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Viability of slow-roll inflation in light of the non-zero $k_{\rm min}$ measured in the CMB power spectrum

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Slow-roll inflation may simultaneously solve the horizon problem and generate a near scale-free fluctuation spectrum $P(k)$. These two processes are intimately connected via the initiation and duration of the inflationary phase. But a recent study based on the latest *Planck* release suggests that $P(k)$ has a hard cutoff, $k_{\rm min} \neq 0$, inconsistent with this conventional picture. Here we demonstrate quantitatively that most—perhaps all—slow-roll inflationary models fail to accommodate this minimum cutoff. We show that the small parameter ϵ must be $\gtrsim 0.9$ throughout the inflationary period to comply with the data, seriously violating the slow-roll approximation. Models with such an ϵ predict extremely red spectral indices, at odds with the measured value. We also consider extensions to the basic picture (suggested by several earlier workers) by adding a kinetic-dominated or radiation-dominated phase preceding the slow-roll expansion. Our approach differs from previously published treatments principally because we require these modifications to—not only fit the measured fluctuation spectrum, but to simultaneously also—fix the horizon problem. We show, however, that even such measures preclude a joint resolution of the horizon problem and the missing correlations at large angles.

1. Introduction

The lack of large-angle correlation in the cosmic microwave background (CMB) anisotropies, confirmed by three independent satellite missions [1–3], raises

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serious questions concerning the viability of basic slow-roll inflation [4,5]. A reliance on cosmic variance [6] for the missing correlations cannot avoid the correspondingly small probabilities ($\lesssim 0.24\%$) that disfavor the conventional picture at $\gtrsim 3\sigma$. This growing tension between the theoretical predictions and the CMB observations was recently put on a much more rigorous, formal footing with a detailed analysis of the recent *Planck* data release [3], showing quite robustly that the absence of large-angle correlation in the CMB is due to a non-zero minimum wavenumber, $k_{\rm min}$, in the fluctuation power spectrum $P(k)$ [7].

The inflationary paradigm posits that quantum fluctuations were generated shortly after the Big Bang [8] with a power-law power spectrum $P(k)$ distributed over an indeterminate range of wavenumbers k. But the latest *Planck* measurements are precise enough for us to question whether or not $k_{\rm min}$ is in fact zero. Ref. [7] demonstrated that the lack of large-angle correlation in the CMB is due to a cutoff $k_{\rm min} \neq 0$, and measured its value by optimizing the theoretical fits to the measured angular-correlation function. These authors provided compelling evidence that the *Planck* data clearly rule out a zero $k_{\rm min}$ at a very high level of confidence—*exceeding* 8σ. This measurement is critically important because—given an inflaton potential, $V(\phi)$, and the notion that a minimum wavenumber corresponds to the first mode leaving the horizon—$k_{\rm min}$ signals a precise cosmic time, $t_{\rm start}$, at which slow-roll inflation is supposed to have started.

Unconstrained slow-roll inflation would have stretched all fluctuations beyond the horizon, resulting in a $P(k)$ with $k_{\rm min} = 0$, which would have produced strong correlations in the CMB at all angles, θ, in contrast to what is actually seen, i.e., an angular correlation function that essentially goes to zero at $\theta \gtrsim 60^\circ$. The *measured* minimum wavenumber is instead

$$k_{\rm min} = \frac{4.34 \pm 0.50}{r_{\rm dec}} \,, \tag{1.1}$$

where $r_{\rm dec}$ is the comoving distance between us and redshift $z_{\rm dec} = 1080$, at which decoupling in standard ΛCDM cosmology is thought to have occurred. Therefore, for the latest *Planck* parameters (see below), one finds $r_{\rm dec} \approx 13,804$ Mpc, and a corresponding minimum wavenumber

$$k_{\rm min} = (3.14 \pm 0.36) \times 10^{-4}\,{\rm Mpc}^{-1}\,. \tag{1.2}$$

In the conventional inflationary picture, mode k exited the horizon at time t_*, satisfying the simple condition [8]

$$\frac{\lambda_k(t_*)}{2\pi} = \frac{c}{H_*}\,, \tag{1.3}$$

where $\lambda_k(t_*) = 2\pi a(t_*)/k$ is its wavelength, $a(t_*)$ is the expansion factor in the Friedmann-Lemaître-Robertson-Walker metric (FLRW), and H_* is the Hubble constant at that moment. This strong observational constraint therefore implies that standard slow-roll inflation must satisfy the initial condition

$$a(t_{\rm start})H_{\rm start} = 94.3 \pm 10.9\,{\rm km\,s^{-1}\,Mpc^{-1}}\,. \tag{1.4}$$

But as we shall show in this paper, at least some inflationary models fail to solve the horizon problem in light of this new measurement. We shall first consider pure slow-roll inflation on its own, but then also demonstrate that the introduction of a kinetic-dominated (KD) or radiation-dominated (RD) phase preceeding the slow-roll expansion cannot produce consistency with the data either.

The missing angular correlation at large angles is related to the unexpectedly low power measured in the small ℓ multiple moments. Several workers have previously attempted to resolve this issue by introducing such an RD or KD phase preceding the flatenning of the inflaton potential. We shall summarize several of these efforts in § 3 below, and provide a set of pertinent references to this previously published work. Our analysis in this *Letter* differs from many of these treatments principally because we require such modifications to—not only account for the missing angular correlation at large angles, but simultaneously to also—fix the horizon problem. This caveat is critical to our conclusion: that the measurement of $k_{\rm min}$ impacts both the measured fluctuation spectrum and the ability of standard slow-roll inflation to equilibrate the CMB temperatue across the visible Universe.

2. Pure Slow-Roll Inflation

We may clearly see the impact of this measurement by considering the simplest case of a pure exponential (i.e., de Sitter) expansion. To ensure that the CMB temperature seen today is equilibrated across the sky, a photon must have traversed a comoving distance prior to decoupling at least twice $r_{\rm dec}$. That is, the minimal condition for inflation is

$$r_{\rm preCMB} = 2r_{\rm dec} \equiv 2c \int_{t_{\rm dec}}^{t_0} \frac{dt}{a(t)}\,, \tag{2.1}$$

where $a(t)$ is the aforementioned expansion factor. In terms of $H = \dot{a}/a$, we may also put

$$r_{\rm dec} = c \int_{a_{\rm dec}}^{a_0} \frac{da}{a^2 H}\,, \tag{2.2}$$

where $H(a)$ is the Hubble parameter as a function of a, and a_0 is the expansion factor today. The latest cosmological measurements all seem to be consistent with a spatially flat Universe [3], for which a_0 may be normalized to 1.

From the Friedmann equation, we have

$$H^2(a) = H_0^2 \left(\frac{\Omega_{\rm m}}{a^3} + \frac{\Omega_{\rm r}}{a^4} + \Omega_\Lambda \right)\,. \tag{2.3}$$

Thus, for the *Planck* optimized values $H_0 = 66.99 \pm 0.92$ km s^{-1} Mpc^{-1}, $\Omega_{\rm m} = 0.321 \pm 0.013$, $\Omega_\Lambda = 0.679 \pm 0.013$, and $\Omega_{\rm r} = 9.3 \times 10^{-5}$ [3], for the Hubble parameter, and fractional matter and cosmological constant energy densities, respectively, one finds $r_{\rm dec} \approx 13,804$ Mpc. By comparison, $r_{\rm preCMB}$ is calculated from the start of inflation, $a_{\rm start} \equiv a(t_{\rm start})$, to decoupling and is mostly due to the expansion up to $a_{\rm end}$, when the inflaton field becomes sub-dominant. Thus,

$$r_{\rm preCMB} \approx c \int_{a_{\rm start}}^{a_{\rm end}} \frac{da}{a^2 H}\,. \tag{2.4}$$

In simple exponential (i.e., pure de Sitter) expansion, $H(a) = H_{\rm start}$ is constant during inflation, so

$$r_{\rm preCMB} = \frac{c}{H_{\rm start}} \left(\frac{1}{a_{\rm start}} - \frac{1}{a_{\rm end}} \right)\,, \tag{2.5}$$

and since $a_{\rm start} \ll a_{\rm end}$, we may also put

$$r_{\rm preCMB} = \frac{c}{H_{\rm start} a_{\rm start}}\,. \tag{2.6}$$

The newly measured constraint in Equation (1.4) therefore implies that $r_{\rm preCMB} \approx 3,181$ Mpc, much smaller than the required comoving distance $2r_{\rm dec} \approx 27,608$ Mpc. This factor 9 disparity therefore rules out pure exponential inflationary models, because they could not solve the horizon problem given the measured value of $k_{\rm min}$.

But the focus today is on *slow-roll* inflation, for which $H(a)$ due to the inflaton field is very nearly—though not exactly—constant. It is not difficult to see that when the small parameter ϵ (see Eq. 2.9 below) is monotonic [9], $H(a) \le H_{\rm start}$ for all $a \ge a_{\rm start}$. As such, one should expect $r_{\rm preCMB}$ to be bigger than that in Equation (2.6) (corresponding to pure exponential expansion) if the starting condition (Eq. 1.4) remains the same.

To quantify the difference, let us define a new variable

$$\beta(a) \equiv \frac{1}{Ha^2} \tag{2.7}$$

(i.e., the integrand in Eq. 2.4). The boundaries relevant to the run of $\beta(a)$ with a are shown schematically in figure 1. The measured cutoff $k_{\rm min}$ corresponds to the solid blue hyperbola,

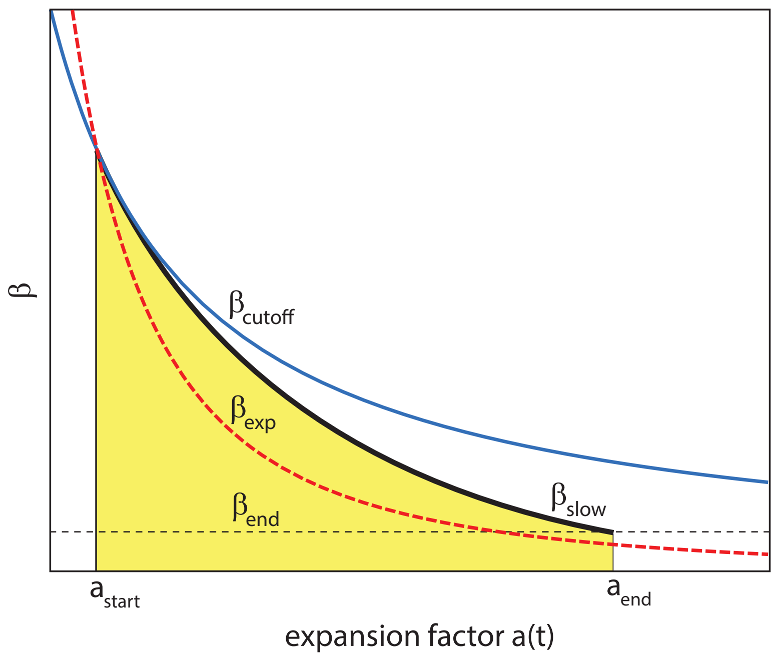

Figure 1. Phase space of permitted $\beta(a)$ versus a trajectories for slow-roll inflationary models. Here, $\beta_{\rm cutoff} = (H_{\rm start} a_{\rm start})^{-1}\, a^{-1} \propto 1/a$ (blue solid); $\beta_{\rm exp} = (H_{\rm start})^{-1} a^{-2} \propto 1/a^2$ (red dashed); and $\beta_{\rm end} = (H_{\rm end} a_{\rm end}^2)^{-1} =$ constant (black dashed). The shaded (yellow) area is the dominant contribution to the integral for $r_{\rm preCMB}$, for a specific slow-roll model with $\beta_{\rm slow}(a)$, and should therefore be compared with the comoving distance $r_{\rm dec}$ to decoupling.

on which $\beta_{\rm cutoff} = (H_{\rm start} a_{\rm start})^{-1}\, a^{-1} \propto 1/a$. Inflation must begin at $a_{\rm start}$ somewhere on this curve. For example, if H is constant (red dashed curve), inflation initiates at the point where the solid and dashed curves intersect, after which $\beta_{\rm exp} \propto 1/a^2$. Also, the Universe is believed to have been radiation dominated right after inflation ended, for which

$$H_{\rm end}^2 = H_0^2 \left(\frac{\Omega_{\rm r}}{a_{\rm end}^4} \right) . \tag{2.8}$$

Again, for exponential inflation with $H(a) = H_{\rm start} = H_{\rm end}$, Equation (2.8) corresponds to the horizontal (black) short-dash line, with $\beta_{\rm end} = (H_{\rm end} a_{\rm end}^2)^{-1} =$ constant near the bottom of the plot. Any slow-roll inflationary model (with H not exactly constant) would then follow a trajectory $\beta_{\rm slow}(a)$ (shown as solid black) somewhere between the $\beta_{\rm cutoff}$ and $\beta_{\rm exp}$ curves. It could never cross the hyperbola because H can never be bigger than its starting value $H_{\rm start}$.

The small parameter ϵ is defined according to [9]

$$\epsilon \equiv \frac{m_{\rm Pl}^2}{4\pi} \left(\frac{H'}{H} \right)^2 , \tag{2.9}$$

where $m_{\rm Pl}$ is the Planck mass and prime denotes a derivative with respect to the inflaton scalar field, ϕ. It is not difficult to show that

$$H(\phi) = H_{\rm start} \exp \left(- \int_{\phi_{\rm start}}^{\phi} \sqrt{\frac{4\pi\epsilon(\phi)}{m_{\rm Pl}^2}}\, d\phi \right) , \tag{2.10}$$

where the subscript 'start' has its usual meaning. It is also useful to introduce the number of e-folds during inflation,

$$N(\phi_{\rm start}, \phi) \equiv \ln\left(\frac{a}{a_{\rm start}}\right) = \int_{\phi_{\rm start}}^{\phi} \sqrt{\frac{4\pi}{m_{\rm Pl}^2\,\epsilon(\phi)}}\, d\phi\,. \tag{2.11}$$

Clearly, $\epsilon = 0$ if H is strictly constant. It is non-zero, but small, if H changes slowly (hence the designation 'slow-roll'). Thus, inflation in slow-roll models must end when ϵ increases to 1, at which point the slow-roll approximation breaks down.

Let us therefore first consider the extreme case in which $\epsilon = 1$ throughout the inflationary phase, for which

$$H(a) = H_{\rm start}\exp(-N) = \frac{H_{\rm start}a_{\rm start}}{a}\,. \tag{2.12}$$

This is in fact the solid black hyperbola shown in figure 1. Therefore,

$$r_{\rm preCMB}^{\epsilon=1} = \frac{c}{H_{\rm start}a_{\rm start}}\ln\left(\frac{a_{\rm end}}{a_{\rm start}}\right)\,. \tag{2.13}$$

This comoving distance is bigger by a factor $\ln(a_{\rm end}/a_{\rm start})$ than that for pure de Sitter expansion (Eq. 2.6), and would be sufficient to account for the required value of $2r_{\rm rec}$. As we shall discuss shortly, however, there are compelling reasons why such a persistently large value of ϵ is inconsistent with the data. Typically, slow-roll models have a very tiny ϵ during most of inflation, approaching 1 only towards the end, when the inflaton field is believed to somehow dissolve into standard model particles, so that the magnitude of H' becomes very large.

To more accurately represent such models, we therefore define another new parameter $0 < b < 1$ such that ϵ^2 is restricted to values $\leq b$ during most of the inflationary expansion, breaking down only at the very end. Then we have

$$\begin{aligned} H(\phi) &> H_{\rm start}\exp\left(-\int_{\phi_{\rm start}}^{\phi}\sqrt{\frac{4\pi b}{m_{\rm Pl}^2\epsilon(\phi)}}\,d\phi\right) \\ &= H_{\rm start}\exp(-\sqrt{b}N) \\ &= H_{\rm start}\left(\frac{a}{a_{\rm start}}\right)^{-\sqrt{b}}\,, \end{aligned} \tag{2.14}$$

so that, assuming $a_{\rm end} >> a_{\rm start}$,

$$r_{\rm preCMB}^{\epsilon^2<b} < r_{\rm preCMB}^{\epsilon^2=b} \equiv \frac{1}{(1-\sqrt{b})H_{\rm start}a_{\rm start}} \tag{2.15}$$

and, combining this with Equation (2.2), we find that $\sqrt{b} > 0.875$ in order for the right-hand side of Equation (2.15) to exceed $2r_{\rm dec}$ and solve the horizon problem.

In other words, ϵ must be quite large compared to typical values required in commonly studied slow-roll models. Indeed, scenarios with $\epsilon \sim 1$ during the whole of inflation have already been considered and eliminated on observational grounds [10], because either (i) inflation would not have lasted long enough to fix the horizon problem, or (ii) the predicted extremely red spectral index ($n_s \ll 1$) in $P(k)$ would be substantially different from its observed value 0.9649 ± 0.0042 [3]. Inflationary models with $\epsilon^2 > b$ are therefore not at all practical.

To demonstrate this general result more practically, let us examine its impact on four rather well-known, specific types of potential that have been studied thus far, beginning with the evolution of the slow-roll parameter ϵ in so-called 'small-field' inflation models, for which the potential may be approximated locally by the expression

$$V(\phi) = V_0\left[1 - (\phi/\mu)^p\right]\,. \tag{2.16}$$

As an illustration, we take $p = 2$ and $\epsilon(a_{\rm end}) = 1$ (the value of μ is irrelevant for the calculation of $\epsilon[a]$). Higher-order terms in $V(\phi)$ become important only towards the end of inflation. Our numerical solution for ϵ, based on the *Planck* optimized parameter values (see paragraph following Eq. 2.3 above), is shown in figure 2. Indeed, $\epsilon^2 > b$ for $8.5 \times 10^{-29} \lesssim a \lesssim a_{\rm end} = 8.9 \times 10^{-29}$, but is far too small elsewhere for $r_{\rm preCMB}$ to exceed $2r_{\rm rec}$.

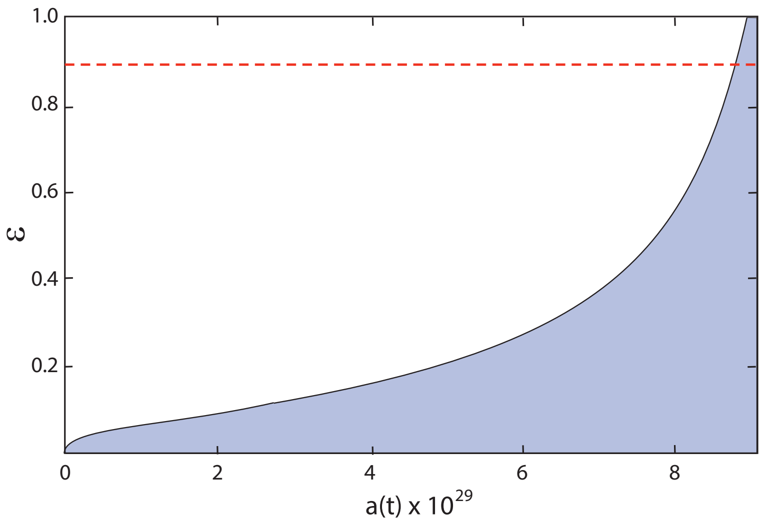

Figure 2. The small parameter ϵ as a function of the expansion factor $a(t)$ for the 'small-field' inflaton potential in Equation (2.16). The (red) dashed line marks the value required for the model to comply with the *Planck* measurement of $k_{\rm min}$.

An alernative characterization of such an evolution may be written in terms of the number of e-folds (Eq. 2.11) required during inflation in order to overcome the horizon problem, compared to the actual number permitted by the $k_{\rm min}$ constraint in Equation (1.4). We have numerically calculated ϵ_V (see Eq. 2.19 below) and N, subject to this constraint, for the following three slow-roll potentials:

$$\begin{aligned}
V_Q(\phi) &= \frac{1}{2}m\phi^2 && \text{(Quadratic)} \\
V_H(\phi) &= V_0[1-(\frac{\phi}{\mu})^2]^2 && \text{(Higgs − like)} \\
V_N(\phi) &= V_0[cos(\frac{\phi}{f})+1] && \text{(Natural)}\,.
\end{aligned} \tag{2.17}$$

For specificity, we have continued to use the *Planck* optimized parameters. The principal difference between our calculation and those carried out in previous work is the inclusion of Equation (1.4) as an initial condition. In addition, to this constraint, the other inputs informing the calculation include: (1) the observed value of the scalar spectral index, $n_s = 0.96$, which was measured at the pivot point $k_{\rm pivot} \equiv 0.05$ Mpc^{-1} [3]; (2) an endpoint of inflation at $\epsilon_V = 1$ (see Eq. 2.19 below); and (3) a smooth transition from this inflated expansion to one driven by a radiation-dominated equation-of-state, as shown in Equation (2.8).

At the early stage of inflation, these potentials may be used to define an alternative set of 'small parameters' η_V and ϵ_V [9], such that

$$n_s \approx 2\eta_V - 6\epsilon_V\,, \tag{2.18}$$

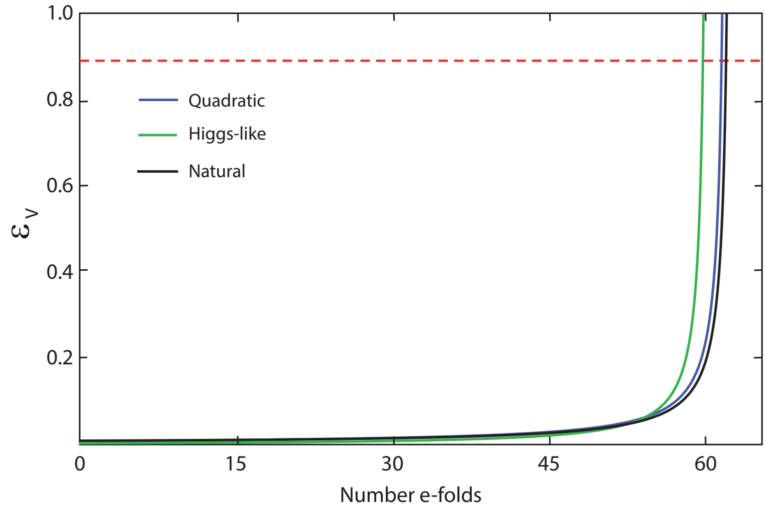

Figure 3. The small parameter ϵ_V as a function of the number of e-folds (Eq. 2.11) for three illustrative slow-roll potentials given in Equation (2.17). The (red) dashed line marks the value required for the model to comply with the *Planck* measurement of $k_{\rm min}$.

where

$$\begin{aligned}\epsilon_V &= \frac{m_{\rm Pl}^2}{16\pi}\left(\frac{V'}{V}\right)^2 \\ \eta_V &= \frac{m_{\rm Pl}^2}{8\pi}\frac{V''}{V}\,. \end{aligned} \tag{2.19}$$

The approximation breaks down when ϵ_V is large, which is conventionally taken to indicate the end of the inflated expansion. The key results of our simulations are as follows:

- *Quadratic:* $r_{\rm preCMB} = 5,547$ Mpc, which is still a factor ~ 5 too small compared to $2r_{\rm dec} = 27,608$ Mpc. This potential would have expanded the Universe by 62 e-folds (see fig. 3), but 64 e-folds would have been required to fix the horizon problem. The difference of 2 e-folds accounts for the factor 5 difference between $r_{\rm preCMB}$ and $2r_{\rm dec}$.
- *Higgs-like:* $r_{\rm preCMB} = 3,339$ Mpc, which is a factor ~ 8 too small. In this case, the Universe would have expanded by 60 e-folds (fig. 3), but a little over 62 e-folds would have been required to mitigate the horizon problem.
- *Natural:* $r_{\rm preCMB} = 3,650$ Mpc, which is also a factor ~ 8 too small. The Universe would have expanded by 63 e-folds, but a little over 65 e-folds would have been required to completely mitigate the horizon problem.

For direct comparison, the small parameter ϵ_V is shown as a function of N for each of these three inflaton potentials in figure 3. As discussed earlier, inflation would have ended when $\epsilon_V \to 1$. As was the case in figure 2, the horizontal red (dashed) line indicates the approximate value ϵ_V requires to comply with the *Planck* measurement of $k_{\rm min}$, and we see that, while ϵ_V does cross this mark in each case, it is not sustained at this high level long enough for the Universe to have expanded sufficiently to remove the horizon problem.

3. Slow-roll Inflation Preceeded by a KD or RD Fast-roll Phase

It appears, therefore, that to simultaneously resolve both the horizon problem and the missing correlations at large angles, one must consider additional inflationary phases coupled to the standard slow-roll expansion. A closely related problem to the missing correlations at large angles is the observed lack of power on the largest scales. Several authors have previously attempted to mitigate this problem by introducing additional features to inflation, such as the aforementioned KD and RD phases. For example, ref. [11] showed that an early fast-roll inflation can lead to a depression of the cosmic microwave background quadrupole moment, with a characteristic scale $k_1 \sim (3,759\,\mathrm{Mpc})^{-1}$ of the implied attractive potential. This is consistent with our previously measured minimum cutoff $k_{\rm min} = (3,442\,\mathrm{Mpc})^{-1}$. These authors did not, however, simultaneously calculate the comoving distances $r_{\rm preCMB}$ and $r_{\rm dec}$ to ensure that $r_{\rm preCMB} \geq 2r_{\rm dec}$. Subsequent work by these authors [12] to include both a decelerated fast-roll and an inflationary fast-roll phase similarly did not address the horizon problem in terms of the required comoving distances. In addition, this work appears to rely on the Bunch-Davies initial conditions, which may be problematic in the context of trans-Planckian physics.

This general approach was followed by other authors [13], who found that a fast-rolling KD initial phase improves the primordial power spectral fit to the data, but they similarly did not consider the impact of this treatment on $r_{\rm preCMB}$ versus $r_{\rm dec}$. Likewise, the Planck Collaboration [14] considered the impact of a cutoff on the spectrum, though not the angular-correlation function. Their treatment apparently also lacks a discussion of the possible impact of such a cutoff on $r_{\rm preCMB}$ and $r_{\rm dec}$.

The work of ref. [15] was published after our measurement of $k_{\rm min}$ [7], and they too considered the impact of a sharp cutoff to the fluctuation spectrum. They concluded that the standard power-law is preferred by the data, but made no mention of the horizon problem and the lack of correlations at large angles, however, and the impact of this approach on $r_{\rm preCMB}$ versus $r_{\rm dec}$.

An early phase of KD inflation was also introduced in ref. [16], though restricted to only polynomial and exponential potentials. These authors confirmed that such a transition exhibits a generic damping of power on large scales, but did not explicitly consider its impact on the angular correlation function and $r_{\rm preCMB}$ versus $r_{\rm dec}$.

The work that comes closest in spirit to our analysis in this paper is that reported in ref. [17]. These authors, however, considered specifically the $\lambda\phi^4$ potential and imposed the condition of "just-enough" inflation. They found that the slow-roll conditions are violated at the largest scales, and that this approach cannot explain the lack of power at the largest angles. In subsequent work [18], this treatment was expanded to include quadratic and hybrid-type potentials, but still without a consideration of their impact on the angular correlation function.

Finally, ref. [19] analyzed how much inflation one should expect for a given energy scale of order 10^{16} GeV. But this work lacks direct relevance to our proposed coupling of $k_{\rm min}$ measured from the angular correlation function to the number of e-folds itself, and its bearing on $r_{\rm preCMB}$ versus $r_{\rm dec}$.

Quite clearly, many authors have by now noted the glaring inconsistency associated with low power in the CMB fluctuations on large scales, which is closely related to their lack of correlation at large angles. Our work amplifies this general view by providing a much stronger argument for a cutoff $k_{\rm min}$ in the primoridal fluctuation spectrum, and its direct impact also on the horizon problem itself. To complete this discussion, we shall now consider whether a KD or RD modification to the basic slow-roll inflationary picture can help mitigate the inconsistency between $r_{\rm preCMB}$ and $r_{\rm dec}$ when a cutoff $k_{\rm min}$ is invoked to suppress the correlation at large angles.

We shall first follow a simplified approach in which we gauge the impact of a KD or RD modification to the horizon problem based solely on the previously measured hard cutoff $k_{\rm min}$. It is well known, however, that the angular power C_ℓ of each multipole ℓ, from which the angular correlation function $C(\theta)$ is calculated, depends on the entire fluctuation spectrum $P(k)$ [7]. Thus, any modification to the power spectrum produced during the KD or RD phase alters $C(\theta)$ from that expected under pure slow-roll conditions. Following our initial discussion of the impact of KD or RD on the horizon problem using the previously measured $k_{\rm min}$, we shall therefore quantitatively assess how much the cutoff wavenumber changes when the angular correlation function is re-optimized for a representative inflaton potential that contains a KD phase transitioning into slow-roll at $k_{\rm start}$. We shall find that $k_{\rm start}$, signalling the start of inflated expansion, can differ fractionally from $k_{\rm min}$ when $C(\theta)$ is fit to the *Planck* data, though insufficiently to qualitatively alter any of the results.

We begin with a radiation-dominated Universe from the Big Bang to the onset of inflation, during which

$$H = Qa^{-2}\,, \tag{3.1}$$

where Q is a constant. Solving for the scale factor, one therefore has

$$a^2 = 2Qt\,, \tag{3.2}$$

so that

$$dt = \frac{a\,da}{Q}\,. \tag{3.3}$$

The comoving distance traveled by a photon during this period is therefore

$$r_{\rm RD} = c\int_0^{t_{\rm start}} \frac{dt}{a} = \frac{c a_{\rm start}}{Q}\,. \tag{3.4}$$

Thus, combining this with Equation (3.1), we have

$$r_{\rm RD} = \frac{c}{H_{\rm start}\, a_{\rm start}}\,. \tag{3.5}$$

The addition of an RD period preceeding slow-roll inflation can therefore double the comoving distance travelled by a photon prior to the end of the inflation. Even this, however, is still far too small to solve the horizon problem, which requires the comoving distance to be at least 10 times bigger.

The addition of a KD fast-roll expansion may hold more promise. For such a scalar field-dominated Universe, we have [9]:

$$H(\phi)^2 = \frac{8\pi}{3m_{\rm Pl}^2}\left(\frac{1}{2}\dot\phi^2 + V(\phi)\right)\,, \tag{3.6}$$

and

$$\ddot\phi + 3H\dot\phi + V' = 0\,. \tag{3.7}$$

From these two expressions, we derive

$$\dot H = -\frac{4\pi}{m_{\rm Pl}^2}\dot\phi^2\,, \tag{3.8}$$

and

$$\dot\phi = -\frac{m_{\rm Pl}^2}{4\pi}H'\,. \tag{3.9}$$

For a KD scalar-field potential, Equation (3.6) reduces to

$$H(\phi)^2 \approx \frac{8\pi}{6m_{\rm Pl}^2}\dot\phi^2 \tag{3.10}$$

and, solving for H, we find that

$$H(\phi) = H_{\rm start}\, e^{\frac{2\sqrt{3\pi}}{m_{\rm Pl}}(\phi-\phi_{\rm start})}\,, \tag{3.11}$$

for which

$$H' = \frac{2\sqrt{3\pi}}{m_{\rm Pl}} H_{\rm start}\, e^{\frac{2\sqrt{3\pi}}{m_{\rm Pl}}(\phi-\phi_{\rm start})}\,. \tag{3.12}$$

With Equation (3.9), we therefore find that

$$\frac{dt}{d\phi} = -\sqrt{\frac{\pi}{3}}\frac{2}{m_{\rm Pl}H_{\rm start}}\, e^{\frac{2\sqrt{3\pi}}{m_{\rm Pl}}(\phi_{\rm start}-\phi)}\,, \tag{3.13}$$

so that

$$t - t_i = \frac{1}{3H_{\text{start}}} e^{\frac{2\sqrt{3\pi}}{m_{\text{Pl}}}(\phi_{\text{start}} - \phi)}, \tag{3.14}$$

where t_i is the time at which the KD expansion begins. Thus, with

$$\tilde{t} \equiv t - t_i \,, \tag{3.15}$$

we also have

$$\tilde{t} = \frac{1}{3H} \,, \tag{3.16}$$

during this phase prior to the onset of slow-roll inflation.

We may now solve for the scale factor $a(t)$, finding that

$$a = M\tilde{t}^{1/3} \,, \tag{3.17}$$

so that

$$dt = d\tilde{t} = \frac{3a^2}{M^3}\, da \,, \tag{3.18}$$

where M is another constant. Therefore, the comoving distance travelled by a photon during this period is

$$r_{\text{KD}} = c \int_{\tilde{t}_i}^{\tilde{t}_{\text{start}}} \frac{d\tilde{t}}{a} = \frac{3c}{2M^3}(a_{\text{start}}^2 - a_i^2) \,. \tag{3.19}$$

As long as the KD period begins right after the Big Bang, we may therefore approximate this expression as

$$r_{\text{KD}} \approx \frac{3c}{2M^3} a_{\text{start}}^2 \,, \tag{3.20}$$

and therefore we find, with the use of Equations (3.11) and (3.12), that

$$r_{\text{KD}} \approx \frac{c}{2a_{\text{start}} H_{\text{start}}} \,. \tag{3.21}$$

Clearly, even combining this comoving distance with that from the slow-roll inflationary period, we find that r_{preCMB} is still far too small to solve the horizon problem.

Finally, we consider all three phases together, beginning with an RD period, followed by a KD Universe and a subsequent slow-roll expansion. It is not difficult to show that

$$r_{\text{RD+KD}} = \frac{3c}{2M^3}\left(a_{\text{start}}^2 - a_\star^2\right) + \frac{ca_\star}{Q} \,, \tag{3.22}$$

where $a_\star$ is the scale factor at the RD to KD transition. Thus

$$r_{\text{RD+KD}} = \left(1 + \frac{a_\star^2}{a_{\text{start}}^2}\right)\left(\frac{c}{2a_{\text{start}} H_{\text{start}}}\right) < \frac{c}{a_{\text{start}} H_{\text{start}}} \,. \tag{3.23}$$

In the last step, we examine the possibility that a more careful calculation of the angular correlation function $C(\theta)$ with a modified $P(k)$ from the KD phase may yield an optimized wavenumber k_{start} (signalling the start of inflation) differing from the hard cutoff k_{min} we have been using in this analysis. It is not difficult to show from Equations (3.10-3.12) that the fluctuation spectrum produced during KD is $P(k) \sim k^3$. To estimate the change one should expect to see with this more detailed approach, we therefore now proceed to re-optimize $C(\theta)$ with

$$P(k) = \begin{cases} A_s(k/k_0)^{n_s - 1} & \text{if} k \geq k_{\text{start}} \\ A_s(k_{\text{start}}/k_0)^{n_s - 4}(k/k_0)^3 & \text{if} k < k_{\text{start}} \,, \end{cases} \tag{3.24}$$

invoking the usual pivot scale k_0.

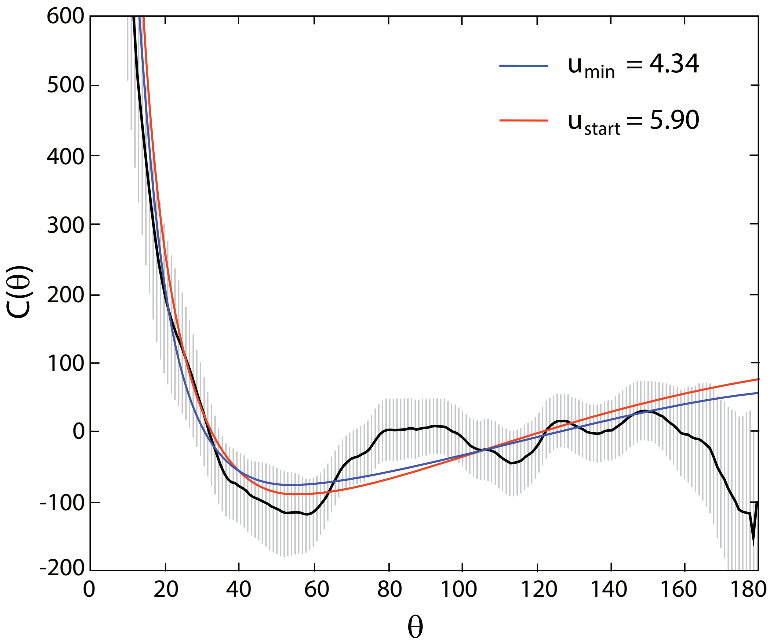

Figure 4. The best-fit angular correlation functions for $P(k)$ with a hard cutoff $k_{\rm min}$ (blue) and for $P(k)$ given in Equation (3.24) (red), with an optimized value $k_{\rm start} = 4.12 \times 10^{-4}\ {\rm Mpc}^{-1}$. These theoretical curves are compared to the angular correlation function measured with *Planck* (dark solid curve) [14], and associated 1σ errors (grey). (Adapted from ref. [7])

We follow the procedure outlined in ref. [7], and infer that the angular power of multipole ℓ relevant to the Sachs-Wolfe domain of fluctuations may be approximated as

$$C_\ell = B \int_0^{u_{\rm start}} \left(\frac{u}{u_{\rm start}}\right)^3 \frac{j_\ell^2(u)}{u}\, du + B \int_{u_{\rm start}}^{\infty} \frac{j_\ell^2(u)}{u}\, du\,, \tag{3.25}$$

where B is a normalization constant encompassing A_s and several other factors; the variable u is defined by the expression $u \equiv k r_{\rm dec}$, in terms of the comoving distance $r_{\rm dec}$ to the decoupling surface; and j_ℓ is the spherical Bessel function of order ℓ. The angular correlation function itself is then given by the expression

$$C(\theta) = \sum_\ell \frac{(2\ell+1)}{4\pi} C_\ell\, P_\ell(\cos\theta)\,, \tag{3.26}$$

where $P_\ell(\cos\theta)$ are the Legendre polynomials [20].

Using Equation (3.26) to refit the angular correlation function measured by *Planck* [7,14], we find that the optimized fit corresponds to the value $u_{\rm start} = 5.9$. Thus, according to the definition of u, we find that

$$k_{\rm start} = 4.12 \times 10^{-4}\ {\rm Mpc}^{-1}\,. \tag{3.27}$$

In figure 4, we show a comparison of the optimized angular correlation functions for $P(k)$ with a hard cutoff $k_{\rm min}$ (blue) and $P(k)$ given in Equation (3.24) (red) with this $k_{\rm start}$. The curves are almost indistinguishable, though the blue one is a slightly better fit to the *Planck* data at both small ($\theta \lesssim 45^\circ$) and large ($\theta \gtrsim 120^\circ$) angles [7]. This difference, however, is too small for us to decide which of these fluctuation distributions is preferred by the *Planck* data. Instead, the principal outcome of this comparison is the change in wavenumber signalling the initiation of inflated expansion: from $k_{\rm min}$ in Equation (1.2) for the hard cutoff, to $k_{\rm start}$ in Equation (3.27) for the KD plus slow-roll potential.

Thus, replacing $k_{\rm min}$ in Equation (1.3) with $k_{\rm start}$, and using Equations (2.6) and (3.23), we find that $r_{\rm preCMB} < 4,848$ Mpc, which is still much smaller than the value (i.e., $27,608$ Mpc) required to solve the horizon problem. In effect, the more detailed treatment of $P(k)$ has increased $r_{\rm preCMB}$ by about 50%, but nowhere near the factor ~ 9 required for this purpose.

No matter when the transition from RD to KD would have occurred, we find that no such modification to the basic slow-roll scenario can render inflation consistent with the measured $k_{\rm min}$ cutoff in the primordial fluctuation spectrum. The key point here is that, while introducing a cutoff to the fluctuation distribution can account for the observed CMB anisotropies, it cannot simultaneously solve the horizon problem.

4. Conclusion

The most recent *Planck* data have affirmed the absence of large-angle correlations in the CMB anisotropies, seen previously with several instruments over several decades. A prevailing view is that this feature may simply be due to 'cosmic variance,' based on the reasonable argument that we have only one Universe to observe, and that a variation away from its most probable configuration should not be unexpected. Certainly none of the work reported in this paper can completely eliminate that possibility. Nevertheless, seeking to find alternative explanations, as we have attempted to do here, is motivated by the presumed low probability of cosmic variance being the sole answer. The analysis reported in ref. [7] shows that a more probable explanation for the lack of large-angle correlations in the CMB is the presence of a hard cutoff $k_{\rm min}$ in the $P(k)$ spectrum. If true, this cutoff has profound consequences on the viability of slow-roll inflationary models because $k_{\rm min}$ points to a well-defined time at which inflation could have started. Quantifying this impact on the possible form of the inflaton potential has been the main goal of this paper.

The constraint implied by $k_{\rm min}$ allows inflation to simultaneously solve the horizon problem and produce a near power-law fluctuation spectrum only if $\epsilon \approx 1$ throughout the inflationary expansion. But such a scenario then predicts an extremely red spectral index completely at odds with the measured value. Here, we have examined in detail the consequences of $k_{\rm min}$ on four well-studied slow-roll inflationary models proposed thus far, showing that, if our interpretation of $k_{\rm min}$ is correct, the *Planck* CMB data rule out such slow-roll potentials at a very high level of confidence.

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