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Construction of the type 2 Poly-Frobenius-Genocchi Polynomials with their certain applications

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Abstract

Motivated by the definition of the type 2 poly-Bernoulli polynomials introduced by Kim-Kim [15], in the present paper, we consider a class of new generating function for the Frobenius-Genocchi polynomials, called the type 2 poly-Frobenius-Genocchi polynomials, by means of the polyexponential function. Then, we derive some useful relations and properties. We show that the type 2 poly-Frobenius-Genocchi polynomias equal a linear combination of the classical Frobenius-Genocchi polynomials and Stirling numbers of the first kind. In a special case, we give a relation between the type 2 poly-Frobenius-Genocchi polynomials and Bernoulli polynomials of order k. Moreover, inspired by the definition of the unipoly-Bernoulli polynomials introduced by Kim-Kim [15], we introduce the unipoly-Frobenius-Genocchi polynomials by means of unipoly function and give multifarious properties including derivative and integral properties. Furthermore, we provide a correlation between the unipoly-Frobenius-Genocchi polynomials and the classical Frobenius-Genocchi polynomials.

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1. Introduction

Special polynomials have their origin in the solution of the differential equations (or partial differential equations) under some conditions. Special polynomials can be defined in a various ways such as by generating functions, by recurrence relations, by p-adic integrals in the sense of fermionic and bosonic, by degenerate versions, etc.

Kim-Kim have introduced polyexponential function in [15] and its degenerate version in [17],[18]. By making use of aforementioned function, they have introduced a new class of some special polynomials. This idea provides a powerfool tool in order to define special numbers and polynomials by making use of polyexponential function. One may see that the notion of polyexponential function form a special class of polynomials because of their great applicability. The importance of these polynomials would be to find applications in analytic number theory, applications in classical analysis and statistics.

Throughout of the paper we make use of the following notations: $\mathbb{N} := \{1, 2, 3, \cdots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Here, as usual, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers.

The Bernoulli $B_n(x)$, Euler $E_n(x)$ and Genocchi $G_n(x)$ polynomials are defined by the following exponential generating functions, respectively:

$$\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}\qquad(|t|<2\pi),\quad \frac{2}{e^t+1}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}\qquad(|t|<\pi) \tag{1.1}$$

and

$$\frac{2t}{e^t+1}=\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!}\qquad(|t|<\pi).$$

One may look at the references [3-7, 10-14, 19, 20, 24] to see the various applications of Bernoulli, Euler and Genocchi polynomials.

Frobenius studied the polynomials $F_n(x\mid u)$ given by

$$\frac{1-u}{e^t-u}e^{xt}=\sum_{n=0}^{\infty}F_n(x\mid u)\frac{t^n}{n!}\quad(u\in\mathbb{C}\setminus\{1\}). \tag{1.2}$$

When $u=-1$, it becomes

$$F_n(x\mid -1)=E_n(x).$$

Owing to relationship with the Euler polynomials as well as their important properties, and in the honor of Frobenius, the aforementioned polynomials denoted by $F_n(x\mid u)$ are called the Frobenius-Euler polynomials, *cf.* [2, 3].

Parallel to (1.2), Yaşar and Özarslan [27] introduced the Frobenius-Genocchi polynomials $G_n^F(x;u)$ given by

$$\frac{(1-u)t}{e^t-u}e^{xt}=\sum_{n=0}^{\infty}G_n^F(x;u)\frac{t^n}{n!} \tag{1.3}$$

since

$$G_n^F(x;-1)=G_n(x).$$

The case $x=0$ in (1.3), $G_n^F(0;u):=G_n^F(u)$ stands for the Frobenius-Genocchi numbers. Several recurrence relations and differential equations are also investigated in [27].

Khan and Srivastava [12] introduced a new class of the generalized Apostol type Frobenius-Genocchi polynomials and investigated some properties and relations including implicit summation formulae and various symmetric identities. Moreover a relation in between Array-type polynomials, Apostol-Bernoulli polynomials and generalized Apostol-type Frobenius-Genocchi polynomials is also given in [12]. Wani *et al.* [26] considered Gould-Hopper based Frobenius-Genocchi polynomials and then, summation formulae and operational rule for these polynomials.

The Bernoulli polynomials of the second kind are defined by means of the following generating function

$$\sum_{n=0}^{\infty}b_n(x)\frac{t^n}{n!}=\frac{t}{\log(1+t)}(1+t)^x. \tag{1.4}$$

When $x=0$, $b_n(0):=b_n$ are called the Bernoulli numbers of the second kind, *cf.* [17].

It is well-known from (1.4) that

$$\frac{t}{\log(1+t)}(1+t)^{x-1}=\sum_{n=0}^{\infty}B_n^{(n)}(x)\frac{t^n}{n!}, \tag{1.5}$$

where $B_n^{(k)}(x)$ are the Bernoulli polynomials of order k which are given by the following generating function

$$\sum_{n=0}^{\infty}B_n^{(k)}(x)\frac{t^n}{n!}=\left(\frac{t}{e^t-1}\right)^k e^{xt}.$$

By (1.4) and (1.5), it is clear that $B_n^{(n)}(x+1)=b_n(x)$, see [17].

Very recently, Kim-Kim [15] performed to generalize the Bernoulli polynomials by using polyexponential function

$$e_k(t)=\sum_{n=1}^{\infty}\frac{t^n}{(n-1)!n^k} \tag{1.6}$$

as inverse to the polylogarithm function

$$Li_k(t)=\sum_{n=1}^{\infty}\frac{t^n}{n^k} \qquad (|t|<1;k\in\mathbb{Z}) \tag{1.7}$$

given by

$$\sum_{n=0}^{\infty}\beta_n^{(k)}(x)\frac{t^n}{n!}=\frac{e_k(\log(1+t))}{e^t-1}e^{xt} \qquad (k\in\mathbb{Z}). \tag{1.8}$$

Upon setting $x=0$ in (1.8), $\beta_n^{(k)}(0):=\beta_n^{(k)}$ are called the type 2 poly-Bernoulli numbers.

Since

$$e_1(t)=e^t-1,$$

it is worthy to note that

$$\beta_n^{(1)}(x):=B_n(x).$$

Kim-Kim [15] also introduced unipoly function $u_k(x|p)$ attached to p being any arithmetic function that is a real or complex valued function defined on the set of positive integer as follows:

$$u_k(x|p)=\sum_{n=1}^{\infty}\frac{p(n)}{n^k}x^n, \qquad (k\in\mathbb{Z}). \tag{1.9}$$

It follows from (1.9) that

$$u_k(x|1)=\sum_{n=1}^{\infty}\frac{x^n}{n^k}=Li_k(x)$$

is the polylogarithm function as given in (1.7). The unipoly function attached to p satisfies the following properties for $k\geq 2$,

$$\frac{d}{dx}u_k(x|p)=\frac{1}{x}u_{k-1}(x|p)$$

and

$$u_k(x|p)=\int_0^x\frac{1}{t}\underbrace{\int_0^t\frac{1}{t}\cdots\int_0^t\frac{1}{t}}_{(k-2)\text{ times}}u_1(x|p)\,dtdt\cdots dt.$$

By means of the unipoly function, Kim-Kim [15] defined unipoly-Bernoulli polynomials as follows:

$$\sum_{n=0}^{\infty}B_{n,p}^{(k)}(x)\frac{t^n}{n!}=\frac{u_k(1-e^{-t}|p)}{1-e^{-t}}e^{xt}. \tag{1.10}$$

They provide several formulae and relations for these polynomials, see [15].

Kwon and Jang [21] defined the type 2 poly-Apostol-Bernoulli polynomials and provided some properties for them. Moreover, by making use of unipoly function, they considered the type 2 unipoly-Apostol-Bernoulli numbers and proved some basic properties.

The Stirling numbers of the first kind $S_1(n,k)$ and the Stirling numbers of the second kind $S_2(n,k)$ are defined by means of the following generating functions:

$$\frac{(\log(1+t))^k}{k!}=\sum_{n=0}^{\infty}S_1(n,k)\frac{t^n}{n!} \text{ and } \frac{(e^t-1)^k}{k!}=\sum_{n=0}^{\infty}S_2(n,k)\frac{t^n}{n!}. \tag{1.11}$$

From (1.11), we get the following relations for $n \geq 0$:

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)\, x^k \text{ and } x^n = \sum_{k=0}^{n} S_1(n,k)\,(x)_k\,, \tag{1.12}$$

where $(x)_0 = 1$ and $(x)_n = x(x-1)(x-2)\cdots(x-n+1)$, *cf.* [2, 4, 5].

An outline of this paper is as follows. Section 2 deals with the contruction of a class of new generating function for the Frobenius-Genocchi polynomials, called the type 2 poly-Frobenius-Genocchi polynomials, by means of the polyexponential function and also provides some useful relations and properties. In addition, this section shows that the type 2 poly-Frobenius-Genocchi polynomias equal a linear combination of the classical Frobenius-Genocchi polynomials and Stirling numbers of the first kind. Section 3 gives the definition of the unipoly-Frobenius-Genocchi polynomials by means of unipoly function and includes several properties including derivative and integral properties. Furthermore, a correlation between the unipoly-Frobenius-Genocchi polynomials and the classical Frobenius-Genocchi polynomials is stated in Section 3. In the last section, the results obtained in this paper are examined.

2. The type 2 Poly-Frobenius-Genocchi Polynomials

Motivated and inspired by the definition of the type 2 poly-Bernoulli polynomials in (1.8) introduced by Kim-Kim [15], in this paper, we consider the following Definition 1 by means of the polyexponential function.

Definition 1. *Let $k \in \mathbb{Z}$. The type 2 poly-Frobenius-Genocchi polynomials are defined via the following exponential generating function (in a suitable neigbourhood of $t = 0$) including the polyexponential function as given below:*

$$\frac{e_k(\log(1+(1-u)t))}{e^t - u} e^{xt} = \sum_{n=0}^{\infty} G_n^{(F,k)}(x;u)\frac{t^n}{n!}. \tag{2.1}$$

At the value $x = 0$ in (2.1), $G_n^{(F,k)}(0;u) := G_n^{(F,k)}(u)$ will be called type 2 poly-Frobenius-Genocchi numbers.

Remark 1. *Taking $k = 1$ in (2.1) yields $G_n^{(F,1)}(x;u) := G_n^{F}(x;u)$.*

Remark 2. *Taking $k = 1$ and $u = -1$ in (2.1) gives $G_n^{(F,1)}(x;-1) := G_n(x)$.*

By Definition 1, we consider that

$$\begin{aligned}\sum_{n=0}^{\infty} G_n^{(F,k)}(x;u)\frac{t^n}{n!} &= \frac{e_k(\log(1+(1-u)t))}{e^t-u} e^{xt} \\ &= \left(\sum_{n=0}^{\infty} G_n^{(F,k)}(u)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} x^n\frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l} G_{n-l}^{(F,k)}(u)\, x^l\right)\frac{t^n}{n!}.\end{aligned}$$

Hence, we give the following theorem.

Theorem 1. *The following relation*

$$G_n^{(F,k)}(x;u) = \sum_{l=0}^{n}\binom{n}{l} G_{n-l}^{(F,k)}(u)\, x^l \tag{2.2}$$

is valid for $k \in \mathbb{Z}$ and $n \geq 0$.

A relation between the type 2 poly-Frobenius-Genocchi polynomials and the classical Frobenius-Genocchi polynomials is stated in the following theorem.

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Theorem 2. *For $k \in \mathbb{Z}$ and $n \geq 0$, we have*

$$G_n^{(F,k)}(x;u) = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} \frac{1}{(m+1)^{k-1}} S_1(l+1,m+1) \frac{(1-u)^l}{l+1} G_{n-l}^F(x;u). \tag{2.3}$$

Proof. From (1.6), (1.11) and (2.1), we observe that

$$\begin{aligned}
\sum_{n=0}^{\infty} G_n^{(F,k)}(x;u) \frac{t^n}{n!} &= \frac{e_k(\log(1+(1-u)t))}{e^t-u} e^{xt} \\
&= \frac{e^{xt}}{e^t-u} \sum_{m=1}^{\infty} \frac{(\log(1+(1-u)t))^m}{(m-1)!m^k} \\
&= \frac{e^{xt}}{e^t-u} \sum_{m=0}^{\infty} \frac{1}{(m+1)^k} \frac{(\log(1+(1-u)t))^{m+1}}{m!} \\
&= \frac{e^{xt}}{e^t-u} \sum_{m=0}^{\infty} \frac{1}{(m+1)^{k-1}} \sum_{n=m+1}^{\infty} S_1(n,m+1) \frac{((1-u)t)^n}{n!} \\
&= \frac{(1-u)t}{e^t-u} e^{xt} \sum_{m=0}^{\infty} \frac{1}{(m+1)^{k-1}} \sum_{n=m}^{\infty} S_1(n+1,m+1) \frac{(1-u)^n}{n+1} \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} G_n^F(x;u) \frac{t^n}{n!} \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \frac{1}{(m+1)^{k-1}} \sum_{n=m}^{\infty} S_1(n+1,m+1) \frac{(1-u)^n}{n+1} \right) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} \frac{1}{(m+1)^{k-1}} S_1(l+1,m+1) \frac{(1-u)^l}{l+1} G_{n-l}^F(x;u) \right) \frac{t^n}{n!},
\end{aligned}$$

which means the asserted result in (2.3). □

The immediate results of the Theorem 2 are stated below.

Corollary 1. *For $k \in \mathbb{Z}$ and $n \geq 0$, we have*

$$G_n^{(F,k)}(u) = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} \frac{1}{(m+1)^{k-1}} S_1(l+1,m+1) \frac{(1-u)^l}{l+1} G_{n-l}^F(u). \tag{2.4}$$

Corollary 2. *Taking $k=1$ in Theorem 2 gives*

$$G_n^F(x;u) = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} S_1(l+1,m+1) \frac{(1-u)^l}{l+1} G_{n-l}^F(x;u).$$

Corollary 3. *Taking $k=1$ and $u=-1$ in Theorem 2 reduces*

$$G_n(x) = \sum_{l=0}^{n} \sum_{m=0}^{l} \frac{\binom{n}{l}}{l+1} 2^l \; S_1(l+1,m+1) G_{n-l}(x)$$

and

$$\sum_{l=1}^{n} \sum_{m=0}^{l} \frac{\binom{n}{l}}{l+1} 2^l \; S_1(l+1,m+1) G_{n-l}(x) = 0.$$

Let $s \in \mathbb{C}$ and $k \in \mathbb{Z}$ with $k \geq 1$. We consider the function $\eta_{k,u}$ by the representation of improper integral as follows

$$\eta_{k,u}(s) := \frac{(1-u)^{s-1}}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t-u} e_k(\log(1+(1-u)t))\, dt, \tag{2.5}$$

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where $\Gamma(s)$ is the well-known gamma function defined by

$$\Gamma(s)=\int_0^\infty t^{s-1}e^t dt \qquad (\Re(s)>0).$$

By (2.5), we observe that

$$\begin{aligned}\eta_{1,u}(s) &= \frac{(1-u)^{s-1}}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}}{e^t-u}e_1(\log(1+(1-u)t))\,dt\\ &= \frac{(1-u)^{s}}{\Gamma(s)}\int_0^\infty \frac{t^{s}}{e^t-u}dt\\ &= \frac{(1-u)^{s}}{\Gamma(s)}\int_0^\infty \frac{t^{s}e^{-t}}{1-ue^{-t}}dt\\ &= (1-u)^s\Phi(u,s+1,1),\end{aligned}$$

where

$$\begin{aligned}\Phi(z,s,a) &= \sum_{n=0}^\infty \frac{z^n}{(n+a)^s}\\ &= \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-at}}{1-ze^{-t}}dt\end{aligned}$$

with $\mathrm{Re}(a)>0$; $\mathrm{Re}(s)>0$ when $|z|\leq 1$ $(z\neq 1)$; $\mathrm{Re}(s)>1$ when $|z|=1$ is the Hurwitz-Lerch zeta function, *cf.* [9] and [23]. Some special cases of $\Phi(z,s,a)$ are listed below:

- the Riemann zeta function
$$\Phi(1,s,1)=\zeta(s), \quad (\Re(s)>1)$$
- the Euler-zeta function
$$\Phi(-1,s,1)=\zeta_E(s), \ (\Re(s)>0)$$
- the polylogarithm function
$$z\Phi(z,k,1)=Li_k(z),$$
see [1] and [25] for details.

Hence, we state the following corollary.

Corollary 4. *The following equality holds true:*

$$\eta_{1,u}(s)=(1-u)^s\Phi(u,s+1,1). \tag{2.6}$$

In view of calculations above, we observe that $\eta_{k,u}(s)$ is holomorphic function for $\mathrm{Re}(s)>0$ because of the comparison test as $e_k(\log(1+(1-u)t))\leq e_1(\log(1+(1-u)t))$ with the assumption $(1-u)t\geq 0$. By (2.5), we have

$$\begin{aligned}\eta_{k,u}(s) &= \frac{(1-u)^{s-1}}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}}{e^t-u}e_k(\log(1+(1-u)t))\,dt\\ &= \frac{(1-u)^{s-1}}{\Gamma(s)}\int_0^1 \frac{t^{s-1}}{e^t-u}e_k(\log(1+(1-u)t))\,dt \qquad (2.7)\\ &\quad+\frac{(1-u)^{s-1}}{\Gamma(s)}\int_1^\infty \frac{t^{s-1}}{e^t-u}e_k(\log(1+(1-u)t))\,dt.\end{aligned}$$

The second integral in (2.7) converges absolutely for any $s\in\mathbb{C}$ and thus, the second term on the right hand side vanishes at non-positive integers. Hence, we get

$$\lim_{s\to -m}\left|\frac{(1-u)^{s-1}}{\Gamma(s)}\int_1^\infty \frac{t}{e^t-u}e_k(\log(1+(1-u)t))\,dt\right|\leq \frac{(1-u)^{-m-1}}{\Gamma(-m)}M=0 \tag{2.8}$$

since
$$\Gamma(s)\Gamma(1-s)=\frac{\pi}{\sin(\pi s)}.$$
Moreover, for $\mathrm{Re}(s)>0$, the first integral in (2.7) can be written as
$$\begin{aligned}
&\frac{(1-u)^{s-1}}{\Gamma(s)}\int_0^1\frac{e_k(\log(1+(1-u)t))}{e^t-u}t^{s-1}dt\\
&=\frac{(1-u)^{s-1}}{\Gamma(s)}\sum_{n=0}^{\infty}\frac{G_n^{(F,k)}(u)}{n!}\int_0^1 t^{n+s-1}dt\\
&=\frac{(1-u)^{s-1}}{\Gamma(s)}\sum_{n=0}^{\infty}\frac{G_n^{(F,k)}(u)}{n!}\frac{1}{n+s},
\end{aligned}\tag{2.9}$$
which defines an entire function of s. Hence, we arrive that $\eta_{k,u}(s)$ can be continued to an entire function of s. From (2.7) and (2.8), we attain
$$\begin{aligned}
\eta_{k,u}(-m) &= \lim_{s\to -m}\frac{(1-u)^{s-1}}{\Gamma(s)}\int_0^1\frac{e_k(\log(1+(1-u)t))}{e^t-u}t^{s-1}dt\\
&= \lim_{s\to -m}\frac{(1-u)^{s-1}}{\Gamma(s)}\sum_{n=0}^{\infty}\frac{G_n^{(F,k)}(u)}{n!(n+s)}\\
&= \cdots+0+\cdots+0+\lim_{s\to -m}\frac{(1-u)^{s-1}}{\Gamma(s)}\frac{G_m^{(F,k)}(u)}{m!(m+s)}+0+0+\cdots\\
&= \lim_{s\to -m}\frac{(1-u)^{s-1}}{m+s}\frac{\Gamma(1-s)\sin(\pi s)}{\pi}\frac{G_m^{(F,k)}(u)}{m!}\\
&= (1-u)^{-m-1}\Gamma(1+m)\cos(\pi m)\frac{G_m^{(F,k)}(u)}{m!}\\
&= (1-u)^{-m-1}(-1)^m G_m^{(F,k)}(u).
\end{aligned}$$
Thus, we get the following theorem.

Theorem 3. *Let $k\in\mathbb{N}$. The function $\eta_{k,u}(s)$ has an analytic continuation to an function of $s\in\mathbb{C}$, and the special values at nonpositive integers are given by*
$$\eta_{k,u}(-m)=(1-u)^{-m-1}(-1)^m G_m^{(F,k)}(u)\qquad(m\in\mathbb{N}_0).$$
Taking $k=1$ in Theorem 3 and by (2.6), we have the following corollary.

Corollary 5. *The following identity holds true:*
$$\Phi(u,-m+1,1)=\frac{(-1)^m}{1-u}G_m^F(u).$$
Corollary 6. *Upon setting $k=1$ and $u=-1$ in Theorem 3 gives*
$$\zeta_E(1-m)=\frac{(-1)^m}{2}G_m(u).$$
The following derivate property holds true (*cf.* [15])
$$\frac{d}{dx}e_k(x)=\frac{1}{x}e_{k-1}(x)\tag{2.10}$$
and the following integral representations also holds true for $k>1$
$$e_k(x)=\int_0^x\frac{1}{t}\underbrace{\int_0^t\frac{1}{t}\cdots\int_0^t\frac{1}{t}}_{(k-2)\text{ times}}\left(e^t-1\right)dtdt\cdots dt.\tag{2.11}$$
Now, we give the following theorem.

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Theorem 4. *For $n \in \mathbb{N}_0$, we have*

$$G_n^{(F,2)}(u) = \sum_{l=0}^{n} \binom{n}{l} (1-u)^l \frac{B_l^{(l)}}{l+1} G_{n-l}^F(u).$$

Proof. By (2.10), we first consider that

$$\begin{aligned} \frac{d}{dx} e_k(\log(1+(1-u)x)) &= \sum_{n=1}^{\infty} \frac{(\log(1+(1-u)x))^n}{(n-1)!n^k} \\ &= \frac{1-u}{1+(1-u)x} \sum_{n=1}^{\infty} \frac{(\log(1+(1-u)x))^{n-1}}{(n-1)!n^{k-1}} \\ &= \frac{1-u}{(1+(1-u)x)\log(1+(1-u)x)} e_{k-1}(\log(1+(1-u)x)). \end{aligned} \tag{2.12}$$

From (2.11) and (2.12), for $k > 1$, we can write

$$\begin{aligned} \sum_{n=0}^{\infty} G_n^{(F,k)}(u) \frac{t^n}{n!} &= \frac{(1-u)^{k-1}}{e^t - u} \\ &\times \int_0^x \frac{1}{(1+(1-u)t)\log(1+(1-u)t)} \\ &\times \underbrace{\int_0^t \frac{1}{(1+(1-u)t)\log(1+(1-u)t)} \cdots \int_0^t \frac{(1-u)t}{(1+(1-u)t)\log(1+(1-u)t)}}_{(k-2)\text{ times}} dt dt \cdots dt. \end{aligned}$$

Hence, we acquire

$$\begin{aligned} \sum_{n=0}^{\infty} G_n^{(F,2)}(u) \frac{x^n}{n!} &= \frac{1-u}{e^x - u} \int_0^x \frac{(1-u)t}{(1+(1-u)t)\log(1+(1-u)t)} dt \\ &= \frac{1-u}{e^x - u} \int_0^x \sum_{n=0}^{\infty} (1-u)^n B_n^{(n)} \frac{t^n}{n!} dt \\ &= \frac{(1-u)x}{e^x - u} \sum_{n=0}^{\infty} \frac{(1-u)^n B_n^{(n)}}{n+1} \frac{x^n}{n!} \\ &= \left(\sum_{n=0}^{\infty} G_n^F(u) \frac{x^n}{n!} \right) \left(\sum_{n=0}^{\infty} \frac{(1-u)^n B_n^{(n)}}{n+1} \frac{x^n}{n!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} (1-u)^l \frac{B_l^{(l)}}{l+1} G_{n-l}^F(u) \right) \frac{x^n}{n!}. \end{aligned}$$

Thus, we have

$$G_n^{(F,2)}(u) = \sum_{l=0}^{n} \binom{n}{l} (1-u)^l \frac{B_l^{(l)}}{l+1} G_{n-l}^F(u).$$

This finalizes the proof of the theorem. □

3. The Unipoly-Frobenius-Genocchi Polynomials

Motivated and inspired by the definition of the unipoly-Bernoulli polynomials in (1.10) given by Kim-Kim [15], we introduce unipoly-Frobenius-Genocchi polynomials by means of the unipoly function attached to p given in (1.9) as follows:

$$\sum_{n=0}^{\infty} G_{n,p}^{(F,k)}(x;u) \frac{t^n}{n!} = \frac{u_k(\log(1+(1-u)t)|p)}{e^t - u} e^{xt}. \tag{3.1}$$

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Note that taking $x=0$ in (3.1), $G_{n,p}^{(F,k)}(0;u):=G_{n,p}^{(F,k)}(u)$ are called the unipoly-Frobenius-Genocchi numbers.

By (3.1), we consider that

$$\begin{aligned}\sum_{n=0}^{\infty}G_{n,p}^{(F,k)}(x;u)\frac{t^n}{n!} &= \frac{u_k(\log(1+(1-u)t)\,|p)}{e^t-u}e^{xt}\\ &= \sum_{n=0}^{\infty}G_{n,p}^{(F,k)}(u)\frac{t^n}{n!}\sum_{n=0}^{\infty}\frac{x^nt^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}G_{n-l,p}^{(F,k)}(u)\,x^l\right)\frac{t^n}{n!}.\end{aligned}$$

Hence, we give the following theorem.

Theorem 5. *The following relation*

$$G_{n,p}^{(F,k)}(x;u)=\sum_{l=0}^{n}\binom{n}{l}G_{n-l,p}^{(F,k)}(u)\,x^l$$

is ture for $k\in\mathbb{Z}$ and $n\geq 0$.

We observe that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{d}{dx}G_{n,p}^{(F,k)}(x;u)\frac{t^n}{n!} &= \frac{u_k(\log(1+(1-u)t)\,|p)}{e^t-u}\frac{d}{dx}e^{xt}\\ &= \sum_{n=0}^{\infty}G_{n,p}^{(F,k)}(x;u)\frac{t^{n+1}}{n!}.\end{aligned}$$

Therefore, we give the following theorem.

Theorem 6. *Let $k\in\mathbb{Z}$ and $n\geq 0$. We have the following derivative rule*

$$\frac{d}{dx}G_{n,p}^{(F,k)}(x;u)=nG_{n-1,p}^{(F,k)}(x;u). \tag{3.2}$$

By Theorem 6, we consider that

$$\begin{aligned}\int_{\alpha}^{\beta}G_{n,p}^{(F,k)}(x;u)\,dx &= \frac{1}{n+1}\int_{\alpha}^{\beta}\frac{d}{dx}G_{n+1,p}^{(F,k)}(x;u)\,dx\\ &= \frac{G_{n+1,p}^{(F,k)}(\beta;u)-G_{n+1,p}^{(F,k)}(\alpha;u)}{n+1}.\end{aligned}$$

Thus, we provide the following theorem.

Theorem 7. *Let $k\in\mathbb{Z}$ and $n\geq 0$. We have the following integral rule*

$$\int_{\alpha}^{\beta}G_{n,p}^{(F,k)}(x;u)\,dx=\frac{G_{n+1,p}^{(F,k)}(\beta;u)-G_{n+1,p}^{(F,k)}(\alpha;u)}{n+1}.$$

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Upon setting $p(n)=\frac{1}{\Gamma(n)}$ in (3.1), we acquire

$$\begin{aligned}\sum_{n=0}^{\infty}G_{n,\frac{1}{\Gamma}}^{(F,k)}(u)\frac{t^n}{n!} &= \frac{1}{e^t-u}u_k\left(\log(1+(1-u)t)\,\Big|\,\frac{1}{\Gamma}\right)\\ &= \frac{1}{e^t-u}\sum_{m=1}^{\infty}\frac{(\log(1+(1-u)t))^m}{m^k(m-1)!}\\ &= \frac{1}{e^t-u}e_k(\log(1+(1-u)t))\\ &\sum_{n=0}^{\infty}G_{n}^{(F,k)}(u)\frac{t^n}{n!},\end{aligned}$$

which gives the following relation

$$G_{n,\frac{1}{\Gamma}}^{(F,k)}(u)=G_{n}^{(F,k)}(u). \tag{3.3}$$

From (1.9) and (3.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty}G_{n,p}^{(F,k)}(u)\frac{t^n}{n!} &= \frac{1}{e^t-u}\sum_{m=1}^{\infty}\frac{p(m)}{m^k}(\log(1+(1-u)t))^m\\ &= \frac{1}{e^t-u}\sum_{m=0}^{\infty}\frac{p(m+1)(m+1)!}{(m+1)^k}\frac{(\log(1+(1-u)t))^{m+1}}{(m+1)!}\\ &= \frac{1}{e^t-u}\sum_{m=0}^{\infty}\frac{p(m+1)(m+1)!}{(m+1)^k}\sum_{n=m+1}^{\infty}S_1(n,m+1)(1-u)^n\frac{t^n}{n!}\\ &= \frac{(1-u)t}{e^t-u}\sum_{m=0}^{\infty}\frac{p(m+1)(m+1)!}{(m+1)^k}\sum_{n=m}^{\infty}S_1(n+1,m+1)(1-u)^n\frac{t^n}{(n+1)!}\\ &= \sum_{n=0}^{\infty}G_n^F(u)\frac{t^n}{n!}\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\frac{p(m+1)(m+1)!}{(m+1)^k}\frac{S_1(n+1,m+1)}{n+1}(1-u)^n\right)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}\frac{p(m+1)(m+1)!}{(m+1)^k}\frac{S_1(l+1,m+1)}{l+1}(1-u)^l G_{n-l}^F(u)\right)\frac{t^n}{n!},\end{aligned}$$

which yields the following theorem.

Theorem 8. *For $k\in\mathbb{Z}$ and $n\geq 0$, we have*

$$G_{n,p}^{(F,k)}(u)=\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}\frac{p(m+1)(m+1)!}{(m+1)^k}\frac{S_1(l+1,m+1)}{l+1}(1-u)^l G_{n-l}^F(u). \tag{3.4}$$

Particularly, for $p(n)=\frac{1}{\Gamma(n)}$,

$$G_{n,\frac{1}{\Gamma}}^{(F,k)}(u)=\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}\frac{m+1}{(m+1)^k}\frac{S_1(l+1,m+1)}{l+1}(1-u)^l G_{n-l}^F(u).$$

4. Conclusion

Motivated by the definition of the type 2 poly-Bernoulli polynomials introduced by Kim-Kim [15], in the present paper, we have considered a class of new generating function for the Frobenius-Genocchi polynomials, called the type 2 poly-Frobenius-Genocchi polynomials, by means of the polyexponential function. Then, we have derived some useful relations and properties. We have showed that the type 2 poly-Frobenius-Genocchi polynomias equal a linear combination of the classical Frobenius-Genocchi polynomials and Stirling numbers of the first kind. In a special case, we have given a relation between the type 2 poly-Frobenius-Genocchi polynomials and Bernoulli polynomials of order k. Moreover, inspired by the definition of the unipoly-Bernoulli polynomials introduced by Kim-Kim [15] we have introduced the unipoly-Frobenius-Genocchi polynomials by means of unipoly function and have given multifarious properties including derivative and integral properties. Furthermore, we have provided a correlation between the unipoly-Frobenius-Genocchi polynomials and the classical Frobenius-Genocchi polynomials.

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