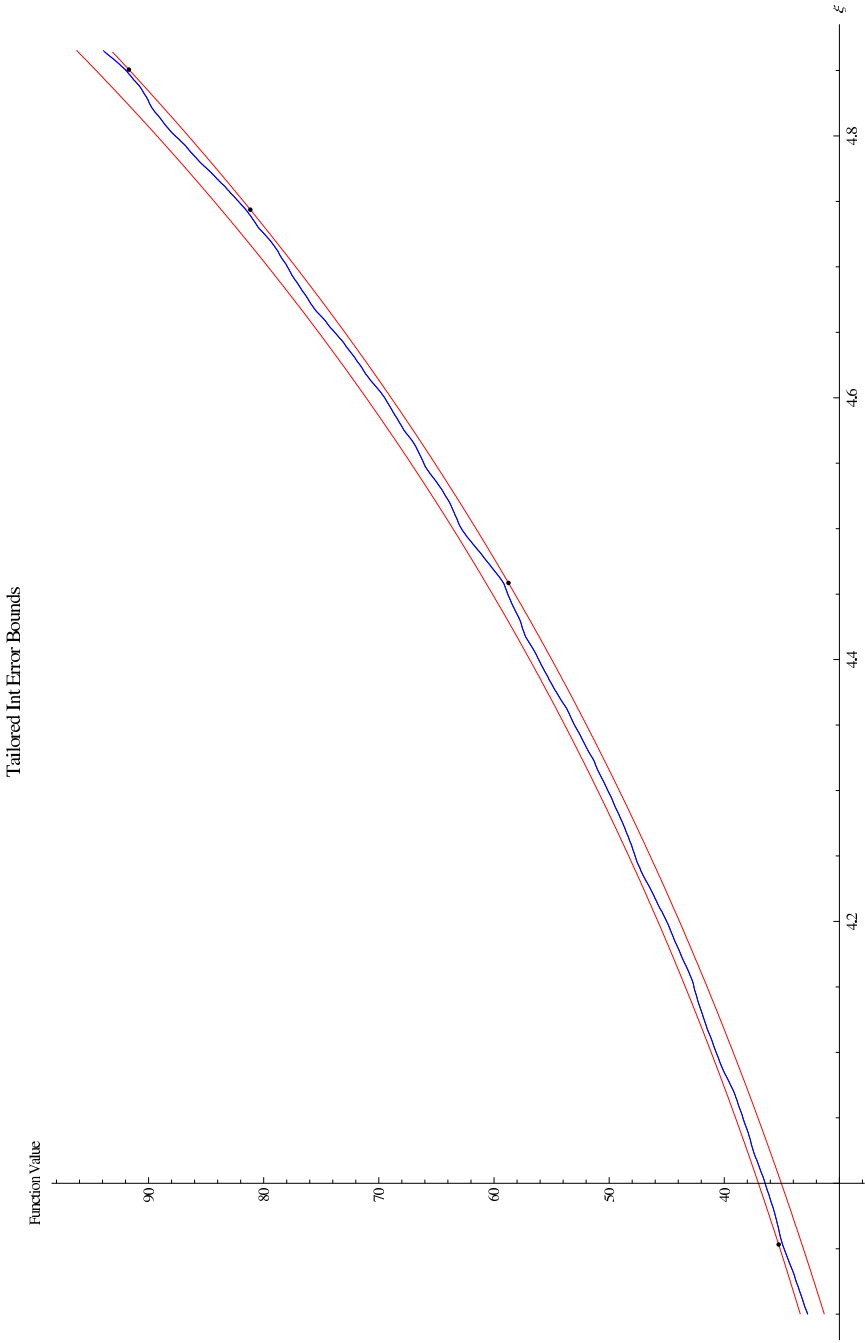
FIGURE 10. The figure is drawn with respect to ξ, which gives the range $n \in \mathbb{N} \mid 5\,799\,991 \leq n \leq 124\,499\,941$.
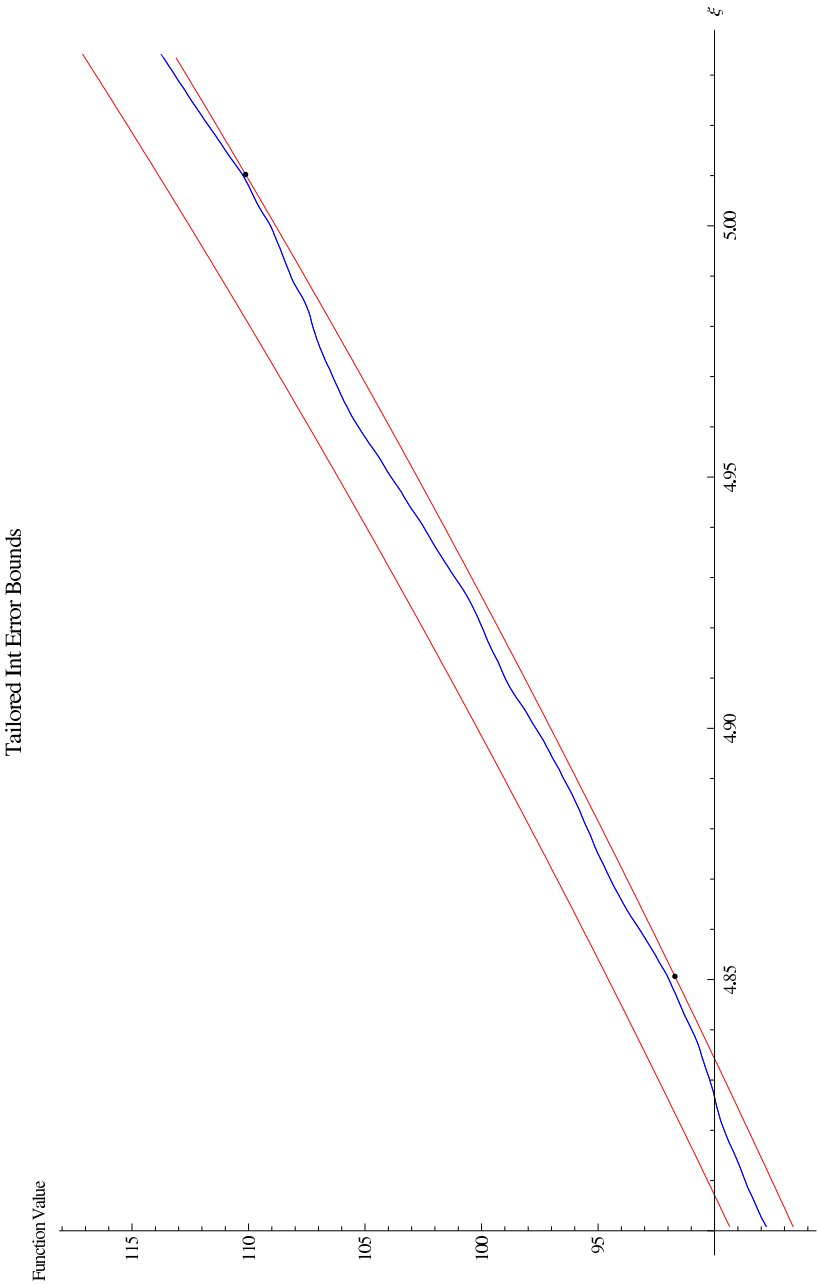
FIGURE 11. The figure is drawn with respect to ξ, which gives the range $n \in \mathbb{N} \mid 101\,499\,997 \le n \le 212\,999\,959$. The graph includes the global minimum at $n = 197\,430\,720$, $\xi \approx 5.010250084067886$. Please refer to Table 6.
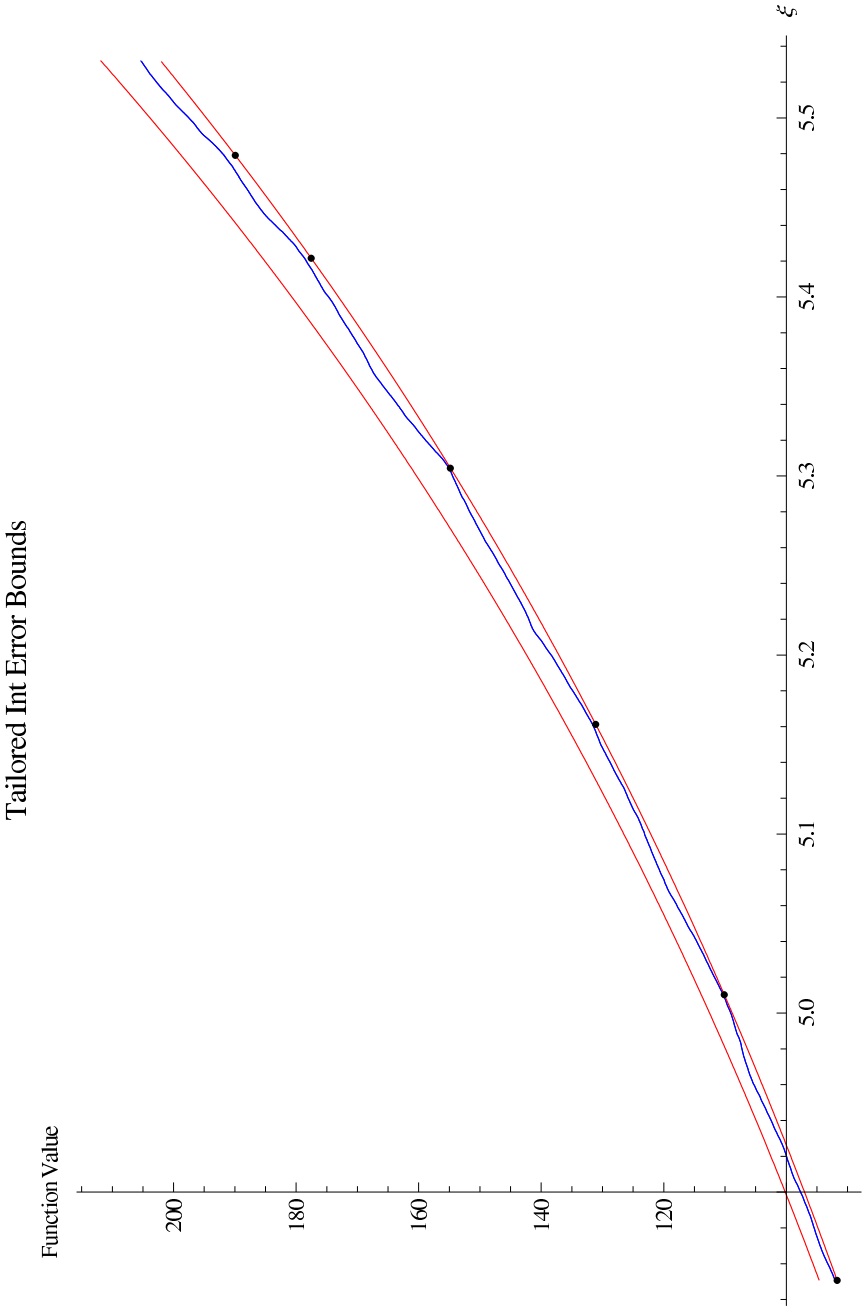
FIGURE 12. The figure is drawn with respect to ξ, which gives the range $n \in \mathbb{N} \mid 118\,888\,261 \le n \le 1\,036\,120\,507$. The graph includes the global minimum at $n = 197\,430\,720$, $\xi \approx 5.010250084067886$, as well as other minima given in Table 6.
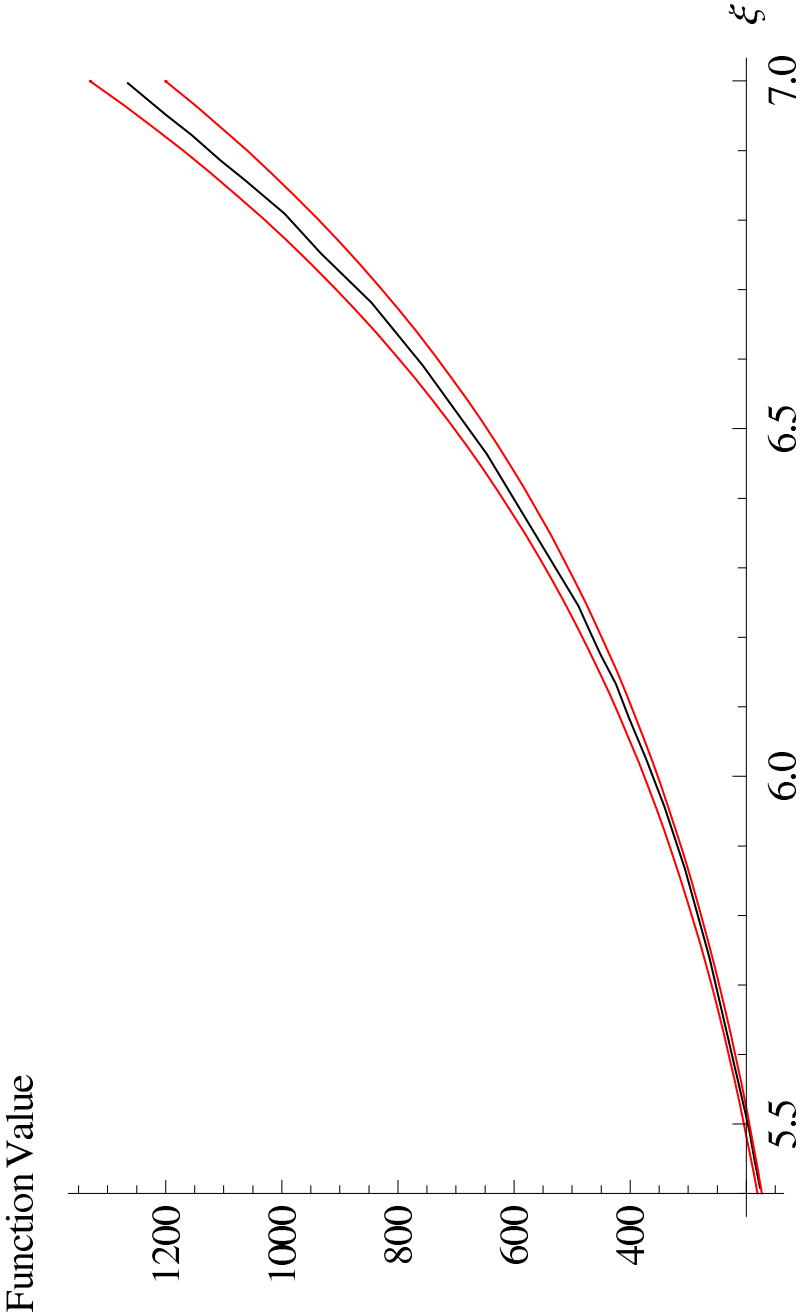
FIGURE 13. The figure is drawn with respect to ξ, which gives the range $n \in \mathbb{N} \mid 700\,000\,001 \le n \le 108\,965\,188\,117$.
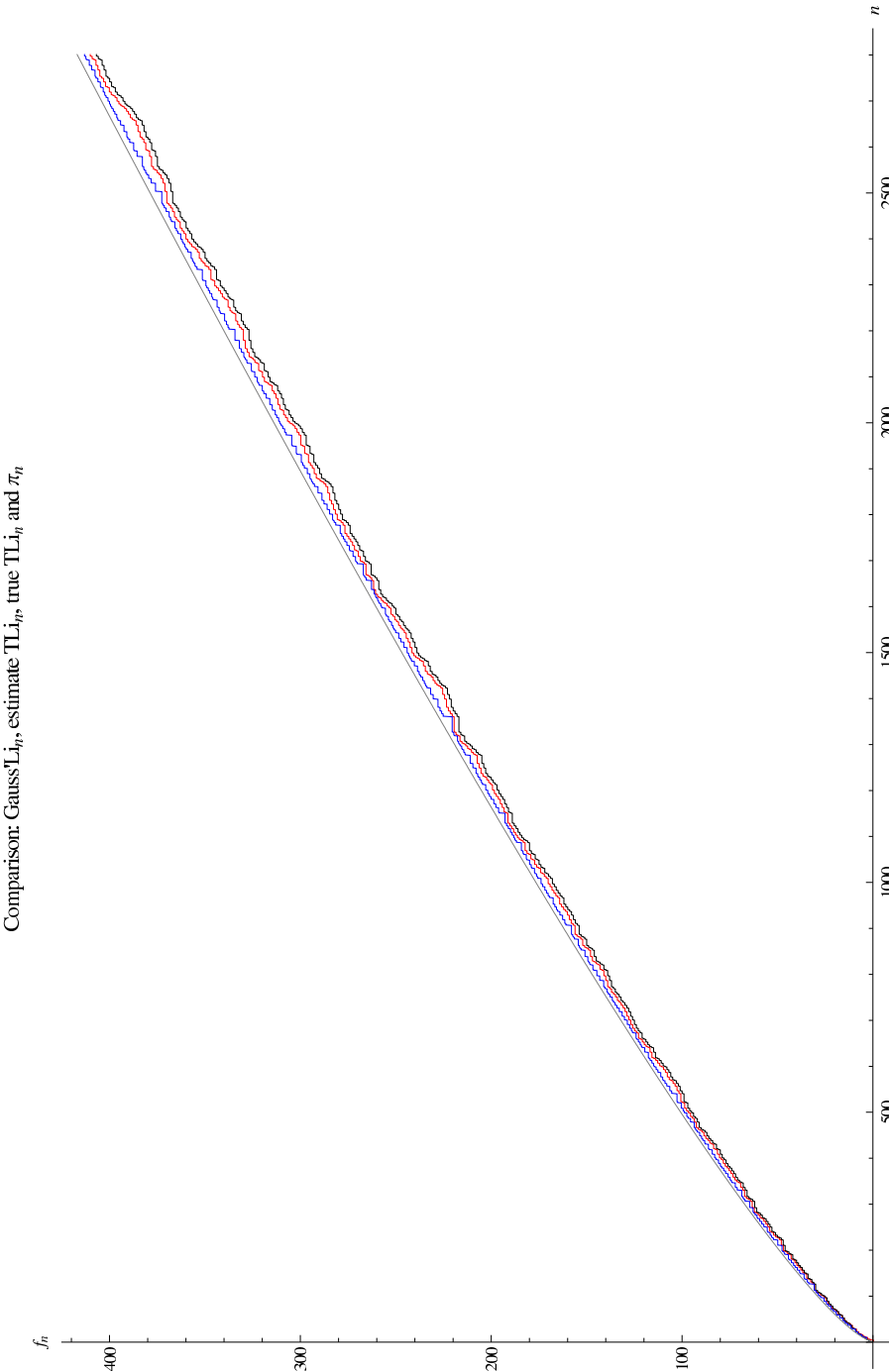
FIGURE 14. Characteristic behaviour of $EstTLi_{(n)}$ (blue) in places where $\pi_{(n)}$ (black) and $TLi_{(n)}$ (red) tend to "sag"

7.2. **Tabular data.**

TABLE 7. $Li_n - \pi_n$ less the $\sum_{p \leq n} 1/p$

Primary Lower Bound			
n	$Li_n - \pi_n - \sum 1/p$	n	$Li_n - \pi_n - \sum 1/p$
47	-0.0110464	467	2.32106
53	0.504175	479	3.26732
59	0.978044	487	3.55977
61	0.450138	491	3.20369
67	0.878096	499	3.49106
71	0.808767	503	3.13251
73	0.262727	509	3.09417
79	0.635637	521	4.01405
83	0.533859	523	3.33175
89	0.86971	541	5.1977
97	1.62457	547	5.14841
101	1.48517	557	5.73052
103	0.907901	563	5.67692
107	0.758054	569	5.62174
109	0.176037	571	4.93517
113	0.0165432	577	4.87793
127	1.93345	587	5.44696
131	1.7489	593	5.38569
137	1.96666	599	5.32296
139	1.36537	601	4.63394
149	2.37092	607	4.56927
151	1.76345	613	4.50318
157	1.9483	617	4.12445
163	2.12441	619	3.43405
167	1.90183	631	4.29647
173	2.06433	641	4.84405
179	2.2192	643	4.15188
181	1.59881	647	3.76864
191	2.50724	653	3.69347
193	1.88247	659	3.617
197	1.63597	661	2.92355
199	1.00914	673	3.76743
211	2.25885	677	3.37995
223	3.48497	683	3.29843
227	3.2191	691	3.52166
229	2.5831	701	4.04804
233	2.31378	709	4.26648
239	2.40774	719	4.78696
241	1.76851	727	5.00076
251	2.58098	733	4.90944
Continued . . .			

TABLE 7. Continued

Primary Lower Bound (Continued)			
257	2.66064	739	4.81701
263	2.73585	743	4.42099
269	2.80674	751	4.62884
271	2.16029	757	4.53313
277	2.22561	761	4.13495
281	1.93238	769	4.33849
283	1.28333	773	3.93891
293	2.0458	787	5.03997
307	3.49711	797	5.53695
311	3.19157	809	6.32988
313	2.53662	811	5.62728
317	2.22881	821	6.11762
331	3.64767	823	5.41439
337	3.6772	827	5.00883
347	4.38819	829	4.30529
349	3.72707	839	4.79081
353	3.40674	853	5.86664
359	3.42525	857	5.45796
367	3.77975	859	4.75289
373	3.7917	863	4.34362
379	3.80094	877	5.41089
383	3.47141	881	4.99983
389	3.47626	883	4.29359
397	3.81292	887	3.88195
401	3.47833	907	5.82244
409	3.80836	911	5.40851
419	4.46549	919	5.58063
421	3.79423	929	6.04396
431	4.4436	937	6.21276
433	3.77086	941	5.79608
439	3.75581	947	5.67092
443	3.41047	953	5.54496
449	3.3918	967	6.58268
457	3.69768	971	6.16336
461	3.34814	977	6.03425
463	2.67195	983	5.90437
The End			

TABLE 8. Theorem 3.6

Maximal Prime Gaps		
Gap start p_i	Actual gap $\mathfrak{g}$	Gap estim. $\mathcal{G}_{(p_i)}$
7	3	3
23	5	9
89	7	19
113	13	21
523	17	36
887	19	43
1129	21	46
1327	33	48
9551	35	79
15683	43	88
19609	51	92
31397	71	101
155921	85	134
360653	95	154
370261	111	155
492113	113	161
1349533	117	187
1357201	131	188
2010733	147	198
4652353	153	222
17051707	179	261
20831323	209	267
47326693	219	294
122164747	221	326
189695659	233	342
191912783	247	343
387096133	249	368
436273009	281	373
1294268491	287	415
1453168141	291	419
2300942549	319	438
3842610773	335	459
4302407359	353	464
10726904659	381	503
20678048297	383	532
22367084959	393	535
25056082087	455	540
42652618343	463	564
127976334671	467	616
182226896239	473	634
241160624143	485	647
297501075799	489	658
Continued . . .		

TABLE 8. Continued

Maximal Prime Gaps (Continued)		
303371455241	499	659
304599508537	513	659
416608695821	515	675
461690510011	531	680
614487453523	533	694
738832927927	539	704
1346294310749	581	735
1408695493609	587	737
1968188556461	601	755
2614941710599	651	770
7177162611713	673	826
13829048559701	715	863
19581334192423	765	883
42842283925351	777	929
90874329411493	803	974
171231342420521	805	1012
218209405436543	905	1028
1189459969825483	915	1136
1686994940955803	923	1159
1693182318746371	1131	1159
43841547845541059	1183	1384
55350776431903243	1197	1401
80873624627234849	1219	1429
203986478517455989	1223	1498
218034721194214273	1247	1503
305405826521087869	1271	1528
352521223451364323	1327	1539
401429925999153707	1355	1549
418032645936712127	1369	1552
804212830686677669	1441	1603
1425172824437699411	1475	1647
The End		