Further results on a matrix equality and matrix set inclusions for generalized inverses of matrix products

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Abstract. This note reconsiders a matrix equality $A_1A_2^-A_3A_4^-A_5 = A$ composed by six matrices of appropriate sizes, where A_2^- and A_4^- are generalized inverses of A_2 and A_4, respectively, and solves a selection of matrix set inclusion problems associated with various mixed reverse order laws for generalized inverses of products of two, three, and four matrices by means of this equality and its variations.

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1 Introduction

Throughout, let $\mathbb{C}^{m\times n}$ denote the set of all $m\times n$ complex matrices, $r(A)$, $\mathscr{R}(A)$, and $\mathscr{N}(A)$ denote the rank, the range, and the null space of a matrix $A\in\mathbb{C}^{m\times n}$, respectively; I_m denote the identity matrix of order m, $[A, B]$ denote a columnwise partitioned matrix consisting of two submatrices A and B. The Moore–Penrose inverse of $A\in\mathbb{C}^{m\times n}$, denoted by $A^{\dagger}$, is the unique matrix $X\in\mathbb{C}^{n\times m}$ that satisfies the four Penrose equations

$$(1)\ AXA = A,\quad (2)\ XAX = X,\quad (3)\ (AX)^{*} = AX,\quad (4)\ (XA)^{*} = XA. \tag{1.1}$$

A matrix X is called a $\{i,\dots,j\}$-generalized inverse of A, denoted by $A^{(i,\dots,j)}$, if it satisfies the ith,…, jth equations in (1.1). The collection of all $\{i,\dots,j\}$-generalized inverses of A is denoted by $\{A^{(i,\dots,j)}\}$. A matrix X is called a generalized inverse of A if it satisfies $AXA = A$, and is denoted by $A^{(1)} = A^{-}$. See [2, 3] for more exposition on generalized inverses of matrices.

In matrix calculus, one can construct various algebraic expressions (functions) that involve known and unknown matrices through the conventional additions and multiplications of matrices, and take them as fundamental objects of study from theoretical and applied points of view. This kind of matrix expressions can generally be written as $f(A_1^{(i_1,\dots,j_1)}, A_2^{(i_2,\dots,j_2)}, \dots, A_k^{(i_k,\dots,j_k)})$, where $A_1, A_2, \dots, A_k$ are a family of given matrices of appropriate sizes. One of this kind of matrix expressions is

$$f(A_2^{(i_2,\dots,j_2)}, A_4^{(i_4,\dots,j_4)}) = A_1A_2^{(i_2,\dots,j_2)}A_3A_4^{(i_4,\dots,j_4)}A_5 - A, \tag{1.2}$$

where $A_1\in\mathbb{C}^{m_1\times m_2}$, $A_2\in\mathbb{C}^{m_3\times m_2}$, $A_3\in\mathbb{C}^{m_3\times m_4}$, $A_4\in\mathbb{C}^{m_5\times m_4}$, and $A_5\in\mathbb{C}^{m_5\times m_6}$, and $A\in\mathbb{C}^{m_1\times m_6}$ are given. This expression is informative because there are six matrices in (1.2), and many matrix expressions in matrix calculus can be written its special cases. Setting $f(A_2^{(i_2,\dots,j_2)}, A_4^{(i_4,\dots,j_4)}) = 0$ leads to the following matrix equality

$$A_1A_2^{(i_2,\dots,j_2)}A_3A_4^{(i_4,\dots,j_4)}A_5 = A. \tag{1.3}$$

This equality may or may not hold for some or all $A_2^{(i_2,\dots,j_2)}$ and $A_4^{(i_4,\dots,j_4)}$. In a recent article [5], Jiang and Tian obtained the following result on the equality in (1.3) for $\{1\}$-generalized inverses of A_2 and A_4.

Lemma 1.1. *Let A_i and A be as given in* (1.3), $i = 1, 2, \dots, 5$. *The following three statements are equivalent*:

(a) *Eq.* (1.3) *holds for all A_2^- and A_4^-, which is denoted by $A_1A_2^-A_3A_4^-A_5 \equiv A$.*

(b) *$A_1A_2^-A_3A_4^-A_5$ is invariant with respect to the choice of A_2^- and A_4^-, and $A = A_1A_2^{\dagger}A_3A_4^{\dagger}A_5$.*

(c) *One of the following six assertions holds*:

(i) $A_1 = 0$ *and* $A = 0$.

(ii) $A_3 = 0$ *and* $A = 0$.

(iii) $A_5 = 0$ *and* $A = 0$.

(iv) $A = 0$ *and* $r\begin{bmatrix} A_2 & A_3 \\ A_1 & 0 \end{bmatrix} = r(A_2)$.

(v) $A = 0$ *and* $r\begin{bmatrix} A_4 & A_5 \\ A_3 & 0 \end{bmatrix} = r(A_4)$.

(vi) $r\begin{bmatrix} -A & 0 & A_1 \\ 0 & A_3 & A_2 \\ A_5 & A_4 & 0 \end{bmatrix} = r(A_2) + r(A_4)$.

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Note that the assertions in Lemma 1.1(c) involve no generalized inverses, and the three block matrices in the rank equalities in (iv), (v), and (vi) are composed by the given matrices. Then these conditions are easy to verify and simplify when the matrices in (1.3) are given with specified cases, thus Lemma 1.1 provides a convenient tool to establish matrix identities that involve mixed products of matrices and their generalized inverses; see e.g., some applications of Lemma 1.1 to generalized inverse problems in [5]. In addition to the above results, the present author proposed and solved many set inclusion problems associated with reverse order laws for generalized inverses of the mixed matrix products in two recent papers [10, 11]. As a continuation of this kind of work, the purpose of this study is to provide several new groups of matrix set inclusions associated with reverse order laws for generalized inverses of the matrix products AB, ABC, and $ABCD$. products.

Lemma 1.2 ([6]). *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, and $C \in \mathbb{C}^{p\times q}$. Then*

$$r(AB) = r(A) + r(B) - n + r[(I_n - BB^-)(I_p - A^-A)], \tag{1.4}$$
$$r(ABC) = r(AB) + r(BC) - r(B) + r[(I_n - (BC)(BC)^-)B(I_p - (AB)^-(AB))] \tag{1.5}$$

hold for all A^-, B^-, $(AB)^-$, and $(BC)^-$. In particular, the following results hold.

(a) *The rank of AB satisfies following inequalities*

$$\max\{0,\ r(A) + r(B) - n\} \leq r(A) + r(B) - r[A^*,\ B] \leq r(AB) \leq \min\{r(A),\ r(B)\}. \tag{1.6}$$

(b) *The rank of ABC satisfies following inequalities*

$$\begin{aligned} r(ABC) &\leq \min\{r(AB),\ r(BC)\} \\ &\leq \min\{r(A),\ r(B),\ r(C)\} \\ &\leq \min\{\, m,\ r(B),\ q\} \leq \min\{\, m,\ n,\ p,\ q\}, \end{aligned} \tag{1.7}$$
$$\begin{aligned} r(ABC) &\geq \max\{0,\ r(AB) + r(BC) - r(B)\} \\ &\geq \max\{0,\ r(A) + r(B) + r(C) - r[\,A^*,\ B\,] - r[\,B^*,\ C\,]\} \\ &\geq \max\{0,\ r(A) + r(B) + r(C) - n - p\}, \end{aligned} \tag{1.8}$$
$$\begin{aligned} r(ABC) &\geq r(AB) + r(C) - r[\,(AB)^*,\ C\,] \\ &\geq \max\{0,\ r(AB) + r(C) - p\} \\ &\geq \max\{0,\ r(A) + r(B) + r(C) - n - p\}, \end{aligned} \tag{1.9}$$
$$\begin{aligned} r(ABC) &\geq r(A) + r(BC) - r[\,A^*,\ BC\,] \\ &\geq \max\{0,\ r(A) + r(BC) - n\} \\ &\geq \max\{0,\ r(A) + r(B) + r(C) - n - p\}. \end{aligned} \tag{1.10}$$

(c) $r(ABC) = r(B) \Leftrightarrow r(AB) = r(BC) = r(B)$.

(d) $r(ABC) = r(A) + r(B) + r(C) - n - p \Leftrightarrow r(ABC) = r(AB) + r(C) - p$ *and* $r(AB) = r(A) + r(B) - n$.

(e) $r(ABC) = r(A) + r(B) + r(C) - n - p \Leftrightarrow r(ABC) = r(A) + r(BC) - n$ *and* $r(BC) = r(B) + r(C) - p$.

Lemma 1.3 ([7, 8]). *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, $C \in \mathbb{C}^{l\times n}$, and $D \in \mathbb{C}^{l\times k}$ be given. Then*

$$\max_{A^-\in\{A^-\}} r(D - CA^-B) = \min\left\{ r[\,C,\ D\,],\ \ r\begin{bmatrix} B \\ D \end{bmatrix},\ \ r\begin{bmatrix} A & B \\ C & D \end{bmatrix} - r(A) \right\}. \tag{1.11}$$

Thus

$$CA^-B = D \text{ for all } A^- \Leftrightarrow [\,C,\ D\,] = 0 \text{ or } \begin{bmatrix} B \\ D \end{bmatrix} = 0 \text{ or } \begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A). \tag{1.12}$$

2 Set inclusions for generalized inverses of mixed matrix products

One can construct matrix equalities that are composed by inverses and generalized inverses of matrices from theoretical and applied points of view. In comparison, one class of the most attractive forms of these matrix equalities are various types of reverse order laws for generalized inverses of matrix products. In this section, the author first presents two lemmas on links between matrix identity problems and matrix set inclusion issues of products of matrices and their generalized inverses.

Lemma 2.1. *Let A_i and A be as given in* (1.3), $i = 1, 2, \ldots, 5$. *Then the following five statements are equivalent:*

(a) $\{A_1A_2^-A_3A_4^-A_5\} \subseteq \{A^-\}$, *i.e.,* $AA_1A_2^-A_3A_4^-A_5A \equiv A$.

(b) $\{AA_1A_2^-A_3A_4^-A_5\} \subseteq \{AA^-\}$.

(c) $\{A_1A_2^-A_3A_4^-A_5A\} \subseteq \{A^-A\}$.

(d) $A = 0$ *or* $r\begin{bmatrix} -A & 0 & AA_1 \\ 0 & A_3 & A_2 \\ A_5A & A_4 & 0 \end{bmatrix} = r(A_2) + r(A_4)$.

(e) $A = 0$ *or* $r\begin{bmatrix} A_3 & A_2 \\ A_4 & A_5AA_1 \end{bmatrix} = r(A_2) + r(A_4) - r(A)$.

Lemma 2.2. *Let A_i and A be as given in* (1.3), $i = 2, 3, 4$. *Then the following four statements are equivalent:*

(a) $\{A_2^- A_3 A_4^-\} \subseteq \{A^-\}$, *i.e.,* $AA_2^- A_3 A_4^- A \equiv A$.

(b) $\{AA_2^- A_3 A_4^-\} \subseteq \{AA^-\}$.

(c) $\{A_2^- A_3 A_4^- A\} \subseteq \{A^- A\}$.

(d) $A = 0$ *or* $r\begin{bmatrix} A_3 & A_2 \\ A_4 & A \end{bmatrix} = r(A_2) + r(A_4) - r(A)$.

It is well known that one of the most fundamental reverse order laws for a matrix product AB is $(AB)^- = B^-A^-$; see e.g., [4, 12]. The authors next reconsiders reverse order laws and present a group of known and new results as follows.

Theorem 2.3. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$ be given. Then the following 23 statements are equivalent:*

(i) $\{(AB)^-\} \supseteq \{B^-A^-\}$, *i.e.,* $ABB^-A^-AB \equiv AB$.

(ii) $\{(AB)^-\} \supseteq \{B^*(BB^*)^-(A^*A)^-A^*\}$.

(iii) $\{(AB)^-\} \supseteq \{B^-(BB^-)^-(A^-A)^-A^-\}$.

(iv) $\{AB(AB)^-\} \supseteq \{ABB^-A^-\}$.

(v) $\{AB(AB)^-\} \supseteq \{ABB^*(BB^*)^-(A^*A)^-A^*\}$.

(vi) $\{AB(AB)^-\} \supseteq \{ABB^-(BB^-)^-(A^-A)^-A^-\}$.

(vii) $\{(AB)^-AB\} \supseteq \{B^-A^-AB\}$.

(viii) $\{(AB)^-AB\} \supseteq \{B^*(BB^*)^-(A^*A)^-A^*AB\}$.

(ix) $\{(AB)^-AB\} \supseteq \{B^-(BB^-)^-(A^-A)^-A^-AB\}$.

(x) $\{B(AB)^-A\} \supseteq \{BB^-A^-A\}$.

(xi) $\{(BAB)^-A\} \supseteq \{BB^-(BB^-)^-(A^-A)^-A^-A\}$.

(xii) $\{(A^-ABB^-)^-\} \supseteq \{(BB^-)^-(A^-A)^-\}$.

(xiii) $\{(B^*A^*)^-\} \supseteq \{(A^*)^-(B^*)^-\}$.

(xiv) $\{(A^*ABB^*)^-\} \supseteq \{(BB^*)^-(A^*A)^-\}$.

(xv) $\{(BB^*A^*A)^-\} \supseteq \{(A^*A)^-(BB^*)^-\}$.

(xvi) $\{[(A^*A)^{1/2}(BB^*)^{1/2}]^-\} \supseteq \{[(BB^*)^{1/2}]^-[(A^*A)^{1/2}]^-\}$.

(xvii) $\{[(BB^*)^{1/2}(A^*A)^{1/2})]^-\} \supseteq \{[(A^*A)^{1/2}]^-[(BB^*)^{1/2}]^-\}$.

(xviii) $\{(AA^*ABB^*B)^-\} \supseteq \{(BB^*B)^-(AA^*A)^-\}$.

(xix) $\{(B^*BB^*A^*AA^*)^-\} \supseteq \{(A^*AA^*)^-(B^*BB^*)^-\}$.

(xx) $AB = 0$ *or* $r(AB) = r(A) + r(B) - n$.

(xxi) $AB = 0$ *or* $(I_n - BB^-)(I_n - A^-A) = 0$ *for some/all* A^- *and* B^-.

(xxii) $\mathcal{N}(A) \supseteq \mathcal{R}(B)$ *or* $\mathcal{N}(A) \subseteq \mathcal{R}(B)$.

(xxiii) $\mathcal{R}(A^*) \supseteq \mathcal{N}(B^*)$ *or* $\mathcal{R}(A^*) \subseteq \mathcal{N}(B^*)$.

Proof. By (a) and (d) in Lemma 2.2, Result (i) holds iff $AB = 0$ or

$$r(A) + r(B) - r(AB) = r\begin{bmatrix} I_n & B \\ A & AB \end{bmatrix} = r\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} = n,$$

establishing the equivalence of Results (i) and (xx).

By (a) and (e) in Lemma 2.1, Result (ii) holds iff $AB = 0$ or

$$r(A) + r(B) - r(AB) = r(A^*A) + r(BB^*) - r(AB) = r\begin{bmatrix} I_n & BB^* \\ A^*A & A^*ABB^* \end{bmatrix} = r\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} = n,$$

establishing the equivalence of Results (ii) and (xx).

By (a) and (e) in Lemma 2.1, Result (iii) holds iff $AB = 0$ or

$$r(A) + r(B) - r(AB) = r(A^-A) + r(BB^-) - r(AB) = r\begin{bmatrix} I_n & BB^- \\ A^-A & A^-ABB^- \end{bmatrix} = r\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} = n,$$

establishing the equivalence of Results (iii) and (xx).

The equivalences of Results (i) and (xiii)–(xx) follow from the following rank equalities

$$r(A) = r(AA^-) = r(AA^*) = r(AA^*A), \tag{2.1}$$
$$r(B) = r(B^-B) = r(B^*B) = r(BB^*B), \tag{2.2}$$
$$\begin{aligned} r(AB) &= r(B^*A^*) = r(A^-ABB^-) = r(A^*ABB^*) = r(BB^*A^*A) \\ &= r[(A^*A)^{1/2}(BB^*)^{1/2}] = r[(BB^*)^{1/2}(A^*A)^{1/2}] \\ &= r(AA^*ABB^*B) = r(B^*BB^*A^*AA^*). \end{aligned} \tag{2.3}$$

The equivalences of Results (i)–(xii) follows from Lemma 2.1(a), (b), and (c).

The equivalences of (i) and (xx)–(xxiii) were first proved in [12]. □

Theorem 2.4. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. Then the following 16 statements are equivalent:*

(i) $\{(AB)^-\} \ni B^\dagger A^\dagger$, *i.e.*, $ABB^\dagger A^\dagger AB = AB$.

(ii) $\{(AB)^-\} \supseteq \{(B^*B)^-B^*A^*(AA^*)^-\}$.

(iii) $\{AB(AB)^-\} \ni ABB^\dagger A^\dagger$.

(iv) $\{AB(AB)^-\} \supseteq \{AB(B^*B)^-B^*A^*(AA^*)^-\}$.

(v) $\{(AB)^-AB\} \ni B^\dagger A^\dagger AB$.

(vi) $\{(AB)^-AB\} \supseteq \{(B^*B)^-B^*A^*(AA^*)^-AB\}$.

(vii) $\{B(AB)^-A\} \ni BB^\dagger A^\dagger A$.

(viii) $\{B(AB)^-A\} \supseteq \{B(B^*B)^-B^*A^*(AA^*)^-A\}$.

(ix) $\{(A^-ABB^-)^-\} \ni (BB^-)^\dagger(A^-A)^\dagger$.

(x) $\{(A^*ABB^*)^-\} \ni (BB^*)^\dagger(A^*A)^\dagger$.

(xi) $\{(BB^*A^*A)^-\} \ni (A^*A)^\dagger(BB^*)^\dagger$.

(xii) $\{[(A^*A)^{1/2}(BB^*)^{1/2}]^-\} \ni [(BB^*)^{1/2}]^\dagger[(A^*A)^{1/2}]^\dagger$.

(xiii) $\{[(BB^*)^{1/2}(A^*A)^{1/2})]^-\} \ni [(A^*A)^{1/2}]^\dagger[(BB^*)^{1/2}]^\dagger$.

(xiv) $\{(AA^*ABB^*B)^-\} \ni (BB^*B)^\dagger(AA^*A)^\dagger$.

(xv) $\{(B^*BB^*A^*AA^*)^-\} \ni (A^*AA^*)^\dagger(B^*BB^*)^\dagger$.

(xvi) $r(AB) = r(A) + r(B) - r[A^*,\, B]$.

Proof. The equivalence of Results (i) and (xiv) follows from the well-known rank formula

$$r(AB - ABB^\dagger A^\dagger AB) = r[A^*,\, B] - r(A) - r(B) - r(AB);$$

see [1,9]. By (a) and (d) in Lemma 2.2, Result (ii) holds iff $AB = 0$ or

$$\begin{aligned} r(A) + r(B) - r(AB) = r(AA^*) + r(B^*B) - r(AB) &= r\begin{bmatrix} B^*A^* & B^*B \\ AA^* & AB \end{bmatrix} \\ &= r\left([A^*,\, B]^*[A^*,\, B]\right) = r[A^*,\, B]. \end{aligned}$$

However, the second rank equality implies $AB = 0$, thus establishing the equivalence of Results (ii) and (xvi).

The equivalences of Results (i) and (ix)–(xvi) follow from (2.1), (2.2), (2.3), and

$$r[A^*,\, B] = r[(A^-A)^*,\, BB^-] = r[A^*A,\, BB^*] = r[(A^*A)^{1/2},\, (BB^*)^{1/2}] = r[A^*AA^*,\, BB^*B].$$

The equivalences of Results (i)–(viii) follow from Lemma 2.2(a), (b), and (c). □

We next present two groups of results on set inclusions associated with the two reverse order laws $(ABC)^- = (BC)^-B(AB)^-$ and $(ABC)^- = C^-B^-A^-$ and their variations.

Theorem 2.5. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, and $C \in \mathbb{C}^{p\times q}$ be given, and denote $M = ABC$. Then the following 36 statements are equivalent:*

(i) $\{M^-\} \supseteq \{(BC)^-B(AB)^-\}$.

(ii) $\{M^-\} \supseteq \{C^*(BCC^*)^-B(A^*AB)^-A^*\}$.

(iii) $\{M^-\} \supseteq \{(B^*BC)^-B^*BB^*(ABB^*)^-\}$.

(iv) $\{M^-\} \supseteq \{C^*(B^*BCC^*)^-B^*BB^*(A^*ABB^*)^-A^*\}$.

(v) $\{M^-\} \supseteq \{C^-(BCC^-)^-B(A^-AB)^-A^-\}$.

(vi) $\{M^-\} \supseteq \{(B^-BC)^-B^-BB^-(ABB^-)^-\}$.

(vii) $\{M^-\} \supseteq \{C^-(B^-BCC^-)^-B^-BB^-(A^-ABB^-)^-A^-\}$.

(viii) $\{MM^-\} \supseteq \{M(BC)^-B(AB)^-\}$.

(ix) $\{MM^-\} \supseteq \{MC^*(BCC^*)^-B(A^*AB)^-A^*\}$.

(x) $\{MM^-\} \supseteq \{M(B^*BC)^-B^*BB^*(ABB^*)^-\}$.
(xi) $\{MM^-\} \supseteq \{MC^*(B^*BCC^*)^-B^*BB^*(A^*ABB^*)^-A^*\}$.
(xii) $\{MM^-\} \supseteq \{MC^-(BCC^-)^-B(A^-AB)^-A^-\}$.
(xiii) $\{MM^-\} \supseteq \{M(B^-BC)^-B^-BB^-(ABB^-)^-\}$.
(xiv) $\{MM^-\} \supseteq \{MC^-(B^-BCC^-)^-B^-BB^-(A^-ABB^-)^-A^-\}$.
(xv) $\{M^-M\} \supseteq \{(BC)^-B(AB)^-M\}$.
(xvi) $\{M^-M\} \supseteq \{C^*(BCC^*)^-B(A^*AB)^-A^*M\}$.
(xvii) $\{M^-M\} \supseteq \{(B^*BC)^-B^*BB^*(ABB^*)^-M\}$.
(xviii) $\{M^-M\} \supseteq \{C^*(B^*BCC^*)^-B^*BB^*(A^*ABB^*)^-A^*M\}$.
(xix) $\{M^-M\} \supseteq \{C^-(BCC^-)^-B(A^-AB)^-A^-M\}$.
(xx) $\{M^-M\} \supseteq \{(B^-BC)^-B^-BB^-(ABB^-)^-M\}$.
(xxi) $\{M^-M\} \supseteq \{C^-(B^-BCC^-)^-B^-BB^-(A^-ABB^-)^-A^-M\}$.
(xxii) $\{CM^-A\} \supseteq \{C(BC)^-B(AB)^-A\}$.
(xxiii) $\{CM^-A\} \supseteq \{CC^*(BCC^*)^-B(A^*AB)^-A^*A\}$.
(xxiv) $\{CM^-A\} \supseteq \{C(B^*BC)^-B^*BB^*(ABB^*)^-A\}$.
(xxv) $\{CM^-A\} \supseteq \{CC^*(B^*BCC^*)^-B^*BB^*(A^*ABB^*)^-A^*A\}$.
(xxvi) $\{CM^-A\} \supseteq \{CC^-(BCC^-)^-B(A^-AB)^-A^-A\}$.
(xxvii) $\{CM^-A\} \supseteq \{C(B^-BC)^-B^-BB^-(ABB^-)^-A\}$.
(xxviii) $\{CM^-A\} \supseteq \{CC^-(B^-BCC^-)^-B^-BB^-(A^-ABB^-)^-A^-A\}$.
(xxix) $\{(A^-MC^-)^-\} \supseteq \{(BCC^-)^-B(A^-AB)^-\}$.
(xxx) $\{(A^*MC^*)^-\} \supseteq \{(BCC^*)^-B(A^*AB)^-\}$.
(xxxi) $\{(AA^*MC^*C)^-\} \supseteq \{(BCC^*C)^-B(AA^*AB)^-\}$.
(xxxii) $\{[(AB)^-M(BC)^-]^-\} \supseteq \{[(BC)(BC)^-]^-B[(AB)^-(AB)]^-\}$.
(xxxiii) $\{[(AB)^*M(BC)^*]^-\} \supseteq \{[(BC)(BC)^*]^-B[(AB)^*(AB)]^-\}$.
(xxxiv) $\{([(AB)^*(AB)]^{1/2}B^-[(BC)(BC)^*]^{1/2})^-\} \supseteq \{([(BC)(BC)^*]^{1/2})^-B([(AB)^*(AB)]^{1/2})^-\}$.
(xxxv) $M = 0$ *or* $[I_n - (BC)(BC)^-]B[I_n - (AB)^-(AB)] = 0$ *for some/all* $(AB)^-$ *and* $(BC)^-$.
(xxxvi) $M = 0$ *or* $r(M) = r(AB) + r(BC) - r(B)$.

Proof. By (a) and (d) in Lemma 2.2, Result (i) holds iff

$$M = 0 \quad \text{or} \quad r(AB) + r(BC) - r(M) = r\begin{bmatrix} B & BC \\ AB & ABC \end{bmatrix} = r\begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} = r(B),$$

establishing the equivalence of Results (i) and (xxxvi). The equivalences of Results (i)–(xxviii) can also be shown by Lemmas 2.1 and 2.2, and therefore the details are omitted.

The equivalences of Results (i) and (xxix)–(xxxiv) and (xxxvi) follow from

$$\begin{aligned} r(AB) &= r(A^-AB) = r(A^*B) = r(AA^*AB), \\ r(BC) &= r(BCC^-) = r(BCC^*) = r(BCC^*C), \\ r(M) &= r(A^-MC^-) = r(A^*MC^*) = r(AA^*MC^*C) \\ &= r[(AB)^-M(BC)^-] = r[(AB)^*M(BC)^*]. \end{aligned}$$

The equivalence of Results (xxxv) and (xxxvi) follow from (1.5). □

Theorem 2.6. *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, *and* $C \in \mathbb{C}^{p\times q}$ *be given, and denote* $M = ABC$. *Then the following 27 statements are equivalent:*

(i) $\{M^-\} \supseteq \{C^-B^-A^-\}$.
(ii) $\{M^-\} \supseteq \{C^*(CC^*)^-B^-(A^*A)^-A^*\}$.
(iii) $\{M^-\} \supseteq \{C^-(CC^-)^-B^-(A^-A)^-A^-\}$.
(iv) $\{M^-\} \supseteq \{(BC)^-A^-\}$ *and* $\{(BC)^-\} \supseteq \{C^-B^-\}$.
(v) $\{M^-\} \supseteq \{C^-(AB)^-\}$ *and* $\{(AB)^-\} \supseteq \{B^-A^-\}$.
(vi) $\{MM^-\} \supseteq \{MC^-B^-A^-\}$.
(vii) $\{MM^-\} \supseteq \{MC^*(CC^*)^-B^-(A^*A)^-A^*\}$.
(viii) $\{MM^-\} \supseteq \{MC^-(CC^-)^-B^-(A^-A)^-A^-\}$.
(ix) $\{MM^-\} \supseteq \{M(BC)^-A^-\}$ *and* $\{BC(BC)^-\} \supseteq \{BCC^-B^-\}$.

(x) $\{MM^-\} \supseteq \{MC^-(AB)^-\}$ *and* $\{AB(AB)^-\} \supseteq \{ABB^-A^-\}$.

(xi) $\{M^-M\} \supseteq \{C^-B^-A^-M\}$.

(xii) $\{M^-M\} \supseteq \{C^*(CC^*)^-B^-(A^*A)^-A^*M\}$.

(xiii) $\{M^-M\} \supseteq \{C^-(CC^-)^-B^-(A^-A)^-A^-M\}$.

(xiv) $\{M^-M\} \supseteq \{(BC)^-A^-M\}$ *and* $\{(BC)^-BC\} \supseteq \{C^-B^-BC\}$.

(xv) $\{M^-M\} \supseteq \{C^-(AB)^-M\}$ *and* $\{(AB)^-\} \supseteq \{B^-A^-AB\}$.

(xvi) $\{CM^-A\} \supseteq \{CC^-B^-A^-A\}$.

(xvii) $\{CM^-A\} \supseteq \{CC^*(CC^*)^-B^-(A^*A)^-A^*A\}$.

(xviii) $\{CM^-A\} \supseteq \{CC^-(CC^-)^-B^-(A^-A)^-A^-A\}$.

(xix) $\{BCM^-A\} \supseteq \{BC(BC)^-A^-A\}$ *and* $\{C(BC)^-B\} \supseteq \{CC^-B^-B\}$.

(xx) $\{CM^-AB\} \supseteq \{CC^-(AB)^-AB\}$ *and* $\{B(AB)^-A\} \supseteq \{BB^-A^-A\}$.

(xxi) $\{(C^*B^*A^*)^-\} \supseteq \{(A^*)^-(B^*)^-(C^*)^-\}$.

(xxii) $\{(A^-MC^-)^-\} \supseteq \{(CC^-)^-B^-(A^-A)^-\}$.

(xxiii) $\{(A^*MC^*)^-\} \supseteq \{(CC^*)^-B^-(A^*A)^-\}$.

(xxiv) $\{(AA^*MC^*C)^-\} \supseteq \{(CC^*C)^-B^-(AA^*A)^-\}$.

(xxv) $M = 0$ *or* $r(M) = r(A) + r(B) + r(C) - n - p$.

(xxvi) $M = 0$ *or* $\{r(M) = r(A) + r(BC) - n$ *and* $r(BC) = r(B) + r(C) - p\}$.

(xxvii) $M = 0$ *or* $\{r(M) = r(AB) + r(C) - p$ *and* $r(AB) = r(A) + r(B) - n\}$.

Proof. By (a) and (d) in Lemma 2.2, Result (i) holds iff

$$M = 0 \quad or \quad r(A) + r(C) - r(M) = r\begin{bmatrix} B^- & C \\ A & M \end{bmatrix} \tag{2.4}$$

holds for all B^-, where by (1.11),

$$\begin{aligned}
\max_{B^-} r\begin{bmatrix} B^- & C \\ A & M \end{bmatrix} &= \max_{B^-} r\left(\begin{bmatrix} I_p \\ 0 \end{bmatrix} B^-[I_n,\, 0] + \begin{bmatrix} 0 & C \\ A & M \end{bmatrix}\right) \\
&= \min\left\{ r\begin{bmatrix} I_p & 0 & C \\ 0 & A & M \end{bmatrix},\ r\begin{bmatrix} I_n & 0 \\ 0 & C \\ A & M \end{bmatrix},\ r\begin{bmatrix} -B & I_n & 0 \\ I_p & 0 & C \\ 0 & A & M \end{bmatrix} - r(B) \right\} \\
&= \min\left\{ r\begin{bmatrix} I_p & 0 & 0 \\ 0 & A & 0 \end{bmatrix},\ r\begin{bmatrix} I_n & 0 \\ 0 & C \\ 0 & 0 \end{bmatrix},\ r\begin{bmatrix} 0 & I_n & 0 \\ I_p & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - r(B) \right\} \\
&= \min\{p + r(A),\ \ n + r(C),\ \ n + p - r(B)\}.
\end{aligned} \tag{2.5}$$

Hence the second equality in (2.4) is equivalent to

$$\min\{p - r(C) + r(M),\ \ n - r(A) + r(M),\ \ n + p - r(A) - r(B) - r(C) + r(M)\} = 0, \tag{2.6}$$

where where by Lemma 2.1(b),

$$\begin{aligned}
&p - r(C) + r(M) \geq r(M) \geq 0, \\
&n - r(A) + r(M) \geq r(M) \geq 0, \\
&r(M) \geq r(A) + r(B) + r(C) - n - p \geq 0.
\end{aligned}$$

Combining (2.6) with the first condition $M = 0$ in (2.4) leads to the equivalence of Results (i) and (xxv).

The remaining equivalences of Results (i)–(xx) can be shown from Lemmas 2.1 and 2.2 by similar approaches, and therefore the proofs are omitted.

By Lemma 2.1(b),

$$\begin{aligned}
r(M) &\geq r(A) + r(BC) - n \geq r(A) + r(B) + r(C) - n - p, \\
r(MM) &\geq r(AB) + r(C) - p \geq r(A) + r(B) + r(C) - n - p.
\end{aligned}$$

Hence,

$$\begin{aligned}
&r(MM) = r(A) + r(B) + r(C) - n - p \\
&\Leftrightarrow r(M) = r(A) + r(BC) - n \quad \text{and} \quad r(BC) = r(B) + r(C) - p \\
&\Leftrightarrow r(M) = r(AB) + r(C) - p \quad \text{and} \quad r(AB) = r(A) + r(B) - n,
\end{aligned}$$

so that Results (xxv), (xxvi), and (xxvii) are equivalent.

The equivalences of Results (i) and (xxi)–(xxiv) follow from $r(M) = r(A^-M) = r(A^*M) = r(AA^*M)$, $r(M) = r(MC^-) = r(MC^*) = r(MC^*C)$, and $r(M) = r(A^-MC^-) = r(A^*MC^*) = r(AA^*MC^*C)$. □

Theorem 2.7. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, $C \in \mathbb{C}^{p\times q}$, and $D \in \mathbb{C}^{q\times s}$ be given, and denote $N = ABCD$. Then the following 36 statements are equivalent:*

(i) $\{N^-\} \supseteq \{(CD)^-C(BC)^-B(AB)^-\}$.

(ii) $\{N^-\} \supseteq \{(C^*CD)^-C^*C(BC)^-BB^*(ABB^*)^-\}$.

(iii) $\{N^-\} \supseteq \{(CD)^-CC^*(B^*BCC^*)^-B^*B(AB)^-\}$.

(iv) $\{N^-\} \supseteq \{D^*(C^*CDD^*)^-C^*C(BC)^-BB^*(A^*ABB^*)^-A^*\}$.

(v) $\{N^-\} \supseteq \{(C^*CD)^-C^*CC^*(B^*BCC^*)^-B^*BB^*(ABB^*)^-\}$.

(vi) $\{N^-\} \supseteq \{D^*(C^*CDD^*)^-C^*CC^*(B^*BCC^*)^-B^*BB^*(A^*ABB^*)^-A^*\}$.

(vii) $\{N^-\} \supseteq \{(C^-CD)^-C^-C(BC)^-BB^-(ABB^-)^-\}$.

(viii) $\{N^-\} \supseteq \{(CD)^-CC^-(B^-BCC^-)^-B^-B(AB)^-\}$.

(ix) $\{N^-\} \supseteq \{D^-(C^-CDD^-)^-C^-C(BC)^-BB^-(A^-ABB^-)^-A^-\}$.

(x) $\{N^-\} \supseteq \{(C^-CD)^-C^-CC^-(B^-BCC^-)^-B^-BB^-(ABB^-)^-\}$.

(xi) $\{N^-\} \supseteq \{D^-(C^-CDD^-)^-C^-CC^-(B^-BCC^-)^-B^-BB^-(A^-ABB^-)^-A^-\}$.

(xii) $\{NN^-\} \supseteq \{N(CD)^-C(BC)^-B(AB)^-\}$.

(xiii) $\{NN^-\} \supseteq \{N(C^*CD)^-C^*C(BC)^-BB^*(ABB^*)^-\}$.

(xiv) $\{NN^-\} \supseteq \{N(CD)^-CC^*(B^*BCC^*)^-B^*B(AB)^-\}$.

(xv) $\{NN^-\} \supseteq \{ND^*(C^*CDD^*)^-C^*C(BC)^-BB^*(A^*ABB^*)^-A^*\}$.

(xvi) $\{NN^-\} \supseteq \{N(C^*CD)^-C^*CC^*(B^*BCC^*)^-B^*BB^*(ABB^*)^-\}$.

(xvii) $\{NN^-\} \supseteq \{ND^*(C^*CDD^*)^-C^*CC^*(B^*BCC^*)^-B^*BB^*(A^*ABB^*)^-A^*\}$.

(xviii) $\{NN^-\} \supseteq \{N(C^-CD)^-C^-C(BC)^-BB^-(ABB^-)^-\}$.

(xix) $\{NN^-\} \supseteq \{N(CD)^-CC^-(B^-BCC^-)^-B^-B(AB)^-\}$.

(xx) $\{NN^-\} \supseteq \{ND^-(C^-CDD^-)^-C^-C(BC)^-BB^-(A^-ABB^-)^-A^-\}$.

(xxi) $\{NN^-\} \supseteq \{N(C^-CD)^-C^-CC^-(B^-BCC^-)^-B^-BB^-(ABB^-)^-\}$.

(xxii) $\{NN^-\} \supseteq \{ND^-(C^-CDD^-)^-C^-CC^-(B^-BCC^-)^-B^-BB^-(A^-ABB^-)^-A^-\}$.

(xxiii) $\{N^-N\} \supseteq \{(CD)^-C(BC)^-B(AB)^-N\}$.

(xxiv) $\{N^-N\} \supseteq \{(C^*CD)^-C^*C(BC)^-BB^*(ABB^*)^-N\}$.

(xxv) $\{N^-N\} \supseteq \{(CD)^-CC^*(B^*BCC^*)^-B^*B(AB)^-N\}$.

(xxvi) $\{N^-N\} \supseteq \{D^*(C^*CDD^*)^-C^*C(BC)^-BB^*(A^*ABB^*)^-A^*N\}$.

(xxvii) $\{N^-N\} \supseteq \{(C^*CD)^-C^*CC^*(B^*BCC^*)^-B^*BB^*(ABB^*)^-N\}$.

(xxviii) $\{N^-N\} \supseteq \{D^*(C^*CDD^*)^-C^*CC^*(B^*BCC^*)^-B^*BB^*(A^*ABB^*)^-A^*N\}$.

(xxix) $\{N^-N\} \supseteq \{(C^-CD)^-C^-C(BC)^-BB^-(ABB^-)^-N\}$.

(xxx) $\{N^-N\} \supseteq \{(CD)^-CC^-(B^-BCC^-)^-B^-B(AB)^-N\}$.

(xxxi) $\{N^-N\} \supseteq \{D^-(C^-CDD^-)^-C^-C(BC)^-BB^-(A^-ABB^-)^-A^-N\}$.

(xxxii) $\{N^-N\} \supseteq \{(C^-CD)^-C^-CC^-(B^-BCC^-)^-B^-BB^-(ABB^-)^-N\}$.

(xxxiii) $\{N^-N\} \supseteq \{D^-(C^-CDD^-)^-C^-CC^-(B^-BCC^-)^-B^-BB^-(A^-ABB^-)^-A^-N\}$.

(xxxiv) $N = 0$ *or* $r(N) = r(AB) + r(BC) + r(CD) - r(B) - r(C)$.

(xxxv) $N = 0$ *or* $\{r(N) = r(ABC) + r(CD) - r(C)$ *and* $r(ABC) = r(AB) + r(BC) - r(B)\}$.

(xxxvi) $N = 0$ *or* $\{r(N) = r(AB) + r(BCD) - r(B)$ *and* $r(BCD) = r(BC) + r(CD) - r(C)\}$.

Proof. By (a) and (d) in Lemma 2.2, Result (i) holds iff

$$N = 0 \quad \text{or} \quad r(AB) + r(CD) - r(N) = r\begin{bmatrix} C(BC)^-B & CD \\ AB & N \end{bmatrix} \tag{2.7}$$

holds for all $(BC)^-$, where by (1.11),

$$\begin{aligned}
&\max_{(BC)^-} r\begin{bmatrix} C(BC)^-B & CD \\ AB & ABCD \end{bmatrix} \\
&= \max_{(BC)^-} r\left(\begin{bmatrix} C \\ 0 \end{bmatrix}(BC)^-[B, 0] + \begin{bmatrix} 0 & CD \\ AB & ABCD \end{bmatrix}\right) \\
&= \min\left\{ r\begin{bmatrix} C & 0 & CD \\ 0 & AB & ABCD \end{bmatrix},\ r\begin{bmatrix} B & 0 \\ 0 & CD \\ AB & ABCD \end{bmatrix},\ r\begin{bmatrix} -BC & B & 0 \\ C & 0 & CD \\ 0 & AB & ABCD \end{bmatrix} - r(BC) \right\} \\
&= \min\left\{ r(AB) + r(C),\quad r(CD) + r(B),\quad r\begin{bmatrix} 0 & B & 0 \\ C & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - r(BC) \right\} \\
&= \min\left\{ r(AB) + r(C),\quad r(CD) + r(B),\quad r(B) + r(C) - r(BC) \right\}.
\end{aligned} \tag{2.8}$$

Hence the second equality in (2.7) is equivalent to

$$\begin{aligned} \min\{ & r(N)+r(C)-r(CD), \quad r(N)+r(B)-r(AB), \\ & r(N)-r(AB)-r(BC)-r(CD)+r(B)+r(C)\} = 0, \end{aligned} \tag{2.9}$$

where by Lemma 2.1(b),

$$\begin{aligned} & r(N)+r(C)-r(CD) \geq r(N) \geq 0, \\ & r(N)+r(B)-r(AB) \geq r(N) \geq 0, \\ & r(N)-r(AB)-r(BC)-r(CD)+r(B)+r(C) \geq 0. \end{aligned}$$

Combining (2.9) with the first condition $N = 0$ in (2.7) leads to the equivalence of Results (i) and (xxxiv). The equivalences of Results (i)–(xxxiii) can be shown by similar approaches and therefore the details are omitted.

The equivalence of Results (xxxiv) and (xxxvi) follow from Lemma 1.2(b). □

Some applications of the preceding results are given below.

Corollary 2.8. *Let $A \in \mathbb{C}^{m\times m}$ be given. Then the following results hold*

$$\begin{aligned} & \{(A-A^2)^-\} \supseteq \{A^-(I_m-A)^-\}, \\ & \{(A-A^2)^-\} \supseteq \{(I_m-A)^-A^-\}, \\ & \{(I_m-A^2)^-\} \supseteq \{(I_m+A)^-(I_m-A)^-\}, \\ & \{(I_m-A^2)^-\} \supseteq \{(I_m-A)^-(I_m+A)^-\}, \\ & \{(A-A^3)^-\} \supseteq \{A^-(I_m+A)^-(I_m-A)^-\}, \\ & \{(A-A^3)^-\} \supseteq \{(I_m+A)^-A^-(I_m-A)^-\}, \\ & \{(A-A^3)^-\} \supseteq \{(I_m+A)^-(I_m-A)^-A^-\}. \end{aligned}$$

Proof. It follows from the following three well-known rank formulas

$$\begin{aligned} & r(A-A^2) = r(A)+r(I_m-A)-m, \\ & r(I_m-A^2) = r(I_m+A)+r(I_m-A)-m, \\ & r(A-A^3) = r(A)+r(I_m+A)+r(I_m-A)-2m, \end{aligned}$$

and Lemmas 2.1 and 2.2. □

Theorem 2.9. *Let $A,\ B \in \mathbb{C}^{m\times n}$ be given. Then the following 6 statements are equivalent*:

(i) $\{(A+B)^-\} \supseteq \left\{ \begin{bmatrix} A \\ B \end{bmatrix}^- \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} [A,\ B]^- \right\}$.

(ii) $\{(A+B)^-\} \supseteq \left\{ \begin{bmatrix} A^*A \\ B^*B \end{bmatrix}^- \begin{bmatrix} A^*AA^* & 0 \\ 0 & B^*BB^* \end{bmatrix} [AA^*,\ BB^*]^- \right\}$.

(iii) $\{(A+B)^-\} \supseteq \left\{ \begin{bmatrix} A^{\dagger}A \\ B^{\dagger}B \end{bmatrix}^- \begin{bmatrix} A^{\dagger} & 0 \\ 0 & B^{\dagger} \end{bmatrix} [AA^{\dagger},\ BB^{\dagger}]^- \right\}$.

(iv) $\{(A+B)^-\} \supseteq \left\{ [I_n,\ I_n] \begin{bmatrix} A^{\dagger}A & A^{\dagger}A \\ B^{\dagger}B & B^{\dagger}B \end{bmatrix}^- \begin{bmatrix} A^{\dagger} & 0 \\ 0 & B^{\dagger} \end{bmatrix} \begin{bmatrix} AA^{\dagger} & BB^{\dagger} \\ AA^{\dagger} & BB^{\dagger} \end{bmatrix}^- \begin{bmatrix} I_m \\ I_m \end{bmatrix} \right\}$.

(v) $\{(A+B)^-\} \supseteq \left\{ [I_n,\ I_n] \begin{bmatrix} A^*A & A^*A \\ B^*B & B^*B \end{bmatrix}^- \begin{bmatrix} A^*AA^* & 0 \\ 0 & B^*BB^* \end{bmatrix} \begin{bmatrix} AA^* & BB^* \\ AA^* & BB^* \end{bmatrix}^- \begin{bmatrix} I_m \\ I_m \end{bmatrix} \right\}$.

(vi) $A+B=0$ *or* $r(A+B) = r\begin{bmatrix} A \\ B \end{bmatrix} + r[A,\ B] - r(A) - r(B)$.

Proof. Writing the sum $A+B$ as a triple matrix product

$$A+B = [I_m,\ I_m] \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} I_n \\ I_n \end{bmatrix},$$

and applying Theorem 2.5 to it yield the desired results. □

It is no doubt that the preceding results can be extended to general settings. For example, the following two groups of results can be shown by Lemmas 2.1 and 1.3.

Theorem 2.10. *Let $A_1 \in \mathbb{C}^{m_1\times m_2}$, $A_2 \in \mathbb{C}^{m_3\times m_2}$, $A_3 \in \mathbb{C}^{m_3\times m_4}$, $A_4 \in \mathbb{C}^{m_5\times m_4}$, and $A_5 \in \mathbb{C}^{m_5\times m_6}$, $A_6 \in \mathbb{C}^{m_7\times m_6}$, $A_7 \in \mathbb{C}^{m_7\times m_8}$, and $A \in \mathbb{C}^{m_8\times m_1}$ are given. Then the following fact holds Then the following 4 statements are equivalent*:

(i) $\{A_1A_2^-A_3A_4^-A_5A_6^-A_7\} \subseteq \{A^-\}$.

(ii) $\{AA_1A_2^-A_3A_4^-A_5A_6^-A_7\} \subseteq \{AA^-\}$.

(iii) $\{A_1A_2^-A_3A_4^-A_5A_6^-A_7A\} \subseteq \{A^-A\}$.

(iv) $A = 0$ or $r\begin{bmatrix} 0 & A_3 & A_2 \\ A_5 & A_4 & 0 \\ A_6 & 0 & -A_7AA_1 \end{bmatrix} = r(A_2) + r(A_4) + r(A_6) - r(A)$.

Theorem 2.11. *let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, $C \in \mathbb{C}^{p\times q}$, and $D \in \mathbb{C}^{q\times s}$ be given, and denote $N = ABCD$. Then the following 8 statements are equivalent:*

(i) $\{N^-\} \supseteq \{D^-C^-B^-A^-\}$.

(ii) $\{N^-\} \supseteq \{(BCD)^-A^-\}$, $\{BCD)^-\} \supseteq \{(CD)^-B^-\}$, *and* $\{(CD)^-\} \supseteq \{D^-C^-\}$.

(iii) $\{N^-\} \supseteq \{D^-(ABC)^-\}$, $\{(ABC)^-\} \supseteq \{C^-(AB)^-\}$, *and* $\{(AB)^-\} \supseteq \{B^-A^-\}$.

(iv) $\{N^-\} \supseteq \{(CD)^-(AB)^-\}$, $\{(AB)^-\} \supseteq \{B^-A^-\}$, *and* $\{(CD)^-\} \supseteq \{D^-C^-\}$.

(v) $N = 0$ *or* $r(N) = r(A) + r(B) + r(C) + r(D) - n - p - q$.

(vi) $N = 0$ *or* $\{r(N) = r(A) + r(BCD) - n,\ r(BCD) = r(B) + r(CD) - p,\ r(CD) = r(A) + r(B) - q\}$.

(vii) $N = 0$ *or* $\{r(N) = r(ABC) + r(D) - q,\ r(ABC) = r(AB) + r(C) - p,\ r(AB) = r(A) + r(B) - n\}$.

(viii) $N = 0$ *or* $\{r(N) = r(AB) + r(CD) - p,\ r(AB) = r(A) + r(B) - n,$ *and* $r(CD) = r(C) + r(D) - p\}$.

Note that the preceding results and facts concern only with a matrix equality for $\{1\}$-generalized inverses of mixed products of matrices. As extensions, it would be of interest but challenging to approach various matrix equalities for $\{i, \ldots, j\}$-generalized inverses of multiple matrix products.

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