

Analysis on the evolution of the effective G_{eff} from Linear Nash-Green fluctuations

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From the linear Nash-Green fluctuations of background metric, we present the perturbation equations in an embedded four space-time. In the context of a five-dimensional bulk, we show that the perturbations are only propagated by the gravitational tensorial field equation. In a Newtonian conformal gauge, we study the matter density evolution in sub-horizon regime and on how such scale may be affected by the extrinsic curvature. We use the “extended Gold 2018” growth dataset with 25 datapoints and the best fit Planck2018/ Λ CDM parameters. Hence, we determine the evolution equation for the density growth $\delta(a)$ as a function of the expansion factor a as a result from the embedded equations of the background geometry. By using solar constraints, we analyse the evolution of the effective Newtonian constant G_{eff} and showing that applying Taylor expansion to the $G_{eff}(a)$ function under the constraint of time-derivative of $G_{eff,0} = G$, G is the bare Newtonian constant, at $a = 1$ in matter domination era, we get an agreement with Big Bang Nucleosynthesis (BBN) and also an alleviation of the $3\text{-}\sigma$ tension to $1\text{-}\sigma$ contour between $(\sigma_8\text{-}\Omega_m)$ of the observations from Cosmic Microwave Background (CMB) and Large Scale Structure (LSS) probes.

I. INTRODUCTION

The Occam’s razor is one of the cornerstone philosophical principles in science that states that the simplest solution of a problem must be followed in detriment of the complex ones. In this realization, the Λ CDM model has been the simplest and successful solution to deal with the accelerated expansion of the universe as corroborated for several independent observations in the last two decades [1–11]. Despite its success, the Λ CDM model has important drawbacks that must be taken into account. For instance, the main components of Λ CDM model lack a fundamental explanation about the nature of the cosmological constant Λ and also the (Cold) Dark Matter (CDM) problem [12–18].

In this paper, we focus on studying the $3\text{-}\sigma$ tension in the σ_8 -contours revealed by the mismatch of the data inferred from Planck Cosmic Microwave Background (CMB) radiation and Large Scale Structure (LSS) observations considering the concordance Λ CDM model as

a background. The σ_8 is the r.m.s amplitude of matter density at a scale of a radius $R \sim 8h\text{Mpc}^{-1}$ within a enclosed mass of a sphere [19]. The main problem apparently resides in the fuzzy origin of such mismatch, which could be a result of systematics or due to deviations of gravity. The situation still resists in both early and late universe landscapes even if one does not consider the Planck CMB data [20] and evidences that similar tensions may also occur in the matter distributions around $2\text{-}\sigma$ [11, 21–23]. In particular, to avoid a biased dependence of σ the quantity $f\sigma_8(z)$ is a good model-independent discriminator for mapping the growth rate of matter. This alleged tension opens an interesting arena for testing gravitational models, once the possibility to alleviate it may come from modified gravity [24–30].

In the context that gravity may be modified departed from Einstein gravity or other fundamental principle, we explore the embedding of geometries (or hypersurfaces) to elaborate a model independent based on seminal works on the subject [31–33] in order to tackle the aforementioned σ_8 -issue in the problem of explanation of the accelerating expansion of the universe. In hindsight, the seminal problem of embedding theories lies in the hierarchy problem of fundamental interactions. The possibility that gravity may access extra-dimensions is taken as a principle for solving the huge ratio of the Planck

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masses (M_{Pl}) to the electroweak energy scale M_{EW} in such $M_{Pl}/M_{EW} \sim 10^{16}$. This option has been explored more vigorously in the last two decades as a solution to the dark energy paradigm. Most of these models have been Kaluza-Klein or/and string inspired, such as, for instance, the works of the Arkani-Hamed, Dvali and Dimopolous (ADD) model [34], the Randall-Sundrum model [35, 36] and the Dvali-Gabadadze-Porrati model (DPG) [37]. Differently from these models with specific conditions, and apart from the braneworld standards and variants, we have explored the embedding as a fundamental guidance for elaboration of a gravitational physical model. Until then, several authors explored embedding of geometries and its physical consequences as a mathematical structure to apply to gravitational problems [31–33, 38–51].

The plan of the paper is organized in sections. In the second section we revise the embedding of geometries and how it may be used to construct a physical framework. In this context, the Nash-Green theorem is discussed. The third and forth sections verse on the background Friedmann–Lemaître–Robertson–Walker (FLRW) metric, transformations and gauge variables involving the extrinsic curvature, respectively. The fifth section shows the resulting conformal Newtonian gauge equations. In the sixth section, we show the contrast matter density $\delta_m(a)$ as a result of the Nash fluctuations and an effective Newtonian constant G_{eff} is also determined that carries a resulting signature of extrinsic curvature. In the seventh section, we compare our model with the Λ CDM model in $f\sigma_8(z)$ measurements and use the “extended Gold 2018” compilation to the Planck 2018 (TT, TE, EE+lowE) within the 68% intervals best fit parameters. We perform our analysis on the growth factor and on G_{eff} using the data points of SDSS [52–54], 6dFGS [55], IRAS [56, 57], 2MASS [56, 58], 2dFGRS [59], GAMA [60], BOSS[61], WiggleZ [62], Vipers [63], FastSound [64], BOSS Q [65] and additional points from the 2018 SSSD-IV [26, 66–68]. In the final section, we present our remarks and prospects.

It is noteworthy to point out that we adopt the Landau time-like convention (− − − +) for the signature of the four dimensional embedded metric and speed of light $c = 1$. Concerning notation, capital Latin indices run from 1 to 5. Small case Latin indices refer to the only one extra dimension considered. All Greek indices refer to the embedded space-time counting from 1 to 4. Hereon we indicate the non-perturbed (background) quantities by the upper-script symbol “0”.

II. THE INDUCED DIMENSIONAL EQUATIONS IN A EMBEDDED SPACE-TIME

Although embedding can be made in an arbitrary number of dimensions (see [31–33, 41, 42, 45–48, 50, 51]), the current alternative models of gravitation are normally stated in five dimensions at most. Then, we start with a

model defined by a gravitational action functional in the presence of confined matter field on a four-dimensional embedded space-time embedded in a five-dimensional larger space that has the form

$$S = -\frac{1}{2\kappa_5^2} \int \sqrt{|\mathcal{G}|} {}^5\mathcal{R} d^5x - \int \sqrt{|\mathcal{G}|} \mathcal{L}_m^* d^5x, \quad (1)$$

where κ_5^2 is a fundamental energy scale on the embedded space, ${}^5\mathcal{R}$ denotes the five dimensional Ricci scalar of the bulk and \mathcal{L}_m^* denotes the confined matter Lagrangian in such the matter energy momentum tensor fulfills a finite hypervolume with constant radius l along the fifth-dimension. Thus, the variation of Einstein-Hilbert action in Eq.(1) with respect to the bulk metric \mathcal{G}_{AB} leads to the Einstein equations

$${}^5\mathcal{R}_{AB} - \frac{1}{2}\mathcal{G}_{AB} = \alpha^* \mathcal{T}_{AB}, \quad (2)$$

where α^* is the energy scale parameter and \mathcal{T}_{AB} is the energy-momentum tensor for the bulk [31–33, 42]. In accordance with the Nash-Green theorem [69, 70], that verses on orthogonal perturbations of the metric in which induce the appearance of the second curvature in that direction, an embedded space-time results from the background fluctuations. To our purposes, we are restricted to the four-dimensionality of the space-time embedded in a five dimensional bulk following the confinement hypothesis [71, 72] such dimensionality will suffice based on experimentally high-energy tests [73].

This model can be regarded as a four-dimensional hypersurface dynamically evolving in a five-dimensional bulk with constant curvature whose related Riemann tensor is

$${}^5\mathcal{R}_{ABCD} = K_* (\mathcal{G}_{AC}\mathcal{G}_{BD} - \mathcal{G}_{AD}\mathcal{G}_{BC}), \quad A...D = 1...5,$$

where \mathcal{G}_{AB} denotes the bulk metric components in arbitrary coordinates and the constant curvature K_* is either zero (flat bulk) or it can have positive (deSitter) or negative (anti-deSitter) constant curvatures.

In accordance with recent observations [11], with a very small value of the cosmological constant Λ , we do not consider any dynamical contribution of such quantity. Then, we chose $K_* = 0$, although our results also hold for any other choice of K_* . The bulk geometry is actually defined by the Einstein-Hilbert principle in Eq.(1), which leads to Einstein’s equations as shown in Eq.(2). The confinement condition implies that $K_* = \Lambda_*/6 = 0$ and the confined components of \mathcal{T}_{AB} are proportional to the energy-momentum tensor of General Relativity (GR): $\alpha_* T_{\mu\nu} = 8\pi G T_{\mu\nu}$, where G is the bare gravitational Newtonian constant. On the other hand, since only gravity propagates in the bulk we have $T_{\mu a} = 0$ and $T_{ab} = 0$. In this sense, it is possible to search a more general physical theory based on the geometries of embedding. Although it is not explicitly showed here, depending on the type of the embedding (e.g., local or

global, isometric, analytic or differentiable, etc.), brane-world models may be an example of this framework [32]. Another important aspect of the original Nash embedding is that it was applied to a flat D-dimensional Euclidean space and explored by J. Rosen [74] with an analysis on pseudo-Euclidean spaces. Its generalization to pseudo-Riemannian manifolds to non-positive signatures and the result that the embedding of the space-times may need a larger number of dimensions was made only two decades later by Greene [70]. Hereon, we simply call the Nash-Green theorem.

In a nutshell, the smoothness of the embedding is the cornerstone of the Nash-Green theorem, once this embedding results from a differentiable mapping of functions of the manifolds. On the other hand, it is not capable of telling us about the physical dynamic equations or evolution of the gravitational field. Thus, a natural choice for the bulk is that its metric satisfies the Einstein-Hilbert principle. By design, it represents the variation of the Ricci scalar and the related curvature must be “smoother” as possible [41]. It warrants that the embedded geometry and their deformations will be differentiable too.

Let be a Riemannian manifold V_4 endowed with a non-perturbed metric ${}^{(0)}g_{\mu\nu}$ being locally and isometrically embedded in a five-dimensional Riemannian manifold V_5 . Given a differentiable and regular map $\mathcal{X} : V_4 \rightarrow V_5$, one imposes the embedding equations

$$\mathcal{X}_{,\alpha}^A \mathcal{X}_{,\beta}^B \mathcal{G}_{AB} = {}^{(0)}g_{\alpha\beta}, \quad (3)$$

$$\mathcal{X}_{,\alpha}^A {}^0\eta_a^B \mathcal{G}_{AB} = 0, \quad (4)$$

$${}^0\eta_a^A {}^0\eta_b^B \mathcal{G}_{AB} = 1, \quad (5)$$

where we have denoted \mathcal{X}^A the non-perturbed embedding coordinate, \mathcal{G}_{AB} the metric components of V_5 in arbitrary coordinates, and ${}^0\eta$ denotes the non-perturbed unit vector field orthogonal to V_4 . This mechanism avoids possible coordinate gauges that may drive to false perturbations. The colons denote ordinary derivatives.

The meaning of those former set of equations is that Eq.(3) represents the isometry condition between the bulk and the embedded space-time. The orthogonality between the embedding coordinates \mathcal{X} and ${}^0\eta$ is represented in Eq.(4). Moreover, Eq.(5) denotes the set of vector normalization ${}^0\eta$. As a result, the integration of the set of Eqs. (3), (4) and (5) gives the embedding map \mathcal{X} .

The main concern of our work is provide a complement to the Einstein gravity by adding a second curvature, i.e., the extrinsic curvature and to study its implications to a physical theory. As defined in traditional textbooks [75], the extrinsic curvature of the embedded space-time V_4 is the projection of the variation of the vector ${}^0\eta$ onto the tangent plane

$$k_{\mu\nu}^{(0)} = -\mathcal{X}_{,\mu}^A {}^0\eta_{,\nu}^B \mathcal{G}_{AB} = \mathcal{X}_{,\mu\nu}^A {}^0\eta^B \mathcal{G}_{AB}. \quad (6)$$

A geometric object $\bar{\Omega}$ always can be constructed in

V_4 in any direction ${}^0\eta$ by the Lie transport along the flow for a certain small distance δy . It is worth noting that it is irrelevant if the distance δy is time-like or not, nor it is positive or negative. Then, the Lie transport is given by $\bar{\Omega} = \bar{\Omega} + \delta y \mathcal{L}_{{}^0\eta} \bar{\Omega}$, where $\mathcal{L}_{{}^0\eta}$ denotes the Lie derivative with respect to the normal vector ${}^0\eta$. In this sense, the Lie transport of the Gaussian coordinates vielbein $\{\mathcal{X}_{\mu}^A, {}^0\eta_a^A\}$, defined on V_4 , can be written as

$$\mathcal{Z}_{,\mu}^A = \mathcal{X}_{,\mu}^A + \delta y \mathcal{L}_{{}^0\eta} \mathcal{X}_{,\mu}^A = \mathcal{X}_{,\mu}^A + \delta y {}^0\eta_{,\mu}^A, \quad (7)$$

$$\eta^A = {}^0\eta^A + \delta y [{}^0\eta, {}^0\eta]^A = {}^0\eta^A. \quad (8)$$

Interestingly, from Eq.(8), it is straightforward the derivative of ${}^0\eta$ is affected by perturbations in a sense $\eta_{,\mu} \neq {}^0\eta_{,\mu}$.

Concerning perturbations of the embedded geometry V_4 , there is a set of perturbed coordinates \mathcal{Z}^A to satisfy the embedding equations likewise Eqs.(3), (4) and (5), as

$$\mathcal{Z}_{,\mu}^A \mathcal{Z}_{,\nu}^B \mathcal{G}_{AB} = g_{\mu\nu}, \quad \mathcal{Z}_{,\mu}^A \eta^B \mathcal{G}_{AB} = 0, \quad \eta^A \eta^B \mathcal{G}_{AB} = 1. \quad (9)$$

As seen in the non-perturbed case, the perturbed coordinate \mathcal{Z} defines a coordinate chart between the bulk and the embedded space-time.

Replacing Eqs.(7) and (8) in Eqs.(9) and (6), for instance, we obtain the fundamentals objects of the new manifold in linear perturbation

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + \delta g_{\mu\nu} + \dots = g_{\mu\nu}^{(0)} - 2y k_{\mu\nu}^{(0)} + \dots, \quad (10)$$

$$k_{\mu\nu} = k_{\mu\nu}^{(0)} + \delta k_{\mu\nu} + \dots = k_{\mu\nu}^{(0)} - 2y {}^0g^{\rho\sigma} k_{\mu\rho}^{(0)} k_{\nu\sigma}^{(0)} + \dots \quad (11)$$

It is important to note that the Nash-Green fluctuations on the perturbed metric $g_{\mu\nu} = g_{\mu\nu}^{(0)} + \delta g_{\mu\nu} + \delta^2 g_{\mu\nu} + \dots$ are continuously smooth and naturally go on adding small increments $\delta g_{\mu\nu}$ of the background metric.

In addition, Nash's deformation formula can be obtained by the derivative of Eq.(10) with respect to y coordinate given by

$$k_{\mu\nu}^{(0)} = -\frac{1}{2} \frac{\partial g_{\mu\nu}^{(0)}}{\partial y}. \quad (12)$$

It is noteworthy to point out that the ADM formulation gives a similar expression later discovered by Choquet-Bruhat and J. York [76]. In a physical context, the interpretation of Eq.(12) reinforces the confinement of matter as a consequence of the well established experimental structure of special relativity, particle physics and quantum field theory, using only the observables which interact with the standard gauge fields and their dual properties. It imposes a geometric constraint that it localizes the matter in the embedded space-time [31, 32].

To avoid redundances, the parameter y does not appear explicitly in the line elements once the perturbation process is triggered. It also holds true for any perturbations resulting from a n -parameter families of embedded submanifolds extended to a larger set of y^a . The resulting perturbed geometry can be bent and/or stretch

without ripping the embedded space-time. This feature is exclusive to embedding geometries with dynamical embeddings which it is not possible to do in the Riemannian realm as acknowledged by Riemann himself [77].

The integrability conditions for these equations are given by the non-trivial components of the Riemann tensor of the embedding space given by

$${}^5\mathcal{R}_{ABCD}\mathcal{Z}^A{}_{,\alpha}\mathcal{Z}^B{}_{,\beta}\mathcal{Z}^C{}_{,\gamma}\mathcal{Z}^D{}_{,\delta} = R_{\alpha\beta\gamma\delta} + ({}^0k_{\alpha\gamma}{}^0k_{\beta\delta} - {}^0k_{\alpha\delta}{}^0k_{\beta\gamma}), \quad (13)$$

$${}^5\mathcal{R}_{ABCD}\mathcal{Z}^A{}_{,\alpha}\mathcal{Z}^B{}_{,\beta}\mathcal{Z}^C{}_{,\gamma}\eta^D = {}^0k_{\alpha[\beta;\gamma]}, \quad (14)$$

where ${}^5\mathcal{R}_{ABCD}$ is the five-dimensional Riemann tensor. The semicolon denotes covariant derivative with respect to the metric. The brackets apply the covariant derivatives to the adjoining indices only.

The first equation is called Gauss equation that shows that Riemann curvature of the embedding space acts as a reference for the Riemann curvature of the embedded space-time. The second equation (Codazzi equation) evinces the projection of the Riemann tensor of the embedding space along the normal direction that is given by the tangent variation of the extrinsic curvature. This guarantees to reconstruct the five-dimensional geometry and to understand its properties from the dynamics of the four-dimensional embedded space-time V_4 . As a result, we can write in embedded vielbein $\{\mathcal{Z}^A{}_{,\mu}, \eta^A\}$ for the metric of the bulk in the vicinity of V_4

$$\mathcal{G}_{AB} = \begin{pmatrix} g_{\mu\nu}^{(0)} & 0 \\ 0 & 1 \end{pmatrix}. \quad (15)$$

III. BACKGROUND FLRW METRIC

The basic familiar line element of FLRW four-dimensional metric is given by

$$ds^2 = dt^2 - a^2 (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2), \quad (16)$$

where the expansion factor is denoted by $a \equiv a(t)$. The coordinate t denotes the physical time. In the Newtonian frame, the former equations turns out to be

$$ds^2 = dt^2 - a^2 (dx^2 + dy^2 + dz^2). \quad (17)$$

A. Non perturbed field equation in a embedded space-time

Using Eqs. (2), (15) and (17), we can obtain the induced field equations in a five-dimensional bulk

$$G_{\mu\nu}^{(0)} + Q_{\mu\nu}^{(0)} = 8\pi G T_{\mu\nu}^{(0)}, \quad (18)$$

$$k_{\mu[\nu;\rho]}^{(0)} = 0, \quad (19)$$

where the energy-momentum tensor of the confined perfect fluid is denoted by $T_{\mu\nu}^{(0)}$ and G is the bare gravita-

tional Newtonian constant. Here $G_{\mu\nu}^{(0)}$ denotes the four dimensional Einstein tensor and $Q_{\mu\nu}^{(0)}$ is called *deformation tensor*.

By direct calculation of Eq.(17) in Eq.(19), the components of $G_{\mu\nu}^{(0)}$ are given as usual [78, 79]:

$$G_{ij}^{(0)} = \frac{1}{a^2} (H^2 + 2\dot{H}) \delta_{ij},$$

$$G_{4j}^{(0)} = 0,$$

$$G_{44}^{(0)} = \frac{3}{a^2} H^2,$$

where the Hubble parameter is defined in the standard way by $H \equiv H(t) = \frac{\dot{a}}{a}$.

The non-perturbed extrinsic term $Q_{\mu\nu}^{(0)}$ in Eq.(19) is given by

$$Q_{\mu\nu}^{(0)} = k_{\mu}^{(0)\rho} k_{\rho\nu}^{(0)} - k_{\mu\nu}^{(0)} h - \frac{1}{2} (K^2 - h^2) g_{\mu\nu}^{(0)}, \quad (20)$$

where we denote the mean curvature $h^2 = h \cdot h$ and $h = {}^0g^{\mu\nu} {}^0k_{\mu\nu}$. The term $K^2 = k^{\mu\nu(0)} k_{\mu\nu}^{(0)}$ is the Gaussian curvature. The equation (20) is readily conserved in the sense that

$$Q_{\mu\nu;\mu}^{(0)} = 0. \quad (21)$$

Since the extrinsic curvature is diagonal in FLRW space-time, one can find the components of extrinsic curvature using Eq.(19) that can be split into spatial and time parts:

$$k_{ij,k}^{(0)} - \Gamma_{ik}^a k_{aj}^{(0)} = k_{ik,j}^{(0)} - \Gamma_{ij}^a k_{ak}^{(0)}.$$

In the Newtonian frame, the spatial components are also symmetric and using the former relation one can obtain $k_{11}^{(0)} = k_{22}^{(0)} = k_{33}^{(0)} = b \equiv b(t)$, and straightforwardly

$$k_{ij}^{(0)} = \frac{b}{a^2} g_{ij}, \quad i, j = 1, 2, 3, \quad k_{44}^{(0)} = \frac{-1}{a} \frac{d}{dt} \frac{b}{a}, \quad (22)$$

and the following objects can be determined:

$$k_{44}^{(0)} = -\frac{b}{a^2} \left(\frac{B}{H} - 1 \right), \quad (23)$$

$$K^2 = \frac{b^2}{a^4} \left(\frac{B^2}{H^2} - 2 \frac{B}{H} + 4 \right), \quad h = \frac{b}{a^2} \left(\frac{B}{H} + 2 \right), \quad (24)$$

$$Q_{ij}^{(0)} = \frac{b^2}{a^4} \left(2 \frac{B}{H} - 1 \right) g_{ij}^{(0)}, \quad Q_{44}^{(0)} = -\frac{3b^2}{a^4}, \quad (25)$$

$$Q^{(0)} = -(K^2 - h^2) = \frac{6b^2}{a^4} \frac{B}{H}, \quad (26)$$

where we define the function $B = B(t) \equiv \frac{\dot{b}}{b}$ in analogy with the Hubble parameter.

Since the conservation equation of the deformation tensor in Eq.(21) is identically satisfied in its covariant form, the only new information gained comes from calculating the components of $Q_{\mu;i}^{\mu} = 0$ with respect to spa-

tial coordinates in the FLRW Newtonian frame, likewise the procedure of obtaining Euler equations in GR. Using the relations in Eqs.(23), (24), (25) and (26), we obtain $\frac{B}{H} = \alpha_0$, where α_0 is an integration constant. Without any further ado, it can be easily solved by quadrature and gives the relation

$$b(t) = b_0 a(t)^{\alpha_0}. \quad (27)$$

It is worthy noting that the former relation completes univocally the set of the components of the extrinsic curvature in Eq.(22), as shown in previous works [32, 42].

B. Hydrodynamical equations

The stress energy tensor in a non-perturbed co-moving fluid is given by

$$T_{\mu\nu}^{(0)} = \left(\rho^{(0)} + p^{(0)} \right) u_\mu u_\nu - p^{(0)} g_{\mu\nu}^{(0)}; u_\mu = \delta_\mu^4.$$

The conservation of $T_{\mu\nu;\mu}^{(0)} = 0$ leads to

$$\rho^{(0)} + 3H \left(\rho^{(0)} + p^{(0)} \right) = 0, \quad (28)$$

and the resulting Friedmann equation turns

$$H^2 = \frac{8}{3} \pi G \rho^{(0)} + \frac{b^2}{a^4}, \quad (29)$$

where $\rho^{(0)}$ is the present value of the non-perturbed matter density (hereon $\rho^{(0)} \equiv \rho_m^{(0)}(t)$). For a pressureless fluid, it is a standard way to write the matter density in terms of redshift as

$$\rho_m^{(0)}(t) = \rho_{m(0)}^{(0)} a^{-3} = \rho_{m(0)}^{(0)} (1+z)^3,$$

and we can rewrite Eq.(29) in terms of redshift as

$$H^2 = \frac{8}{3} \pi G \rho_{m(0)}^{(0)} (1+z)^3 + b_0^2 (1+z)^{4-2\alpha_0}. \quad (30)$$

Using the definition of the cosmological parameter $\Omega_i = \frac{8\pi G}{3H_0^2} \rho_{i(0)}^{(0)}$, we finally have

$$\left(\frac{H}{H_0} \right)^2 = \Omega_{m(0)} (1+z)^3 + (1 - \Omega_{m(0)}) (1+z)^{4-2\alpha_0}, \quad (31)$$

where $\Omega_{m(0)}$ is the current cosmological parameter for matter content and for a flat universe $\Omega_{ext(0)} = 1 - \Omega_{m(0)}$ and H_0 is the current value of Hubble constant in units of $\text{km.s}^{-1} \text{Mpc}^{-1}$. It is worth noting that Eq.(31) with the α_0 -parameter nearly resembles the w CDM model in terms of comparison with their Friedmann equations at background level, where w is a dimensionless parameter of the fluid equation of state $w = \frac{p}{\rho}$ [80]. As we are going to show, the strikingly differences will appear at perturbation level. In this particular study, we ne-

glected the radiation term since it can be neglected for late times. Likewise, in conformal time η in such a way that $dt = a(\eta)d\eta$ and $\mathcal{H} \equiv aH$, we can write the Friedmann equation in this frame as

$$\mathcal{H}^2 = \frac{k_0}{3} a^2 \left(\rho_m^{(0)}(t) + \frac{b_0^2}{k_0} a^{2\alpha_0-4} \right), \quad (32)$$

where $k_0 \equiv \frac{8}{3} \pi G$ and the conformal Hubble parameter is $\mathcal{H} = \frac{a'}{a}$. The prime symbol represents the conformal time derivative. Hence, the conformal time derivative of Hubble parameter is given by

$$\mathcal{H}' \equiv \frac{d\mathcal{H}}{d\eta} = -\frac{k_0}{6} a^2 \left(\rho_m^{(0)} + 3p^{(0)} + (\alpha_0 - 4) \frac{b_0^2}{k_0} a^{2\alpha_0-4} \right), \quad (33)$$

and completes the set of equations for a non-perturbed fluid in a conformal Newtonian frame.

IV. TRANSFORMATIONS AND GAUGE VARIABLES

Using the standard line element of FLRW metric in Euclidean coordinates in Eq.(17), one finds

$$ds^2 = a^2 (d\eta^2 - \delta_{ij} dx^i dx^j), \quad (34)$$

where $a = a(\eta)$ is the expansion parameter in conformal time. We start with the standard process as known in GR [78, 79]. The novelty of this approach is the inclusion of the extrinsic curvature in the theoretical framework. Thus, let be the coordinate transformation $x^\alpha \rightarrow \tilde{x}^\alpha = x^\alpha + \xi^\alpha$ such as $\xi^\alpha \ll 1$, then we have for a second order tensor

$$\tilde{g}_{\alpha\beta}(\tilde{x}^\rho) = \frac{\partial x^\gamma}{\partial \tilde{x}^\alpha} \frac{\partial x^\delta}{\partial \tilde{x}^\beta} g_{\gamma\delta}(x^\rho).$$

Henceforth, we can write the perturbed metric tensor in the new coordinates $\delta\tilde{g}_{\alpha\beta}$ as

$$\delta\tilde{g}_{\alpha\beta} = \delta g_{\alpha\beta} - g_{\alpha\beta,\gamma}^{(0)} \xi^\gamma - g_{\alpha\delta}^{(0)} \xi_{,\beta}^\delta - g_{\beta\delta}^{(0)} \xi_{,\alpha}^\delta, \quad (35)$$

where the infinitesimally vector function $\xi^\alpha = \xi^{(4)} + \xi^i$ can be split into two parts

$$\xi^i = \xi^{i\perp} + \zeta^i,$$

in which $\xi^{i\perp}$ is the orthogonal part decomposition and ζ is a scalar function. The prime symbol denotes the derivative with respect to conformal time η . As a result,

we can obtain

$$\delta\tilde{g}_{ij} = \delta g_{ij} + a^2 \left[2 \frac{a'}{a} \delta_{ij} \xi^{(4)} + 2\zeta_{,ij} + \xi_{i,j}^\perp + \xi_{j,i}^\perp \right], \quad (36)$$

$$\delta\tilde{g}_{4i} = \delta g_{4i} + a^2 \left(\xi_{\perp i}' + [\zeta' - \xi^{(4)}]_{,i} \right), \quad (37)$$

$$\delta\tilde{g}_{44} = \delta g_{44} - 2a(a\xi^{(4)})'. \quad (38)$$

Using Eq.(35), we obtain a similar transformation for $k_{\mu\nu}$ as

$$\delta\tilde{k}_{\alpha\beta} = \delta k_{\alpha\beta} - k_{\alpha\beta,\gamma}^{(0)} \xi^\gamma - k_{\alpha\delta}^{(0)} \xi_{,\beta}^\delta - k_{\beta\delta}^{(0)} \xi_{,\alpha}^\delta. \quad (39)$$

And taking into account the Nash-Greene theorem

$$k_{\mu\nu}^{(0)} = -\frac{1}{2} g_{\mu\nu}^{\bullet(0)}, \quad (40)$$

where we denote $g_{\mu\nu}^{\bullet(0)} = \frac{\partial g_{\mu\nu}^{(0)}}{\partial y}$, and y is the coordinate of direction of perturbations from the background to the extra-dimensions (in this case, just one extra-dimension). Thus, we can rewrite Eq.(39) as

$$\delta\tilde{k}_{\alpha\beta} = \delta k_{\alpha\beta} + \frac{1}{2} g_{\alpha\beta,\gamma}^{\bullet(0)} \xi^\gamma + \frac{1}{2} g_{\alpha\delta}^{\bullet(0)} \xi_{,\beta}^\delta + \frac{1}{2} g_{\beta\delta}^{\bullet(0)} \xi_{,\alpha}^\delta, \quad (41)$$

and we get straightforwardly

$$\delta\tilde{k}_{ij} = \delta k_{ij} - (a^2)^\bullet \left[\frac{1}{2} \frac{((a^2)^\bullet)}{(a^2)^\bullet} \delta_{ij} \xi^{(4)} + 2\zeta_{,ij} + \xi_{i,j}^\perp + \xi_{j,i}^\perp \right],$$

$$\delta\tilde{k}_{4i} = \delta k_{4i} + \frac{1}{2} (a^2)^\bullet \left(\xi_{\perp i}' + [\zeta' - \xi^{(4)}]_{,i} \right),$$

$$\delta\tilde{k}_{44} = \delta k_{44} + \left((a^2)^\bullet \xi^{(4)} \right)_{,4} - \frac{1}{2} ((a^2)^\bullet)_{,4} \xi^{(4)}.$$

Taking the previous expressions and to avoid the implications of the ambiguity of two “times” coordinates, likewise the Rosen bi-metric theory [2, 81] that led to erroneous results such as a dipole gravitational waves, we adopt y as a set of space-like coordinates. Then, one obtains

$$\delta\tilde{k}_{ij} = \delta k_{ij}, \quad (42)$$

$$\delta\tilde{k}_{4i} = \delta k_{4i}, \quad (43)$$

$$\delta\tilde{k}_{44} = \delta k_{44}. \quad (44)$$

For scalar perturbations the metric takes the form

$$ds^2 = a^2 [(1+2\phi)d\eta^2 + 2B_{,i} dx^i d\eta - ((1-2\psi)\delta_{ij} - 2E_{,ij}) dx^i dx^j], \quad (45)$$

where $\phi = \phi(\vec{x}, \eta)$, $\psi = \psi(\vec{x}, \eta)$, $B = B(\vec{x}, \eta)$ and $E = E(\vec{x}, \eta)$ are scalar functions.

For the tensors $G_{\mu\nu}$, $T_{\mu\nu}$ and $Q_{\mu\nu}$, one can use the same set of transformations. In this sense, for small perturbations, we can write the Einstein tensor in a coordinate system \tilde{x} as

$$\tilde{G}_{\mu\nu} = G_{\mu\nu}^{(0)} + \delta\tilde{G}_{\mu\nu},$$

where $\delta\tilde{G}_{\mu\nu}$ denotes linear perturbations in the new coordinate system

$$\delta\tilde{G}_{\alpha\beta} = \delta G_{\alpha\beta} - G_{\alpha\beta,\gamma}^{(0)} \xi^\gamma - G_{\alpha\delta}^{(0)} \xi_{,\beta}^\delta - G_{\beta\delta}^{(0)} \xi_{,\alpha}^\delta. \quad (46)$$

Immediately, we have a similar expression for $T_{\mu\nu}$

$$\tilde{T}_{\mu\nu} = T_{\mu\nu}^{(0)} + \delta\tilde{T}_{\mu\nu},$$

that leads to

$$\delta\tilde{T}_{\alpha\beta} = \delta T_{\alpha\beta} - T_{\alpha\beta,\gamma}^{(0)} \xi^\gamma - T_{\alpha\delta}^{(0)} \xi_{,\beta}^\delta - T_{\beta\delta}^{(0)} \xi_{,\alpha}^\delta, \quad (47)$$

and also, for the deformation tensor

$$\tilde{Q}_{\mu\nu} = Q_{\mu\nu}^{(0)} + \delta\tilde{Q}_{\mu\nu},$$

that leads to

$$\delta\tilde{Q}_{\alpha\beta} = \delta Q_{\alpha\beta} - Q_{\alpha\beta,\gamma}^{(0)} \xi^\gamma - Q_{\alpha\delta}^{(0)} \xi_{,\beta}^\delta - Q_{\beta\delta}^{(0)} \xi_{,\alpha}^\delta. \quad (48)$$

And using Eq.(45), we obtain for $\delta\tilde{G}_{\mu\nu}$:

$$\delta\tilde{G}_j{}^i = \delta G_j{}^i - ({}^{(0)}G_j{}^i)'(B - E'), \quad (49)$$

$$\delta\tilde{G}_i{}^4 = \delta G_i{}^4 - ({}^{(0)}G_i{}^4 - \frac{1}{3}({}^{(0)}G_k{}^k)(B - E')_{,i}, \quad (50)$$

$$\delta\tilde{G}_4{}^4 = \delta G_4{}^4 - ({}^{(0)}G_4{}^4)'(B - E'). \quad (51)$$

And for $\delta\tilde{T}_{\mu\nu}$ and $\delta\tilde{Q}_{\mu\nu}$, one obtains the set of equations, respectively:

$$\delta\tilde{T}_j{}^i = \delta T_j{}^i - ({}^{(0)}T_j{}^i)'(B - E') \quad (52)$$

$$\delta\tilde{T}_i{}^4 = \delta T_i{}^4 - ({}^{(0)}T_i{}^4 - \frac{1}{3}({}^{(0)}T_k{}^k)(B - E')_{,i}, \quad (53)$$

$$\delta\tilde{T}_4{}^4 = \delta T_4{}^4 - ({}^{(0)}T_4{}^4)'(B - E'). \quad (54)$$

$$\delta\tilde{Q}_j{}^i = \delta Q_j{}^i - ({}^{(0)}Q_j{}^i)'(B - E') \quad (55)$$

$$\delta\tilde{Q}_i{}^4 = \delta Q_i{}^4 - ({}^{(0)}Q_i{}^4 - \frac{1}{3}({}^{(0)}Q_k{}^k)(B - E')_{,i}, \quad (56)$$

$$\delta\tilde{Q}_4{}^4 = \delta Q_4{}^4 - ({}^{(0)}Q_4{}^4)'(B - E'). \quad (57)$$

V. SCALAR PERTURBATIONS IN NEWTONIAN GAUGE

In longitudinal conformal Newtonian gauge, the main condition resides in the vanishing functions of $B = B(\vec{x}, \eta)$ and $E = E(\vec{x}, \eta)$, as well as the quantities $\xi^{(4)}$, ξ' , ζ . Hence, the metric in Eq.(45) turns to be

$$ds^2 = a^2 [(1 + 2\Phi)d\eta^2 - ((1 - 2\Psi)\delta_{ij} dx^i dx^j)], \quad (58)$$

where $\Phi = \Phi(\vec{x}, \eta)$ and $\Psi = \Psi(\vec{x}, \eta)$ denotes the Newtonian potential and the Newtonian curvature, respectively. In addition, we obtain a simplification of the previous transformations of the curvature-related quantities and

the set of following outcomes:

$$\begin{aligned} \delta\tilde{g}_{44} &= \delta g_{44} ; \delta\tilde{g}_{4i} = \delta g_{4i} = 0 ; \delta\tilde{g}_{ij} = \delta g_{ij} , \\ \delta\tilde{G}_4^4 &= \delta G_4^4 ; \delta\tilde{G}_i^4 = \delta G_i^4 ; \delta\tilde{G}_j^i = \delta G_j^i , \\ \delta\tilde{T}_4^4 &= \delta T_4^4 ; \delta\tilde{T}_i^4 = \delta T_i^4 ; \delta\tilde{T}_j^i = \delta T_j^i , \\ \delta\tilde{Q}_4^4 &= \delta Q_4^4 ; \delta\tilde{Q}_i^4 = \delta Q_i^4 ; \delta\tilde{Q}_j^i = \delta Q_j^i . \end{aligned} \quad (59)$$

A. Perturbed gravitational equations

Taking into account all the former results of Eqs.(59), we can write the perturbed field equations as

$$\delta G_\nu^\mu = 8\pi G \delta T_\nu^\mu - \delta Q_\nu^\mu , \quad (60)$$

$$\delta k_{\mu\nu;\rho} = \delta k_{\mu\rho;\nu} . \quad (61)$$

Using the Nash-Green theorem, we notice that Codazzi equations in Eq.(61) do not propagate perturbations in which are confined to the background. In other words, Codazzi equations maintain their background form. First, we calculate the linear perturbations for five-dimensions that the new geometry $\tilde{g}_{\mu\nu} = g_{\mu\nu}^{(0)} + \delta g_{\mu\nu}$ generated by Nash's fluctuations are given by

$$\tilde{g}_{\mu\nu} = g_{\mu\nu}^{(0)} - 2\delta y k_{\mu\nu}^{(0)} , \quad (62)$$

and the related perturbed extrinsic curvature

$$\tilde{k}_{\mu\nu} = k_{\mu\nu}^{(0)} - 2\delta y^{(0)} g^{\sigma\rho} k_{\mu\sigma}^{(0)} k_{\nu\rho}^{(0)} , \quad (63)$$

where we can identify $\delta k_{\mu\nu} = {}^{(0)}g^{\sigma\rho} k_{\mu\sigma}^{(0)} k_{\nu\rho}^{(0)}$ and using the Nash relation $\delta g_{\mu\nu} = -2k_{\mu\nu}^{(0)} \delta y$, we obtain

$$\delta k_{\mu\nu} = {}^{(0)}g^{\sigma\rho} k_{\mu\sigma}^{(0)} \delta g_{\nu\rho} . \quad (64)$$

Applying Eq.(64) to Eq.(61), we obtain the background equation as in Eq.(19). In this sense, we have to look for the effects of the Nash-Green fluctuations on the perturbed gravi-tensor equation and verify if the propagations of cosmological perturbations occur. Using the conformal metric in Eq.(58), we can write the components of Eq.(60) as

$$\begin{aligned} \delta G_j^i &= 8\pi G \delta T_j^i - \delta Q_j^i , \\ \delta G_i^4 &= 8\pi G \delta T_i^4 - \delta Q_i^4 , \\ \delta G_4^4 &= 8\pi G \delta T_4^4 - \delta Q_4^4 , \end{aligned}$$

and we have respectively the components in the conformal frame,

$$\begin{aligned} \left[\Psi'' + \mathcal{H}(2\Psi + \Phi)' + (\mathcal{H}^2 + 2\mathcal{H}')\Phi + \frac{1}{2}\nabla^2(\Psi - \Phi) \right] \delta_{ij} & (65) \\ = \frac{1}{2}(\Psi - \Phi)_{,ij} + \frac{1}{2}a^2\delta Q_j^i - 4\pi G a^2 \delta T_j^i , \end{aligned}$$

$$[\Psi' + \mathcal{H}\Phi]_{,i} = 4\pi G a^2 \delta T_i^4 - \frac{1}{2}a^2 \delta Q_i^4 , \quad (66)$$

$$\nabla^2\Psi - 3\mathcal{H}(\Psi' + \Phi\mathcal{H}) = 4\pi G a^2 \delta T_4^4 - \frac{1}{2}a^2 \delta Q_4^4 . \quad (67)$$

Finally, the perturbation of the deformation tensor $Q_{\mu\nu}$ can be made from its background form in Eq.(20) and the resulting $k_{\mu\nu}$ perturbations from the Nash fluctuations of Eq.(64) such as

$$\delta Q_{\mu\nu} = -\frac{3}{2}(K^2 - h^2)\delta g_{\mu\nu} . \quad (68)$$

Moreover, from Eqs.(60) and (68), we can check straightforwardly that $\delta Q_{\mu\nu}$ is not independently conserved under a covariant analysis. Since it depends on the nature of the fluid, it seems to allow room for a non-conservative process in a perturbation theory eventually (in the background it does not hold true). It may happen with a consideration of exotic fluids which received particular attention in recent years [82–87]. For an exotic fluid $\delta T_{\mu;\nu}^{(exotic)} \neq 0$, thus $\delta Q_{\mu;\nu} \neq 0$.

Then using the background relations, we can determine the components of $\delta Q_{\mu\nu}$

$$\delta Q_j^i = 18\alpha_0 b_0^2 a^{2\alpha_0-2} \Psi \delta_j^i , \quad (69)$$

$$\delta Q_4^i = 0 , \quad (70)$$

$$\delta Q_4^4 = 18\alpha_0 b_0^2 a^{2\alpha_0-2} \Phi \delta_4^4 , \quad (71)$$

and we get the basic gauge invariant field equations in the conformal Newtonian frame:

$$\begin{aligned} \left[\Psi'' + \mathcal{H}(2\Psi + \Phi)' + (\mathcal{H}^2 + 2\mathcal{H}')\Phi + \frac{1}{2}\nabla^2(\Psi - \Phi) \right] \delta_{ij} & (72) \\ = 9\gamma_0 a^{2\alpha_0} \Psi \delta_{ij} + \frac{1}{2}(\Psi - \Phi)_{,ij} - 4\pi G a^2 \delta T_j^i , \end{aligned}$$

$$[\Psi' + \mathcal{H}\Phi]_{,i} = 4\pi G a^2 \delta T_i^4 , \quad (73)$$

$$\nabla^2\Psi - 3\mathcal{H}(\Psi' + \Phi\mathcal{H}) = 4\pi G a^2 \delta T_4^4 - 9\gamma_0 a^{2\alpha_0} \Phi , \quad (74)$$

where we denote $\gamma_0 = \alpha_0 b_0^2$.

B. Hydrodynamical gravitational perturbed equations

For a perturbed fluid with pressure p and density ρ , one can write the perturbed components of the related stress-tensor

$$\delta\tilde{T}_4^4 = \delta\rho , \quad (75)$$

$$\delta\tilde{T}_i^4 = \frac{1}{a}(\rho_0 + p_0)\delta u_{||i} , \quad (76)$$

$$\delta\tilde{T}_j^i = -\delta p \delta_j^i , \quad (77)$$

where $\delta u_{||i}$ denotes the tangent velocity potential and ρ_0 and p_0 denote the non-perturbed components of den-

sity and pressure, respectively. Hence, we can rewrite Eqs.(72), (73) and (74) as

$$\nabla^2 \Psi - 3\mathcal{H}(\Psi' + \Phi\mathcal{H}) = 4\pi G a^2 \delta\rho - 9\gamma_0 a^{2\alpha_0} \Phi, \quad (78)$$

$$[\Psi' + \mathcal{H}\Phi]_{,i} = 4\pi G a(\rho_0 + p_0)\delta u_{||i}, \quad (79)$$

$$\left[\Psi'' + \mathcal{H}(2\Psi + \Phi)' + (\mathcal{H}^2 + 2\mathcal{H}')\Phi + \frac{1}{2}\nabla^2(\Psi - \Phi) \right] \delta_{ij} (80)$$

$$= \frac{1}{2}(\Psi - \Phi)_{,ij} + [4\pi G a^2 \delta p + 9\gamma_0 a^{2\alpha_0} \Psi] \delta_{ij}.$$

Those set of equations can be better understood in the Fourier k -space wave modes. Taking the Fourier transform of each main quantity (with subscript “ k ”), we obtain a new set of equations:

$$k^2 \Psi_k + 3\mathcal{H}(\Psi'_k + \Phi_k \mathcal{H}) = -4\pi G a^2 \delta\rho_k + 9\gamma_0 a^{2\alpha_0} \Phi_k, \quad (81)$$

$$\Psi'_k + \mathcal{H}\Phi_k = -4\pi G a^2(\rho_0 + p_0)\theta, \quad (82)$$

where $\theta = ik^j \delta u_{||j}$ denotes the divergence of fluid velocity in k -space. Finally, the third equation is given by

$$\Psi''_k + \mathcal{H}(2\Psi_k + \Phi_k)' + (\mathcal{H}^2 + 2\mathcal{H}')\Phi_k + \frac{1}{2}k^2(\Psi_k - \Phi_k) \quad (83)$$

$$= \frac{1}{2}\hat{k}^i \cdot \hat{k}_i(\Psi_k - \Phi_k) + 4\pi G a^2 \delta p + 9\gamma_0 a^{2\alpha_0} \Psi_k.$$

VI. MATTER DENSITY EVOLUTION UNDER SUBHORIZON REGIME

A. Simplest case for matter perturbations $\Psi = \Phi$

Using the simplest condition for perturbations $\Psi = \Phi$, we obtain the following set of equations:

$$k^2 \Phi_k + 3\mathcal{H}(\Phi'_k + \Phi_k \mathcal{H}) = -4\pi G a^2 \delta\rho_k + 9\gamma_0 a^{2\alpha_0} \Phi_k, \quad (84)$$

$$(a\Phi_k)' = -4\pi G a^2(\rho_0 + p_0)\theta. \quad (85)$$

The third equation is given by

$$\Phi''_k + 3\mathcal{H}\Phi'_k + (\mathcal{H}^2 + 2\mathcal{H}')\Phi_k - 9\gamma_0 a^{2\alpha_0} \Phi_k \quad (86)$$

$$= 4\pi G a^2(c_s^2 \delta\rho_k + \tau \delta S_k),$$

where we use the relation $\delta p_k = c_s^2 \delta\rho_k + \tau \delta S$ in which c_s denotes the sound speed and S is the entropy. Hence, combining the former equations, we obtain the gravitational potential formula in k -space:

$$\Phi''_k + 3(1 + c_s^2)\mathcal{H}\Phi'_k + [k^2 c_s^2 + (1 + 3c_s^2)\mathcal{H}^2 + 2\mathcal{H}']\Phi_k \quad (87)$$

$$= 9\gamma_0 a^{2\alpha_0}(1 + c_s^2)\Phi_k + 4\pi G a^2 \tau \delta S_k.$$

B. Subhorizon regime and the master equation for evolution of perturbations

It is important to notice that when $\gamma_0 \rightarrow 0$ in Eq.(87), the standard GR equations are obtained. Hence, we can recover the subhorizon approximation with $k^2 \gg \mathcal{H}^2$ or $k^2 \gg a^2 H^2$ which means $\Phi'_k, \mathcal{H}\Phi'_k, \mathcal{H}\Phi'_k \sim 0$, then Eq.(87) turns the expected Newtonian formula $\Phi_k \sim \frac{\delta\rho_k}{k^2}$. The adiabatic solutions are obtained for $\delta S_k = 0$. To determine the gravitational potential Φ , as in GR, we need to work with the continuity and Euler equations from calculating the components $\delta T_{\mu;4}^\mu$ and $\delta T_{\mu;i}^\mu$ to give, respectively

$$\delta\rho' + (p_0 + \rho_0)(\nabla^2 u_i - 3\Phi') + 3\mathcal{H}(\delta p + \delta\rho) = 0, \quad (88)$$

$$\frac{d}{d\eta}[(p_0 + \rho_0)u_i] + (p_0 + \rho_0)(4\mathcal{H}u_i + \Phi) + \delta p = 0, \quad (89)$$

where the former expressions can be also written in terms of the fluid parameter $w = \frac{p_0}{\rho_0}$.

Taking Eq.(88) under a Fourier transform, we obtain the following equation in the k -space:

$$\delta\rho'_k - k^2 \rho_0 u_k - 3\rho_0 \Phi' + 3\mathcal{H}\delta\rho_k = 0,$$

which in subhorizon approximation gives

$$\delta\rho'_k - k^2 \rho_0 u_k \simeq 0. \quad (90)$$

For the pressureless form of Eq.(89), we have

$$\rho'_k u_k + \rho_k u'_k + 4\mathcal{H}u_k \rho_k + \rho_k \Phi_k = 0,$$

and using the background formula from conservation equation of Eq.(28), we have

$$k^2 \rho_0 u'_k = -k^2 \mathcal{H}u_k - k^2 \Phi_k. \quad (91)$$

Performing the definition of the “contrast” matter density $\delta_m \equiv \frac{\delta\rho}{\rho_0}$, and using Eqs. (90), (91) and (84), we obtain a relation with Φ_k and δ_m as

$$k^2 \Phi_k = -4\pi G_{eff} a^2 \rho_0 \delta_m, \quad (92)$$

where G_{eff} is the effective Newtonian constant and is given by

$$G_{eff}(a, k) = \frac{G}{1 - \frac{9\gamma_0}{k^2} a^{2\alpha_0}}. \quad (93)$$

Taking the conformal time derivative of Eq.(90) and writing Eq.(91) in terms of δ_m and using Eq.(92), we obtain the equation of evolution of the contrast matter density $\delta_m(\eta)$ in conformal longitudinal Newtonian frame

$$\delta''_m + \mathcal{H}\delta'_m - 4\pi G_{eff} a^2 \rho_0 \delta_m = 0. \quad (94)$$

To express Eq.(94) in terms of the physical time, we use the notation $\dot{\delta}_m \equiv \frac{\delta'_m}{a}$ and obtain the useful relation

$\delta_m'' = a^2 \ddot{\delta}_m + a^2 H \dot{\delta}_m$. We are leading to

$$\ddot{\delta}_m(t) + 2H\dot{\delta}_m(t) - 4\pi G_{eff}\rho_0\delta_m(t) = 0. \quad (95)$$

Finally, we can obtain an alternatively way to express the former equation in terms of the expansion factor $a(t)$. We use the notation $\frac{d\delta_m(a)}{da} = \delta_m^\circ(a)$ and $\frac{d^2\delta_m(a)}{da^2} = \delta_m^{\circ\circ}(a)$, and obtain useful relations $\delta_m^\circ(a) = \frac{1}{a}\dot{\delta}_m(t)$, $\delta_m^{\circ\circ}(a) = \frac{1}{a^2}\ddot{\delta}_m(t)$ and $\frac{H^\circ(a)}{H(a)}$. Hence, the contrast matter density $\delta_m(a)$ is governed by the equation

$$\delta_m^{\circ\circ}(a) + \left(\frac{3}{a} + \frac{H^\circ(a)}{H(a)}\right)\delta_m^\circ(a) - \frac{3\Omega_{m0}G_{eff}/G}{2(H^2(a)/H_0^2)}\delta_m(a) = 0. \quad (96)$$

which solutions are possible only numerically. For instance, in the context of GR, where $G_{eff} = G$ that turns $\delta_m(a)$ independent of the scale k , with the fluid parameter w , one has the following solution

$$\delta(a) = a.2F_1\left(-\frac{1}{3w}, \frac{1}{2} - \frac{1}{2w}; 1 - \frac{5}{6w}; a^{-3w}(1 - \Omega_m^{-1})\right) \quad (97)$$

where $2F_1(a, b; c; z)$ is a hypergeometric function.

VII. ANALYSIS ON EVOLUTION OF G_{eff} AND $f\sigma_8$

The form for the effective Newtonian constant is given by Eq.(93) as a result of the linear Nash fluctuations of the metric and the extrinsic curvature as shown in Eq.(11). To alleviate the tension of $f\sigma_8$, one possible way is to modify gravity in some sense [24–30] and to this intent, the G_{eff} function plays a paramount role. Firstly, we adopt the minimum value of expansion parameter $a_{min} = 0.001$ and for sub-horizon scales we fix the value of $k = 300 H_0 \sim 0.1h\text{Mpc}^{-1}$. This consideration holds our analysis in the linear matter growth landscape. Since gravity has been tested in several experiments at solar scale, the G_{eff} function must be constrained by $G_{eff}|_{a=0} = 1$, $G_{eff}|_{a=1} = 1$ and at solar scale by the ordinary time-derivative $\frac{dG_{eff}}{dt}|_{a=1} = 0$. The latter relation is satisfied when $\gamma_0 \rightarrow 0$ for $b_0 \rightarrow 0$, once $\gamma_0 = \alpha_0 b_0^2$ that means that the extrinsic curvature must vanish at solar scale and only at cosmological range the extrinsic curvature may induce topological changing of the universe [31, 32, 42, 47, 48]. As familiar in most modified gravity models, we also assume that G_{eff} is scale independent and does not depend on the matter density [25, 26]. Thus, we expand Eq.(93) around $a = 1$ in a Taylor series and once we have fixed the wave-number k , we can blend all constants (b_0 is an arbitrary constant) and simplify G_{eff} in terms of two non-dimensional parameters (α_0, β_0) as

$$\frac{G_{eff}}{G} = 1 - \alpha_0\beta_0(a-1)^{2\alpha_0} + \alpha_0\beta_0(a-1)^{4\alpha_0}. \quad (98)$$

Accordingly, the related graphic behavior is shown in Fig.(1). We show five curves with the values of α_0 varying from 0.5 to 4, where we have summarized the results of minimizing data in Table I. Error estimates were calculated using Fisher Matrix around the related best-fit values for each model. In the course of this study, we

TABLE I. The set of values for the parameters α_0 and β_0 and related curves of Fig.(1).

α_0	Curve color	β_0	χ^2
0.5	red dashed line	-0.401 ± 0.127	16.17
1	black thick line	1.048 ± 0.325	15.45
2	blue dashed line	0.71 ± 0.265	18.64
3	black thick dashed line	0.636 ± 0.266	20.16
4	green thick dashed line	0.638 ± 0.268	20.81

realized that the values of α_0 parameter seem to obey a narrower window of validity on the range $\alpha_0 = [1, 3]$, and outside of this range, the G_{eff} constraints are not satisfied and the related models are ruled out. Also, this set of models defined by α_0 matches the Big Bang Nucleosynthesis (BBN) expectations $\frac{G_{eff}}{G} = 1.09 \pm 0.2$ [25].

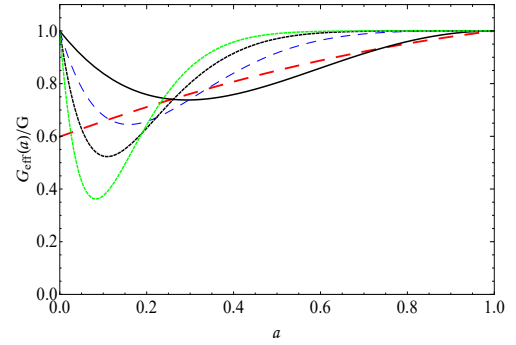


FIG. 1. Resulting curves of G_{eff} resulting from Nash's fluctuations for selected models as shown in Table I.

Another test consist to use the σ_8 parameter that measures the amplitude of growth of r.m.s fluctuations on the scale of $8h^{-1}\text{Mpc}$ and is an important reference for selection of dark energy and/or modified gravity models. A biased-free analysis can be performed by the measure of the quantity

$$f\sigma_8(a) \equiv f(a).\sigma_8(a). \quad (99)$$

where $f(a) = \frac{\ln \delta}{\ln a}$ is the growth rate and the growth factor $\delta(a)$ is given by Eq.(95). In the χ^2 -statistics, one must consider the observed growth parameter $f(a_{obs})$ in minimization due to the Alcock-Paczynski effect to take into account redshift-space distortions. We use the “extended Gold-2018” compilation to the Planck 2018 (TT, TE, EE+lowE) best fit parameters as shown in Table II on the data points of SDSS [52–54], 6dFGS [55], IRAS [56, 57], 2MASS [56, 58], 2dFGRS [59], GAMA [60], BOSS [61], WiggleZ [62], Vipers [63], FastSound [64], BOSS Q [65]

and an additional points from the 2018 SSSD-IV [26, 66–68]. These additional data points provide growth rate at relatively higher redshifts. Moreover, as pointed out in Refs.[25, 26], to compatibilize the data dependence from the fiducial cosmology and another cosmological surveys, it is necessary to rescale the growth-rate data by the ratio $r(z)$ of the Hubble parameter $H(z)$ and the angular distance $D_A(z)$ by

$$r(z) = \frac{H(z)D_A(z)}{H_f(z)D_fA(z)}. \quad (100)$$

where the subscript “ f ” corresponds a quantity of fiducial cosmology. Similarly, the compatibilization of the related χ^2 statistics is also necessary. It can be done using the expression

$$\chi^2(\Omega_{0m}, w, \sigma_8) = V^i C_{ij}^{-1} V_j, \quad (101)$$

where $V^i \equiv f\sigma_{8,i} - r(z_i)f\sigma_8(z_i, \Omega_{0m}, w, \sigma_8)$ denotes a set of vectors that go up to i th-datapoints at redshift z_i for each $i = 1 \dots N$. N is the total number of datapoints of a related collection of a data. The set of $f\sigma_{8,i}$ datapoints come from theoretical predictions [25]. The set of C_{ij}^{-1} denotes the inverse covariance matrix. A final important correction concerns the necessity to disentangle the datapoints related to WiggleZ dark energy survey which are at first correlated. Then, the covariant matrix C_{ij} [62] is given by

$$C_{ij}^{wigglez} = 10^{-3} \begin{bmatrix} 6.400 & 2.570 & 0.000 \\ 2.570 & 3.969 & 2.540 \\ 0.000 & 2.540 & 5.184 \end{bmatrix} \quad (102)$$

and the resulting total matrix C_{ij}^{tot}

$$C_{ij}^{tot} = 10^{-3} \begin{bmatrix} \sigma_1^2 & 0 & 0 & \dots \\ 0 & C_{ij}^{wigglez} & 0 & \dots \\ 0 & 0 & \dots & \sigma_N^2 \end{bmatrix} \quad (103)$$

where the set of σ^2 's denote the N -variances.

In Fig.(2), we present the σ_8 -contours with 68.3%, 95.4% and 99.7% confidence levels (C.L.) in the $(\sigma_8 - \Omega_m)$ plane. In the left panel, as expected, we identify the 3- σ tension for the model $\alpha_0 = 0$ that means that the extrinsic curvature vanishes (and Einstein equations of GR are restored) and $G_{eff} = G$ that reproduces Planck2018/ Λ CDM discrepancy with Λ CDM model. The black point denotes the best fit of the models for the values of the σ_8 parameter. On the other hand, the right panel shows the model for $\alpha_0 = 2$, $\beta_0 = 0.71 \pm 0.265$ and $\chi^2 = 18.64$ with Planck2018 background and exhibits an alleviation for the 3- σ tension falling well within the 1- σ level. In both panels, the red points indicate the best fit of Planck2018/ Λ CDM model in $(\sigma_8 - \Omega_m)$ plane.

In addition, we present in Fig.(3) the resulting plot of the $f\sigma_8$ from the datapoints of Table II. The lines

TABLE II. Datapoints the “extended Gold-2018” compilation to the Planck 2018 (TT, TE, EE+lowE) best fit parameters [25] with additional points from BOSS Q [65] and SSSD-IV [26, 66–68].

Dataset	redshift	$f\sigma_8(z)$	Ω_m
6dFGS+SnIa	0.02	0.428 ± 0.0465	0.3
SnIa+IRAS	0.02	0.398 ± 0.065	0.3
2MASS	0.02	0.314 ± 0.048	0.266
SDSS-veloc	0.10	0.370 ± 0.130	0.3
SDSS-MGS	0.15	0.490 ± 0.145	0.31
2dFGRS	0.17	0.510 ± 0.060	0.3
GAMMA	0.18	0.360 ± 0.090	0.27
GAMMA	0.38	0.440 ± 0.090	0.27
SDSS-LRG-200	0.25	0.3512 ± 0.0583	0.25
SDSS-LRG-200	0.37	0.4602 ± 0.0378	0.25
BOSS-LOWZ	0.32	0.384 ± 0.095	0.274
SDSS-CMASS	0.59	0.488 ± 0.060	0.30711
WiggleZ	0.44	0.413 ± 0.080	0.27
WiggleZ	0.60	0.390 ± 0.063	0.27
WiggleZ	0.73	0.437 ± 0.072	0.27
Vipers PDR-2	0.60	0.550 ± 0.120	0.3
Vipers PDR-2	0.86	0.400 ± 0.110	0.3
FastSound	1.40	0.482 ± 0.116	0.270
BOSS-Q	1.52	0.426 ± 0.077	0.31
SDSS-IV	1.52	0.420 ± 0.076	0.26479
SDSS-IV	1.52	0.396 ± 0.079	0.31
SDSS-IV	0.978	0.379 ± 0.176	0.31
SDSS-IV	1.23	0.385 ± 0.099	0.31
SDSS-IV	1.526	0.342 ± 0.070	0.31
SDSS-IV	1.944	0.364 ± 0.106	0.31

are identified as follows: the black thick-dashed line denotes the best fit of the Λ CDM model ($\Omega_m = 0.21$, $\sigma_8 = 0.88$, $\chi^2 = 12.73$); the red thin-dashed line denotes the Planck2018/ Λ CDM ($\chi^2 = 23.9$). To reinforce the results, a self-contained information can be obtained from the σ -distances D_σ which are computed by

$$D_\sigma = \sqrt{2} \text{Inverf} \left[0, 1 - \Gamma\left(1, \frac{\Delta\chi^2}{2}\right) \right], \quad (104)$$

where $\text{Inverf}(x)$ is the inverse of the error function $\text{Erf}(x)$ and $\Gamma(1, \frac{\Delta\chi^2}{2})$ is the incomplete gamma function, with $\Delta\chi^2 = \chi_{model(2)}^2 - \chi_{model(1)}^2$. The resulting σ -distance $D_\sigma = 1.79$ results from the distance of the best-fit values of our model and Planck2018 results, but within the 1- σ contour. On the other hand, we obtain $D_\sigma = 2.898$ for Λ CDM and Planck18/ Λ CDM which reinforces that the σ_8 tension persists in the Planck2018 dataset.

The three additional lines represent the best-fit of G_{eff} evolution for the models defined by the value of $\alpha_0 = (1, 2, 3)$ as presented in Table I. In Fig.(03), they are represented by the blue thick, black dot-dashed and red thin lines, respectively. In particular, the growth rate fluctuations represented by the blue thick line $\alpha_0 = 2$ is

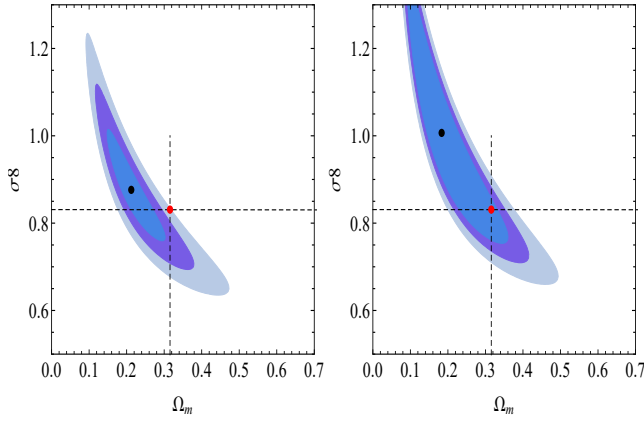


FIG. 2. The σ_8 -contours with 68.3%, 95.4% and 99.7% C.L. in the $(\sigma_8 - \Omega_m)$. The left panel shows the $3\text{-}\sigma$ tension for the model $\alpha_0 = 0$ that reproduces the tension of Planck2018/ Λ CDM cosmology. The right panel shows the contours for $\alpha_0 = 2$ and $\beta_0 = 0.71 \pm 0.265$ with a reduction of the σ_8 tension. In comparison, the red points indicate the best fit of Planck2018/ Λ CDM model.

the best accommodated curve considering the constraints from various redshift surveys [11] and reinforces the previous results as shown in Fig.(1) and (2) consisting a promising result for further studies.

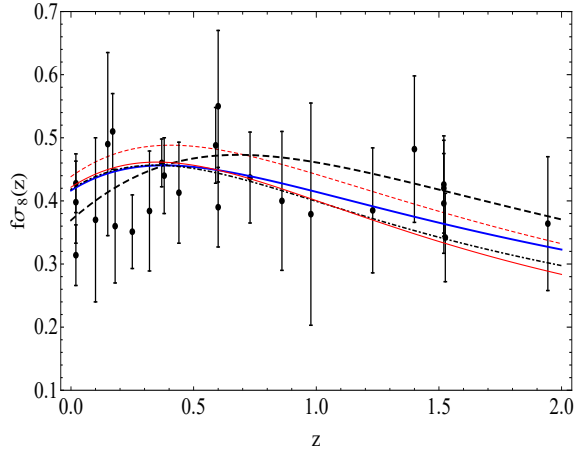


FIG. 3. Curves of the $f\sigma_8$ evolution that show a comparison with Λ CDM model (black thick dashed) and Planck18/ Λ CDM (red thin-dashed). The three last curves top-to-bottom are the models from G_{eff} .

VIII. REMARKS

In this paper we studied cosmic perturbations of matter in a search of understanding if the contribution of the extrinsic curvature to complement Einsteinian gravity is rather than semantics and relies on physical reality. It may pinpoint a renewed consideration of the concept of curvature, a paramount element of contemporary physics, as a fundamental physical agent itself. From the linear Nash-Green perturbations of metric, we have shown how to transpose the initial process in the background metric of the embedding of geometries to trigger the perturbations. In this sense, we have shown the perturbed field equations. An interesting fact resides that in five dimensions the gravitational tensor equation is indeed a perturbed equation, once the perturbation of the Codazzi equation does not propagate cosmological perturbations being hampered by linear Nash's fluctuations. Moreover, the perturbed deformation equation $\delta Q_{\mu\nu}$ is not conserved independently which may open an interesting arena to add exotic interactions. This landscape can be dramatically different in $dim \geq 6$ with appearance of new geometric objects, such as the third fundamental form $A_{\mu\nu a}$ that is associated to gauge fields. We also calculated the longitudinal Newtonian gauge of this framework in the simplest case that the gravitational potentials coincide $\Psi = \Phi$. Moreover, we have obtained in the subhorizon scale the contrast matter density ignited by the embedding equations. The finding of the matter overdensity equation δ_m is a paramount quantity for latter studies to identify any signature of modifications of gravity due to cosmic acceleration. We also have shown the determination of effective Newtonian constant G_{eff} that matches the Big Bang Nucleosynthesis (BBN) constraint. Furthermore, the $3\text{-}\sigma$ tension paradigm may be solved as a result from a modification of gravity with a narrow set of values for the parameters of the model and a consistent behaviour of $f\sigma_8$ growth descriptor. These results pose an interesting scenario since the model seems to provide a necessary gravitational strength to correct the σ_8 discrepancy. This was obtained by the inclusion of the extrinsic curvature as a pivot element to modify standard Einstein's gravity. As prospects, the integrated Sachs-Wolfe (ISW) effect will be analysed for the G_{eff} model in the light of large surveys on dark energy and to study the impact of the model on CMB power spectrum.

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