Article

Orthogonal gyrodecompositions of real inner product gyrogroups

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Abstract: In this article, we prove an orthogonal decomposition theorem for real inner product gyrogroups, which unify some well-known gyrogroups in the literature: Einstein, Möbius, Proper Velocity, and Chen's gyrogroups. This leads to the study of left (right) coset partition of a real inner product gyrogroup induced from a subgyrogroup that is a finite dimensional subspace. As a result, we obtain gyroprojectors onto the subgyrogroup and its orthogonal complement. We construct also quotient spaces and prove an associated isomorphism theorem. The left (right) cosets are characterized using gyrolines (cogyrolines) together with automorphisms of the subgyrogroup. With the algebraic structure of the decompositions, we study fiber bundles and sections inherited by the gyroprojectors. Finally, the general theory is exemplified for the aforementioned gyrogroups.

Keywords: Real inner product gyrogroup; orthogonal decomposition; gyroprojection; coset space; partitions; quotient space; gyrolines; cogyrolines; fiber bundles.

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Gyrogroups are a suitable generalization of groups, arising from the study of the parametrization of the Lorentz transformation group made by Ungar in [1]. A vast and comprehensive collection of results about gyrogroup theory was compiled by Ungar in the books [2] and [3] (see also the vast list of references therein). Gyrogroups share remarkable analogies with groups. In fact, every group forms a gyrogroup under the same operation. The three main gyrogroups associated with analytic hyperbolic geometry are Möbius, Einstein, and Proper Velocity gyrogroups (see [3]). Each of them has a gyrovector space structure that enables to treat analytic hyperbolic geometry similarly as in the Euclidean case. Factorizations of Möbius gyrogroups were first studied in [4,5]. These factorizations were used for defining continuous wavelet transforms on the unit sphere, associated to appropriate sections in the quotient Möbius gyrogroup [6]. This makes use of strong connections between Möbius transformations, Clifford algebras, and gyrogroups. It can be seen that the gyrogroup structure is important in order to factorize the unit ball of n-dimensional Euclidean space $\mathbb{R}^n$ and to obtain sections that allow defining dilation operators on the unit sphere [6, Section 4]. Recently, in [7], the authors rediscovered gyrodecompositions in the Möbius gyrogroup and incorporated the gyrovector structure in their study, however, they did not refer to previous results obtained in [5]. Using the notion of L-subgyrogroups, Suksumran and Wiboonton studied the factorization of abstract gyrogroups, similar to the case of groups [8], and applied the theory to a finite gyrogroup. As it is shown in this paper, their theory does not apply here.

The aim of this paper is to give a general theory for real inner product gyrogroups that encode a real gyrovector space structure imposing only two conditions (see Section 2). In this way, we unify the factorization theory for some well-known gyrogroups in the literature that are examples of real inner product gyrogroups such as Einstein, Möbius, Proper Velocity, and Chen's gyrogroups. Our results can be used for constructing orthogonal gyroexpansions with respect to any orthogonal basis in a real inner product gyrogroup, or in the construction of integral transforms such as the wavelet transform on some manifolds, such as the sphere, the ball, or the hyperboloid.

The structure of the article is organized as follows. In Section 1, we review the basic theory of gyrogroups and gyrovector spaces. In Section 2, we examine orthogonal decompositions of real inner product gyrogroups and construct left and right coset spaces, gyroprojectors, quotient gyrogroups, and give a geometric realization of the cosets using gyrolines and cogyrolines of the gyrogroup. In Section 3, we examine fiber bundles and sections of real inner product gyrogroups inherited by the gyroprojectors. In Section 4, we present concrete prominent examples of real inner product gyrogroups: Einstein, Möbius, Proper Velocity, and Chen's gyrogroups and state several theorems obtained as consequences of the general theorems given in previous sections.

1. Preliminaries

Definition 1 (Gyrogroups). *A nonempty set G, together with a binary operation $\oplus$ on G, is called a gyrogroup if it satisfies the following axioms.*

(G1) *There exists an element $0 \in G$ such that $0 \oplus a = a$ for all $a \in G$.*
(G2) *For each $a \in G$, there exists an element $b \in G$ such that $b \oplus a = 0$.*
(G3) *For all $a, b \in G$, there is an automorphism $\mathrm{gyr}[a, b] \in \mathrm{Aut}\, G$ such that*

$$a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b]c \qquad \text{(left gyroassociative law)}$$

    for all $c \in G$.
(G4) *For all $a, b \in G$, $\mathrm{gyr}[a \oplus b, b] = \mathrm{gyr}[a, b]$.*   *(left loop property)*

It can be proved that the element 0 in (G1) is also a right identity and is unique, called the gyrogroup identity. Further, the element b in (G2) is also a right inverse of a and is unique, called the inverse of a, written $\ominus a$. A gyrogroup $(G, \oplus)$ is said to be *gyrocommutative* if it satisfies the following gyrocommutative law:

$$a \oplus b = \mathrm{gyr}[a, b](b \oplus a) \tag{1}$$

for all $a, b \in G$.

To capture useful analogies between gyrogroups and groups there is defined as a second binary operation in G, called the *coaddition*, which is denoted by $\boxplus$ and given by

$$a \boxplus b = a \oplus \mathrm{gyr}[a, \ominus b]b \tag{2}$$

for all $a, b \in G$ (cf. Definition 2.7 of [2]). We have also that $a \boxminus b = a \boxplus (\ominus b)$.

The main cancellation laws in a gyrogroup $(G, \oplus)$ are

$$a \oplus (\ominus a \oplus b) = b \qquad \text{left cancellation law} \tag{3}$$

$$(b \ominus a) \boxplus a = b \qquad \text{first right cancellation law} \tag{4}$$

$$(b \boxminus a) \oplus a = b \qquad \text{second right cancellation law.} \tag{5}$$

Using the left gyroassociative law given in (G3) and the left cancellation law (3) we can write the following gyrator identity

$$\mathrm{gyr}[a, b]c = \ominus(a \oplus b) \oplus (a \oplus (b \oplus c)) \tag{6}$$

from which it follows that $\text{gyr}[a, b]0 = 0$, i.e. gyrations are gyroautomorphisms of G that preserve the identity element.

The next theorem shows the unique solution of the two basic gyrogroup equations.

Theorem 1. *(see [2]) Let* $(G, \oplus)$ *be a gyrogroup, and let* $a, b \in G$. *The unique solution of the equation* $a \oplus x = b$ *is* $x = \ominus a \oplus b$, *and the unique solution of the equation* $x \oplus a = b$ *is* $x = b \boxminus a$.

Definition 2 (Real inner product gyrovector spaces, [2]). *A real inner product gyrovector space* $(G, \oplus, \otimes)$ *(gyrovector space, in short) is a gyrocommutative gyrogroup* $(G, \oplus)$ *that obeys the following axioms:*

(A1) *G is a subset of a real inner product vector space* V, $G \subseteq V$, *from which it inherits its inner product* $\cdot$ *and norm* $\| \cdot \|$, *which are invariant under gyroautomorphisms; that is,*

$$\text{gyr}[u, v]a \cdot \text{gyr}[u, v]b = a \cdot b$$

for all $a, b, u, v \in G$.

(A2) *G admits a scalar multiplication* $\otimes$ *satisfying for all* $r, r_1, r_2 \in \mathbb{R}, a \in G$,

(V1) $1 \otimes a = a$
(V2) $(r_1 + r_2) \otimes a = r_1 \otimes a \oplus r_2 \otimes a$
(V3) $(r_1 r_2) \otimes a = r_1 \otimes (r_2 \otimes a)$
(V4) $\dfrac{|r| \otimes a}{\|r \otimes a\|} = \dfrac{a}{\|a\|}$
(V5) $\text{gyr}[u, v](r \otimes a) = r \otimes \text{gyr}[u, v]a$
(V6) $\text{gyr}[r_1 \otimes v, r_2 \otimes v] = I, \ I$ *is the identity map*

(A3) *The set* $\|G\| = \{\pm\|a\| : a \in G\}$ *admits a real vector space structure* $(\|G\|, \oplus, \otimes)$ *such that*

(V7) $\|r \otimes a\| = |r| \otimes \|a\|$
(V8) $\|a \oplus b\| \leq \|a\| \oplus \|b\|$

for all $a, b \in G, r \in \mathbb{R}$.

Unlike in the vector case, gyroaddition $\oplus$ does not in general distribute with scalar multiplication. However, gyrovector spaces possesses a monodistributive law given by

$$r \otimes (r_1 \otimes a \oplus r_2 \otimes a) \ = \ r \otimes (r_1 \otimes a) \oplus r \otimes (r_2 \otimes a)$$

for all $r, r_1\, r_2 \in \mathbb{R}, a \in G$.

The motions of a gyrovector space $(G, \oplus, \otimes)$ are all its left gyrotranslations $L_a : x \mapsto a \oplus x$, where $a, x \in G$, and its automorphisms $\tau \in \text{Aut}(G, \oplus, \otimes)$ satisfying

$$\tau(a \oplus b) = \tau(a) \oplus \tau(b) \tag{7}$$

$$\tau(r \otimes a) = r \otimes \tau(a) \tag{8}$$

$$\tau(a \cdot b) = \tau(a) \cdot \tau(b) \tag{9}$$

for all $a, b \in G, r \in \mathbb{R}$.

A gyrovector space is a gyrometric space with a gyrodistance given by

$$d_\oplus(a, b) = \|b \ominus a\|, \quad \forall a, b \in G \tag{10}$$

that satisfies the gyrotriangle inequality (see Theorem 6.9 in [2]):

$$\|c \ominus a\| \leq \|c \ominus b\| \oplus \|b \ominus a\|, \quad \forall a, b, c \in G. \tag{11}$$

Moreover, a gyrovector space is also cogyrometric with a cogyrodistance $d_\boxplus(a, b)$ given by

$$d_\boxplus(a, b) = \|b \boxminus a\|, \quad \forall a, b \in G \tag{12}$$

that satisfies the cogyrotriangle inequality (see Theorem 6.11 in [2]):

$$\|a \boxminus \mathrm{gyr}[a \boxminus b, b \boxminus c]c\| \le \|a \boxminus b\| \boxplus \|b \boxminus c\|, \quad \forall a, b, c \in G. \tag{13}$$

Curves on which the gyrotriangle inequality reduces to an equality are called gyrolines, while curves on which the cogyrotriangle inequality reduces to an equality are called cogyrolines.

Gyrolines and cogyrolines play an important role in hyperbolic analytic geometry regulated by the gyrovector space structure. For distinct elements a and b in G, the two distinct hyperbolic line expressions are defined by

$$
\begin{aligned}
L^g &= \{a \oplus (t \otimes b) : t \in \mathbb{R}\} && \text{Gyroline or the hyperbolic line,} \\
L^c &= \{(t \otimes b) \oplus a : t \in \mathbb{R}\} && \text{Cogyroline or the hyperbolic dual line.}
\end{aligned}
\tag{14}
$$

Gyrolines and cogyrolines are uniquely determined by any two distinct points contained by them. Therefore, expressions in (14) can be replaced by

$$
\begin{aligned}
L^g_{a,b} &= \{a \oplus (t \otimes (\ominus a \oplus b)) : t \in \mathbb{R}\} && \text{Gyroline passing through } a \text{ at } t = 0 \text{ and } b \text{ at } t = 1, \\
L^c_{a,b} &= \{(t \otimes (b \boxminus a)) \oplus a : t \in \mathbb{R}\} && \text{Cogyroline passing through } a \text{ at } t = 0 \text{ and } b \text{ at } t = 1.
\end{aligned}
$$

Several properties of gyrolines and cogyrolines are discussed in [2, Chapter 6]. Among them we mention the following properties of gyrolines:

P1. Two gyrolines that share two distinct points are coincident. (Theorem 6.20)
P2. A left gyrotranslation of a gyroline is again a gyroline. (Theorem 6.21)
P3. For $\tau \in \mathrm{Aut}(G)$, it holds that $\tau(a \oplus (t \otimes b)) = \tau a \oplus (t \otimes \tau b)$. (Theorem 6.37)
P4. Gyrolines are gyrogeodesics. (Theorem 6.50)

Cogyrolines are natural objects in gyrovector spaces and also have important properties. We mention the following:

P5. Two cogyrolines that share two distinct points are coincident. (Theorem 6.53)
P6. Cogyrolines admit parallelism. (Theorem 6.65)
P7. The cogyrotranslation $\boxminus d$ of the cogyroline $(t \otimes b) \oplus a$ is again a cogyroline, that is, $((t \otimes b) \oplus a) \boxminus d$ is a cogyroline. Moreover, the cogyroline and the cogyrotranslated cogyroline are parallel. (Theorem 6.66)
P8. Cogyrolines are cogyrogeodesics. (Theorem 6.78)

For the basic theory of gyrogroups and gyrovector spaces the interested reader is referred to [2,3,9]. Next we present the definition of subgyrogroups and L-subgyrogroups.

Definition 3. *[8] Let $(G, \oplus)$ be a gyrogroup. A nonempty subset H of G is called a subgyrogroup, denoted by $H \le G$, if H forms a gyrogroup under the restriction of the operation inherited from G and for all $a, b \in H$, the restriction of $\mathrm{gyr}[a, b]$ to H is an automorphism of H.*

Theorem 2. *[8] (The subgyrogroup criterion) A nonempty subset H of G is a subgyrogroup if and only if $a \oplus b \in H$ and $\ominus a \in H$ for all $a, b \in H$.*

The notion of L-subgyrogroups was introduced in [8].

Definition 4. *A subgyrogroup H of G is said to be an L-subgyrogroup, denoted by $H \leq_L G$, if $\mathrm{gyr}[a,h]H = H$ for all $a \in G$ and $h \in H$.*

For more details about factorization of gyrogroups by L-subgyrogroups, Cayley's Theorem, and isomorphisms theorems, see [8].

2. Orthogonal decompositions

Trying to be as general as possible we define next a *real inner product gyrogroup* contained in a real inner product space using the general addition (15).

Definition 5 (Real inner product gyrogroups). *A gyrogroup $(G, \oplus)$ contained in a real inner product space $(V, +, \cdot)$ is called a real inner product gyrogroup if it contains the zero of V as the gyrogroup identity and the binary operation $\oplus$ is given by*

$$a \oplus b = \phi_1(a,b)a + \phi_2(a,b)b \tag{15}$$

for all $a, b \in G$, where $\phi_i \colon G \times G \to \mathbb{R}$ is a map with $\phi_i(a, \ominus a) \neq 0$ for all $a \in G$, $i = 1, 2$, and $\oplus$ satisfies the gyrogroup axioms. Moreover, we say that the real inner product gyrogroup $(G, \oplus)$ admits orthogonal decompositions if the following additional conditions are satisfied:

(H1) *For all $v \in V$, if $\|v\| \leq \|a\|$ for some $a \in G$, then $v \in G$.*
(H2) *For all $c_1, c_2 \in G$ such that $c_1 + c_2 \in G$ and $c_1 \cdot c_2 = 0$, there exist nonzero scalars $\lambda_1 = \lambda_1(c_1, c_2) \in \mathbb{R}$ and $\lambda_2 = \lambda_2(c_1, c_2) \in \mathbb{R}$ such that $(x = \lambda_1 c_1, y = \lambda_2 c_2) \in G \times G$ is the unique solution of the system of equations:*

$$\begin{cases} \phi_1(x,y)x &= c_1 \\ \phi_2(x,y)y &= c_2 \end{cases}. \tag{16}$$

We remark that since the general addition $a \oplus b$ defined in (15) satisfies the gyrogroup axioms of Definition 1 then the functions ϕ_1 and ϕ_2 have to satisfy some functional equalities and relations. In particular, since, $0 \oplus b = b$ and $a \oplus 0 = a$, for all $a, b \in G$, then it implies that $\phi_1(a, 0) = 1$ and $\phi_2(0, b) = 1$, for all $a, b \in G$. Regarding the conditions (H1) and (H2), the first condition says essentially that G consists of a ball with finite or infinite radius in the carrier space V, while the second condition allows us to obtain orthogonal decompositions of G, and corresponding left and right cosets of G. Concrete prominent examples of real inner product gyrogroups that admit orthogonal decompositions are exhibited in Section 4: Euclidean Einstein, Möbius, Proper Velocity, and Chen's gyrogroups.

Throughout the remainder of this article, we assume that $(G, \oplus)$ is a real inner product gyrogroup contained in the carrier inner product space $(V, +, \cdot)$ satisfying conditions (H1) and (H2).

2.1. Unique decomposition and orthogonal gyroprojections

Let P be a finite-dimensional linear subspace of V and let $P^\perp$ be its orthogonal complement in V (that is, $V = P \oplus P^\perp$ is an orthogonal direct sum). Define

$$P_G = P \cap G \qquad \text{and} \qquad P_G^\perp = P^\perp \cap G. \tag{17}$$

Theorem 3. *If P is a finite-dimensional subspace of V, then P_G and $P_G^\perp$ are subgyrogroups of G.*

Proof. Note that $P_G \neq \varnothing$ since $0 \in P_G$. Let $a, b \in P_G$. Then $a, b \in P$ and $a, b \in G$. By (15), $a \oplus b \in P$ since P is closed under vector addition. By the closure property of G, $a \oplus b \in G$. Hence, $a \oplus b \in P_G$. Note that $0 = a \oplus (\ominus a) = \phi_1(a, \ominus a)a + \phi_2(a, \ominus a) \ominus a$, which implies $\ominus a = -\dfrac{\phi_1(a, \ominus a)}{\phi_2(a, \ominus a)}a$ since $\phi_2(a, \ominus a) \neq 0$. Hence, $\ominus a \in P$ and so $\ominus a \in P_G$. This proves that P_G is a subgyrogroup of G. Since the proof does not make use of the assumption of being finite-dimensional, we conclude that $P_G^\perp$ is a subgyrogroup of G as well. $\square$

According to Definition 4, P_G is an L-subgyrogroup if $\text{gyr}[a,h]P_G = P_G$, for all $a \in G$ and $h \in P_G$. Since gyrations are isometries of $(G, \oplus, \otimes)$ that preserve the norm inherited from $(V, +, \cdot)$, the only gyrations that preserve P_G are from the automorphism group of P_G. Hence, P_G is not an L-subgyrogroup.

A first fundamental result that we are going to show is the conditions under which the addition $\oplus$ in a real inner product gyrogroup is associative.

Theorem 4. *Let $a, b, c \in (G, \oplus, \otimes)$. Then $a \oplus (b \oplus c) = (a \oplus b) \oplus c$ if and only if $a \cdot c = 0$ and $b \cdot c = 0$ or $a = \lambda \otimes b$ for some $\lambda \in \mathbb{R}$.*

Proof. By the left gyroassociative law we have

$$a \oplus (b \oplus c) = (a \oplus b) \oplus \text{gyr}[a,b]c.$$

We know that isometries of a real inner product space $(V, +, \cdot)$ that fixes the origin and an arbitrary element c are the isometries that belong to a linear subspace orthogonal to c or the trivial isometry (identity isometry) that belongs to a one-dimensional linear subspace of V. Therefore, since gyrations of $(G, \oplus, \otimes)$ are isometries in the carrier space V, it follows that $\text{gyr}[a,b]c = c$ if and only if $a \cdot c = 0$ and $b \cdot c = 0$ or $a = \lambda \otimes b$ for some $\lambda \in \mathbb{R}$, i.e., either a and b belong to a linear subspace orthogonal to c or a and b are in a given one dimensional linear subspace. Note that by axiom (V6) in Definition 2 it follows that $\text{gyr}[a, \lambda \otimes a] = I$. $\square$

The following theorem shows that any real inner product gyrogroup G can be decomposed into gyrosums of P_G and $P_G^\perp$, where P is a finite-dimensional subspace of the carrier space V.

Theorem 5 (Unique Decomposition). *Let P be a finite-dimensional subspace of V. For all $c \in G$, there are unique elements $a, u \in P_G$, $b, v \in P_G^\perp$ such that*

$$c = a \oplus b \qquad and \qquad c = v \oplus u. \qquad (18)$$

Proof. Let $c \in G$ be arbitrary. Since $G \subseteq V$ and $V = P \oplus P^\perp$, it follows that $c = c_1 + c_2$ with $c_1 \in P$ and $c_2 \in P^\perp$. Note that $\|c\|^2 = \|c_1 + c_2\|^2 = \|c_1\|^2 + \|c_2\|^2$. Hence, $\|c_1\| \leq \|c\|$ and $\|c_2\| \leq \|c\|$. By hypothesis (H1), $c_1 \in G$ and $c_2 \in G$. Since $c_1 + c_2 = c \in G$ and $c_1 \cdot c_2 = 0$, we have by hypothesis (H2) that there exist scalars $\lambda_1 = \lambda_1(c_1, c_2), \lambda_2 = \lambda_2(c_1, c_2) \in \mathbb{R}$ such that $(x = \lambda_1 c_1, y = \lambda_2 c_2) \in G \times G$ is the unique solution of the system of equations:

$$\begin{cases} \phi_1(x,y)x = c_1 \\ \phi_2(x,y)y = c_2 \end{cases}. \qquad (19)$$

Set $a = \lambda_1 c_1$ and $b = \lambda_2 c_2$. Then $a \in P_G$ and $b \in P_G^\perp$. Furthermore, we obtain

$$\begin{aligned} a \oplus b &= \phi_1(a,b)a + \phi_2(a,b)b \\ &= \phi_1(\lambda_1 c_1, \lambda_2 c_2)\lambda_1 c_1 + \phi_2(\lambda_1 c_1, \lambda_2 c_2)\lambda_2 c_2 \\ &= c_1 + c_2 \\ &= c. \end{aligned}$$

To prove uniqueness of the factorization, suppose that $c = a' \oplus b'$ with $a' \in P_G$, $b' \in P_G^\perp$. Then

$$c_1 + c_2 = \phi_1(a', b')a' + \phi_2(a', b')b'$$

and so $c_1 = \phi_1(a', b')a'$ and $c_2 = \phi_2(a', b')b'$ since the sum $P \oplus P^\perp$ is direct. It follows that (a', b') is a solution of (19). Hence, $a' = \lambda_1 c_1 = a$ and $b' = \lambda_2 c_2 = b$.

Similarly, $c_2 + c_1 = c \in G$ and $c_2 \cdot c_1 = 0$ together imply by hypothesis (H2) that there exist scalars $\lambda_3 = \lambda_3(c_1, c_2) \in \mathbb{R}$ and $\lambda_4 = \lambda_4(c_1, c_2) \in \mathbb{R}$ such that $(x = \lambda_3 c_2, y = \lambda_4 c_1) \in G \times G$ is the unique solution of the system of equations (19). Regarding the first decomposition it is immediate to see that $\lambda_3(c_1, c_2) = \lambda_1(c_2, c_1)$ and $\lambda_4(c_1, c_2) = \lambda_2(c_2, c_1)$, with the order of c_1 and c_2 being changed in λ_1 and λ_2. Set $v = \lambda_3 c_2$ and $u = \lambda_4 c_1$. Then $u \in P_G, v \in P_G^\perp$, and $c = v \oplus u$. Uniqueness of the factorization can be proved in the same way. $\square$

We remark that the proof of the uniqueness of Theorem 5 could also be made using Theorem 4.

Corollary 1 (Orthogonal Decompositions). *The following orthogonal decompositions of G hold:*

$$G = P_G \oplus P_G^\perp \quad and \quad G = P_G^\perp \oplus P_G.$$

Using Theorem 5, we can define orthogonal gyroprojections of the gyrogroup $(G, \oplus)$ onto the subgyrogroups P_G and $P_G^\perp$, with respect to the decompositions $G = P_G \oplus P_G^\perp$ and $G = P_G^\perp \oplus P_G$. In fact, each element $c \in G$ admits the unique decomposition $c = c_1 + c_2 = c_2 + c_1$, with $c_1 \in P_G$ and $c_2 \in P_G^\perp$ and according to Theorem 5 four orthogonal gyroprojectors can be defined:

$$
\begin{aligned}
\mathcal{P}^\ell &: G = P_G \oplus P_G^\perp \to P_G, & \mathcal{P}^\ell(c) &= \lambda_1(c_1, c_2)c_1, \\
\mathcal{Q}^r &: G = P_G \oplus P_G^\perp \to P_G^\perp, & \mathcal{Q}^r(c) &= \lambda_2(c_1, c_2)c_2, \\
\mathcal{Q}^\ell &: G = P_G^\perp \oplus P_G \to P_G^\perp, & \mathcal{Q}^\ell(c) &= \lambda_3(c_1, c_2)c_2 = \lambda_1(c_2, c_1)c_2, \\
\mathcal{P}^r &: G = P_G^\perp \oplus P_G \to P_G, & \mathcal{P}^r(c) &= \lambda_4(c_1, c_2)c_1 = \lambda_2(c_2, c_1)c_1,
\end{aligned}
\tag{20}
$$

such that

$$c = \mathcal{P}^\ell(c) \oplus \mathcal{Q}^r(c) \qquad and \qquad c = \mathcal{Q}^\ell(c) \oplus \mathcal{P}^r(c), \tag{21}$$

where the superscripts ℓ and r stand for "left" and "right" and indicate that the gyroprojection is on the left or on the right of each decomposition. The following identities hold immediately:

$$(\mathcal{P}^\ell)^2 = \mathcal{P}^\ell, \qquad (\mathcal{P}^r)^2 = \mathcal{P}^r, \qquad (\mathcal{Q}^\ell)^2 = \mathcal{Q}^\ell, \qquad (\mathcal{Q}^r)^2 = \mathcal{Q}^r. \tag{22}$$

Moreover, we obtain the following orthogonal gyrodecompositions:

$$\mathcal{I} = \mathcal{P}^\ell \oplus \mathcal{Q}^r \qquad and \qquad \mathcal{I} = \mathcal{Q}^\ell \oplus \mathcal{P}^r. \tag{23}$$

Corollary 2. *For all $a \in P_G$ and $b \in P_G^\perp$, we have*

$$
\begin{aligned}
\mathcal{P}^\ell(a \oplus b) &= a \quad and \quad \mathcal{Q}^r(a \oplus b) = b, \\
\mathcal{P}^r(b \oplus a) &= a \quad and \quad \mathcal{Q}^\ell(b \oplus a) = b.
\end{aligned}
\tag{24}
$$

Proof. By (15) and Theorem 5 we have

$$a \oplus b = \phi_1(a,b)a + \phi_2(a,b)b = \lambda_1(\phi_1(a,b)a, \phi_2(a,b)b)\phi_1(a,b)a \oplus \lambda_2(\phi_1(a,b)a, \phi_2(a,b)b)\phi_2(a,b)b$$

which implies by the Unique Decomposition Theorem (Theorem 5) that

$$
\begin{cases}
\lambda_1(\phi_1(a,b)a, \phi_2(a,b)b)\phi_1(a,b)a = a \\
\lambda_2(\phi_1(a,b)a, \phi_2(a,b)b)\phi_2(a,b)b = b
\end{cases}.
\tag{25}
$$

Hence, using the definition of the gyroprojections (20) and (25) we obtain the identities

$$\mathcal{P}^\ell(a \oplus b) = a \quad and \quad \mathcal{Q}^r(a \oplus b) = b.$$

Using similar reasonings we obtain also

$$\mathcal{P}^r(b \oplus a) = a \quad \text{and} \quad \mathcal{Q}^\ell(b \oplus a) = b. \qquad \square$$

By Corollary 1 the gyrogroup $(G, \oplus)$ has two unique decompositions $G = P_G \oplus P_G^\perp$ and $G = P_G^\perp \oplus P_G$. The relation between them is given more precisely in the next theorem.

Theorem 6. *Let $a, b \in (G, \oplus)$ such that $a \cdot b = 0$. Then there exist nonzero scalars $\mu_1(a, b)$ and $\mu_2(a, b)$ such that*

$$a \oplus b = \mu_1(a, b)b \oplus \mu_2(a, b)a. \qquad (26)$$

Proof. Let $a, b \in G$ such that $a \perp b$. Then by (15) we have

$$a \oplus b = \phi_1(a, b)a + \phi_2(a, b)b.$$

Consider now $c_1 = \phi_1(a, b)a$ and $c_2 = \phi_2(a, b)b$. Then $c_1 \perp c_2$ and by Theorem 5 there exist nonzero scalars $\lambda_3 = \lambda_3(c_1, c_2) \in \mathbb{R}$ and $\lambda_4 = \lambda_4(c_1, c_2) \in \mathbb{R}$ such that

$$\begin{aligned}
a \oplus b &= \lambda_3(c_1, c_2)c_2 \oplus \lambda_4(c_1, c_2)c_1 \\
&= \lambda_3(\phi_1(a, b)a, \phi_2(a, b)b)\phi_2(a, b)b + \lambda_4(\phi_1(a, b)a, \phi_2(a, b)b)\phi_1(a, b)a.
\end{aligned}$$

Taking

$$\begin{cases} \mu_1(a, b) = \lambda_3(\phi_1(a, b)a, \phi_2(a, b)b)\, \phi_2(a, b) \\ \mu_2(a, b) = \lambda_4(\phi_1(a, b)a, \phi_2(a, b)b)\, \phi_1(a, b) \end{cases} \qquad (27)$$

we obtain (26). $\square$

Corollary 3. *For all $a \in P_G$ and $b \in P_G^\perp$, we have*

$$\begin{aligned}
\mathcal{P}^\ell(b \oplus a) = \mu_1(b, a)a \quad &\text{and} \quad \mathcal{Q}^r(b \oplus a) = \mu_2(b, a)b, \\
\mathcal{P}^r(a \oplus b) = \mu_2(a, b)a \quad &\text{and} \quad \mathcal{Q}^\ell(a \oplus b) = \mu_1(a, b)b.
\end{aligned} \qquad (28)$$

Proof. From Theorem 6 we have:

$$\begin{aligned}
\mathcal{P}^\ell(b \oplus a) &= \mathcal{P}^\ell(\mu_1(b, a)a \oplus \mu_2(b, a)b) &=& \mu_1(b, a)a, \\
\mathcal{Q}^r(b \oplus a) &= \mathcal{Q}^r(\mu_1(b, a)a \oplus \mu_2(b, a)b) &=& \mu_2(b, a)b, \\
\mathcal{P}^r(a \oplus b) &= \mathcal{P}^r(\mu_1(a, b)b \oplus \mu_2(a, b)a) &=& \mu_2(a, b)a, \\
\mathcal{Q}^\ell(a \oplus b) &= \mathcal{Q}^\ell(\mu_1(a, b)b \oplus \mu_2(a, b)a) &=& \mu_1(a, b)b. \qquad \square
\end{aligned}$$

2.2. Left and right coset spaces

Let H be a subgyrogroup of G. In contrast to groups, the relation

$$a \sim b \qquad \text{if and only if} \qquad \ominus a \oplus b \in H$$

does not, in general, define an equivalence relation on G. In [8] this relation was modified to

$$a \sim b \qquad \text{if and only if} \qquad \ominus a \oplus b \in H \text{ and } \mathrm{gyr}[\ominus a, b]H = H$$

obtaining an equivalence relation on G and leading to the notion of L-subgyrogroups. Since in our case P_G is not an L-subgyrogroup, we cannot apply the results in [8]. In the case of real inner product gyrogroups that have attached a gyrovector space structure we are going to show that the

gyroprojection $\mathcal{Q}^\ell$ (respectively, $\mathcal{Q}^r$) induces an equivalence relation on G so that G can be written as a disjoint union of left (respectively, right) cosets with representatives from $P_G^\perp$.

Let P be a finite-dimensional linear subspace of V. Define $\sim_\ell$ by

$$a \sim_\ell b \qquad \text{if and only if} \qquad \ominus \mathcal{Q}^\ell(b) \oplus a \in P_G \tag{29}$$

for all $a, b \in G$.

Theorem 7. *The relation $\sim_\ell$ defined by* (29) *is an equivalence relation on G.*

Proof. Let $a, b, c \in G$.

(*Reflexive property*) Since $a = \mathcal{Q}^\ell(a) \oplus \mathcal{P}^r(a)$, it follows from Theorem 1 we have $\ominus \mathcal{Q}^\ell(a) \oplus a = \mathcal{P}^r(a) \in P_G$. Hence, $a \sim_\ell a$.

(*Symmetric property*) Suppose that $a \sim_\ell b$. Then $\ominus \mathcal{Q}^\ell(b) \oplus a \in P_G$ and so there is an element $c \in P_G$ such that $\ominus \mathcal{Q}^\ell(b) \oplus a = c$. This implies $a = \mathcal{Q}^\ell(b) \oplus c$. It follows that $\mathcal{Q}^\ell(a) = \mathcal{Q}^\ell(\mathcal{Q}^\ell(b) \oplus c) = \mathcal{Q}^\ell(b)$. Since $b = Q^\ell(b) \oplus P^r(b)$, then we obtain from Theorem 1 that $\ominus Q^\ell(b) \oplus b = P^r(b)$. Therefore,

$$\ominus Q^\ell(a) \oplus b = \ominus Q^\ell(b) \oplus b = P^r(b) \in P_G.$$

Hence, $b \sim_\ell a$.

(*Transitive property*) Suppose that $a \sim_\ell b$ and $b \sim_\ell c$. It follows that $\mathcal{Q}^\ell(a) = \mathcal{Q}^\ell(b)$ and $\mathcal{Q}^\ell(b) = \mathcal{Q}^\ell(c)$, as proved in the previous paragraph. Since $\ominus \mathcal{Q}^\ell(a) \oplus a = \mathcal{P}^r(a)$, we obtain that $\ominus \mathcal{Q}^\ell(c) \oplus a = \ominus \mathcal{Q}^\ell(a) \oplus a = \mathcal{P}^r(a)$ belongs to P_G. Hence, $a \sim_\ell c$. □

For each $b \in G$, we denote by $[b]_\ell$ the equivalence class containing b determined by the relation $\sim_\ell$.

Lemma 1. *For all $b \in G$, $[b]_\ell = Q^\ell(b) \oplus P_G$, where*

$$Q^\ell(b) \oplus P_G = \{Q^\ell(b) \oplus u \colon u \in P_G\}.$$

Proof. Let $x \in Q^\ell(b) \oplus P_G$. Then $x = Q^\ell(b) \oplus u$ for some $u \in P_G$. By the left cancellation law we have $\ominus Q^\ell(b) \oplus x = u \in P_G$, that is, $x \sim_\ell b$, which implies $x \in [b]_\ell$. To prove the reverse inclusion, let $x \in [b]_\ell$. Then $x \sim_\ell b$, that is, $\ominus Q^\ell(b) \oplus x \in P_G$. Then, there exists an element $u \in P_G$ such that $\ominus Q^\ell(b) \oplus x = u$. Hence, by the left cancellation law we have $x = Q^\ell(b) \oplus u$, that is, $x \in Q^\ell(b) \oplus P_G$. Thus, $[b]_\ell = Q^\ell(b) \oplus P_G$. □

Theorem 8. *Let P be a finite-dimensional linear subspace of V. Then the collection*

$$\{b \oplus P_G \colon b \in P_G^\perp\}$$

is a disjoint partition of G, that is,

$$G = \bigcup_{b \in P_G^\perp}^{\cdot} (b \oplus P_G). \tag{30}$$

Proof. According to Lemma 1, we know that for all $b, v \in P_G^\perp$, either $b \oplus P_G = v \oplus P_G$ or $(b \oplus P_G) \cap (v \oplus P_G) = \varnothing$ since $b \oplus P_G = [b]_\ell$ and $v \oplus P_G = [v]_\ell$. Hence, the union in (30) is disjoint. It is clear that $b \oplus P_G \neq \varnothing$ since $b \in b \oplus P_G$ for all $b \in P_G^\perp$. Let $x \in G$. By Theorem 5, $x = v \oplus u$ for some $v \in P_G^\perp$, $u \in P_G$. Hence, $x \in v \oplus P_G \subseteq \bigcup_{b \in P_G^\perp} (b \oplus P_G)$. This proves $G \subseteq \bigcup_{b \in P_G^\perp} (b \oplus P_G)$ and so equality holds. □

Denote by $(G/P_G, \sim_\ell)$ the set of all equivalence classes obtained by the equivalence relation $\sim_\ell$. By Lemma 1 and Theorem 8 the coset space $(G/P_G, \sim_\ell)$ represents the collection $\{b \oplus P_G : b \in P_G^\perp\}$.

Corollary 4. *The set $(G/P_G, \sim_\ell)$ is a left coset space of G whose cosets are of the form $b \oplus P_G$ with $b \in P_G^\perp$.*

Next, we present the right counterpart of Theorem 8 defining first the equivalence relation that leads to the construction of the right cosets.

Let P be a finite-dimensional linear subspace of V. Define $\sim_r$ by

$$a \sim_r b \qquad \text{if and only if} \qquad a \boxminus \mathcal{Q}^r(b) \in P_G \tag{31}$$

for all $a, b \in G$, where $a \boxminus \mathcal{Q}^r(b) = a \boxplus (\ominus \mathcal{Q}^r(b))$ and $\boxplus$ stands for the coaddition in G defined in (2).

Theorem 9. *The relation $\sim_r$ defined by (31) is an equivalence relation on G.*

Proof. Let $a, b, c \in G$.

(*Reflexive property*) Since $a = \mathcal{P}^\ell(a) \oplus \mathcal{Q}^r(a)$, it follows from Theorem 1 that $\mathcal{P}^\ell(a) = a \boxminus \mathcal{Q}^r(a)$. Hence, $a \boxminus \mathcal{Q}^r(a) \in P_G$ and so $a \sim_r a$.

(*Symmetric property*) Suppose that $a \sim_r b$. Then $a \boxminus \mathcal{Q}^r(b) \in P_G$ and so there is an element $c \in P_G$ such that $a \boxminus \mathcal{Q}^r(b) = c$. This implies $a = c \oplus \mathcal{Q}^r(b)$. It follows that $\mathcal{Q}^r(a) = \mathcal{Q}^r(c \oplus \mathcal{Q}^r(b)) = \mathcal{Q}^r(b)$. Since $b = \mathcal{P}^\ell(b) \oplus \mathcal{Q}^r(b)$, we obtain from Theorem 1 that $\mathcal{P}^\ell(b) = b \boxminus \mathcal{Q}^r(b)$. Therefore, $b \boxminus Q^r(a) = b \boxminus Q^r(b) = P^r(b) \in P_G$. Hence, $b \sim_r a$.

(*Transitive property*) Suppose that $a \sim_r b$ and $b \sim_r c$. It follows that $\mathcal{Q}^r(a) = \mathcal{Q}^r(b)$ and $\mathcal{Q}^r(b) = \mathcal{Q}^r(c)$, as proved in the previous paragraph. Since $a \boxminus \mathcal{Q}^r(a) = \mathcal{P}^\ell(a)$, we obtain that

$$a \boxminus \mathcal{Q}^r(c) = a \boxminus \mathcal{Q}^r(a) = \mathcal{P}^\ell(a)$$

belongs to P_G. Hence, $a \sim_r c$. $\square$

For each $b \in G$, we denote by $[b]_r$ the equivalence class containing b determined by the relation $\sim_r$.

Lemma 2. *For all $b \in G$, $[b]_r = P_G \oplus Q^r(b)$, where*

$$P_G \oplus Q^r(b) = \{u \oplus Q^r(b) : u \in P_G\}.$$

Proof. Let $x \in P_G \oplus Q^r(b)$. Then $x = u \oplus Q^r(b)$ for some $u \in P_G$. Then, by the right cancellation law we have $x \boxminus Q^r(b) = u \in P_G$, that is, $x \sim_r b$, which implies $x \in [b]_r$. To prove the reverse inclusion let $x \in [b]_r$. Then $x \sim_r b$, that is, $x \boxminus Q^r(b) \in P_G$. Then, there exists an element $u \in P_G$ such that $x \boxminus Q^r(b) = u$. Hence, by the right cancellation law II (cf. Equation (2.63) of [3]) we have $x = u \oplus Q^r(b)$, that is, $x \in P_G \oplus Q^r(b)$. Thus, $[b]_r = P_G \oplus Q^r(b)$. $\square$

Theorem 10. *Let P be a finite-dimensional linear subspace of V. Then the collection*

$$\{P_G \oplus b : b \in P_G^\perp\}$$

is a disjoint partition of G, that is,

$$G = \bigcup_{b \in P_G^\perp}^{\cdot} (P_G \oplus b). \tag{32}$$

Proof. According to Lemma 2, we know that for all $b, v \in P_G^\perp$, either $P_G \oplus b = P_G \oplus v$ or $(P_G \oplus b) \cap (P_G \oplus v) = \varnothing$ since $P_G \oplus b = [b]_r$ and $P_G \oplus v = [v]_r$. Hence, the union in (32) is disjoint. It is clear that

$P_G \oplus b \neq \varnothing$ since $b \in P_G \oplus b$ for all $b \in P_G^\perp$. Let $x \in G$. By Theorem 5, $x = a \oplus b$ for some $a \in P_G$, $b \in P_G^\perp$. Hence, $x \in P_G \oplus b \subseteq \bigcup_{b \in P_G^\perp} (P_G \oplus b)$. This proves $G \subseteq \bigcup_{b \in P_G^\perp} (P_G \oplus b)$ and so equality holds. $\square$

Denote by $(G/P_G, \sim_r)$ the set of all equivalence classes obtained by the equivalence relation $\sim_r$.

Corollary 5. *The set $(G/P_G, \sim_r)$ is a right coset space of G whose cosets are of the form $P_G \oplus b$ with $b \in P_G^\perp$.*

2.3. Quotient gyrogroups and the isomorphism theorem

The coset spaces $(G/P_G, \sim_\ell)$ and $(G/P_G, \sim_r)$ turn out to be gyrogroups, called left and right *quotient gyrogroups* by P_G, respectively.

Lemma 3. *Let P be an arbitrary finite-dimensional subspace of V.*

(1) *For all $a, b \in P_G^\perp$, $a \oplus P_G = b \oplus P_G$ if and only if $a = b$.*
(2) *For all $a, b \in P_G^\perp$, $P_G \oplus a = P_G \oplus b$ if and only if $a = b$.*

Proof. Let $a, b \in P_G^\perp$.

To prove item (1), suppose that $a \oplus P_G = b \oplus P_G$. Then $a \sim_\ell b$ and so $\ominus \mathcal{Q}^\ell(b) \oplus a \in P_G$. Since $b \in P_G^\perp$, we have $\mathcal{Q}^\ell(b) = b$. Hence, $\ominus b \oplus a \in P_G$. Since $P_G^\perp$ is a subgyrogroup of G, we also have $\ominus b \oplus a \in P_G^\perp$. It follows that $\ominus b \oplus a \in P_G \cap P_G^\perp = \{0\}$. Hence, $\ominus b \oplus a = 0$, which implies $a = b$.

To prove item (2), suppose that $P_G \oplus a = P_G \oplus b$. Then $a \sim_r b$ and so $a \boxminus \mathcal{Q}^r(b) \in P_G$. Since $b \in P_G^\perp$, we have $\mathcal{Q}^r(b) = b$. Hence, $a \boxminus b \in P_G$. Since $P_G^\perp$ is a subgyrogroup of G, we also have $a \boxminus b = a \oplus \mathrm{gyr}[a, b](\ominus b) \in P_G^\perp$. It follows that $a \boxminus b \in P_G \cap P_G^\perp = \{0\}$. Hence, $a \boxminus b = 0$, which implies $a = b$. $\square$

From Lemma 3 and the fact that $P_G^\perp$ is a subgyrogroup of G, we obtain that the following operations on the coset spaces $(G/P_G, \sim_\ell)$ and $(G/P_G, \sim_r)$ are well defined:

$$(a \oplus P_G) \oplus (b \oplus P_G) = (a \oplus b) \oplus P_G, \qquad a, b \in P_G^\perp; \tag{33}$$

$$(P_G \oplus a) \oplus (P_G \oplus b) = P_G \oplus (a \oplus b), \qquad a, b \in P_G^\perp. \tag{34}$$

In fact, the left and right coset spaces form gyrogroups, as shown in the following theorems.

Theorem 11. *The left coset space $(G/P_G, \sim_\ell)$ forms a gyrogroup under the operation defined by (33).*

Proof. The coset $0 \oplus P_G$ is a left identity in G/P_G. For each $a \oplus P_G \in G/P_G$ with $a \in P_G^\perp$, the coset $(\ominus a) \oplus P_G$ lies in G/P_G and is a left inverse of $a \oplus P_G$. For $X = a \oplus P_G, Y = b \oplus P_G$ with $a, b \in P_G^\perp$, define

$$\mathrm{gyr}[X, Y](c \oplus P_G) = (\mathrm{gyr}[a, b]c) \oplus P_G, \qquad c \in P_G^\perp. \tag{35}$$

If $d \in P_G^\perp$ and if $d \oplus P_G = c \oplus P_G^\perp$, then $d = c$ by Lemma 3 and so

$$(\mathrm{gyr}[a, b]d) \oplus P_G = (\mathrm{gyr}[a, b]c) \oplus P_G,$$

which proves that $\mathrm{gyr}[X, Y]$ is well defined. The proof that $\mathrm{gyr}[X, Y]$ fits the definition of a gyrogroup follows the same steps as in the proof of Theorem 29 of [9] with appropriate modifications. $\square$

Proceeding in a similar fashion, we obtain the right counterpart of Theorem 11.

Theorem 12. *The right coset space $(G/P_G, \sim_r)$ forms a gyrogroup under the operation defined by (34).*

Definition 6. *The gyrogroup* $(G/P_G, \sim_\ell)$ *in Theorem* 11 *is called a left quotient gyrogroup and the gyrogroup* $(G/P_G, \sim_r)$ *in Theorem* 12 *is called a right quotient gyrogroup. We define the canonical projection mappings* π_ℓ *and* π_r *from G to the left and right coset spaces by*

$$\pi_\ell(c) = [c]_\ell = \mathcal{Q}^\ell(c) \oplus P_G, \qquad c \in G, \tag{36}$$

$$\pi_r(c) = [c]_r = P_G \oplus \mathcal{Q}^r(c), \qquad c \in G. \tag{37}$$

Theorem 13 (The Isomorphism Theorem). *Let P be an arbitrary finite-dimensional linear subspace of* V. *Then* π_ℓ *restricts to a gyrogroup isomorphism from* $P_G^\perp$ *to* $(G/P_G, \sim_\ell)$ *and* π_r *restricts to a gyrogroup isomorphism from* $P_G^\perp$ *to* $(G/P_G, \sim_r)$. *Therefore, the following gyrogroup isomorphisms hold:*

$$(G/P_G, \sim_\ell) \cong P_G^\perp \cong (G/P_G, \sim_r).$$

Proof. That $\pi_\ell|_{P_G^\perp}$ is surjective is clear. That $\pi_\ell|_{P_G^\perp}$ is injective follows from Lemma 3. According to (33), $\pi_\ell|_{P_G^\perp}$ preserves the gyrogroup operations. This shows that $\pi_\ell|_{P_G^\perp}: P_G^\perp \to (G/P_G, \sim_\ell)$ is a gyrogroup isomorphism. The proof for $\pi_r|_{P_G^\perp}$ is similar. It follows that $(G/P_G, \sim_\ell) \cong P_G^\perp$ and that $(G/P_G, \sim_r) \cong P_G^\perp$. $\square$

2.4. Geometric characterization of cosets

In this section, we assume that $(G, \oplus)$ is a real inner product gyrogroup together with a scalar multiplication $\otimes$, turning $(G, \oplus, \otimes)$ a gyrovector space . The next theorem gives a geometric characterization of the left equivalence classes $b \oplus P_G$ with $b \in P_G^\perp$ in terms of the automorphisms of $(P_G, \oplus, \otimes)$ acting on a given gyroline L^g.

Theorem 14. *Let* $b \in P_G^\perp$ *and let* $0 \neq c \in P_G$ *be fixed. Then*

$$b \oplus P_G = \{\tau L^g : \tau \in \mathrm{Aut}(P_G, \oplus, \otimes)\},$$

where the gyroline L^g *is the gyrogeodesic given by*

$$L^g = \{b \oplus (t \otimes c) : t \in \mathbb{R}\}.$$

Proof. Let $b \in P_G^\perp$ and $0 \neq c \in P_G$. Since for any $t \in \mathbb{R}$,

$$b \oplus (t \otimes c) = \phi_1(b, t \otimes c)b + \phi_2(b, t \otimes c)(t \otimes c),$$

we know that L^g is a gyroline in G in the plane defined by the vectors b and c. Now, we consider $a \in P_G$ arbitrary. Since it is always possible to find $t \in \mathbb{R}$ such that $\|t \otimes c\| = \|a\|$, then there exists an automorphism $\tau \in \mathrm{Aut}(G, \oplus, \otimes)$ such that $a = \tau(t \otimes c)$ and τ leaves P_G invariant and takes each element of $P_G^\perp$ as a fixed point. This means that $\tau \in \mathrm{Aut}(P_G, \oplus, \otimes)$. In particular, since $b \in P_G^\perp$ then $\tau b = b$, and consequently, $\tau^{-1}b = b$. Then, by the invariance property P3 of gyrolines (described in Section 1) we have

$$b \oplus a = b \oplus (\tau(t \otimes c)) = \tau(\tau^{-1}b \oplus (t \otimes c)) = \tau(b \oplus (t \otimes c)).$$

Thus, we get $b \oplus P_G = \{\tau L^g : \tau \in \mathrm{Aut}(P_G, \oplus, \otimes)\}$. $\square$

In an analogous way, we can characterize the right equivalence classes $P_G \oplus b$ with $b \in P_G^\perp$ in terms of the automorphisms of $(P_G, \oplus, \otimes)$ acting on a given cogyroline L^c.

Theorem 15. *Let $b \in P_G^\perp$ and $0 \neq c \in P_G$ be fixed. Then*

$$P_G \oplus b = \{\tau L^c : \tau \in \mathrm{Aut}(P_G, \oplus, \otimes)\},$$

where the cogyroline L^c is the cogyrogeodesic given by

$$L^c = \{(t \otimes c) \oplus b : t \in \mathbb{R}\}.$$

Proof. Let $b \in P_G^\perp$ and $0 \neq c \in P_G$. Since for any $t \in \mathbb{R}$,

$$(t \otimes c) \oplus b = \phi_1(t \otimes c, b)(t \otimes c) + \phi_2(t \otimes c, b)b,$$

we know that L^c is a cogyroline in G in the plane defined by the vectors b and c. Now, we consider $a \in P_G$ arbitrary. Then there exist $t \in \mathbb{R}$ and $\tau \in \mathrm{Aut}(P_G, \oplus, \otimes)$ such that $a = \tau(t \otimes c)$ and τ leaves P_G invariant. Thus, by property P3 of gyrolines since $\tau^{-1}b = b$, we have

$$a \oplus b = (\tau(t \otimes c)) \oplus b = \tau((t \otimes c) \oplus \tau^{-1}b) = \tau((t \otimes c) \oplus b).$$

Hence, we get $P_G \oplus b = \{\tau L^c : \tau \in \mathrm{Aut}(P_G, \oplus, \otimes)\}$. $\square$

3. Fiber bundles and sections of real inner product gyrogroups

We denote (G, X, π, Y) as a fiber bundle with base space X, fiber Y, and bundle map $\pi : G \to X$. A global section of the fiber bundle (G, X, π, Y) is a continuous map $f : X \to G$ such that $\pi(f(y)) = y$ for all $y \in X$, while a local section is a map $f : U \to G$, where U is an open set in X and $\pi(f(x)) = x$ for all $x \in U$.

According with the two unique decompositions $G = P_G \oplus P_G^\perp$ and $G = P_G^\perp \oplus P_G$, and using Lemma 1, Lemma 2, (24), and (28), we can define four different fiber bundle structures on $(G, \oplus)$ with fiber bundle mappings given by

$$
\begin{aligned}
\pi_1 : P_G \oplus P_G^\perp &\to (G/P_G, \sim_r) & \pi_2 : P_G^\perp \oplus P_G &\to (G/P_G, \sim_\ell) \\
a \oplus b &\mapsto [a \oplus b]_r = [b]_r = P_G \oplus b & b \oplus a &\mapsto [b \oplus a]_\ell = [b]_\ell = b \oplus P_G
\end{aligned}
$$

$$
\begin{aligned}
\pi_3 : P_G^\perp \oplus P_G &\to (G/P_G, \sim_r) & \pi_4 : P_G \oplus P_G^\perp &\to (G/P_G, \sim_\ell) \\
b \oplus a &\mapsto [b \oplus a]_r = P_G \oplus \mu_2(b, a)b & a \oplus b &\mapsto [a \oplus b]_\ell = \mu_1(a, b)b \oplus P_G .
\end{aligned}
$$

$$\tag{38}$$

We remark that the mappings π_3 and π_4 defined in (38) correct the definitions presented in [5].

It is easy to see that the first and the second bundles are trivial ones. The first bundle π_1 is isomorphic to the trivial bundle $(P_G \times P_G^\perp, P_G^\perp, Q^r, P_G)$, where Q^r is the projection onto the second factor defined by:

$$Q^r : P_G \oplus P_G^\perp \to P_G^\perp, \quad a \oplus b \mapsto b.$$

Hence, the following diagram commutes:

$$
\begin{array}{ccc}
G = P_G \oplus P_G^\perp & \xrightarrow{\ \pi_1\ } & (G/P_G, \sim_r) \\
\downarrow id & & \downarrow \Phi_1 \\
P_G \oplus P_G^\perp & \xrightarrow{\ Q^r\ } & P_G^\perp
\end{array}
$$

where Φ_1 is the isomorphism between $(G/P_G, \sim_r)$ and $P_G^\perp$ given by $\Phi_1(P_G \oplus b) = b$, for any $b \in P_G^\perp$. All global sections of the first fiber bundle are given by

$$f(P_G \oplus b) = g(b) \oplus b,$$

for any continuous map $g : P_G^\perp \to P_G$. In fact, $\pi_1(f(P_G \oplus b)) = \pi_1(g(b) \oplus b) = P_G \oplus b$, for all $b \in P_G^\perp$.

The second bundle π_2 is isomorphic to the trivial bundle $(P_G^\perp \times P_G, P_G^\perp, \mathcal{Q}^\ell, P_G)$, where $\mathcal{Q}^\ell$ is the projection onto the first factor defined by:

$$\mathcal{Q}^\ell : P_G^\perp \oplus P_G \to P_G^\perp, \quad b \oplus a \mapsto b.$$

Indeed, the following diagram commutes

$$
\begin{array}{ccc}
G = P_G^\perp \oplus P_G & \xrightarrow{\ \pi_2\ } & (G/P_G, \sim_\ell) \\
\downarrow id & & \downarrow \Phi_2 \\
P_G^\perp \oplus P_G & \xrightarrow{\ \mathcal{Q}^\ell\ } & P_G^\perp
\end{array}
$$

where Φ_2 is the isomorphism between $(G/P_G, \sim_\ell)$ and $P_G^\perp$ given by $\Phi_2(b \oplus P_G) = b$, for any $b \in P_G^\perp$. All global sections of the second fiber bundle are given by

$$f(b \oplus P_G) = b \oplus g(b),$$

for any continuous map $g : P_G^\perp \to P_G$. Note that $\pi_2(f(b \oplus P_G)) = \pi_2(b \oplus g(b)) = b \oplus P_G$, for all $b \in P_G^\perp$.

In the third and fourth bundles we will consider the sections obtained from the quotient spaces $(G/P_G^\perp, \sim_r)$ and $(G/P_G^\perp, \sim_\ell)$.

In the third case if we consider for any $a \in P_G$ fixed the map $\tau_a^{(1)}$ defined by

$$
\begin{aligned}
\tau_a^{(1)} : (G/P_G, \sim_r) &\to G \\
[b]_r &\mapsto a \oplus b
\end{aligned}
\tag{39}
$$

with $b \in P_G^\perp$, which means that $\tau_a^{(1)}(G/P_G, \sim_r) = a \oplus P_G^\perp$ is a left coset in $(G/P_G^\perp, \sim_\ell)$, we obtain a global section. In fact, $\pi_3(\tau_a^{(1)}([b]_r)) = \pi_3(a \oplus b) = P_G \oplus b = [b]_r$, for any $b \in P_G^\perp$. This means that the left cosets of $(G/P_G^\perp, \sim_\ell)$ are global sections for $(G/P_G, \sim_r)$.

However, if we consider the map $\tau_a^{(2)}$ defined for any $a \in P_G \backslash \{0\}$ by

$$
\begin{aligned}
\tau_a^{(2)} : (G/P_G, \sim_r) &\to G \\
[b]_r &\mapsto b \oplus a
\end{aligned}
\tag{40}
$$

for $b \in P_G^\perp$, which means that $\tau_a^{(2)}(G/P_G, \sim_r) = P_G^\perp \oplus a$ is a right coset in $(G/P_G^\perp, \sim_r)$, then by (38) we have

$$\pi_3(\tau_a^{(2)}([b]_r)) = \pi_3(b \oplus a) = P_G \oplus \mu_2(b,a)b = [\mu_2(b,a)b]_r$$

with $\mu_2(b,a)$ given by (27) with the order of a and b being changed. Depending on the properties of the mapping μ_2 we can have global or local sections. If for $a \in P_G$ fixed we have $\{\mu_2(b,a)b : b \in P_G^\perp\} = P_G^\perp$ then $\tau_a^{(2)}$ is a global section while if $\{\mu_2(b,a)b : b \in P_G^\perp\} \subsetneq P_G^\perp$ then $\tau_a^{(2)}$ is only a local section for the fiber bundle defined by π_3. The case $a = 0$ gives a global section since

$$\pi_3(\tau_0^{(2)}([b]_r)) = \pi_3(b \oplus 0) = \pi_3(0 \oplus b) = [b]_r, \quad \text{for any } b \in P_G^\perp. \tag{41}$$

The conclusions obtained for the sections $\tau_a^{(1)}$ and $\tau_a^{(2)}$ are valid for the fiber bundles associated to the projection mappings π_1 and π_3.

In the fourth case we consider the sections $\tau_a^{(3)}$ and $\tau_a^{(4)}$ defined for any $a \in P_G$ by

$$
\begin{aligned}
\tau_a^{(3)} : (G/P_G, \sim_\ell) &\to G \\
[b]_\ell &\mapsto b \oplus a
\end{aligned}
\tag{42}
$$

and

$$
\begin{aligned}
\tau_a^{(4)} : (G/P_G, \sim_\ell) &\to G \\
[b]_\ell &\mapsto a \oplus b.
\end{aligned}
\tag{43}
$$

It is easy to see that $\tau_a^{(3)}$ is a global section for π_4 since

$$
\pi_4(\tau_a^{(3)}([b]_\ell)) = \pi_4(b \oplus a) = b \oplus P_G = [b]_\ell
$$

for any $b \in P_G^\perp$. Since $\tau_a^{(3)}(G/P_G, \sim_\ell) = P_G^\perp \oplus a$ is a right coset of $(G/P_G^\perp, \sim_r)$ then the right cosets of $(G/P_G^\perp, \sim_r)$ are global sections for $(G/P_G, \sim_\ell)$.

Concerning $\tau_a^{(4)}$ we have

$$
\pi_4(\tau_a^{(4)}([b]_\ell)) = \pi_4(a \oplus b) = \mu_1(a,b)b \oplus P_G = [\mu_1(a,b)b]_\ell
$$

for any $a \in P_G$, where $\mu_1(a,b)$ is given by (27). If for $a \in P_G$ fixed we have $\{\mu_1(a,b)b : b \in P_G^\perp\} = P_G^\perp$ then $\tau_a^{(4)}$ is a global section while if $\{\mu_1(a,b)b : b \in P_G^\perp\} \subsetneq P_G^\perp$ then $\tau_a^{(4)}$ is only a local section for the fiber bundle π_4. The case $a = 0$ gives a global section since

$$
\pi_4(\tau_0^{(4)}([b]_\ell)) = \pi_4(0 \oplus b) = \pi_3(b \oplus 0) = b \oplus P_G = [b]_\ell,
\tag{44}
$$

for any $b \in P_G^\perp$. The conclusions obtained for the sections $\tau_a^{(3)}$ and $\tau_a^{(4)}$ are valid for the fiber bundles associated to the projection mappings π_2 and π_4.

We summarize in the next theorem the duality relations between left and right cosets obtained from the orthogonal factorization of G by P_G and by $P_G^\perp$.

Theorem 16. *Let P_G be a subgyrogroup of $(G, \oplus)$. The following relations hold:*

1. *The cosets of $(G/P_G^\perp, \sim_\ell)$ are global sections for the quotient space $(G/P_G, \sim_r)$.*
2. *The cosets of $(G/P_G^\perp, \sim_r)$ are global sections for the quotient space $(G/P_G, \sim_r)$ if $\{\mu_2(b,a)b : b \in P_G^\perp\} = P_G^\perp$ or local sections if $\{\mu_2(b,a)b : b \in P_G^\perp\} \subsetneq P_G^\perp$.*
3. *The cosets of $(G/P_G^\perp, \sim_r)$ are global sections for the quotient space $(G/P_G, \sim_\ell)$.*
4. *The cosets of $(G/P_G^\perp, \sim_\ell)$ are global sections for the quotient space $(G/P_G, \sim_\ell)$ if $\{\mu_1(a,b)b : b \in P_G^\perp\} = P_G^\perp$ or local sections if $\{\mu_1(a,b)b : b \in P_G^\perp\} \subsetneq P_G^\perp$.*

4. Examples of real inner product gyrogroups

In this section, we show that four standard gyrogroups known in the literature are indeed real inner product gyrogroups that admit orthogonal decompositions.

4.1. Euclidean Einstein gyrogroup

The (Euclidean) *Einstein gyrogroup* [10] consists of the open unit ball in $\mathbb{R}^n$,

$$
\mathbb{B} = \{\mathbf{v} \in \mathbb{R}^n \colon \|\mathbf{v}\| < 1\},
\tag{45}
$$

endowed with *Einstein addition* $\oplus_E$ defined by

$$\mathbf{u} \oplus_E \mathbf{v} = \frac{1}{1 + \langle \mathbf{u}, \mathbf{v} \rangle} \left(\mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}} \mathbf{v} + \frac{\gamma_{\mathbf{u}}}{1 + \gamma_{\mathbf{u}}} \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{u} \right), \tag{46}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{B}$, where $\langle \, , \, \rangle$ denotes the inner product in $\mathbb{R}^n$ and $\gamma_{\mathbf{u}}$ is the *Lorentz factor* given by $\gamma_{\mathbf{u}} = \dfrac{1}{\sqrt{1 - \|\mathbf{u}\|^2}}$. Einstein addition satisfies the gamma identity

$$\gamma_{\mathbf{u} \oplus_E \mathbf{v}} = \gamma_{\mathbf{u}} \gamma_{\mathbf{v}} \left(1 + \langle \mathbf{u}, \mathbf{v} \rangle \right).$$

In view of (46), we define

$$\phi_1(\mathbf{u}, \mathbf{v}) = \frac{1 + \gamma_{\mathbf{u}} + \gamma_{\mathbf{u}} \langle \mathbf{u}, \mathbf{v} \rangle}{(1 + \langle \mathbf{u}, \mathbf{v} \rangle)(1 + \gamma_{\mathbf{u}})} \quad \text{and} \quad \phi_2(\mathbf{u}, \mathbf{v}) = \frac{1}{\gamma_{\mathbf{u}}(1 + \langle \mathbf{u}, \mathbf{v} \rangle)} \tag{47}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{B}$. The gyrogroup identity of $(\mathbb{B}, \oplus_E)$ is the zero vector $\mathbf{0}$. Further, $\ominus_E \mathbf{u} = -\mathbf{u}$ for all $\mathbf{u} \in \mathbb{B}$. Let $\mathbf{u}, \mathbf{v} \in \mathbb{B}$. By the Cauchy–Schwarz inequality,

$$-1 < -\|\mathbf{u}\|\|\mathbf{v}\| \leq \langle \mathbf{u}, \mathbf{v} \rangle \leq \|\mathbf{u}\|\|\mathbf{v}\| < 1.$$

Hence, $1 + \langle \mathbf{u}, \mathbf{v} \rangle > 0$. This implies that $\phi_1(\mathbf{u}, \mathbf{v}) > 0$ and $\phi_2(\mathbf{u}, \mathbf{v}) > 0$.

Clearly, the Einstein gyrogroup satisfies (H1). Suppose that $\mathbf{u}, \mathbf{v} \in \mathbb{B}$, $\mathbf{u} + \mathbf{v} \in \mathbb{B}$, and $\langle \mathbf{u}, \mathbf{v} \rangle = 0$. Then by (47) it is easy to see that the unique solution of the system of equations

$$\begin{cases} \phi_1(\lambda_1 \mathbf{u}, \lambda_2 \mathbf{v}) \lambda_1 \mathbf{u} = \mathbf{u} \\ \phi_2(\lambda_1 \mathbf{u}, \lambda_2 \mathbf{v}) \lambda_2 \mathbf{v} = \mathbf{v} \end{cases}$$

is given by $\lambda_1 = \lambda_1(u, v) = 1$ and $\lambda_2 = \lambda_2(u, v) = \gamma_{\mathbf{u}}$. Thus, $(x = \mathbf{u}, y = \gamma_{\mathbf{u}} \mathbf{v})$ is the solution to the system (16), where $c_1 = \mathbf{u}$ and $c_2 = \mathbf{v}$. We claim that $y \in \mathbb{B}$. To prove the claim, suppose to the contrary that $y \notin \mathbb{B}$. Hence, $\|\gamma_{\mathbf{u}} \mathbf{v}\| \geq 1$. This implies $\gamma_{\mathbf{u}} \|\mathbf{v}\| \geq 1$ and so $\dfrac{\|\mathbf{v}\|^2}{1 - \|\mathbf{u}\|^2} \geq 1$. Hence, $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \geq 1$, contrary to the assumption that $\mathbf{u} + \mathbf{v} \in \mathbb{B}$. Thus, $y \in \mathbb{B}$. Hence, the Einstein gyrogroup satisfies (H2).

Now, let P be a linear subspace of $\mathbb{R}^n$, $P_{\mathbb{B}} = P \cap \mathbb{B}$, and $P_{\mathbb{B}}^{\perp} = P^{\perp} \cap \mathbb{B}$. Then $\mathbb{R}^n = P \oplus P^{\perp}$ and $P_{\mathbb{B}}$ and $P_{\mathbb{B}}^{\perp}$ are subgyrogroups of the Einstein gyrogroup $(\mathbb{B}, \oplus_E)$. The next three theorems are consequences of Theorems 5, 8, 10, 13 obtained in Section 2.

Theorem 17. *Let $P_{\mathbb{B}}$ be a subgyrogroup of $(\mathbb{B}, \oplus_E)$. For all $\mathbf{w} \in \mathbb{B}$ such that $\mathbf{w} = \mathbf{u} + \mathbf{v}$, with $\mathbf{u} \in P_{\mathbb{B}}$ and $\mathbf{v} \in P_{\mathbb{B}}^{\perp}$, the unique orthogonal decompositions of $\mathbf{w}$ according to $\oplus_E$ are given by*

$$\mathbf{w} = \lambda_1(\mathbf{u}, \mathbf{v}) \mathbf{u} \oplus_E \lambda_2(\mathbf{u}, \mathbf{v}) \mathbf{v} \quad \text{and} \quad \mathbf{w} = \lambda_3(\mathbf{u}, \mathbf{v}) \mathbf{v} \oplus_E \lambda_4(\mathbf{u}, \mathbf{v}) \mathbf{u}, \tag{48}$$

where

$$\begin{aligned} \lambda_1(\mathbf{u}, \mathbf{v}) &= 1 \quad and \quad \lambda_2(\mathbf{u}, \mathbf{v}) = \gamma_{\mathbf{u}}, \\ \lambda_3(\mathbf{u}, \mathbf{v}) &= 1 \quad and \quad \lambda_4(\mathbf{u}, \mathbf{v}) = \lambda_2(\mathbf{v}, \mathbf{u}) = \gamma_{\mathbf{v}}. \end{aligned} \tag{49}$$

As a consequence $\mathbb{B} = P_{\mathbb{B}} \oplus_E P_{\mathbb{B}}^{\perp}$ and $\mathbb{B} = P_{\mathbb{B}}^{\perp} \oplus_E P_{\mathbb{B}}$. By (20) the orthogonal gyroprojectors of the Einstein gyrogroup $(\mathbb{B}, \oplus_E)$ onto the subgyrogroups $P_{\mathbb{B}}$ and $P_{\mathbb{B}}^{\perp}$ are given by

$$
\begin{aligned}
\mathcal{P}_E^{\ell} &: \mathbb{B} = P_{\mathbb{B}} \oplus_E P_{\mathbb{B}}^{\perp} \to P_{\mathbb{B}}, & \mathcal{P}_E^{\ell}(\mathbf{w}) &= \mathbf{u}, \\
\mathcal{Q}_E^{r} &: \mathbb{B} = P_{\mathbb{B}} \oplus_E P_{\mathbb{B}}^{\perp} \to P_{\mathbb{B}}^{\perp}, & \mathcal{Q}_E^{r}(\mathbf{w}) &= \gamma_{\mathbf{u}}\mathbf{v}, \\
\mathcal{Q}_E^{\ell} &: \mathbb{B} = P_{\mathbb{B}}^{\perp} \oplus_E P_{\mathbb{B}} \to P_{\mathbb{B}}^{\perp}, & \mathcal{Q}_E^{\ell}(\mathbf{w}) &= \mathbf{v}, \\
\mathcal{P}_E^{r} &: \mathbb{B} = P_{\mathbb{B}}^{\perp} \oplus_E P_{\mathbb{B}} \to P_{\mathbb{B}}, & \mathcal{P}_E^{r}(\mathbf{w}) &= \gamma_{\mathbf{v}}\mathbf{u}.
\end{aligned}
\tag{50}
$$

Theorem 18. *Let $P_{\mathbb{B}}$ be a subgyrogroup of $(\mathbb{B}, \oplus_E)$. Then the sets $\{b \oplus_E P_{\mathbb{B}} \colon b \in P_{\mathbb{B}}^{\perp}\}$ and $\{P_{\mathbb{B}} \oplus_E b \colon b \in P_{\mathbb{B}}^{\perp}\}$ are disjoint partitions of $\mathbb{B}$, that is,*

$$
\mathbb{B} = \bigcup_{b \in P_{\mathbb{B}}^{\perp}}^{\cdot} (b \oplus_E P_{\mathbb{B}}) \quad and \quad \mathbb{B} = \bigcup_{b \in P_{\mathbb{B}}^{\perp}}^{\cdot} (P_{\mathbb{B}} \oplus_E b).
\tag{51}
$$

Moreover, $(\mathbb{B}/P_{\mathbb{B}}, \sim_{\ell}) = \{b \oplus_E P_{\mathbb{B}} \colon b \in P_{\mathbb{B}}^{\perp}\}$ and $(\mathbb{B}/P_{\mathbb{B}}, \sim_r) = \{P_{\mathbb{B}} \oplus_E b \colon b \in P_{\mathbb{B}}^{\perp}\}$.

Theorem 19. *Let $P_{\mathbb{B}}$ be a subgyrogroup of $(\mathbb{B}, \oplus_E)$. Then the following are gyrogroup isomorphisms:*

$$
(\mathbb{B}/P_{\mathbb{B}}, \sim_{\ell}) \cong (P_{\mathbb{B}}^{\perp}, \oplus_E) \cong (\mathbb{B}/P_{\mathbb{B}}, \sim_r).
$$

In order to see a remarkable connection between gyroprojections and Euclidean projections, let us refer to a version of Einstein addition defined on an open ball of $\mathbb{R}^n$ of radius t, where t is a positive number. Set

$$
\mathbb{B}_t = \{\mathbf{v} \in \mathbb{R}^n \colon \|\mathbf{v}\| < t\}.
$$

Recall that the t-Einstein addition is defined by

$$
\mathbf{u} \oplus_{E_t} \mathbf{v} = \frac{1}{1 + \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{t^2}} \left(\mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}} \mathbf{v} + \frac{1}{t^2} \frac{\gamma_{\mathbf{u}}}{1 + \gamma_{\mathbf{u}}} \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{u} \right)
\tag{52}
$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{B}_t$ and $\gamma_{\mathbf{u}}$ is redefined by $\gamma_{\mathbf{u}} = \left(1 - \frac{\|\mathbf{u}\|^2}{t^2} \right)^{-\frac{1}{2}}$. When t is arbitrarily large (that is, $t \to +\infty$), $\mathbb{B}_t$ expands to the whole space $\mathbb{R}^n$, $\oplus_{E_t}$ reduces to ordinary vector addition of $\mathbb{R}^n$, and $\gamma_{\mathbf{u}} \to 1$. Therefore, the orthogonal gyroprojections of the t-Einstein gyrogroup reduce to ordinary projections of $\mathbb{R}^n$:

$$
\mathcal{P}(\mathbf{u} + \mathbf{v}) = \mathbf{u} \quad and \quad \mathcal{Q}(\mathbf{u} + \mathbf{v}) = \mathbf{v},
$$

for all $\mathbf{u} \in P$ and $\mathbf{v} \in P^{\perp}$.

Finally, we analyze the fiber bundles and sections of the Einstein gyrogroup arisen from the two orthogonal decompositions $\mathbb{B} = P_{\mathbb{B}} \oplus_E P_{\mathbb{B}}^{\perp}$ and $\mathbb{B} = P_{\mathbb{B}}^{\perp} \oplus_E P_{\mathbb{B}}$. First we give the relation between these two decompositions according to Theorem 6.

Theorem 20. *Let $\mathbf{u}, \mathbf{v} \in (\mathbb{B}, \oplus_E)$ such that $\langle \mathbf{u}, \mathbf{v} \rangle = 0$. Then*

$$
\mathbf{u} \oplus_E \mathbf{v} = \mu_1(\mathbf{u}, \mathbf{v})\mathbf{v} \oplus_E \mu_2(\mathbf{u}, \mathbf{v})\mathbf{u},
\tag{53}
$$

where

$$
\mu_1(\mathbf{u}, \mathbf{v}) = \frac{1}{\gamma_{\mathbf{u}}} \quad and \quad \mu_2(\mathbf{u}, \mathbf{v}) = \gamma_{\frac{1}{\gamma_{\mathbf{u}}}\mathbf{v}}.
\tag{54}
$$

Proof. By (27), (47), (49), we have

$$\mu_1(\mathbf{u}, \mathbf{v}) = \lambda_3(\phi_1(\mathbf{u},\mathbf{v})\mathbf{u}, \phi_2(\mathbf{u},\mathbf{v})\mathbf{v})\, \phi_2(\mathbf{u},\mathbf{v}) = \frac{1}{\gamma_\mathbf{u}},$$

$$\mu_2(\mathbf{u}, \mathbf{v}) = \lambda_4(\phi_1(\mathbf{u},\mathbf{v})\mathbf{u}, \phi_2(\mathbf{u},\mathbf{v})\mathbf{v})\, \phi_1(\mathbf{u},\mathbf{v}) = \lambda_4\left(\mathbf{u}, \frac{1}{\gamma_\mathbf{u}}\mathbf{v}\right) = \gamma_{\frac{1}{\gamma_\mathbf{u}}\mathbf{v}}. \qquad \square$$

Now, we analyze the sections $\tau_\mathbf{u}^{(2)}$ and $\tau_\mathbf{u}^{(4)}$ defined by (40) and (43), respectively, with the changes $a \leftrightarrow \mathbf{u} \in P_\mathbb{B}$ and $b \leftrightarrow \mathbf{v} \in P_\mathbb{B}^\perp$.

Since for any $\mathbf{u} \in P_\mathbb{B}\backslash\{\mathbf{0}\}$ it turns out that $\mu_1(\mathbf{u},\mathbf{v})\|\mathbf{v}\| = \sqrt{1-\|\mathbf{u}\|^2}\|\mathbf{v}\|$ is a strictly increasing function such that $\mu_1(\mathbf{u},\mathbf{v})\|\mathbf{v}\| \in [0, \sqrt{1-\|\mathbf{u}\|^2}[\subsetneq [0,1[$ for all $\mathbf{v} \in P_\mathbb{B}^\perp$, we conclude that $\{\mu_1(\mathbf{u},\mathbf{v})\mathbf{v} : \mathbf{v} \in P_\mathbb{B}^\perp\} \subsetneq P_\mathbb{B}^\perp$. Then for any $\mathbf{u} \in P_\mathbb{B}\backslash\{\mathbf{0}\}$, the section $\tau_\mathbf{u}^{(2)}$ is a local section for the fiber bundles π_1 and π_3 defined in (38). In the case when $\mathbf{u} = \mathbf{0}$ the section $\tau_\mathbf{0}^{(2)}$ is a global section as seen in (41).

Concerning the section $\tau_\mathbf{u}^{(4)}$, since for each $\mathbf{u} \in P_\mathbb{B}$ we have that $\mu_2(\mathbf{v},\mathbf{u})\|\mathbf{v}\| = \frac{\|\mathbf{v}\|}{\sqrt{1-(1-\|\mathbf{v}\|^2)\|\mathbf{u}\|^2}}$ is a strictly increasing function in the variable $\|\mathbf{v}\|$ such that $\mu_2(\mathbf{0},\mathbf{u})\|\mathbf{0}\| = 0$ and $\lim_{\|\mathbf{v}\|\to 1}\mu_2(\mathbf{v},\mathbf{u})\|\mathbf{v}\| = 1$, we conclude that $\{\mu_2(\mathbf{v},\mathbf{u})\mathbf{v} : \mathbf{v} \in P_\mathbb{B}^\perp\} = P_\mathbb{B}^\perp$. Hence, for any $\mathbf{u} \in P_\mathbb{B}$, it follows that $\tau_\mathbf{u}^{(4)}$ is a global section for the fiber bundles π_2 and π_4 defined in (38).

In Figure 1 we show the plots of $\mu_1(\mathbf{u},\mathbf{v})\|\mathbf{v}\|$ and $\mu_2(\mathbf{v},\mathbf{u})\|\mathbf{v}\|$, with $\|\mathbf{u}\|, \|\mathbf{v}\| < 1$, that explain the different behavior between local and global sections.
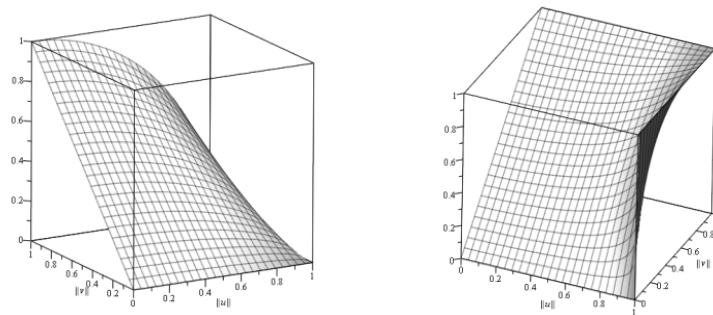

Figure 1. (a) Plot of $\mu_1(\mathbf{u},\mathbf{v})\|\mathbf{v}\|$ with $\|\mathbf{u}\|, \|\mathbf{v}\| < 1$; (b) Plot of $\mu_2(\mathbf{v},\mathbf{u})\|\mathbf{v}\|$ with $\|\mathbf{u}\|, \|\mathbf{v}\| < 1$, for the Einstein gyrogroup.

By the results obtained in Section 3 and the previous conclusions we summarize in the next theorem the duality relations between left and right cosets obtained from the orthogonal factorization of $\mathbb{B}$ by $P_\mathbb{B}$ and by $P_\mathbb{B}^\perp$, for the Einstein gyrogroup.

Theorem 21. *Let $P_\mathbb{B}$ be a subgyrogroup of $(\mathbb{B}, \oplus_E)$. The following duality relations hold:*

1. *The cosets of $(\mathbb{B}/P_\mathbb{B}^\perp, \sim_r)$ are global sections for the quotient spaces $(\mathbb{B}/P_\mathbb{B}, \sim_\ell)$ and $(\mathbb{B}/P_\mathbb{B}, \sim_r)$, and vice versa.*
2. *The cosets of $(\mathbb{B}/P_\mathbb{B}^\perp, \sim_\ell)$ are global sections for the quotient space $(\mathbb{B}/P_\mathbb{B}, \sim_r)$, and vice versa.*
3. *The cosets of $(\mathbb{B}/P_\mathbb{B}^\perp, \sim_\ell)$ are local sections for the quotient space $(\mathbb{B}/P_\mathbb{B}, \sim_\ell)$ except the identity coset $\mathbf{0} \oplus_E P_B^\perp = P_B^\perp$ that is a global section, and vice versa.*

To visualize left and right cosets we restrict now to the 3-dimensional space $\mathbb{R}^3$. In $\mathbb{R}^3$ the nontrivial linear subspaces are of dimension 1 (straight lines passing through the origin) or dimension 2 (planes passing through the origin). Let $L_{e_3} = \{(0,0,x_3) : x_3 \in \mathbb{R}\}$ be the straight line that passes through the origin and the North Pole $e_3 = (0,0,1)$ and let $D_{e_3} = \{(x_1,x_2,0) : x_1, x_2 \in \mathbb{R}\}$ be the plane that passes through the origin and is perpendicular to e_3. The restriction of L_{e_3} and D_{e_3} to $\mathbb{B}^3 = \{x \in \mathbb{R}^3 : \|x\| < 1\}$ will be denoted by $L_{e_3}^{\mathbb{B}^3}$ and $D_{e_3}^{\mathbb{B}^3}$, respectively. Then $L_{e_3}^{\mathbb{B}^3}$ and $D_{e_3}^{\mathbb{B}^3}$ are subgyrogroups of $(\mathbb{B}^3, \oplus_E)$ and

provide orthogonal gyrodecompositions of $(\mathbb{B}^3, \oplus_E)$. In Figures 2 and 3 we show the left and right cosets obtained from the orthogonal decomposition of $\mathbb{B}^3$ by $L_{e_3}^{\mathbb{B}^3}$ and $D_{e_3}^{\mathbb{B}^3}$.
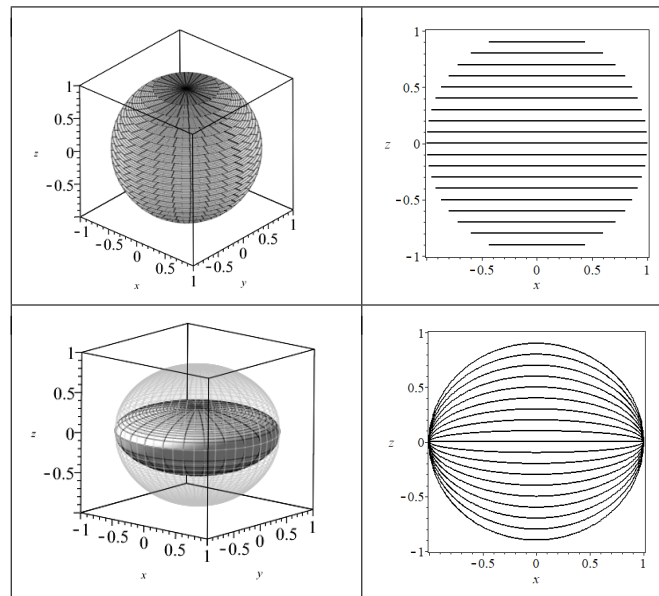

Figure 2. (a) Left cosets $(\mathbb{B}^3/D_{e_3}^{\mathbb{B}^3}, \sim_\ell) = \{\mathbf{v} \oplus_E D_{e_3}^{\mathbb{B}^3}, \ \mathbf{v} \in L_{e_3}^{\mathbb{B}^3}\}$; (b) Gyrolines in the Einstein gyrovector plane (projection in the xz-plane of left cosets); (c) Right cosets $(\mathbb{B}^3/D_{e_3}^{\mathbb{B}^3}, \sim_r) = \{D_{e_3}^{\mathbb{B}^3} \oplus_E \mathbf{u}, \ \mathbf{u} \in L_{e_3}^{\mathbb{B}^3}\}$; (d) Cogyrolines in the Einstein gyrovector plane (projection in the xz-plane of right cosets).
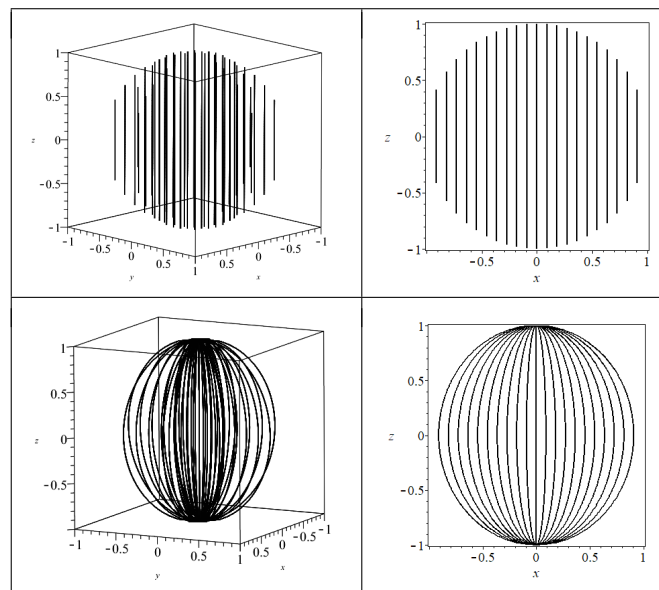

Figure 3. (a) Left cosets $(\mathbb{B}^3/L_{e_3}^{\mathbb{B}^3}, \sim_\ell) = \{\mathbf{v} \oplus_E L_{e_3}^{\mathbb{B}^3}, \ \mathbf{v} \in D_{e_3}^{\mathbb{B}^3}\}$; (b) Gyrolines in the Einstein gyrovector plane (projection in the xz-plane of left cosets); (c) Right cosets $(\mathbb{B}^3/L_{e_3}^{\mathbb{B}^3}, \sim_r) = \{L_{e_3}^{\mathbb{B}^3} \oplus_E \mathbf{v}, \ \mathbf{v} \in D_{e_3}^{\mathbb{B}^3}\}$; (d) Cogyrolines in the Einstein gyrovector plane (projection in the xz-plane of right cosets).

For higher dimensions ($n > 3$) we can have cosets with higher codimension, which are surfaces of revolution obtained from the action of the automorphism group of $(P_{\mathbb{B}}, \oplus, \otimes)$ on the gyrolines and cogyrolines shown in Figures 2 and 3. Thus, all the information of these surfaces is encoded in the projection on an adequate plane.

4.2. Euclidean Möbius gyrogroup

The (Euclidean) *Möbius gyrogroup* [11] consists of the open unit ball $\mathbb{B}$ in $\mathbb{R}^n$ endowed with *Möbius addition* $\oplus_M$ defined by

$$\mathbf{u} \oplus_M \mathbf{v} = \frac{(1 + 2\langle \mathbf{u}, \mathbf{v} \rangle + \|\mathbf{v}\|^2)\mathbf{u} + (1 - \|\mathbf{u}\|^2)\mathbf{v}}{1 + 2\langle \mathbf{u}, \mathbf{v} \rangle + \|\mathbf{u}\|^2\|\mathbf{v}\|^2} \tag{55}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{B}$. The gyrogroup identity of $(\mathbb{B}, \oplus_M)$ is the zero vector $\mathbf{0}$, and $\ominus_M \mathbf{u} = -\mathbf{u}$ for all $\mathbf{u} \in \mathbb{B}$. A generalization of the Möius addition into the ball of any real inner product space was obtained in [12]. Möbius addition satisfies the gamma identity

$$\gamma_{\mathbf{u} \oplus_M \mathbf{v}} = \gamma_{\mathbf{u}} \gamma_{\mathbf{v}} \sqrt{1 + \langle \mathbf{u}, \mathbf{v} \rangle + \|\mathbf{u}\|^2\|\mathbf{v}\|^2}.$$

In view of (55), we define

$$\phi_1(\mathbf{u}, \mathbf{v}) = \frac{1 + 2\langle \mathbf{u}, \mathbf{v} \rangle + \|\mathbf{v}\|^2}{1 + 2\langle \mathbf{u}, \mathbf{v} \rangle + \|\mathbf{u}\|^2\|\mathbf{v}\|^2} \quad \text{and} \quad \phi_2(\mathbf{u}, \mathbf{v}) = \frac{1 - \|\mathbf{u}\|^2}{1 + 2\langle \mathbf{u}, \mathbf{v} \rangle + \|\mathbf{u}\|^2\|\mathbf{v}\|^2} \tag{56}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{B}$. Let $\mathbf{u}, \mathbf{v} \in \mathbb{B}$. Using the Cauchy–Schwarz inequality, we have

$$1 + 2\langle \mathbf{u}, \mathbf{v} \rangle + \|\mathbf{v}\|^2 \geq 1 - 2\|\mathbf{u}\|\|\mathbf{v}\| + \|\mathbf{v}\|^2 = (\|\mathbf{v}\| - \|\mathbf{u}\|)^2 + (1 - \|\mathbf{u}\|^2) \geq 1 - \|\mathbf{u}\|^2 > 0.$$

Hence, $\phi_1(\mathbf{u}, \mathbf{v}) > 0$. It is clear that $\phi_2(\mathbf{u}, \mathbf{v}) > 0$.

Clearly, the Möbius gyrogroup satisfies (H1). Suppose that $\mathbf{u}, \mathbf{v} \in \mathbb{B}$, $\mathbf{u} + \mathbf{v} \in \mathbb{B}$, and $\langle \mathbf{u}, \mathbf{v} \rangle = 0$. In the case when $\mathbf{v} = \mathbf{0}$, set $\lambda_1 = 1$ and $\lambda_2 = \dfrac{1}{1 - \|\mathbf{u}\|^2}$. In the case when $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{u} = \mathbf{0}$, set $\lambda_1 = \dfrac{1}{1 + \|\mathbf{v}\|^2}$ and $\lambda_2 = 1$. In both cases, direct computations show that

$$\phi_1(\lambda_1 \mathbf{u}, \lambda_2 \mathbf{v})\lambda_1 \mathbf{u} = \mathbf{u} \quad \text{and} \quad \phi_2(\lambda_1 \mathbf{u}, \lambda_2 \mathbf{v})\lambda_2 \mathbf{v} = \mathbf{v}.$$

Further, $(\lambda_1 \mathbf{u}, \lambda_2 \mathbf{v}) \in \mathbb{B} \times \mathbb{B}$. Therefore, we may assume that $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{v} \neq \mathbf{0}$. Set

$$\lambda_1 = \frac{2}{1 + \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + \sqrt{(1 - (\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2))^2 + 4\|\mathbf{v}\|^2}} \tag{57}$$

and

$$\lambda_2 = \frac{2}{1 - (\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2) + \sqrt{(1 - (\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2))^2 + 4\|\mathbf{v}\|^2}}. \tag{58}$$

An elaborate computation (see [7] for details) shows that λ_1 and λ_2 satisfy the following equalities:

$$\frac{\lambda_1(1 + \lambda_2^2\|\mathbf{v}\|^2)}{1 + \lambda_1^2\lambda_2^2\|\mathbf{u}\|^2\|\mathbf{v}\|^2} = 1 \quad \text{and} \quad \frac{\lambda_2(1 - \lambda_1^2\|\mathbf{u}\|^2)}{1 + \lambda_1^2\lambda_2^2\|\mathbf{u}\|^2\|\mathbf{v}\|^2} = 1 \tag{59}$$

and that $\lambda_1 \mathbf{u}, \lambda_2 \mathbf{v} \in \mathbb{B}$. Hence, $(x = \lambda_1 \mathbf{u}, y = \lambda_2 \mathbf{v}) \in \mathbb{B} \times \mathbb{B}$ is a solution of the system of equations:

$$\begin{cases} \phi_1(x, y)x = \mathbf{u} \\ \phi_2(x, y)y = \mathbf{v} \end{cases}.$$

Next, we prove that the solution is unique. Suppose that $(\mathbf{x}', \mathbf{y}') \in \mathbb{B} \times \mathbb{B}$ is also a solution. Hence, $\phi_1(\mathbf{x}', \mathbf{y}')\mathbf{x}' = \mathbf{u}$ and $\phi_2(\mathbf{x}', \mathbf{y}')\mathbf{y}' = \mathbf{v}$. Note that

$$0 = \langle \mathbf{u}, \mathbf{v} \rangle = \langle \phi_1(\mathbf{x}', \mathbf{y}')\mathbf{x}', \phi_2(\mathbf{x}', \mathbf{y}')\mathbf{y}' \rangle = \phi_1(\mathbf{x}', \mathbf{y}')\phi_2(\mathbf{x}', \mathbf{y}')\langle \mathbf{x}', \mathbf{y}' \rangle,$$

which implies $\langle \mathbf{x}', \mathbf{y}' \rangle = 0$ since $\phi_1(\mathbf{x}', \mathbf{y}') \neq 0$ and $\phi_2(\mathbf{x}', \mathbf{y}') \neq 0$. It follows that

$$\phi_1(\mathbf{x}', \mathbf{y}') = \frac{1 + \|\mathbf{y}'\|^2}{1 + \|\mathbf{x}'\|^2 \|\mathbf{y}'\|^2} \quad \text{and} \quad \phi_2(\mathbf{x}', \mathbf{y}') = \frac{1 - \|\mathbf{x}'\|^2}{1 + \|\mathbf{x}'\|^2 \|\mathbf{y}'\|^2}. \tag{60}$$

Further, $\phi_1(\mathbf{x}', \mathbf{y}')\|\mathbf{x}'\| = \|\mathbf{u}\|$ and $\phi_2(\mathbf{x}', \mathbf{y}')\|\mathbf{y}'\| = \|\mathbf{v}\|$. If $\mathbf{v} = \mathbf{0}$, we have $\mathbf{y}' = \mathbf{0}$ since $\phi_2(\mathbf{x}', \mathbf{y}') \neq 0$. This implies $\phi_1(\mathbf{x}', \mathbf{y}') = 1$ and so $\mathbf{x}' = \mathbf{u}$. In this case, $(\mathbf{x}', \mathbf{y}') = (\mathbf{u}, \mathbf{0}) = (\lambda_1 \mathbf{u}, \lambda_2 \mathbf{v})$. If $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{u} = \mathbf{0}$, we have $\mathbf{x}' = \mathbf{0}$. Hence $\phi_2(\mathbf{x}', \mathbf{y}') = 1$ and so $\mathbf{y}' = \mathbf{v}$. In this case, $(\mathbf{x}', \mathbf{y}') = (\mathbf{0}, \mathbf{v}) = (\lambda_1 \mathbf{u}, \lambda_2 \mathbf{v})$. Therefore, we may assume that $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{v} \neq \mathbf{0}$. Set

$$\mu_1 = \frac{1}{\phi_1(\mathbf{x}', \mathbf{y}')} \quad \text{and} \quad \mu_2 = \frac{1}{\phi_2(\mathbf{x}', \mathbf{y}')}. \tag{61}$$

Then $\mu_1, \mu_2 > 0$ and $\mathbf{x}' = \mu_1 \mathbf{u}$ and $\mathbf{y}' = \mu_2 \mathbf{v}$. Direct computation shows that $1 + \mu_2 \|\mathbf{v}\|^2 = \dfrac{1}{\mu_1}$. Furthermore, $\phi_1(\mathbf{x}', \mathbf{y}')\mathbf{x}' = \mathbf{u}$, $\mathbf{x}' = \mu_1 \mathbf{u}$, and $\mathbf{y}' = \mu_2 \mathbf{v}$ together imply $\phi_1(\mu_1 \mathbf{u}, \mu_2 \mathbf{v})\mu_1 \mathbf{u} = \mathbf{u}$. This in turn implies $\phi_1(\mu_1 \mathbf{u}, \mu_2 \mathbf{v})\mu_1 = 1$ since $\mathbf{u} \neq \mathbf{0}$. Hence, $\dfrac{\mu_1(1 + \mu_2^2 \|\mathbf{v}\|^2)}{1 + \mu_1^2 \mu_2^2 \|\mathbf{u}\|^2 \|\mathbf{v}\|^2} = 1$ and we obtain that $\|\mathbf{v}\|^2 \mu_2^2 + (1 - (\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2))\mu_2 - 1 = 0$. It follows that μ_2 is a solution to the quadratic equation

$$\|\mathbf{v}\|^2 x^2 + (1 - (\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2))x - 1 = 0.$$

If $\mu_2 = \dfrac{2}{1 - \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2 - \sqrt{(1 - (\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2))^2 + 4\|\mathbf{v}\|^2}}$ were true, we would have

$$\mu_2 = -\frac{2}{-1 + \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + \sqrt{(1 - (\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2))^2 + 4\|\mathbf{v}\|^2}} < 0,$$

which is a contradiction. Hence, it must be the case that

$$\mu_2 = \frac{2}{1 - \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + \sqrt{(1 - (\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2))^2 + 4\|\mathbf{v}\|^2}} = \lambda_2.$$

This in turn implies $\mu_1 = \dfrac{1}{1 + \mu_2 \|\mathbf{v}\|^2} = \dfrac{1}{1 + \lambda_2 \|\mathbf{v}\|^2} = \lambda_1$. Hence, $(\mathbf{x}', \mathbf{y}') = (\lambda_1 \mathbf{u}, \lambda_2 \mathbf{v})$. This proves uniqueness of the solution. Therefore, the Möbius gyrogroup satisfies (H2).

Now, let P be a linear subspace of $\mathbb{R}^n$ and $P_\mathbb{B} = P \cap \mathbb{B}$ and $P_\mathbb{B}^\perp = P^\perp \cap \mathbb{B}$. The next three theorems are consequences of Theorems 5, 8, 10, 13 obtained in Section 2.

Theorem 22. *Let $P_\mathbb{B}$ be a subgyrogroup of $(\mathbb{B}, \oplus_M)$. For all $\mathbf{w} \in \mathbb{B}$ such that $\mathbf{w} = \mathbf{u} + \mathbf{v}$, with $\mathbf{u} \in P_\mathbb{B}$ and $\mathbf{v} \in P_\mathbb{B}^\perp$, the unique orthogonal decompositions of $\mathbf{w}$ according to $\oplus_M$ are given by*

$$\mathbf{w} = \lambda_1(\mathbf{u}, \mathbf{v})\mathbf{u} \oplus_M \lambda_2(\mathbf{u}, \mathbf{v})\mathbf{v} \quad \text{and} \quad \mathbf{w} = \lambda_3(\mathbf{u}, \mathbf{v})\mathbf{v} \oplus_M \lambda_4(\mathbf{u}, \mathbf{v})\mathbf{u}, \tag{62}$$

where

$$\lambda_1(\mathbf{u}, \mathbf{v}) = \frac{2}{1 + \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + \sqrt{(1 - (\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2))^2 + 4\|\mathbf{v}\|^2}} \,,$$

$$\lambda_2(\mathbf{u}, \mathbf{v}) = \frac{2}{1 - \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2 + \sqrt{(1 - (\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2))^2 + 4\|\mathbf{v}\|^2}} \,,$$

$$\lambda_3(\mathbf{u}, \mathbf{v}) = \frac{2}{1 + \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + \sqrt{(1 - (\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2))^2 + 4\|\mathbf{u}\|^2}} \,,$$

$$\lambda_4(\mathbf{u}, \mathbf{v}) = \frac{2}{1 - \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2 + \sqrt{(1 - (\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2))^2 + 4\|\mathbf{u}\|^2}} \,.$$

(63)

We remark that the special cases $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$ are included in the general solution (63). As a consequence of Theorem 22, for each $\mathbf{w} \in \mathbb{B}$, if $\mathbf{w} = \mathbf{u} + \mathbf{v}$, with $\mathbf{u} \in P$ and $\mathbf{v} \in P^\perp$ is the unique decomposition of $\mathbf{w}$, according to the orthogonal direct sum decomposition $\mathbb{R}^n = P \oplus P^\perp$, then the orthogonal gyroprojections of the Möbius gyrogroup $(\mathbb{B}, \oplus_M)$ onto the subgyrogroups $P_{\mathbb{B}}$ and $P_{\mathbb{B}}^\perp$ are given by

$$\mathcal{P}_M^\ell : \mathbb{B} = P_{\mathbb{B}} \oplus_M P_{\mathbb{B}}^\perp \to P_{\mathbb{B}}, \quad \mathcal{P}_M^\ell(\mathbf{w}) = \frac{2}{1 + \|\mathbf{w}\|^2 + \sqrt{(1 - \|\mathbf{w}\|^2)^2 + 4\|\mathbf{v}\|^2}} \, \mathbf{u},$$

$$\mathcal{Q}_M^r : \mathbb{B} = P_{\mathbb{B}} \oplus_M P_{\mathbb{B}}^\perp \to P_{\mathbb{B}}^\perp, \quad \mathcal{Q}_M^r(\mathbf{w}) = \frac{2}{1 - \|\mathbf{w}\|^2 + \sqrt{(1 - \|\mathbf{w}\|^2)^2 + 4\|\mathbf{v}\|^2}} \, \mathbf{v},$$

$$\mathcal{Q}_M^\ell : \mathbb{B} = P_{\mathbb{B}}^\perp \oplus_M P_{\mathbb{B}} \to P_{\mathbb{B}}^\perp, \quad \mathcal{Q}_M^\ell(\mathbf{w}) = \frac{2}{1 + \|\mathbf{w}\|^2 + \sqrt{(1 - \|\mathbf{w}\|^2)^2 + 4\|\mathbf{u}\|^2}} \, \mathbf{v},$$

$$\mathcal{P}_M^r : \mathbb{B} = P_{\mathbb{B}}^\perp \oplus_M P_{\mathbb{B}} \to P_{\mathbb{B}}, \quad \mathcal{P}_M^r(\mathbf{w}) = \frac{2}{1 - \|\mathbf{w}\|^2 + \sqrt{(1 - \|\mathbf{w}\|^2)^2 + 4\|\mathbf{u}\|^2}} \, \mathbf{u} \,.$$

(64)

Theorem 23. *Let $P_{\mathbb{B}}$ be a subgyrogroup of $(\mathbb{B}, \oplus_M)$. Then the sets $\{b \oplus_M P_{\mathbb{B}} \colon b \in P_{\mathbb{B}}^\perp\}$ and $\{P_{\mathbb{B}} \oplus_M b \colon b \in P_{\mathbb{B}}^\perp\}$ are disjoint partitions of $\mathbb{B}$, that is,*

$$\mathbb{B} = \bigcup_{b \in P_{\mathbb{B}}^\perp}^{\cdot} (b \oplus_M P_{\mathbb{B}}) \quad and \quad \mathbb{B} = \bigcup_{b \in P_{\mathbb{B}}^\perp}^{\cdot} (P_{\mathbb{B}} \oplus_M b). \tag{65}$$

Moreover, $(\mathbb{B}/P_{\mathbb{B}}, \sim_\ell) = \{b \oplus_M P_{\mathbb{B}} \colon b \in P_{\mathbb{B}}^\perp\}$ and $(\mathbb{B}/P_{\mathbb{B}}, \sim_r) = \{P_{\mathbb{B}} \oplus_M b \colon b \in P_{\mathbb{B}}^\perp\}$.

Theorem 24. *Let $P_{\mathbb{B}}$ be a subgyrogroup of $(\mathbb{B}, \oplus_M)$. Then the following are gyrogroup isomorphisms:*

$$(\mathbb{B}/P_{\mathbb{B}}, \sim_\ell) \cong (P_{\mathbb{B}}^\perp, \oplus_M) \cong (\mathbb{B}/P_{\mathbb{B}}, \sim_r).$$

For each positive number *t*, recall that the *t-Möbius addition* is defined by

$$\mathbf{u} \oplus_{M_t} \mathbf{v} = \frac{(1 + \frac{2}{t^2}\langle \mathbf{u}, \mathbf{v}\rangle + \frac{1}{t^2}\|\mathbf{v}\|^2)\mathbf{u} + (1 - \frac{1}{t^2}\|\mathbf{u}\|^2)\mathbf{v}}{1 + \frac{2}{t^2}\langle \mathbf{u}, \mathbf{v}\rangle + \frac{1}{t^4}\|\mathbf{u}\|^2\|\mathbf{v}\|^2} \tag{66}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{B}_t$. Let P be a subspace of $\mathbb{R}^n$. For each $\mathbf{w} \in \mathbb{B}_t$, if $\mathbf{w} = \mathbf{u} + \mathbf{v}$ is the unique expression of $\mathbf{w}$, according to the orthogonal direct sum decomposition $\mathbb{R}^n = P \oplus P^\perp$, with $\mathbf{u} \in P$ and $\mathbf{v} \in P^\perp$, then

the orthogonal gyroprojections of the t-Möbius gyrogroup $\left(\text{after rescaling } u \mapsto \frac{u}{t}, v \mapsto \frac{v}{t}, w \mapsto \frac{w}{t}\right)$ are given by

$$
\begin{aligned}
\mathcal{P}^{\ell}_{M_t}(\mathbf{w}) &= \frac{2}{1 + \frac{\|\mathbf{w}\|^2}{t^2} + \sqrt{\left(1 - \frac{\|\mathbf{w}\|^2}{t^2}\right)^2 + 4\frac{\|\mathbf{v}\|^2}{t^2}}} \, \mathbf{u}, \\[2mm]
\mathcal{P}^{r}_{M_t}(\mathbf{w}) &= \frac{2}{1 - \frac{\|\mathbf{w}\|^2}{t^2} + \sqrt{\left(1 - \frac{\|\mathbf{w}\|^2}{t^2}\right)^2 + 4\frac{\|\mathbf{u}\|^2}{t^2}}} \, \mathbf{u}, \\[2mm]
\mathcal{Q}^{\ell}_{M_t}(\mathbf{w}) &= \frac{2}{1 + \frac{\|\mathbf{w}\|^2}{t^2} + \sqrt{\left(1 - \frac{\|\mathbf{w}\|^2}{t^2}\right)^2 + 4\frac{\|\mathbf{u}\|^2}{t^2}}} \, \mathbf{v}, \\[2mm]
\mathcal{Q}^{r}_{M_t}(\mathbf{w}) &= \frac{2}{1 - \frac{\|\mathbf{w}\|^2}{t^2} + \sqrt{\left(1 - \frac{\|\mathbf{w}\|^2}{t^2}\right)^2 + 4\frac{\|\mathbf{v}\|^2}{t^2}}} \, \mathbf{v}.
\end{aligned}
\tag{67}
$$

When $t \to +\infty$, $\oplus_{M_t}$ reduces to ordinary vector addition of $\mathbb{R}^n$, $\mathcal{P}^{\ell}_{M_t}$ and $\mathcal{P}^{r}_{M_t}$ reduce to the ordinary projection $\mathcal{P}(\mathbf{u} + \mathbf{v}) = \mathbf{u}$, and $\mathcal{Q}^{\ell}_{M_t}$ and $\mathcal{Q}^{r}_{M_t}$ reduce to the ordinary projection $\mathcal{Q}(\mathbf{u} + \mathbf{v}) = \mathbf{v}$. Further, the equivalence relation (29) reflects the Euclidean left coset relation:

$$
\mathbf{u} \sim_{\ell} \mathbf{v} \qquad \text{if and only if} \qquad -\mathbf{v} + \mathbf{u} \in P
$$

and the equivalence relation (31) reflects the Euclidean right coset relation:

$$
\mathbf{u} \sim_r \mathbf{v} \qquad \text{if and only if} \qquad \mathbf{u} - \mathbf{v} \in P.
$$

Now, we analyze the fiber bundles and the sections of the Möbius gyrogroup arisen from the two orthogonal decompositions $\mathbb{B} = P_{\mathbb{B}} \oplus_M P_{\mathbb{B}}^{\perp}$ and $\mathbb{B} = P_{\mathbb{B}}^{\perp} \oplus_M P_{\mathbb{B}}$. First we give the relation between these two decompositions according to Theorem 6.

Theorem 25. *Let* $\mathbf{u}, \mathbf{v} \in (\mathbb{B}, \oplus_M)$ *such that* $\langle \mathbf{u}, \mathbf{v} \rangle = 0$. *Then*

$$
\mathbf{u} \oplus_M \mathbf{v} = \mu_1(\mathbf{u}, \mathbf{v})\mathbf{v} \oplus_M \mu_2(\mathbf{u}, \mathbf{v})\mathbf{u},
\tag{68}
$$

where

$$
\begin{aligned}
\mu_1(\mathbf{u}, \mathbf{v}) &= \frac{2(1 - \|\mathbf{u}\|^2)}{(1 + \|\mathbf{u}\|^2)(1 + \|\mathbf{v}\|^2) + \sqrt{(1 - \|\mathbf{u}\|)^2(1 - \|\mathbf{v}\|)^2 + 4(1 + \|\mathbf{v}\|^2)\|u\|^2}}, \\[2mm]
\mu_2(\mathbf{u}, \mathbf{v}) &= \frac{2(1 + \|\mathbf{v}\|^2)}{(1 - \|\mathbf{u}\|^2)(1 - \|\mathbf{v}\|^2) + \sqrt{(1 - \|\mathbf{u}\|)^2(1 - \|\mathbf{v}\|)^2 + 4(1 + \|\mathbf{v}\|^2)\|u\|^2}}.
\end{aligned}
\tag{69}
$$

Proof. By (27), (56), (49), and straightforward computations we have:

$$\mu_1(\mathbf{u}, \mathbf{v}) = \lambda_3(\phi_1(\mathbf{u}, \mathbf{v})\mathbf{u}, \phi_2(\mathbf{u}, \mathbf{v})\mathbf{v})\,\phi_2(\mathbf{u}, \mathbf{v})$$

$$= \lambda_3\left(\frac{1 + \|\mathbf{v}\|^2}{1 + \|\mathbf{u}\|^2\|\mathbf{v}\|^2}\mathbf{u}, \frac{1 - \|\mathbf{u}\|^2}{1 + \|\mathbf{u}\|^2\|\mathbf{v}\|^2}\mathbf{v}\right)\frac{1 - \|\mathbf{u}\|^2}{1 + \|\mathbf{u}\|^2\|\mathbf{v}\|^2}$$

$$= \frac{2(1 - \|\mathbf{u}\|^2)}{(1 + \|\mathbf{u}\|^2)(1 + \|\mathbf{v}\|^2) + \sqrt{(1 - \|\mathbf{u}\|)^2(1 - \|\mathbf{v}\|)^2 + 4(1 + \|\mathbf{v}\|^2)\|\mathbf{u}\|^2}},$$

$$\mu_2(\mathbf{u}, \mathbf{v}) = \lambda_4(\phi_1(\mathbf{u}, \mathbf{v})\mathbf{u}, \phi_2(\mathbf{u}, \mathbf{v})\mathbf{v})\,\phi_1(\mathbf{u}, \mathbf{v})$$

$$= \lambda_4\left(\frac{1 + \|\mathbf{v}\|^2}{1 + \|\mathbf{u}\|^2\|\mathbf{v}\|^2}\mathbf{u}, \frac{1 - \|\mathbf{u}\|^2}{1 + \|\mathbf{u}\|^2\|\mathbf{v}\|^2}\mathbf{v}\right)\frac{1 + \|\mathbf{v}\|^2}{1 + \|\mathbf{u}\|^2\|\mathbf{v}\|^2}$$

$$= \frac{2(1 + \|\mathbf{v}\|^2)}{(1 - \|\mathbf{u}\|^2)(1 - \|\mathbf{v}\|^2) + \sqrt{(1 - \|\mathbf{u}\|)^2(1 - \|\mathbf{v}\|)^2 + 4(1 + \|\mathbf{v}\|^2)\|\mathbf{u}\|^2}}. \qquad \square$$

Now, we analyze the properties of $\mu_1(\mathbf{u}, \mathbf{v})\|\mathbf{v}\|$ and $\mu_2(\mathbf{v}, \mathbf{u})\|\mathbf{v}\|$. For any $\mathbf{u} \in P_{\mathbb{B}}\backslash\{\mathbf{0}\}$ we have that $\mu_1(\mathbf{u}, \mathbf{v})\|\mathbf{v}\|$ is a strictly increasing function such that $\mu_1(\mathbf{u}, \mathbf{0})\|\mathbf{0}\| = 0$ and $\lim_{\|\mathbf{v}\|\to 1}\mu_1(\mathbf{u}, \mathbf{v})\|\mathbf{v}\| = \frac{1 - \|\mathbf{u}\|^2}{1 + \sqrt{2}\|\mathbf{u}\| + \|\mathbf{u}\|^2}$. Therefore, for any $\mathbf{u} \in P_{\mathbb{B}}\backslash\{\mathbf{0}\}$ it follows that $\mu_1(\mathbf{u}, \mathbf{v})\|\mathbf{v}\| \in \left[0, \frac{1 - \|\mathbf{u}\|^2}{1 + \sqrt{2}\|\mathbf{u}\| + \|\mathbf{u}\|^2}\right[\subsetneq [0, 1[$ for all $\mathbf{v} \in P_{\mathbb{B}}^{\perp}$ and, consequently, $\{\mu_1(\mathbf{u}, \mathbf{v})\mathbf{v} : \mathbf{v} \in P_{\mathbb{B}}^{\perp}\} \subsetneq P_{\mathbb{B}}^{\perp}$. Then for any $\mathbf{u} \in P_{\mathbb{B}}\backslash\{\mathbf{0}\}$, $\tau_{\mathbf{u}}^{(2)}$ is a local section for the fiber bundles π_1 and π_3 defined in (38). In the case when $\mathbf{u} = \mathbf{0}$ the section $\tau_{\mathbf{0}}^{(2)}$ is a global section as seen in (41).

Concerning $\mu_2(\mathbf{v}, \mathbf{u})\|\mathbf{v}\|$ it turns out that for each $\mathbf{u} \in P_{\mathbb{B}}$, the function $\mu_2(\mathbf{v}, \mathbf{u})\|\mathbf{v}\|$ is strictly increasing such that $\mu_2(\mathbf{0}, \mathbf{u})\|\mathbf{0}\| = 0$ and $\lim_{\|\mathbf{v}\|\to 1}\mu_2(\mathbf{v}, \mathbf{u})\|\mathbf{v}\| = 1$. Therefore, we conclude that $\{\mu_2(\mathbf{v}, \mathbf{u})\mathbf{v} : \mathbf{v} \in P_{\mathbb{B}}^{\perp}\} = P_{\mathbb{B}}^{\perp}$ and, consequently, for any $\mathbf{u} \in P_{\mathbb{B}}$ the section $\tau_{\mathbf{u}}^{(4)}$ is a global section for the fiber bundles π_2 and π_4 defined in (38).

In Figure 4 we show the plots of $\mu_1(\mathbf{u}, \mathbf{v})\|\mathbf{v}\|$ and $\mu_2(\mathbf{u}, \mathbf{v})\|\mathbf{v}\|$, with $\|\mathbf{u}\|, \|\mathbf{v}\| < 1$, for the case of the Möbius gyrogroup.
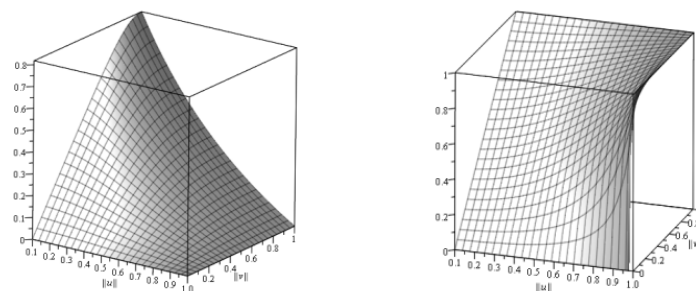

Figure 4. (a) Plot of $\mu_1(\mathbf{u}, \mathbf{v})\|\mathbf{v}\|$ with $\|\mathbf{u}\|, \|\mathbf{v}\| < 1$; (b) Plot of $\mu_2(\mathbf{v}, \mathbf{u})\|\mathbf{v}\|$ with $\|\mathbf{u}\|, \|\mathbf{v}\| < 1$, for the Möbius gyrogroup.

Theorem 21 about the duality relations between left and right cosets obtained from the orthogonal factorization of $\mathbb{B}$ by $P_{\mathbb{B}}$ and by $P_{\mathbb{B}}^{\perp}$, holds also in the case of the Möbius gyrogroup. Analogously to the Einstein case we restrict now to the 3-dimensional space and we show in Figures 5 and 6 the left and right cosets obtained from the orthogonal decompositions of the gyrogroup $(\mathbb{B}^3, \oplus_M)$ by the subgyrogroups $L_{e_3}^{\mathbb{B}^3}$ and $D_{e_3}^{\mathbb{B}^3}$.
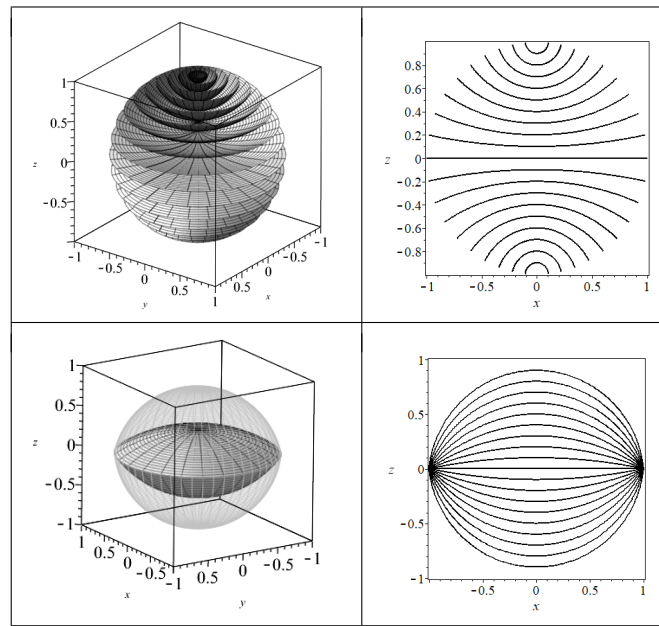
Figure 5. (a) Left cosets $(\mathbb{B}^3/D_{e_3}^{\mathbb{B}^3}, \sim_\ell) = \{\mathbf{u} \oplus_M D_{e_3}^{\mathbb{B}^3}, \mathbf{u} \in L_{e_3}^{\mathbb{B}^3}\}$; (b) Gyrolines in the Möbius gyrovector plane (projection in the xz-plane of left cosets); (c) Right cosets $(\mathbb{B}^3/D_{e_3}^{\mathbb{B}^3}, \sim_r) = \{D_{e_3}^{\mathbb{B}^3} \oplus_M \mathbf{u}, \mathbf{u} \in L_{e_3}^{\mathbb{B}^3}\}$; (d) Cogyrolines in the Möbius gyrovector plane (projection in the xz-plane of right cosets).
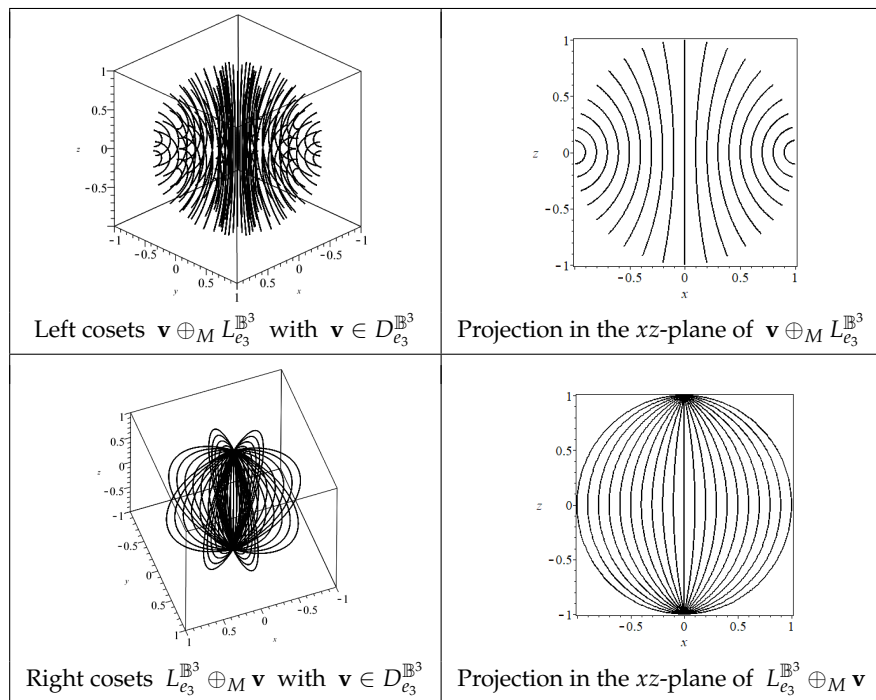

Left cosets $\mathbf{v} \oplus_M L_{e_3}^{\mathbb{B}^3}$ with $\mathbf{v} \in D_{e_3}^{\mathbb{B}^3}$ | Projection in the xz-plane of $\mathbf{v} \oplus_M L_{e_3}^{\mathbb{B}^3}$

Right cosets $L_{e_3}^{\mathbb{B}^3} \oplus_M \mathbf{v}$ with $\mathbf{v} \in D_{e_3}^{\mathbb{B}^3}$ | Projection in the xz-plane of $L_{e_3}^{\mathbb{B}^3} \oplus_M \mathbf{v}$

Figure 6. (a) Left cosets $(\mathbb{B}^3/L_{e_3}^{\mathbb{B}^3}, \sim_\ell) = \{\mathbf{v} \oplus_M L_{e_3}^{\mathbb{B}^3}, \mathbf{v} \in D_{e_3}^{\mathbb{B}^3}\}$; (b) Gyrolines in the Möbius gyrovector plane (projection in the xz-plane of left cosets); (c) Right cosets $(\mathbb{B}^3/L_{e_3}^{\mathbb{B}^3}, \sim_r) = \{L_{e_3}^{\mathbb{B}^3} \oplus_M \mathbf{v}, \mathbf{v} \in D_{e_3}^{\mathbb{B}^3}\}$; (d) Cogyrolines in the Möbius gyrovector plane (projection in the xz-plane of right cosets).

4.3. Proper velocity gyrogroup

The (Euclidean) *Proper Velocity (PV) gyrogroup* [3] consists of the n-dimensional Euclidean space $\mathbb{R}^n$ endowed with *PV addition* $\oplus_U$ defined by

$$\mathbf{u} \oplus_U \mathbf{v} = \left(\frac{\beta_{\mathbf{u}}}{1 + \beta_{\mathbf{u}}} \langle \mathbf{u}, \mathbf{v} \rangle + \frac{1}{\beta_{\mathbf{v}}} \right) \mathbf{u} + \mathbf{v} \tag{70}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, where $\beta_{\mathbf{u}}$ is the *beta factor* given by $\beta_{\mathbf{u}} = \dfrac{1}{\sqrt{1 + \|\mathbf{u}\|^2}}$ for all $\mathbf{u} \in \mathbb{R}^n$. In view of (70), we define

$$\phi_1(\mathbf{u}, \mathbf{v}) = \frac{\beta_{\mathbf{u}}}{1 + \beta_{\mathbf{u}}} \langle \mathbf{u}, \mathbf{v} \rangle + \frac{1}{\beta_{\mathbf{v}}} \qquad \text{and} \qquad \phi_2(\mathbf{u}, \mathbf{v}) = 1 \tag{71}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. The gyrogroup identity of $(\mathbb{R}^n, \oplus_U)$ is the zero vector $\mathbf{0}$. Further, $\ominus_U \mathbf{u} = -\mathbf{u}$ for all $\mathbf{u} \in \mathbb{R}^n$. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. The PV addition satisfies the beta identity given by

$$\frac{1}{\beta_{\mathbf{u} \oplus_U \mathbf{v}}} = \frac{1}{\beta_{\mathbf{u}} \beta_{\mathbf{v}}} + \langle \mathbf{u}, \mathbf{v} \rangle = \frac{1 + \beta_{\mathbf{u}} \beta_{\mathbf{v}} \langle \mathbf{u}, \mathbf{v} \rangle}{\beta_{\mathbf{u}} \beta_{\mathbf{v}}}.$$

Hence, $1 + \beta_{\mathbf{u}} \beta_{\mathbf{v}} \langle \mathbf{u}, \mathbf{v} \rangle > 0$ since $\dfrac{\beta_{\mathbf{u}} \beta_{\mathbf{v}}}{\beta_{\mathbf{u} \oplus_U \mathbf{v}}} > 0$. This implies

$$\phi_1(\mathbf{u}, \mathbf{v}) = \frac{\beta_{\mathbf{u}} + (1 + \beta_{\mathbf{u}} \beta_{\mathbf{v}} \langle \mathbf{u}, \mathbf{v} \rangle)}{(1 + \beta_{\mathbf{u}}) \beta_{\mathbf{v}}} > 0.$$

It is clear that the PV gyrogroup satisfies (H1). Suppose that $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and that $\langle \mathbf{u}, \mathbf{v} \rangle = 0$. It is easy to see that the unique solution of the system of equations

$$\begin{cases} \phi_1(\lambda_1 \mathbf{u}, \lambda_2 \mathbf{v}) \lambda_1 \mathbf{u} = \mathbf{u} \\ \phi_2(\lambda_1 \mathbf{u}, \lambda_2 \mathbf{v}) \lambda_2 \mathbf{v} = \mathbf{v} \end{cases}$$

is given by $\lambda_1 = \beta_{\mathbf{v}}$ and $\lambda_2 = 1$. Thus, the PV gyrogroup satisfies (H2) and we obtain the next three theorems immediately as consequences of Theorems 5, 8, 10, 13 in Section 2.

Theorem 26. *Let P be a subgyrogroup of $(\mathbb{R}^n, \oplus_U)$. For all $\mathbf{w} \in \mathbb{R}^n$ such that $\mathbf{w} = \mathbf{u} + \mathbf{v}$, with $\mathbf{u} \in P$ and $\mathbf{v} \in P^\perp$, the unique orthogonal decompositions of $\mathbf{w}$ according to $\oplus_U$ are given by*

$$\mathbf{w} = \lambda_1(\mathbf{u}, \mathbf{v}) \mathbf{u} \oplus_U \lambda_2(\mathbf{u}, \mathbf{v}) \mathbf{v} \quad \text{and} \quad \mathbf{w} = \lambda_3(\mathbf{u}, \mathbf{v}) \mathbf{v} \oplus_U \lambda_4(\mathbf{u}, \mathbf{v}) \mathbf{u}, \tag{72}$$

where

$$\begin{aligned} \lambda_1(\mathbf{u}, \mathbf{v}) &= \beta_{\mathbf{v}} \quad \text{and} \quad \lambda_2(\mathbf{u}, \mathbf{v}) = 1, \\ \lambda_3(\mathbf{u}, \mathbf{v}) &= \lambda_1(\mathbf{v}, \mathbf{u}) = \beta_{\mathbf{u}} \quad \text{and} \quad \lambda_4(\mathbf{u}, \mathbf{v}) = 1. \end{aligned} \tag{73}$$

For each $\mathbf{w} \in \mathbb{R}^n$, if $\mathbf{w} = \mathbf{u} + \mathbf{v}$ is the unique expression of $\mathbf{w}$, according to the orthogonal direct sum decomposition $\mathbb{R}^n = P \oplus P^\perp$, then the orthogonal gyroprojections of the PV gyrogroup $(\mathbb{R}^n, \oplus_U)$ onto the subgyrogroups P and $P^\perp$ are given by

$$\begin{aligned} \mathcal{P}_U^\ell &: \mathbb{R}^n = P \oplus_U P^\perp \to P, & \mathcal{P}_U^\ell(\mathbf{w}) &= \beta_{\mathbf{v}} \mathbf{u}, \\ \mathcal{Q}_U^r &: \mathbb{R}^n = P \oplus_U P^\perp \to P^\perp, & \mathcal{Q}_U^r(\mathbf{w}) &= \mathbf{v}, \\ \mathcal{Q}_U^\ell &: \mathbb{R}^n = P^\perp \oplus_U P \to P^\perp, & \mathcal{Q}_U^\ell(\mathbf{w}) &= \beta_{\mathbf{u}} \mathbf{v}, \\ \mathcal{P}_U^r &: \mathbb{R}^n = P^\perp \oplus_U P \to P, & \mathcal{P}_U^r(\mathbf{w}) &= \mathbf{u}. \end{aligned} \tag{74}$$

Theorem 27. *Let P be a linear subspace of $\mathbb{R}^n$. Then the sets $\{b \oplus_U P \colon b \in P^\perp\}$ and $\{P \oplus_U b \colon b \in P^\perp\}$ are disjoint partitions of $\mathbb{R}^n$, that is,*

$$\mathbb{R}^n = \overset{\cdot}{\bigcup_{b \in P^\perp}} (b \oplus_U P) \quad and \quad \mathbb{R}^n = \overset{\cdot}{\bigcup_{b \in P^\perp}} (P \oplus_U b). \tag{75}$$

Moreover, $(\mathbb{R}^n/P, \sim_\ell) = \{b \oplus_U P \colon b \in P^\perp\}$ and $(\mathbb{R}^n/P, \sim_r) = \{P \oplus_U b \colon b \in P^\perp\}$.

Theorem 28. *Let P be a linear subspace of $\mathbb{R}^n$. Then the following are gyrogroup isomorphisms:*

$$(\mathbb{R}^n/P, \sim_\ell) \cong (P^\perp, \oplus_U) \cong (\mathbb{R}^n/P, \sim_r).$$

Recall that t-PV addition is defined by

$$\mathbf{u} \oplus_{U_t} \mathbf{v} = \left(\frac{\beta_{\mathbf{u}}}{1 + \beta_{\mathbf{u}}} \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{t^2} + \frac{1}{\beta_{\mathbf{v}}} \right) \mathbf{u} + \mathbf{v}, \tag{76}$$

for all $\mathbf{u}, \mathbf{v}, \in \mathbb{R}^n$, where $\beta_{\mathbf{u}}$ is redefined by $\beta_{\mathbf{u}} = \dfrac{1}{\sqrt{1 + \frac{\|\mathbf{u}\|^2}{t^2}}}$. When t is arbitrarily large (that is, $t \to +\infty$), $\oplus_{U_t}$ reduces to ordinary vector addition of $\mathbb{R}^n$ and $\beta_{\mathbf{w}} \to 1$. Therefore, the orthogonal gyroprojections of the t-PV gyrogroup reduce to ordinary projection of $\mathbb{R}^n$:

$$P(\mathbf{u} + \mathbf{v}) = \mathbf{u} \qquad and \qquad Q(\mathbf{u} + \mathbf{v}) = \mathbf{v}$$

for all $\mathbf{u} \in P, \mathbf{v} \in P^\perp$.

Now, we analyze the fiber bundles and sections of the PV gyrogroup arisen from the two orthogonal decompositions $\mathbb{R}^n = P \oplus_U P^\perp$ and $\mathbb{R}^n = P^\perp \oplus_U P$. First we give the relation between these two decompositions according to Theorem 6.

Theorem 29. *Let $\mathbf{u}, \mathbf{v} \in (\mathbb{R}^n, \oplus_U)$ such that $\langle \mathbf{u}, \mathbf{v} \rangle = 0$. Then*

$$\mathbf{u} \oplus_U \mathbf{v} = \mu_1(\mathbf{u}, \mathbf{v}) \mathbf{v} \oplus_U \mu_2(\mathbf{u}, \mathbf{v}) \mathbf{u}, \tag{77}$$

where

$$\mu_1(\mathbf{u}, \mathbf{v}) = \beta_{\frac{1}{\beta_{\mathbf{v}}} \mathbf{u}} \quad and \quad \mu_2(\mathbf{u}, \mathbf{v}) = \frac{1}{\beta_{\mathbf{v}}}. \tag{78}$$

Proof. By (27), (71), (73), we have

$$\mu_1(\mathbf{u}, \mathbf{v}) = \lambda_3(\phi_1(\mathbf{u}, \mathbf{v}) \mathbf{u}, \phi_2(\mathbf{u}, \mathbf{v}) \mathbf{v}) \phi_2(\mathbf{u}, \mathbf{v}) = \lambda_3\left(\frac{1}{\beta_{\mathbf{v}}} \mathbf{u}, \mathbf{v} \right) = \beta_{\frac{1}{\beta_{\mathbf{v}}} \mathbf{u}},$$

$$\mu_2(\mathbf{u}, \mathbf{v}) = \lambda_4(\phi_1(\mathbf{u}, \mathbf{v}) \mathbf{u}, \phi_2(\mathbf{u}, \mathbf{v}) \mathbf{v}) \phi_1(\mathbf{u}, \mathbf{v}) = \frac{1}{\beta_{\mathbf{v}}}. \qquad \square$$

In Figure 7 we show the plots of $\mu_1(\mathbf{u}, \mathbf{v}) \|\mathbf{v}\|$ and $\mu_2(\mathbf{v}, \mathbf{u}) \|\mathbf{v}\|$, with $\|\mathbf{u}\|, \|\mathbf{v}\| < 1$, for the PV gyrogroup.
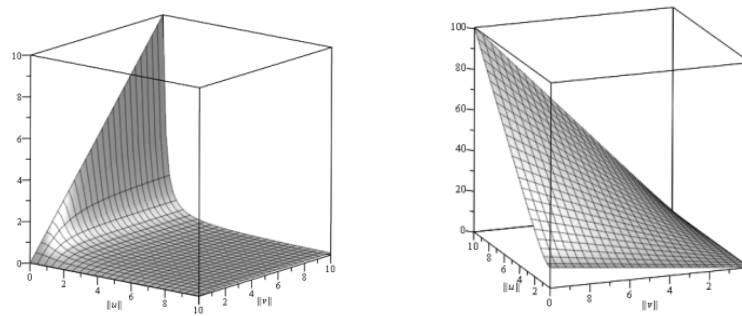
Figure 7. (a) Plot of $\mu_1(\mathbf{u}, \mathbf{v})\|\mathbf{v}\|$ with $\|u\|, \|v\| \in [0, 10]$; (b) Plot of $\mu_2(\mathbf{v}, \mathbf{u})\|\mathbf{v}\|$ with $\|u\|, \|v\| \in [0, 10]$, for the PV gyrogroup.

Since for any $\mathbf{u} \in P \setminus \{\mathbf{0}\}$, $\mu_1(\mathbf{u}, \mathbf{v})\|\mathbf{v}\| = \frac{\|\mathbf{v}\|}{\sqrt{1 + (1 + \|\mathbf{v}\|^2)\|\mathbf{u}\|^2}}$ is a strictly increasing function such that $\mu_1(\mathbf{u}, 0)\|\mathbf{0}\| = 0$ and $\lim_{\|\mathbf{v}\| \to +\infty} \mu_1(\mathbf{u}, \mathbf{v})\|\mathbf{v}\| = \frac{1}{\|u\|}$, we conclude that $\mu_1(\mathbf{u}, \mathbf{v})\|\mathbf{v}\| \in \left[0, \frac{1}{\|\mathbf{u}\|} \right[\subsetneq [0, +\infty[$, for all $\mathbf{v} \in P^{\perp}$. Therefore, for any $\mathbf{u} \in P \setminus \{\mathbf{0}\}$, we have that $\{\mu_1(\mathbf{u}, \mathbf{v})\mathbf{v} : \mathbf{v} \in P^{\perp}\} \subsetneq P^{\perp}$ and, consequently, $\tau_{\mathbf{u}}^{(2)}$ is a local section for the fiber bundles π_1 and π_3 defined in (38). In the case when $\mathbf{u} = \mathbf{0}$ the section $\tau_{\mathbf{0}}^{(2)}$ is a global section as seen in (41).

Concerning the section $\tau_{\mathbf{u}}^{(4)}$, since for each $\mathbf{u} \in P$ we have that $\mu_2(\mathbf{v}, \mathbf{u})\|\mathbf{v}\| = \sqrt{1 + \|\mathbf{v}\|^2}\|\mathbf{v}\|$ is a strictly increasing function in the variable $\|\mathbf{v}\|$ such that $\mu_2(\mathbf{0}, \mathbf{u})\|\mathbf{0}\| = 0$ and $\lim_{\|\mathbf{v}\| \to +\infty} \mu_2(\mathbf{v}, \mathbf{u})\|\mathbf{v}\| = +\infty$, we conclude that $\{\mu_2(\mathbf{v}, \mathbf{u})\mathbf{v} : \mathbf{v} \in P^{\perp}\} = P^{\perp}$. Hence, for any $\mathbf{u} \in P$, it follows that $\tau_{\mathbf{u}}^{(4)}$ is a global section for the fiber bundles π_2 and π_4 defined in (38).

We can finally conclude that the same duality relations as in Theorem 21 happens for the case of the PV gyrogroup.

Theorem 30. *Let P be a subgyrogroup of $(\mathbb{R}^n, \oplus_U)$. The following duality relations hold:*

1. *The cosets of $(\mathbb{R}^n / P^{\perp}, \sim_r)$ are global sections for the quotient spaces $(\mathbb{R}^n / P, \sim_\ell)$ and $(\mathbb{R}^n / P, \sim_r)$, and vice versa.*
2. *The cosets of $(\mathbb{R}^n / P^{\perp}, \sim_\ell)$ are global sections for the quotient space $(\mathbb{R}^n / P, \sim_r)$, and vice versa.*
3. *The cosets of $(\mathbb{R}^n / P^{\perp}, \sim_\ell)$ are local sections for the quotient space $(\mathbb{R}^n / P, \sim_\ell)$ except the identity coset $\mathbf{0} \oplus_U P^{\perp} = P^{\perp}$ that is a global section, and vice versa.*

To visualize the left and right cosets we restrict now to the 3-dimensional space $\mathbb{R}^3$ and we show in Figures 8 and 9 the cosets obtained from the orthogonal decompositions of the gyrogroup $(\mathbb{R}^3, \oplus_V)$ by the subgyrogroups L_{e_3} and D_{e_3}.
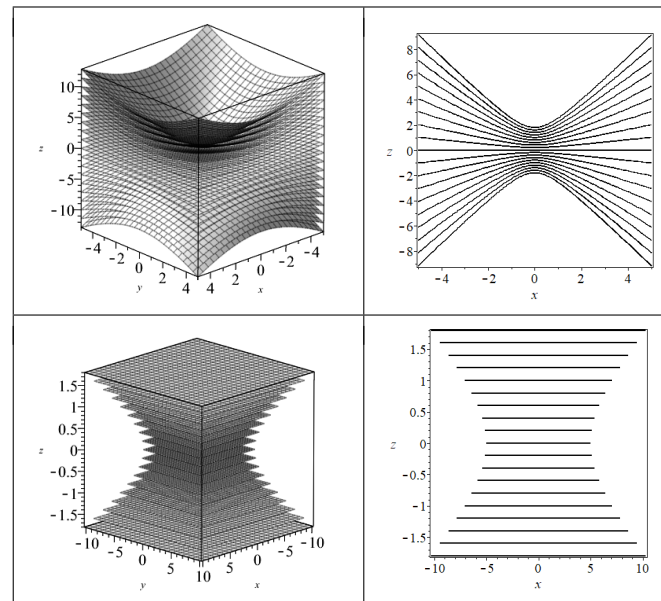
Figure 8. (a) Left cosets $(\mathbb{R}^3/D_{e_3}, \sim_\ell) = \{\mathbf{u} \oplus_U D_{e_3}, \; \mathbf{u} \in L_{e_3}\}$; (b) Gyrolines in the PV gyrovector plane (projection in the xz-plane of left cosets); (c) Right cosets $(\mathbb{R}^3/D_{e_3}, \sim_r) = \{D_{e_3} \oplus_U \mathbf{u}, \; \mathbf{u} \in L_{e_3}\}$; (d) Coyrolines in the PV gyrovector plane (projection in the xz-plane of right cosets).
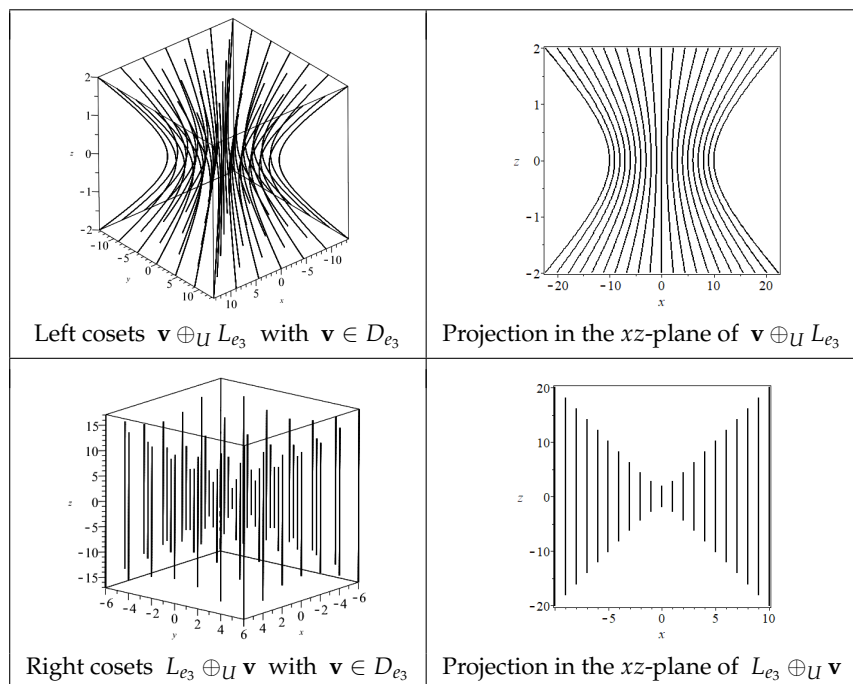

Left cosets $\mathbf{v} \oplus_U L_{e_3}$ with $\mathbf{v} \in D_{e_3}$

Projection in the xz-plane of $\mathbf{v} \oplus_U L_{e_3}$

Right cosets $L_{e_3} \oplus_U \mathbf{v}$ with $\mathbf{v} \in D_{e_3}$

Projection in the xz-plane of $L_{e_3} \oplus_U \mathbf{v}$

Figure 9. (a) Left cosets $(\mathbb{R}^3/L_{e_3}, \sim_\ell) = \{\mathbf{v} \oplus_U L_{e_3}, \; \mathbf{v} \in D_{e_3}\}$; (b) Gyrolines in the PV gyrovector plane (projection in the xz-plane of left cosets); (c) Right cosets $(\mathbb{R}^3/L_{e_3}, \sim_r) = \{L_{e_3} \oplus_U \mathbf{v}, \; \mathbf{v} \in D_{e_3}\}$; (d) Coyrolines in the PV gyrovector plane (projection in the xz-plane of right cosets).

4.4. The SL(2, ℂ) general addition and Chen's gyrogroup

Einstein, Möbius, and PV gyrogroups are three different realizations of hyperbolic geometry associated to the Lorentz group. In [13] it was shown that the algebra of the group $SL(2, \mathbb{C})$ naturally leads to the notion of gyrogroups and gyrovector spaces for dealing with the Lorentz group and its hyperbolic geometry. A general addition for real inner product gyrogroups extended from the group $SL(2, \mathbb{C})$ is given in the next definition.

Definition 7 (SL$(2, \mathbb{C})$ general addition,[13]). *Let $V = (V, +, \cdot)$ be a real inner product space, and let*

$$\phi : V \to \mathbb{R}^+, \qquad \phi : \mathbf{v} \mapsto \phi_{\mathbf{v}}$$

be a continuous map of a subset $V_t \subseteq V$ into the positive real ray $\mathbb{R}^+$, taking $\mathbf{v} \in V_t$ into $\phi(\mathbf{v}) = \phi_{\mathbf{v}} \in \mathbb{R}^+$, normalized by the condition $\phi(\mathbf{0}) = 0$. We assume that $\mathbf{0} \in V_t$, and that V_t is the identity-connected component of the preimage $\{\mathbf{v} \in V : \phi(\mathbf{v}) \in \mathbb{R}^+\}$ of $\mathbb{R}^+$ under ϕ. Furthermore, let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous, bijective self-map of $\mathbb{R}^+$, satisfying the condition

$$f(\phi_{\mathbf{v}}) = \frac{\|\mathbf{v}\|}{t}$$

for an arbitrarily fixed positive constant t, such that

$$\frac{\sinh \phi_{\mathbf{v}}}{f(\phi_{\mathbf{v}})} \neq 0$$

for all $\mathbf{v} \in V_t$. Then the SL$(2, \mathbb{C})$ general binary operation $\oplus$ in V_t is given by the equation

$$\mathbf{u} \oplus \mathbf{v} = \frac{1}{\frac{\sinh(\phi_{\mathbf{u} \oplus \mathbf{v}})}{f(\phi_{\mathbf{u} \oplus \mathbf{v}})}} (C_{\mathbf{u}, \mathbf{v}} \mathbf{u} + C_{\mathbf{v}} \mathbf{v}) \tag{79}$$

for all $\mathbf{u}, \mathbf{v} \in V_t$, where

$$
\begin{aligned}
C_{\mathbf{u}, \mathbf{v}} &= \frac{\sinh \phi_{\mathbf{u}}}{f(\phi_{\mathbf{u}})} \cosh \phi_{\mathbf{v}} + \left(\frac{\sinh \phi_{\mathbf{u}}}{f(\phi_{\mathbf{u}})} \right)^2 \frac{\sinh \phi_{\mathbf{v}}}{f(\phi_{\mathbf{v}})} \frac{1}{1 + \cosh \phi_{\mathbf{u}}} \frac{\mathbf{u} \cdot \mathbf{v}}{t^2} \\
C_{\mathbf{v}} &= \frac{\sinh \phi_{\mathbf{v}}}{f(\phi_{\mathbf{v}})}
\end{aligned}
$$

and

$$\cosh(\phi_{\mathbf{u} \oplus \mathbf{v}}) = \cosh \phi_{\mathbf{u}} \cosh \phi_{\mathbf{v}} + \frac{\sinh \phi_{\mathbf{u}}}{f(\phi_{\mathbf{u}})} \frac{\sinh \phi_{\mathbf{v}}}{f(\phi_{\mathbf{v}})} \frac{\mathbf{u} \cdot \mathbf{v}}{t^2}.$$

Furthermore, the scalar multiplication in V_t is given by the equation

$$r \otimes \mathbf{v} = t f \left(r f^{-1} \left(\frac{\|\mathbf{v}\|}{t} \right) \right) \frac{\mathbf{v}}{\|\mathbf{v}\|}, \quad \mathbf{v} \neq \mathbf{0}$$

and $r \otimes \mathbf{0} = \mathbf{0}$, for all $\mathbf{v} \in V_t$ and $r \in \mathbb{R}$.

The groupoid $(V_t, \oplus)$ is a gyrocommutative gyrogroup. Moreover, it admits scalar multiplication, $\otimes$, turning itself into a gyrovector space $(V_t, \oplus, \otimes)$. Specifying the function f in various ways results in various binary operations in various subsets V_t of V. For instance, t-Einstein addition is recovered from the SL$(2, \mathbb{C})$ addition by choosing the function $f(r) = \tanh r$, $r \in \mathbb{R}^+$, the t-Möbius addition is recovered from the choice $f(r) = \tanh \left(\frac{r}{2} \right)$, and the t-PV addition is recovered from the choice $f(r) = \sinh r$ (see [13] for a detailed discussion).

In view of (79), we define

$$\phi_1(\mathbf{u}, \mathbf{v}) = \frac{1}{\frac{\sinh(\phi_{\mathbf{u} \oplus \mathbf{v}})}{f(\phi_{\mathbf{u} \oplus \mathbf{v}})}} C_{\mathbf{u}, \mathbf{v}} \quad \text{and} \qquad \phi_2(\mathbf{u}, \mathbf{v}) = \frac{1}{\frac{\sinh(\phi_{\mathbf{u} \oplus \mathbf{v}})}{f(\phi_{\mathbf{u} \oplus \mathbf{v}})}} C_{\mathbf{v}} \tag{80}$$

for all $\mathbf{u}, \mathbf{v} \in V_t$. It is immediate to see that $\phi_1(\mathbf{u}, \mathbf{0}) = 1$ and $\phi_2(\mathbf{0}, \mathbf{v}) = 1$ for all $\mathbf{u}, \mathbf{v} \in V_t$. Let us now see the conditions that ϕ_1 and ϕ_2 have to satisfy such that hypothesis (H1) and (H2) hold. By construction, hypothesis (H1) naturally holds. For the hypothesis (H2), consider $\mathbf{u}, \mathbf{v} \in V_t$, with

$\mathbf{u} + \mathbf{v} \in V_t$ and $\mathbf{u} \cdot \mathbf{v} = 0$. Since $\lambda_1 \mathbf{u} \perp \lambda_2 \mathbf{v}$, for all $\lambda_1, \lambda_2 \in \mathbb{R}$ it follows from (80) that the system of equations

$$\phi_1(\lambda_1 \mathbf{u}, \lambda_2 \mathbf{v})\lambda_1 \mathbf{u} = \mathbf{u} \quad \text{and} \quad \phi_2(\lambda_1 \mathbf{u}, \lambda_2 \mathbf{v})\lambda_2 \mathbf{v} = \mathbf{v}$$

is equivalent to

$$\frac{f(\phi_{\lambda_1 \mathbf{u} \oplus \lambda_2 \mathbf{v}})}{\sinh(\phi_{\lambda_1 \mathbf{u} \oplus \lambda_2 \mathbf{v}})} \frac{\sinh(\phi_{\lambda_1 \mathbf{u}})}{f(\phi_{\lambda_1 \mathbf{u}})} \cosh(\phi_{\lambda_2 \mathbf{v}})\lambda_1 = 1 \quad \text{and} \quad \frac{f(\phi_{\lambda_1 \mathbf{u} \oplus \lambda_2 \mathbf{v}})}{\sinh(\phi_{\lambda_1 \mathbf{u} \oplus \lambda_2 \mathbf{v}})} \frac{\sinh(\phi_{\lambda_2 \mathbf{v}})}{f(\phi_{\lambda_2 \mathbf{v}})}\lambda_2 = 1.$$

The solution of the previous system depends on the choice of the functions f and ϕ. In the cases of Einstein, Möbius, and PV gyrogroups we already proved that the system has a unique solution. In [1] it was obtained another example of a real inner product gyrogroup considering the function $f(r) = \sinh \frac{r}{2}, r \in \mathbb{R}^+$ in the general $SL(2, \mathbb{C})$ addition (79). This addition was called *Chen's addition* and is given by

$$\mathbf{u} \oplus_C \mathbf{v} = \frac{\left(\frac{2}{\beta_{\mathbf{v}}} - \beta_{\mathbf{v}} + 2\beta_{\mathbf{u}}\frac{\mathbf{u} \cdot \mathbf{v}}{t^2}\right)\mathbf{u} + \beta_{\mathbf{u}}\mathbf{v}}{\sqrt{\beta_{\mathbf{u}}^2\beta_{\mathbf{v}}^2 - (\beta_{\mathbf{u}}^2 + \beta_{\mathbf{v}}^2) + 2\left(1 + \beta_{\mathbf{u}}\beta_{\mathbf{v}}\frac{\mathbf{u} \cdot \mathbf{v}}{t^2}\right)}} \tag{81}$$

for all $\mathbf{u}, \mathbf{v} \in V_t = V$, where $\beta_{\mathbf{v}} = \dfrac{1}{\sqrt{1 + \frac{\|\mathbf{v}\|^2}{t^2}}}, \mathbf{v} \in V$. Chen's addition satisfies the following beta identity:

$$\frac{1}{\beta_{\mathbf{u} \oplus_C \mathbf{v}}} = \frac{1}{\beta_{\mathbf{u}}}\frac{1}{\beta_{\mathbf{v}}}\sqrt{\beta_{\mathbf{u}}^2\beta_{\mathbf{v}}^2 - (\beta_{\mathbf{u}}^2 + \beta_{\mathbf{v}}^2) + 2\left(1 + \beta_{\mathbf{u}}\beta_{\mathbf{v}}\frac{\mathbf{u} \cdot \mathbf{v}}{t^2}\right)}.$$

The scalar multiplication associated with Chen's addition is given by

$$r \otimes_C \mathbf{v} = t \sinh\left(r \sinh^{-1}\frac{\|\mathbf{v}\|}{t}\right)\frac{\mathbf{v}}{\|\mathbf{v}\|}$$

for $\mathbf{v} \neq \mathbf{0}$ and $r \otimes_C \mathbf{0} = \mathbf{0}$. Thus, $(V, \oplus_C, \otimes_C)$ becomes a gyrovector space. The gyrogroup identity of $(V, \oplus_C)$ is the zero vector $\mathbf{0}$. Further, $\ominus_C \mathbf{u} = -\mathbf{u}$ for all $\mathbf{u} \in V$. In view of (81), we define

$$\phi_1(\mathbf{u}, \mathbf{v}) = \frac{\frac{2}{\beta_{\mathbf{v}}} - \beta_{\mathbf{v}} + 2\beta_{\mathbf{u}}\frac{\mathbf{u} \cdot \mathbf{v}}{t^2}}{\sqrt{\beta_{\mathbf{u}}^2\beta_{\mathbf{v}}^2 - (\beta_{\mathbf{u}}^2 + \beta_{\mathbf{v}}^2) + 2\left(1 + \beta_{\mathbf{u}}\beta_{\mathbf{v}}\frac{\mathbf{u} \cdot \mathbf{v}}{t^2}\right)}}$$
$$\phi_2(\mathbf{u}, \mathbf{v}) = \frac{\beta_{\mathbf{u}}}{\sqrt{\beta_{\mathbf{u}}^2\beta_{\mathbf{v}}^2 - (\beta_{\mathbf{u}}^2 + \beta_{\mathbf{v}}^2) + 2\left(1 + \beta_{\mathbf{u}}\beta_{\mathbf{v}}\frac{\mathbf{u} \cdot \mathbf{v}}{t^2}\right)}} \tag{82}$$

for all $\mathbf{u}, \mathbf{v} \in V$. It is clear that Chen's gyrogroup satisfies (H1) by construction. Now, we suppose that $\mathbf{u}, \mathbf{v} \in V, \mathbf{u} + \mathbf{v} \in V$, and $\mathbf{u} \cdot \mathbf{v} = 0$. Using computer algebra we can find that the unique solution of the system of equations

$$\begin{cases} \phi_1(\lambda_1 \mathbf{u}, \lambda_2 \mathbf{v})\lambda_1 \mathbf{u} = \mathbf{u} \\ \phi_2(\lambda_1 \mathbf{u}, \lambda_2 \mathbf{v})\lambda_2 \mathbf{v} = \mathbf{v} \end{cases}$$

is given by

$$\lambda_1 = \frac{1}{\frac{\sqrt{2}\|\mathbf{u}\|}{t}}\sqrt{\frac{1 + \frac{2}{t^2}(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2)}{\sqrt{1 + 4\left(1 + \frac{\|\mathbf{u}\|^2}{t^2}\right)\frac{\|\mathbf{v}\|^2}{t^2} + 4\frac{\|\mathbf{v}\|^4}{t^4}}} - 1},$$

$$\lambda_2 = \frac{1}{\frac{\sqrt{2}\|\mathbf{v}\|}{t}} \sqrt{\sqrt{1 + 4\left(1 + \frac{\|\mathbf{u}\|^2}{t^2}\right)\frac{\|\mathbf{v}\|^2}{t^2} + 4\frac{\|\mathbf{v}\|^4}{t^4}} - 1}.$$

Now, let P be a linear subspace of V. The next three theorems are consequences of Theorems 5, 8, 10, 13 obtained in Section 2.

Theorem 31. *Let P be a subgyrogroup of $(\mathbb{R}^n, \oplus_C)$. For all $\mathbf{w} \in \mathbb{R}^n$ such that $\mathbf{w} = \mathbf{u} + \mathbf{v}$, with $\mathbf{u} \in P$ and $\mathbf{v} \in P^\perp$, the unique orthogonal decompositions of $\mathbf{w}$ according to $\oplus_C$ are given by*

$$\mathbf{w} = \lambda_1(\mathbf{u}, \mathbf{v})\mathbf{u} \oplus_C \lambda_2(\mathbf{u}, \mathbf{v})\mathbf{v} \quad and \quad \mathbf{w} = \lambda_3(\mathbf{u}, \mathbf{v})\mathbf{v} \oplus_C \lambda_4(\mathbf{u}, \mathbf{v})\mathbf{u}, \tag{83}$$

where

$$\lambda_1(\mathbf{u}, \mathbf{v}) = \frac{1}{\frac{\sqrt{2}\|\mathbf{u}\|}{t}} \sqrt{\frac{1 + \frac{2}{t^2}(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2)}{\sqrt{1 + \frac{4}{\beta_{\mathbf{u}}^2}\frac{\|\mathbf{v}\|^2}{t^2} + 4\frac{\|\mathbf{v}\|^4}{t^4}}} - 1,}$$

$$\lambda_2(\mathbf{u}, \mathbf{v}) = \frac{1}{\frac{\sqrt{2}\|\mathbf{v}\|}{t}} \sqrt{\sqrt{1 + \frac{4}{\beta_{\mathbf{u}}^2}\frac{\|\mathbf{v}\|^2}{t^2} + 4\frac{\|\mathbf{v}\|^4}{t^4}} - 1,}$$

$$\lambda_3(\mathbf{u}, \mathbf{v}) = \frac{1}{\frac{\sqrt{2}\|\mathbf{v}\|}{t}} \sqrt{\frac{1 + \frac{2}{t^2}(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2)}{\sqrt{1 + \frac{4}{\beta_{\mathbf{v}}^2}\frac{\|\mathbf{u}\|^2}{t^2} + 4\frac{\|\mathbf{u}\|^4}{t^4}}} - 1,}$$

$$\lambda_4(\mathbf{u}, \mathbf{v}) = \frac{1}{\frac{\sqrt{2}\|\mathbf{u}\|}{t}} \sqrt{\sqrt{1 + \frac{4}{\beta_{\mathbf{v}}^2}\frac{\|\mathbf{u}\|^2}{t^2} + 4\frac{\|\mathbf{u}\|^4}{t^4}} - 1.} \tag{84}$$

For each $\mathbf{w} \in V$, if $\mathbf{w} = \mathbf{u} + \mathbf{v}$, with $\mathbf{u} \in P$ and $\mathbf{v} \in P^\perp$ is the unique decomposition of $\mathbf{w}$, according to the orthogonal direct sum decomposition $V = P \oplus P^\perp$, then the orthogonal gyroprojections of Chen's gyrogroup $(V, \oplus_C)$ onto P and $P^\perp$ are given by

$$\mathcal{P}_C^\ell : V = P \oplus_C P^\perp \to P, \quad \mathcal{P}_C^\ell(\mathbf{w}) = \sqrt{\frac{1 + \frac{2}{t^2}(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2)}{\sqrt{1 + \frac{4}{\beta_{\mathbf{u}}^2}\frac{\|\mathbf{v}\|^2}{t^2} + 4\frac{\|\mathbf{v}\|^4}{t^4}}} - 1} \frac{1}{\frac{\sqrt{2}\|\mathbf{u}\|}{t}}\mathbf{u},$$

$$\mathcal{Q}_C^r : V = P \oplus_C P^\perp \to P^\perp, \quad \mathcal{Q}_C^r(\mathbf{w}) = \sqrt{\sqrt{1 + \frac{4}{\beta_{\mathbf{u}}^2}\frac{\|\mathbf{v}\|^2}{t^2} + 4\frac{\|\mathbf{v}\|^4}{t^4}} - 1} \frac{1}{\frac{\sqrt{2}\|\mathbf{v}\|}{t}}\mathbf{v},$$

$$\mathcal{Q}_C^\ell : V = P^\perp \oplus_C P \to P^\perp, \quad \mathcal{Q}_C^\ell(\mathbf{w}) = \sqrt{\frac{1 + \frac{2}{t^2}(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2)}{\sqrt{1 + \frac{4}{\beta_{\mathbf{v}}^2}\frac{\|\mathbf{u}\|^2}{t^2} + 4\frac{\|\mathbf{u}\|^4}{t^4}}} - 1} \frac{1}{\frac{\sqrt{2}\|\mathbf{v}\|}{t}}\mathbf{v}, \tag{85}$$

$$\mathcal{P}_C^r : V = P^\perp \oplus_C P \to P, \quad \mathcal{P}_C^r(\mathbf{w}) = \sqrt{\sqrt{1 + \frac{4}{\beta_{\mathbf{v}}^2}\frac{\|\mathbf{u}\|^2}{t^2} + 4\frac{\|\mathbf{u}\|^4}{t^4}} - 1} \frac{1}{\frac{\sqrt{2}\|\mathbf{u}\|}{t}}\mathbf{u}.$$

Theorem 32. *Let P be a linear subspace of V. Then the sets $\{b \oplus_C P \colon b \in P^\perp\}$ and $\{P \oplus_C b \colon b \in P^\perp\}$ are disjoint partitions of V, that is,*

$$V = \bigcup_{b \in P^\perp}^{\cdot} (b \oplus_C P) \quad and \quad V = \bigcup_{b \in P^\perp}^{\cdot} (P \oplus_C b). \tag{86}$$

Moreover, $(V/P, \sim_\ell) = \{b \oplus_C P \colon b \in P^\perp\}$ and $(V/P, \sim_r) = \{P \oplus_C b \colon b \in P^\perp\}$.

Theorem 33. *Let P be a linear subspace of V. Then the following are gyrogroup isomorphisms:*

$$(V/P, \sim_\ell) \cong (P^\perp, \oplus_C) \cong (V/P, \sim_r).$$

Finally, when $t \to +\infty$, $\oplus_C$ reduces to ordinary vector addition of V. Surprisingly, $\mathcal{P}_C^\ell$ and $\mathcal{P}_C^r$ reduce again to the ordinary projection $\mathcal{P}(\mathbf{u} + \mathbf{v}) = \mathbf{u}$, and $\mathcal{Q}_C^\ell$ and $\mathcal{Q}_C^r$ reduce to the ordinary projection $\mathcal{Q}(\mathbf{u} + \mathbf{v}) = \mathbf{v}$. Further, the equivalence relation (29) reflects the left coset relation in $(V, +)$:

$$\mathbf{u} \sim_\ell \mathbf{v} \quad \text{if and only if} \quad -\mathbf{v} + \mathbf{u} \in P$$

and the equivalence relation (31) reflects the right coset relation in $(V, +)$:

$$\mathbf{u} \sim_r \mathbf{v} \quad \text{if and only if} \quad \mathbf{u} - \mathbf{v} \in P.$$

Let us now find $\mu_1(\mathbf{u}, \mathbf{v})$ and $\mu_2(\mathbf{u}, \mathbf{v})$ for Chen's gyrogroup, according to Theorem 6.

Theorem 34. *Let $\mathbf{u}, \mathbf{v} \in (V, \oplus_C)$ such that $\mathbf{u} \cdot \mathbf{v} = 0$. Then*

$$\mathbf{u} \oplus_C \mathbf{v} = \mu_1(\mathbf{u}, \mathbf{v})\mathbf{v} \oplus_C \mu_2(\mathbf{u}, \mathbf{v})\mathbf{u}, \tag{87}$$

where

$$\mu_1(\mathbf{u}, \mathbf{v}) = \frac{1}{\frac{\sqrt{2}\|\mathbf{v}\|}{t}} \sqrt{\frac{1 + \frac{2}{t^2}\left(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\frac{\|\mathbf{u}\|^2\|\mathbf{v}\|^2}{t^2}\right)}{\sqrt{1 + \frac{4}{\beta_{\mathbf{u}}^2}\frac{\|\mathbf{u}\|^2}{t^2}\left(1 + \frac{2\|\mathbf{v}\|^2}{t^2}\right)^2}} - 1},$$

$$\mu_2(\mathbf{u}, \mathbf{v}) = \frac{1}{\frac{\sqrt{2}\|\mathbf{u}\|}{t}} \sqrt{\sqrt{1 + \frac{4}{\beta_{\mathbf{u}}^2}\frac{\|\mathbf{u}\|^2}{t^2}\left(1 + \frac{2\|\mathbf{v}\|^2}{t^2}\right)} - 1}. \tag{88}$$

The proof follows by straightforward computations using (27), (82), and (84).

In Figure 10 we show the plots of $\mu_1(\mathbf{u}, \mathbf{v})\|\mathbf{v}\|$ and $\mu_2(\mathbf{v}, \mathbf{u})\|\mathbf{v}\|$ for Chen's gyrogroup.
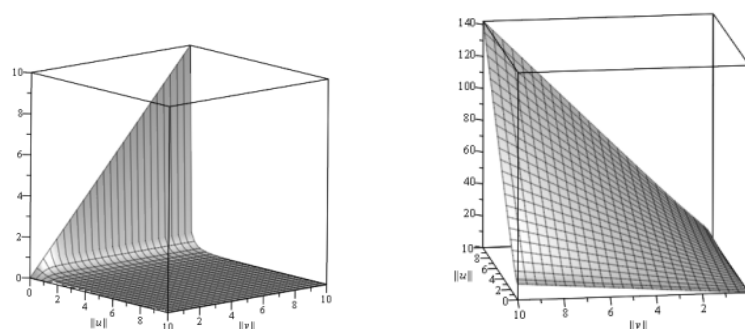

Figure 10. (a) Plot of $\mu_1(\mathbf{u}, \mathbf{v})\|\mathbf{v}\|$ with $t = 1$ and $\|u\|, \|v\| \in [0, 10]$; (b) Plot of $\mu_2(\mathbf{v}, \mathbf{u})\|\mathbf{v}\|$ with $t = 1$ and $\|u\|, \|v\| \in [0, 10]$, for Chen's gyrogroup.

Since $\lim_{\|\mathbf{v}\|\to+\infty}\mu_1(\mathbf{u},\mathbf{v})\|\mathbf{v}\| = \frac{t}{2}\sqrt{\frac{t^2+2\|\mathbf{u}\|^2}{\|\mathbf{u}\|\sqrt{t^2+\|\mathbf{u}\|^2}}-2}$ then for all $\mathbf{u}\in P\backslash\{\mathbf{0}\}$ we have that

$\mu_1(\mathbf{u},\mathbf{v})\|\mathbf{v}\| \in \left[0,\frac{t}{2}\sqrt{\frac{t^2+2\|\mathbf{u}\|^2}{\|\mathbf{u}\|\sqrt{t^2+\|\mathbf{u}\|^2}}-2}\right[\subsetneq [0,+\infty[$, for all $\mathbf{v}\in P^\perp$. This implies that $\tau_\mathbf{u}^{(2)}$ is only a

local section for the fiber bundles π_1 and π_3 defined in (38). In the case when $\mathbf{u}=\mathbf{0}$ the section $\tau_\mathbf{0}^{(2)}$ is a global section as seen in (41).

Regarding $\mu_2(\mathbf{v},\mathbf{u})\|\mathbf{v}\|$ we have that for each $\mathbf{u}\in P$ it is a strictly increasing function in the variable $\|\mathbf{v}\|$ such that $\mu_2(\mathbf{0},\mathbf{u})\|\mathbf{0}\| = 0$ and $\lim_{\|\mathbf{v}\|\to+\infty}\mu_2(\mathbf{v},\mathbf{u})\|\mathbf{v}\| = +\infty$. Therefore, we conclude that $\{\mu_2(\mathbf{v},\mathbf{u})\mathbf{v}:\mathbf{v}\in P^\perp\} = P^\perp$. Hence, for any $\mathbf{u}\in P$, it follows that $\tau_\mathbf{u}^{(4)}$ is a global section for the fiber bundles π_2 and π_4 defined in (38).

In the case of Chen's gyrogroup we have the same duality relations as in Theorem 30. To visualize the left and right cosets in this case we consider V the 3-dimensional space $\mathbb{R}^3$ and we show in Figures 11 and 12 the cosets obtained from the orthogonal decompositions of the gyrogroup $(\mathbb{R}^3,\oplus_C)$ by the subgyrogroups L_{e_3} and D_{e_3}.
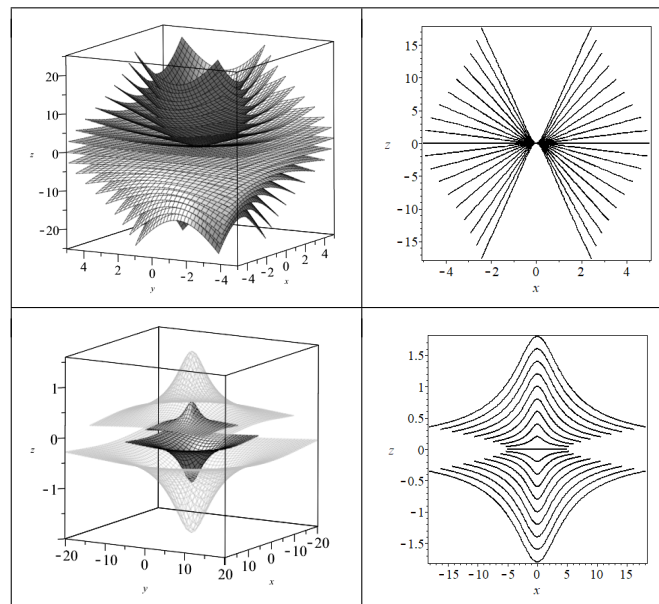

Figure 11. (a) Left cosets $(\mathbb{R}^3/D_{e_3},\sim_\ell) = \{\mathbf{u}\oplus_C D_{e_3}, \mathbf{u}\in L_{e_3}\}$; (b) Gyrolines in the Chen's gyrovector plane (projection in the xz-plane of left cosets); (c) Right cosets $(\mathbb{R}^3/D_{e_3},\sim_r) = \{D_{e_3}\oplus_C\mathbf{u}, \mathbf{u}\in L_{e_3}\}$; (d) Cogyrolines in the Chen's gyrovector plane (projection in the xz-plane of right cosets).
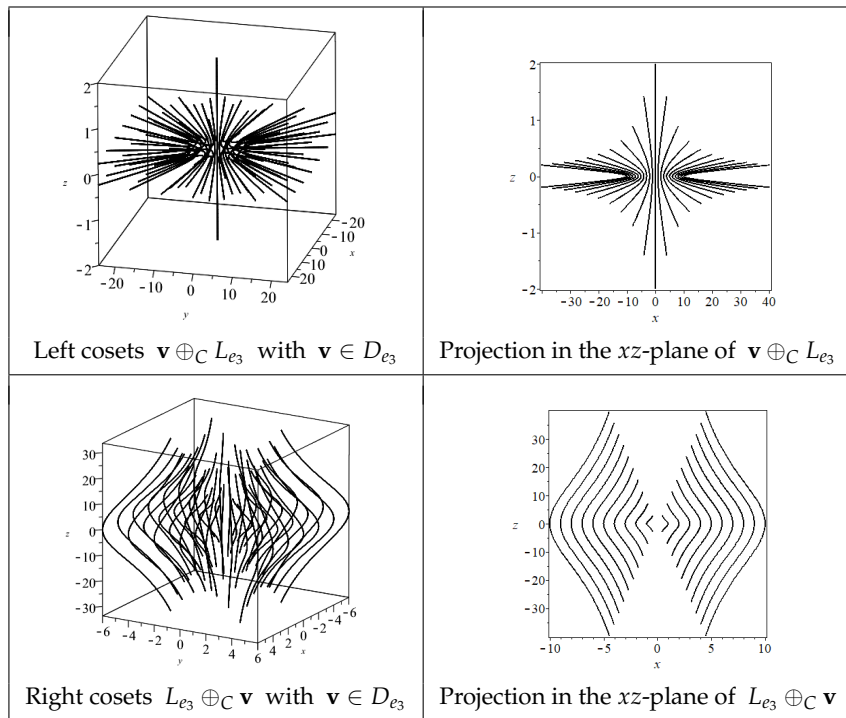
Left cosets $\mathbf{v} \oplus_C L_{e_3}$ with $\mathbf{v} \in D_{e_3}$	Projection in the xz-plane of $\mathbf{v} \oplus_C L_{e_3}$
Right cosets $L_{e_3} \oplus_C \mathbf{v}$ with $\mathbf{v} \in D_{e_3}$	Projection in the xz-plane of $L_{e_3} \oplus_C \mathbf{v}$

Figure 12. (a) Left cosets $(\mathbb{R}^3 / L_{e_3}, \sim_\ell) = \{\mathbf{v} \oplus_C L_{e_3}, \ \mathbf{v} \in D_{e_3}\}$; (b) Gyrolines in the Chen's gyrovector plane (projection in the xz-plane of left cosets); (c) Right cosets $(\mathbb{R}^3 / L_{e_3}, \sim_r) = \{L_{e_3} \oplus_C \mathbf{v}, \ \mathbf{v} \in D_{e_3}\}$; (d) Cogyrolines in the Chen's gyrovector plane (projection in the xz-plane of right cosets).

5. Conclusion

We generalized the study of factorization of Möbius gyrogroups to that of real inner product gyrogroups and proved the Unique Decomposition Theorem. This is the main theorem that leads to other remarkable results proved in this work. It resembles the standard theorem in linear algebra that every inner product space has an orthogonal direct sum decomposition associated to its finite-dimensional subspace. Because of the nonassociativity and the noncommutativity, we defined suitable equivalence relations on real inner product gyrogroups. With the equivalence relations we could partition a real inner product gyrogroup into left and right coset spaces. The four gyrogroups studied to confirm the general theory allow explicit calculations of the gyroprojectors and the left and right cosets. There are several possible applications of our results. We mention, for example, the construction of orthogonal gyroexpansions with respect to an orthogonal basis in a real inner product gyrogroup, or the construction of integral transforms such as the wavelet transform on some manifolds, such as the sphere, the ball, or the hyperboloid (cf. [6,14]). Finally, it would be interesting to generalize these results to complex gyrogroups and to the novel bi-gyrogroups (see [15]), that give a parametrization of generalized Lorentz groups $SO(m,n)$, $m, n \in \mathbb{N}$, in pseudo-Euclidean spaces of signature (m,n).

Author Contributions: All authors contributed equally to this article.

Symbol	Description
Aut G	automorphism group of G
$\beta_{\mathbf{u}}$	beta factor
$\oplus_C$	Chen's addition
$\boxplus$	coaddition
L^c	cogyroline
$\boxminus$	cosubtraction
$\oplus_E$	Einstein addition
$[b]_\ell$	equivalence class containing b determined by the relation $\sim_\ell$
$[b]_r$	equivalence class containing b determined by the relation $\sim_r$
$\sim_\ell$	equivalence relation defined from the left orthogonal gyroprojector $\mathcal{Q}^\ell$
$\sim_r$	equivalence relation defined from the right orthogonal gyroprojector $\mathcal{Q}^r$
gyr $[a, b]$	gyroautomorphism generated by a and b
$\oplus$	gyrogroup addition
$\ominus$	gyrogroup subtraction
L^g	gyroline
$\cdot$	inner product in an arbitrary inner product space
$\langle \cdot, \cdot \rangle$	inner product in $\mathbb{R}^n$
π_ℓ	left canonical projection
$a \oplus P_G$	left coset of P_G with representative a
$(G/P_G, \sim_\ell)$	left coset space of G
L_a	left gyrotranslation
$\mathcal{P}^\ell$	left orthogonal gyroprojector associated with $P_G \oplus P_G^\perp$
$\mathcal{Q}^\ell$	left orthogonal gyroprojector associated with $P_G^\perp \oplus P_G$
$\gamma_{\mathbf{u}}$	Lorentz factor
$\oplus_M$	Möbius addition
$\mathbb{R}^n$	n-dimensional Euclidean space
$\| \cdot \|$	norm of a vector
$\mathbb{B}_t$	open ball in $\mathbb{R}^n$ of radius t
$\mathbb{B}$	open unit ball in $\mathbb{R}^n$
$\mathbb{B}^3$	open unit ball in $\mathbb{R}^3$
$P^\perp$	orthogonal complement in a vector space
D_{e_3}	plane that passes through the origin and is perpendicular to the North Pole
$\oplus_U$	PV addition
π_r	right canonical projection
$P_G \oplus a$	right coset of P_G with representative a
$(G/P_G, \sim_r)$	right coset space of G
$\mathcal{P}^r$	right orthogonal gyroprojector associated with $P_G^\perp \oplus P_G$
$\mathcal{Q}^r$	right orthogonal gyroprojector associated with $P_G \oplus P_G^\perp$
$\otimes$	scalar multiplication in a gyrovector space
L_{e_3}	straight line that passes through the origin and the North Pole
$\oplus_{E_t}$	t-Einstein addition
$\oplus_{M_t}$	t-Möbius addition

Table S1: List of symbols used in the paper.

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